Math 7311, Analysis 1, preparing for the final and comprehensive exam.

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For both tests study all homeworks. Also for the comprehensive exam, study the midterm and final.

Most important material for the final exam

(1) Integration of functions.
(2) Convergence theorems
(3) Fubini's theorem.

Other material for the final exam

(1) Banach spaces.
(2) Duality $(L^p)^* = L^{p'}$ if $1 \leq p < \infty$.
(3) Functions of bounded variation.
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From the book: 5.11, 5,12 (for one direction use Jensen's inequality for concave functions, for the other direction show that $\sqrt{1+f^2} \leq 1 + f$ is $f \geq 0$) (p. 83), 5.23, 5.25 (p. 88), 5.30 (Show first that the set $\{x \mid f(x) > 1\}$ is a zero-set.), 5.34, 5.35 (p. 92), 5.42, 6.9 (you can assume that g is continuous) , 6.10, 6.11, 6.12 (p. 114), 7.6 (p. 128), 7. 7 (p. 131), 8.3 and 8.4 (p. 156).

Solutions to extra problems

5.11) The function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(x) = \sqrt{x}$, is convex. Use Jensen's inequality.

5.12) Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be the function $\varphi(x) = \sqrt{1+x^2}$. Then

$$\varphi'(x) = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \varphi''(x) = \frac{1}{(1+x^2)^{3/2}} > 0.$$ Thus φ is ~~c~~concave.

Use Jensen's inequality. Next note that

$$(1+f)^2 - (1+f^2) = 2f \geq 0$$

as $f \geq 0$. Hence $\sqrt{1+f^2} \leq 1+f$. Integrating, using that $\int h\,d\mu \leq \int g\,d\mu$ if $h \leq g$, implies the other inequality.

5.23) We note first that $f_n(x) = \frac{x}{n} e^{-\frac{x}{n}} \to 0$ for all x (fixed) so $f \equiv 0$. We have

$$|f_x(x)| \leq a$$

for all $0 < a < \infty$ and $x \in [0,a]$. As every constant function is ~~bounded~~ integrable on a closed interval it follows by LDCT that

$$\lim_{n\to\infty} \int_0^a f_n \, d\lambda = 0 = \int f.$$

[This can also be shown directly by evaluating the integral]

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$$\int_{[0,a]} f_n = -a e^{-a/n} - n e^{-a/n} + n \longrightarrow n, \; a \to \infty.$$

Thus $\int_{\mathbb{R}} f_n = n$ (why?). It follows that

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5.25] Let $f(x,\alpha) = f(x)\cos(\alpha x)$. Then

$$|f(x,\alpha)| \leq |f(x)| \in L^1$$

Thus

$$F(\alpha) = \int f(x)\cos(\alpha x)\, d\lambda = \int f(x,\alpha)\, d\lambda(x)$$

is well defined. We have

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Thus F is differentiable and

$$\frac{dF}{d\alpha}(\alpha) = F'(\alpha) = -\int x f(x) \sin(\alpha x)\, d\lambda(x).$$

5.30] Let $A = \{x \mid f(x) > 1\}$. Then

$$A = \bigcup_{n=1}^{\infty} A_n$$

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If A is not a zero set then there exists m s.t. $\lambda(A_m) = \delta > 0$. But then

$$f^m \geq \left(\frac{m+1}{m}\right)^m 1_{A_m}$$

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5.42) You can use that if $f \in L^1$, $f \geq 0$, then

$$\int f = \sup_{\substack{0 \le g \le f \\ g \text{ simple}}} \int g$$

<u>Claim</u>: Let B be a Banach space. A set $A \subseteq B$ is dense (in B) if for all $b \in B$ and $\varepsilon > 0$ there exists $a = a(\varepsilon)$ s.t. $\|b - a\| < \varepsilon$.

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It follows that

$$\|f - (\varphi^+ - \varphi^-)\|_1 = \int f^+ - \varphi^+ + \int f^- - \varphi^- < \varepsilon.$$

Recall from p. 95 in the book, that a function $f: [a,b] \to \mathbb{R}$ is a step function if there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ s.t. $f|_{(x_{i-1}, x_i)} = c_i$ is a constant. Thus up to a set of measure zero $f = \sum c_j 1_{(x_{j-1}, x_j)}$. Note that $\tilde{f} = \sum c_j 1_{[x_{j-1}, x_j)}$ defines the same class in $L^1[a,b]$. Let $A \subseteq [a,b]$ be measurable and $\varepsilon > 0$. Then there exits a disjoint collection $\{[a_i, b_i)\}$ s.t. $A \subseteq \bigcup [a_i, b_i)$ and

$$\lambda(\bigcup [a_i, b_i)) - \varepsilon \leq \lambda(A) \leq \lambda(\bigcup [a_i, b_i)) = \sum_i (b_i - a_i).$$

This implies (fill in the details): Let $f \in L^1[a,b]$ and $\varepsilon > 0$ then there exists a step function g s.t.

$$\| f - g \|_1 < \varepsilon.$$

Thus the space of step functions is dense in $L^1[a,b]$. Let $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$. Then there exist $R > 0$ s.t.

$$\int_{|x| > R} |f| \, d\lambda < \frac{\varepsilon}{2}.$$

The function $f 1_{[-R,R]}$ is integrable on $[-R,R]$. Hence, there exits a step function $g: [-R,R] \to \mathbb{R}$ such that

$$\| f 1_{[-R,R]} - g \|_1 < \frac{\varepsilon}{2}$$

It follows that (because $g(x) = 0$ if $|x| > R$)

$$\|f-g\|_1 = \int_{|x|>R} |f| + \int_{|x|\leq R} |f-g|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Thus every $f \in L^1$ can be approximated by ~~an~~ step function (over some interval $[-R, R]$, that depends on ε).

Now we can solve the problem. First let $f = 1_{(a,b)}$. Then for $|q| \neq 0$

$$\hat{f}(q) = \int_a^b \sin(qx)\,dx$$ (where we use that $\sin(qx)$ is R-integrable on $[a,b]$)

$$= -\frac{1}{q}\cos(qx)\Big]_a^b$$

$$= \frac{1}{q}\left(\cos(qa) - \cos(qb)\right)$$

Thus $|\hat{f}(q)| \leq \frac{2}{q} \to 0$, $q \to \infty$. It follows that $\hat{f}(q) \to 0$, $q \to \infty$, for arbitrary step functions f (on any interval). Let $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$. Let $R > 0$ be so that

$$\int_{|x| \geq R} |f(x)|\,d\lambda < \frac{\varepsilon}{3}$$

Let $g: [-R, R]$ be a step function s.t.

$$\| f|_{[-R,R]} - g \|_1 < \frac{\varepsilon}{3}.$$

Finally Let $M > 0$ be so that $|\hat{g}(\alpha)| < \frac{\varepsilon}{3}$ for all $|\alpha| \geq M$. Then (using $|\sin(\alpha x)| \leq 1$)

$$|\hat{f}(\alpha)| = |\int f(x) \sin(\alpha x) \, d\lambda|$$

$$\leq |\int_{|x| \geq R} f(x) \sin(\alpha x) \, d\lambda| + |\int_{|x| \leq R} f(x) \sin(\alpha x) \, d\lambda|$$

$$< \frac{\varepsilon}{3} + |\int_{|x| \leq R} (f-g)(x) \sin(\alpha x) \, d\lambda| + |\int_{|x| \leq R} g(x) \sin(\alpha x) \, d\lambda|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \hat{g}(\alpha)$$

$$< \varepsilon \quad \text{if } |\alpha| > M. \quad \boxtimes$$

6.9 a) It is given that we can assume that g is continuous. Then, for each $y \in \mathbb{R}$,

$$\lim_{x \to x_0} |g(x-y) - g(x_0 - y)| = 0$$

As $|f(x)(g(x-y) - g(x_0 - y))| \leq 2M |f(x)| \in L^1$ it follows by LDCT that

$$\lim_{x \to x_0} |f * g(x) - f * g(x_0)| = \lim_{x \to x_0} |\int f(y)(g(x-y) - g(x_0-y)) \, d\lambda(y)|$$

$$\leq \lim_{x \to x_0} \int |f(y)(g(x-y) - g(x_0 - y))| \, d\lambda(y)|$$

$$= \int |f(y)| \lim_{x \to x_0} |g(x-y) - g(x_0-y)| \, d\lambda(y)$$

$$= 0.$$

6.10) a) We have $1_{-A}(x-y) = 1_{A+x}(y)$. Hence if A and B are two sets

$$1_A(y)1_{-B}(x-y) = 1_{A\cap(B+x)}(y).$$

It follows that

$$\exists y: \quad 1_A(y)1_{-B}(x-y) \neq 0 \iff \exists a\in A, b\in B: y=a=b+x$$
$$\iff \cancel{y}\ x = a-b \in A-B.$$

We have

$$1_A * 1_{-A}(x) = \int 1_{A\cap(A+x)}(y)\,d\lambda.$$

In particular

$$1_A * 1_{-A}(0) = \int 1_A = \lambda(A) > 0$$

b) As $1_A * 1_{-A}$ is continuous there exists $\delta>0$ s.t.

$1_A * 1_{-A}(x) > 0$ for all $|x| \leq \delta$. But then

$$(-\delta, \delta) \subseteq A-A$$

6.11) Assume first that $\lambda(A) < \infty$. Then $\lambda(-A)<\infty$ &

$$1_{-A} * 1_B(\cancel{y}) = \int 1_{-A}(y)1_{-B+x}(y)\,d\lambda < \lambda(-A) < \infty.$$

By Tonelli's theorem

$$\int 1_{-A} * 1_B(x)\,d\lambda(x) = \int\int 1_{-A}(y)1_{+B}(x-y)\,d\lambda(y)\,d\lambda(x)$$
$$= \int 1_{-A}(y)\int 1_B(x-y)\,d\lambda(x)\,d\lambda(y)$$
$$= \lambda(-A)\lambda(B)\left(\cancel{<\infty}\right)$$
$$> 0$$

If $\lambda(A)=\infty$, then ~~there~~ exists $R>0$ s.t.

$0<\lambda([-R,R]\cap A) < \infty$. But the $\cancel{\exists q}$ $\exists q \in A\cap[-R,R]+B$

$\subseteq -A+B$

Thus there exists x_0 s.t. $1_{-A} * 1_B(x_0) > 0$. As $1_{-A} * 1_B$ is continuous there exists $\delta > 0$ s.t. $1_{-A} * 1_B(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. But then there exists $q \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{Q}$ s.t. $1_{-A} * 1_B(q) > 0$. That implies that

$$q \in -A + B.$$

If $\lambda(A) = \infty$, then there exists $R > 0$ s.t. $0 < \lambda(A \cap [-R, R]) < \infty$. But then the first part shows that there exists $q \in \mathbb{Q}$ s.t.

$$q \in B - A \cap [-R, R] \subseteq B - A.$$

6.12) It follows by Fubini's theorem that

$$x \mapsto g(x,y) = f(x) - f(y)$$

is integrable for almost all y. Let y be so that $x \mapsto f(x) - f(y)$ is integrable. As $\mu(X) < \infty$ it follows that $x \mapsto f(y)$ is integrable. Hence

$$f = (f - f(y)) + f(y)$$

is integrable. By Fubini's theorem we have

$$\int g \, d\mu \otimes \mu = \int_X \left(\int_X f(x) - f(y) \, d\mu(y) \right) d\mu(x)$$

$$= \int_X f(x)\mu(X) - \left(\int f(y) \, d\mu(y) \right) d\mu(x)$$

$$= \left(\int f \right) \mu(X) - \left(\int f \right) d\mu(X)$$

$$= 0.$$

7.6) Let $f = 1_{[0,1] \cap \mathbb{Q}}$. Then $\int_{[0,x]} f \, d\lambda \equiv 0$. Hence $\frac{d}{dx} \int_{[0,x]} f \equiv 0$.

~~7.7~~)