Math 7311, Analysis 1, Homework #12.
Due Monday, Nov 19 at 11:30 in Class

As usually $(X, A, \mu)$ denotes a measurspace. $\lambda_k$ stands for the Lebesgue measure on $\mathbb{R}^k$.

1) Let $X = Y = [0, 1]$, $A$ the Borel $\sigma$-algebra and $B = \mathcal{P}([0, 1])$. Let $\lambda_1$ be the Lebesgue measure on $[0, 1]$ and $\mu$ the counting measure. Finally let $D = \{(x, x) | x \in [0, 1]\}$. Show the following:
   a) $D$ is measurable.
   b) Let $f(x, y) = 1_D(x, y)$. Then
      \[
      \int_X \left( \int_Y f(x, y) \, d\mu(y) \right) \, d\lambda_1(x) \neq \int_Y \left( \int_X f(x, y) \, d\lambda_1(x) \right) \, d\mu(y).
      \]

2) (August 2011) Assume that $\mu(X) = 1$. If $g, f$ are positive measurable functions on $X$ such that $fg \geq 1$ then
   \[
   \int_X f \, d\mu \int_X g \, d\mu \geq 1.
   \]
   (Hint: Use Hölder's inequality and #3 from the last homework.)

3) (From the book, p. 113.) Let $f \in L^1(\mathbb{R}^2, \lambda_2)$.
   a) For $n \in \mathbb{N}$ show that
      \[
      F_n(x) = \int_0^1 f(x, y + n) \, d\lambda_1(y).
      \]
      exists for almost all $x \in \mathbb{R}$.
   b) Prove that $F_n \in L^1(\mathbb{R}, \lambda_1)$. Determine whether or not the sequence $F_n$ has a limit in $L^1(\mathbb{R}, \lambda_1)$.

4) (From the book, p. 113.) Let $p : \mathbb{R}^n \to \mathbb{R}$ by a polynomial in $n$ real variables. Assume that $p$ is not the zero polynomial. Prove that the set $p^{-1}(0)$ is a $\lambda_\sigma$-null set. (Hint: Do first $n = 1$ and then use induction using Fubini.)
Define \( F : [0,1] \times [0,1] \rightarrow \mathbb{R} \), \( F(x,y) = x - y \). Then \( F \) is continuous and hence measurable w.r.t. \( \mathcal{B} \otimes \mathcal{B} \cap \mathcal{P}[0,1] \). Here \( \mathcal{B} \) is the Borel \( \sigma \)-algebra. It follows that \( F \) is measurable. Thus \( D = F^{-1}(\{0\}) \) is measurable.

b) \( \int f(x,y) \, d\mu(y) = 1 \) for all \( x \in [0,1] \). Hence
\[
\int \int f(x,y) \, d\mu(y) \, d\lambda_1(x) = 1.
\]
On the other hand, for fixed \( y \) we have \( f(x,y) = 0 \) for almost all \( x \). Thus
\[
\int f(x,y) \, d\lambda_1(x) = 0
\]
and
\[
\int (\int f(x,y) \, d\lambda_1(x)) \, d\mu = \int 0 \, d\mu = 0.
\]
2) We have
\[
1 \leq (\int g) \frac{1}{2} - \frac{1}{2} \int g^2 \leq \int g.
\]
Hence
\[
1 \leq \int g^{\frac{1}{2}} \leq (\int g) \frac{1}{2} (\int g^2) \frac{1}{2}.
\]
It follows that
\[
1 \leq \left[ (\int g) \frac{1}{2} (\int g^2) \frac{1}{2} \right]^2 = \int f \, d\mu \int g \, d\mu.
\]
3) a) Note first that
\[
F_n(x) = \int \int f(x,y) \, d\lambda_1(y) = \int f(x,y) \, d\lambda_1(y) \leq (\int f(x,y) \, d\lambda_1(y)) (\int g(y) \, d\lambda_1(y))
\]
As \( |f(x,y)| \leq (\int f(x,y) \, d\lambda_1(y)) \) and \( f \) is integrable, it follows that
\[
(x,y) \mapsto f(x,y) \, d\lambda_1(y)
\]
is integrable. Hence Fubini's theorem implies that \( F_n(x) \) exists for almost all \( x \).
b) Fubini's theorem implies that \( F_n \in L^1 \). We also have
\[
\int |F_n(x)| \, d\lambda \leq \int \int |f(x,y)| \, \lambda_{\Sigma_{n+1}}(y) \, d\lambda(x) \, d\lambda(y)
\]
\[
= \int \int |f(x,y)| \, \lambda_{\Sigma_{n+1}}(y) \, d\lambda(x) \, d\lambda(y)
\]
where \( G(y) = \int \int |f(x,y)| \, d\lambda(x) \). As \( G \) is integrable it follows that for \( \varepsilon > 0 \) there exist \( R > 0 \) s.t.
\[
\int \int |G(y)| \, d\lambda(x) < \varepsilon \quad \text{(*)}
\]
Let \( \varepsilon > 0 \) and let \( R \) be so that \( (*) \) holds.
Let \( M \in \mathbb{N} \), \( M > R \). Then for \( n > M \) we have
\[
\int |F_n(x)| \, d\lambda(x) < \int G(y) \, d\lambda < \varepsilon.
\]
(\( |y| > R \))

Hence \( F_n \to 0 \) in \( L^1 \).

4) If \( n = 1 \) then \( P^{-1}(0) = \bigcup_{j=1}^{k} \{ x_j \} \) is a finite union of points. Hence \( \lambda_1(P^{-1}(0)) = 0 \). In general we have
\[
P^{-1}(0) = \bigcap_{n \geq 1} \{ x_n \}
\]
Let \( m > 1 \). For fixed \( (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) we know that \( x_n \mapsto P(x_1, \ldots, x_{n-1}, x_n) \) is a polynomial. Hence
\[
\int \mathtt{sp}(x_1, \ldots, x_{n-1}, x_n) \, d\lambda \leq \lambda_2(0) = 0.
\]
The claim follows by Fubini because
\[
\lambda_2(P^{-1}(0)) = \int \lambda_2(0) = 0.
\]