Math 7311, Analysis 1, Midterm 2012

Name: Solutions

If $X \subset \mathbb{R}$ then the σ-algebra will always be the restriction of the Lebesgue σ-algebra to X and the measure λ is the Lebesgue measure. $(X, \mathcal{A}, \mu)$ will stand for a general measurspace. If $f \in \mathcal{L}^+(X)$ and $A \in \mathcal{A}$ then $\int_A f\,d\mu = \int_X f(x)\mathbf{1}_A\,d\mu(x)$.

1) Let $K \subset \mathbb{R}^n$ be compact. Show that $\lambda(K) < \infty$.

Solution: K compact $\Rightarrow$ K is closed and bounded. In particular there exists $[a,b]$ s.t. $K \subseteq [a,b]$. But then

$$\lambda(K) \leq \lambda([a,b]) = b-a < \infty.$$

2) Let $\sigma : \mathcal{A} \to [0,\infty)$ be a finitely additive set function such that $\sigma(X) < \infty$. Show: If $\sigma(\bigcap_n A_n) = 0$ for all decreasing sequences $\{A_n\}$ in $\mathcal{A}$ such that $\bigcap A_n = \emptyset$ then σ is countably additive.

Solution: Let $\{E_n\}$ be a disjoint sequence in $\mathcal{A}$ s.t. $\bigcup E_n \in \mathcal{A}$. Let $A = \bigcup E_n$, $A_m = \bigcup_{n=m+1}^{\infty} E_n = A \setminus \bigcup_{n=1}^{m} E_n$. Then $\{A_m\}$ is decreasing and

$$\bigcap_{n=1}^{\infty} A_n = A \setminus \bigcup_{n=1}^{\infty} E_n = \emptyset.$$

Note that

$$\sigma(A_m) = \sigma(A) - \sum_{n=1}^{m} \sigma(E_n)$$

because $\sigma(X) < \infty$ and σ is finitely additive. It follows that

$$0 = \lim_{m\to\infty} \sigma(A_m) = \sigma(A) - \lim_{m\to\infty} \sum_{n=1}^{m} \sigma(E_n) = \sigma(A) - \sum_{n=1}^{\infty} \sigma(E_n).$$

Thus

$$\sigma\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \sigma(E_n) \quad \blacksquare$$

3) Let $f \in \mathcal{L}^+$. Show that the set-function $\mu_f : \mathcal{A} \to [0, \infty)$. $\mu_f(A) = \int_A f(x)\, d\mu(x)$ is countably additive. (Hint: Recall the monotone convergence theorem for positive functions.)

<u>Solution</u>: Let $\{E_n\}_{n=1}^{m}$ be a disjoint seq. in $\mathcal{U}$. Then

$$\mu_f\left(\bigcup_{n=1}^{m} E_n\right) = \int f 1_{\cup E_n}$$

$$= \int f \sum 1_{E_n} = \sum_{n=1}^{m} \mu_f(E_n)$$

So μ_f is finitely additive.

Let now $\{E_n\}$ be a countable disj. sequence and let

$$f_m = f 1_{\bigcup_{n=1}^{m} E_n} = \sum_{n=1}^{m} f \cdot 1_{E_n} \nearrow f 1_E \leq f \in \mathcal{L}^+$$

where $E = \bigcup_{n=1}^{\infty} E_n$. It follows that

$$\mu_f(E) = \int_E f = \int \lim f_n = \lim_{m \to \infty} \int f_m$$

$$= \lim_{m \to \infty} \sum_{n=1}^{m} \mu_f(E_n)$$

$$= \sum_{n=1}^{\infty} \mu_f(E_n).$$

∎

4) (From the comprehensive exam spring 2012) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to \mathbb{R}$. Show that the set

$$S = \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists }\}$$

is measurable. (Hint: Consider the measurable functions $\liminf f_n$ and $\limsup f_n$ and use them to describe S.)

Solution: (Here we understand $\lim_{n\to\infty} f_n(x)$ exists as $\lim_{n\to\infty} f_n(x) \in \mathbb{R}$.)

Let $F = \overline{\lim}\, f_n$ and $G = \underline{\lim}\, f_n$. Then $\lim f_n(x)$ exists $\iff F(x) = G(x)$. Note, that both F and G are measurable. Define

$$H(x) = \begin{cases} F(x) - G(x) & \text{if } F(x) \text{ and } G(x) \text{ are finite} \\ \infty & \text{if } F(x) \text{ or } G(x) \text{ are not finite} \end{cases}$$

Note that the functions $F(x) 1_{\{x \mid |F(x)| < \infty\}}$ and $G(x) 1_{\{x \mid |G(x)| < \infty\}}$ are measurable and that the set

$$A = \{x \mid |F(x)| = \infty\} \cup \{x \mid |G(x)| = \infty\} \in \mathcal{A}.$$

Thus

$$H(x) = F(x) 1_{\{x \mid |F(x)| < \infty\}} - G(x) 1_{\{x \mid |G(x)| < \infty\}} + \infty 1_A$$

is measurable. It follows that

$$S = H^{-1}(\{0\})$$

is measurable. $\boxtimes$