Math 7311, Analysis 1, Homework #12.
Due Monday, Nov 19 at 11:30 in Class

As usually $(X, \mathcal{A}, \mu)$ denotes a measurspace. λ_k stands for the Lebesgue measure on $\mathbb{R}^k$.

1) Let $X = Y = [0,1]$, $\mathcal{A}$ the Borel σ-algebra and $\mathcal{B} = \mathcal{P}([0,1])$. Let λ_1 be the Lebesgue measure on $[0,1]$ and μ the counting measure. Finally let $D = \{(x,x) \mid x \in [0,1]\}$. Show the following:

a) D is measurable.

b) Let $f(x,y) = \mathbf{1}_D(x,y)$. Then

$$\int_X \left(\int_Y f(x,y)\, d\mu(y) \right) d\lambda_1(x) \neq \int_Y \left(\int_X f(x,y)\, d\lambda_1(x) \right) d\mu(y)\,.$$

2) (August 2011) Assume that $\mu(X) = 1$. If g, f are positive measurable functions on X such that $fg \geq 1$ then

$$\int_X f\, d\mu \int_X g\, d\mu \geq 1\,.$$

(Hint: Use Hölder's inequalitiy and # 3 from the last homework.)

3) (From the book, p. 113.) Let $f \in L^1(\mathbb{R}^2, \lambda_2)$.

a) For $n \in \mathbb{N}$ show that

$$F_n(x) = \int_0^1 f(x, y+n)\, d\lambda_1(y)\,.$$

exists for almost all $x \in \mathbb{R}$.

b) Prove that $F_n \in L^1(\mathbb{R}, \lambda_1)$. Determine wether or not the sequence F_n has a limit in $L^1(\mathbb{R}, \lambda_1)$.

4) (From the book, p. 113.) Let $p : \mathbb{R}^n \to \mathbb{R}$ by a polynomial in n real variables. Assume that p is **not** the zero polynomial. Prove that the set $p^{-1}(0)$ is a λ_n-null set. (Hint: Do first $n = 1$ and then use induction using Fubini.)

Homework # 12 - Solutions

1) a) Define $f : [0,1]\times[0,1] \to \mathbb{R}$, $f(x,y) = x - y$. Then f is continuous and hence measurable w.r.t. $\tilde{\mathcal{B}} \otimes \tilde{\mathcal{B}} \subset \mathcal{L} \otimes \mathcal{P}[0,1]$. Here $\tilde{\mathcal{B}}$ is the Borel σ-algebra. It follows that f is measurable. Thus $D = f^{-1}(\{0\})$ is measurable.

b)
$$\int f(x,y)\,d\mu(y) = 1 \quad \text{for all } x \in [0,1]. \text{ Hence}$$
$$\int\int f(x,y)\,d\mu(y)\,d\lambda_1(x) = 1.$$
On the other hand, for fixed y we have $f(x,y) = 0$ for almost all x. Thus
$$\int f(x,y)\,d\lambda_1(x) = 0$$
and
$$\int\left(\int f(x,y)\,d\lambda_1(x)\right)d\mu = \int 0\,d\mu = 0.$$

2) We have $1 \leq (fg)^{1/2} = f^{1/2}g^{1/2} \leq fg$. Hence
$$1 \leq \int f^{1/2}g^{1/2} \leq \left(\int f\right)^{\frac{1}{2}}\left(\int g\right)^{1/2}. \text{ It follows that}$$
$$1 \leq \left[\left(\int f\right)^{1/2}\left(\int g\right)^{1/2}\right]^2 = \int f\,d\mu \int g\,d\mu.$$

3) a) Note first that
$$F_n(x) = \int_n^{n+1} f(x,y)\,d\lambda_1(y) = \int f(x,y)1_{[n,n+1]}(y)\,d\lambda(y)$$
As $|f(x,y)1_{[n,n+1]}(y)| \leq |f(x,y)|$ and f is integrable. It follows that
$$(x,y) \mapsto f(x,y)1_{[n,n+1]}(y)$$
is integrable. Hence Fubini's theorem implies that $F_n(x)$ exists for almost all x.

b) Fubini's theorem implies that $F_n \in L^1$. We also have

$$\int |F_n(x)|\,d\lambda \leq \int\int |f(x,y)| 1_{[n,n+1]}(y)\,d\lambda(y)\,d\lambda(x)$$

$$= \int\int |f(x,y)| 1_{[n,n+1]}(y)\,d\lambda(x)\,d\lambda(y)$$

$$= \int G(y) 1_{[n,n+1]}(y)\,d\lambda(y)$$

where $G(y) = \int |f(x,y)|\,d\lambda(x)$. As G is integrable it follows that for $\varepsilon > 0$ there exist $R > 0$ s.t.

$$\int_{|y| \geq R} G(y)\,d\lambda(y) < \varepsilon \qquad (*)$$

Let $\varepsilon > 0$ and let R be so that $(*)$ holds. Let $M \in \mathbb{N}$, $M \geq R$. Then for $n \geq M$ we have

$$\int |F_n(x)|\,d\lambda(x) \leq \int_{|y| \geq n} G(y)\,d\lambda < \varepsilon.$$

Hence $F_n \to 0$ in L^1.

4) If $n = 1$ then $p^{-1}(0) = \bigcup_{j=1}^{k} \{x_j\}$ is a finite union of points. Hence $\lambda_1(p^{-1}(0)) = 0$. In general we have

$$1_{p^{-1}(0)} = 1_{\{0\}} \circ p$$

Let $n > 1$. For fixed $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ we know that $x_n \mapsto p(x_1, \ldots, x_{n-1}, x_n)$ is a polynomial. Hence

$$\int p(x_1, \ldots, x_{n-1}, x_n)\,d\lambda(x_n) = 0.$$

The claim follows by Fubini because

$$\lambda(p^{-1}(0)) = \int_{\mathbb{R}^n} 1_{\{0\}} \circ p(x_1, \ldots, x_n).$$