Math 7311, Analysis 1, Final fall 2012

Name: Solutions

You are allowed to use that if $f \in L^1(\mathbb{R}^d)$ and $r \in \mathbb{R} \setminus \{0\}$, then $r^d \int_{\mathbb{R}^d} f(rx)\, d\lambda(x) = \int_{\mathbb{R}^d} f(x)\, d\lambda(x)$ (but after the test, think about why that is correct). You can also use that if $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$ then $f * g$ is continuous where $f * g(x) = \int_{\mathbb{R}^d} f(y) g(x-y)\, d\lambda(y)$ is the convolution of f and g.

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1) Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and $g \in \mathcal{L}^+(\mathbb{R}^d)$. Assume further that $\int_{\mathbb{R}^d} g(x)\, d\lambda(x) = 1$ and that $g(x) = 0$ for $\|x\| \geq 1$. Show that

$$\lim_{n\to\infty} n^d \int_{\mathbb{R}^d} g(nx) f(x)\, d\lambda(x)$$

exist and find the limit.

Solution: We have

$$n^d \int_{\mathbb{R}^d} g(nx) f(x)\, d\lambda = \int_{\mathbb{R}^d} g(x) f\left(\tfrac{x}{n}\right) d\lambda$$

$$= \int_{\|x\| \leq 1} g(x) f\left(\tfrac{x}{n}\right) d\lambda .$$

If $\|x\| \leq 1$, then $\|\frac{x}{n}\| \leq \frac{1}{n} \leq 1$. As f is continuous it follows that f is bounded on $\{x \mid \|x\| \leq 1\}$, say

$$|f(x)| \leq M, \quad \|x\| \leq 1.$$

Then $|g(x) f(\frac{x}{n})| \leq g(x)$ and $g \in L^1$. By LDCT we get

$$\lim_{n\to\infty} n^d \int_{\mathbb{R}^d} g(nx) f(x)\, d\lambda = \int \lim g(x) f\left(\tfrac{x}{n}\right) d\lambda(x)$$

$$= f(0) \int g(x)\, d\lambda$$

$$= f(0) \quad \blacksquare$$

2) Let $f \in L^1(\mathbb{R}^+)$. Define $F : (0,\infty) \to \mathbb{R}$ by

$$F(t) = \int_0^\infty e^{-tx} f(x)\, d\lambda(x)\,.$$

Show by induction that all the derivatives $F^{(n)}$, $n \in \mathbb{N}$ exists.

Solution:

Let $t_0 > 0$. Let $t_0 > \varepsilon > 0$. Then $|(-x)^n e^{-tx}| \leq$

$\leq x^n e^{-\varepsilon x}$ ($\leftarrow$ bounded on $[\varepsilon, \infty)$)

$\leq M_n$

Hence $x \mapsto |(-x)^n e^{-tx} f(x)| \leq M_n |f(x)|$ and $x \mapsto M_n |f(x)|$ is integrable. Let $F_n(x,t) = (-x)^n e^{-tx} f(x)$.

Then $x \mapsto F_n(x,t)$ is integrable,

$$\frac{\partial}{\partial t} F_n(x,t) = (-x)^{n+1} e^{-tx} f(x)$$
$$= F_{n+1}(x,t)$$

so $|\frac{\partial}{\partial t} F_n(x,t)| \leq M_{n+1} |f(x)| \in L^1$. Thus

$$t \longmapsto \int^\infty (-x)^n e^{-tx} f(x)\, d\lambda = F_n(t)$$

is differentiable and

$$F_n'(t) = \int_0^\infty \frac{\partial}{\partial t} (-x) e^{-tx} f(x)\, d\lambda$$

$$= \int_0^\infty F_{n+1}(x,t)\, d\lambda(x)\,.$$

Induction shows that

$$F^{(n+1)}(t) = \frac{\partial}{\partial t} F_n(t)$$

and hence all the derivatives exists and

$$F^{(n)}(t) = \int_0^\infty (-x)^n e^{-tx} f(x)\, d\lambda(x)$$

3) Let $A, B \subset \mathbb{R}^d$ be measurable such that $0 < \lambda(A)\lambda(B) < \infty$. Show that there exists $z \in \mathbb{R}^d$ such that $\lambda(A \cap (B+z)) > 0$. (Hint: Write $1_A * 1_{-B}(x)$ in terms of $1_{A\cap(B+x)}$ and use Fubini to show that $1_A * 1_{-B} \neq 0$.)

Solution First we note that

$$1_{-B}(x-y) = 1_{B+x}(y)$$

$(x-y \in -B \Leftrightarrow y \in B+x)$. Hence

$$1_A * 1_{-B}(x) = \int 1_A(y) 1_{-B}(x-y)\,d\lambda(y)$$
$$= \int 1_A(y) 1_{B+x}(y)\,d\lambda(y)$$
$$= \int 1_{A\cap(B+x)}(y)\,d\lambda(y)$$
$$= \lambda(A\cap(B+x))$$

By Tornelli/Fubini we have

$$\int 1_A * 1_{-B}(x)\,d\lambda(x) = \int \lambda(A\cap(B+x))\,d\lambda(x)$$
$$= \int\int 1_A(y) 1_{-B}(x-y)\,d\lambda(y)\,d\lambda(x)$$
$$= \int 1_A(y) \int 1_{-B}(x-y)\,d\lambda(x)\,d\lambda(y)$$
$$= \int 1_A(y) \underbrace{\lambda(-B)}_{=\lambda(B)}\,d\lambda(y)$$
$$= \lambda(A)\lambda(B) > 0.$$

It follows that $x \mapsto \lambda(A\cap(B+x))$ is > 0 for some x (in fact for x in a set of positive measure.)

4) Let $f \in L^4([0,1])$. Show that $\int_{[0,1]} \frac{f(x)}{x^{1/4}} \, d\lambda(x)$ is finite.

Solution: Let q be such that

$$\frac{1}{4} + \frac{1}{q} = 1.$$

Then $q = 4/3$. We have

$$\int_{[0,1]} \left(x^{-1/4}\right)^{4/3} d\lambda(x) = \int_{[0,1]} x^{-1/3} \, d\lambda(x)$$

$$= 3/2 < \infty$$

Hence, with $g(x) = x^{-1/4}$ we have $g \in L^{4/3}[0,1]$.
It follows by Hölder's ineq. that

$$\left| \int_{[0,1]} f(x) g(x) \, d\lambda(x) \right| \leq \left(\int_{[0,1]} |f(x)|^4 d\lambda \right)^{\frac{1}{4}} \left(\int_{[0,1]} |g(x)|^{4/3} d\lambda \right)^{\frac{3}{4}}$$

$$< \infty.$$