Math 7311/Fall 2012, Real Analysis 1

Homework set 1, due Monday, August 27, 2012

From the book: 2.1, 2.4, 2.7 and

4) Show that the following subsets of $\mathbb{R}$ generate the same σ-algebra:

1. $\mathcal{A}_1 = \{(a,b) \mid -\infty < a < b < \infty\}$
2. $\mathcal{A}_2 = \{[a,b) \mid -\infty < a < b < \infty\}$
3. $\mathcal{A}_3 = \{(a,\infty) \mid a \in \mathbb{R}\}$
4. $\mathcal{A}_4 = \{(-\infty,a] \mid a \in\}$

Hint: Show that $\mathcal{A}_2 \subset \sigma(\mathcal{A}_1)$ etc. where $\sigma(\mathcal{A}_j)$ denotes the σ-algebra generated by $\mathcal{A}_j$.

Solutions #1

1) (Problem 2.1) We have to show that if we define an algebra to be a non-empty collection of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ then closed under unions and taking complements implies it is also closed under taking an union, $A, B \in \mathcal{A}$, implies $A \cup B \in \mathcal{A}$. But $A \cup B = [A^c \cap B^c]^c \in \mathcal{A}$.

2) (Problem 2.4) Let $f: X \to Y$. Let $\mathcal{A} \subseteq \mathcal{P}(Y)$ be a σ-algebra. Then

$$f^{-1}(\mathcal{A}) = \{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a σ-algebra. [This was done in class.]

Solution: Let $\{E_j\}$ be a sequence in $f^{-1}(\mathcal{A})$. Let $B_j \in \mathcal{A}$ be such that $E_j = f^{-1}(B_j)$. Then

$$\bigcup_i E_j = \bigcup_j f^{-1}(B_j) = f^{-1}(\bigcup_i B_j) \in f^{-1}(\mathcal{A}).$$

Let $A = f^{-1}(B) \in f^{-1}(\mathcal{A})$. Then $A^c = f^{-1}(B^c) \in f^{-1}(\mathcal{A})$ ■

3) (Problem 2.7) Given an example of an infinite set X and $\mathcal{A} \subset \mathcal{P}(X)$ such that $\mathbb{M}(\mathcal{A})$ is not a σ-algebra.

Solution: Let $X = \mathbb{R}$ and let $\mathcal{A}$ be the collection of all finite or countably infinite subsets of X.

4) We will show that

$$\sigma(\mathcal{A}_1) \supseteq \sigma(\mathcal{A}_2) \supseteq \sigma(\mathcal{A}_3) \supseteq \sigma(\mathcal{A}_4) \supseteq \sigma(\mathcal{A}_5).$$

Then all of them have to be the same.

Solutions #1

- We have $[a,b) = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b)$. Hence $[a,b) \in \sigma(\mathcal{A}_1)$
It follows that the σ-algebra $\sigma(\mathcal{A}_2)$ generated by the intervals of the form $[a,b)$ is contained in $\sigma(\mathcal{A}_1)$.

- Let $a \in \mathbb{R}$. Then
$$(a,\infty) = \bigcup_{n=1}^{\infty} [a+\tfrac{1}{n}, a+n) \in \sigma(\mathcal{A}_2).$$

- We have
$$(-\infty, a] = \mathbb{R} \setminus (a,\infty)$$

- Let $a<b$. Then $(a,b] = (-\infty, b] \setminus (-\infty, a] \in \sigma(\mathcal{A}_4)$.
~~But~~ then we also have
$$(a,b) = \bigcup_{\substack{n=1 \\ (b-\frac{1}{n}>a)}}^{\infty} (a, b-\tfrac{1}{n}] \in \sigma(\mathcal{A}_4)$$

and the claim follows.

Math 7311, Analysis 1, Homework due Wednesday, Sept 9, 11:30

If $\mathcal{A}$ is a σ-algebra then by a measure on $\mathcal{A}$ we mean a countable additive measure.

1) Let X be a non-empty set and $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. Let $\emptyset \neq Y \subset X$. Define

$$\mathcal{A}_Y = \{A \cap Y \mid A \in \mathcal{A}\}.$$

Show that $\mathcal{A}|_Y$ is a σ-algebra on Y.

2) Let X and Y be as above. Let $\mathcal{B}$ be a σ-algebra on Y. Define

$$\mathcal{B}^X = \{E \subset X \mid E \cap Y \in \mathcal{B}\}.$$

Show that $\mathcal{B}^X$ is a σ-algebra.

3) We use the same notation as in exercise 2. Let μ be a measure on $\mathcal{B}$. Show that μ_X define by $\mu^X(A) = \mu(A \cap Y)$ is measure defined on $\mathcal{B}^X$.

4) Let $X = \mathbb{N}$, the set of natural numbers. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers (not all equal to zero). Let $\mathcal{A} = \mathcal{P}(X)$ and define $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+ = [0,\infty]$ by

$$\mu(A) = \sum_{n\in A} a_n .$$

Show that μ is a σ-finite measure.

5) Exercise 2.14

Solutions #2

1) Let $\{A_j\}_{j=1}^{\infty}$ be a sequence in $\mathcal{A}_Y$. Take $B_j \in \mathcal{A}$ s.t. $A_j = Y \cap B_j$. Then $Y \cap \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} Y \cap B_j = \bigcup_{j=1}^{\infty} A_j$.

As $\bigcup_{j=1}^{\infty} B_j \in \mathcal{A}$ it follows that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}_Y$.

Let $A = Y \cap B \in \mathcal{A}_Y$. Then $Y \setminus A = Y \cap (X \setminus B) \in \mathcal{A}_Y$.

2) Let $\{E_j\}$ be a sequence in $\mathcal{B}^X$. Then

$$Y \cap \left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcup_{j=1}^{\infty} Y \cap E_j \in \mathcal{B}$$

Hence $\bigcup_{j=1}^{\infty} E_j \in \mathcal{B}^X$. Let $E \in \mathcal{B}^X$. Then

$$Y \cap (X \setminus E) = Y \setminus E \cap Y \in \mathcal{B}$$

Thus $X \setminus E \in \mathcal{B}^X$.

3) It is clear that $\mu^X(A) \geq 0$. Let $\{E_j\}_{j=1}^{\infty}$ be a disjoint sequence in $\mathcal{B}^X$. Then

$$\mu^X\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\left(\bigcup_{j=1}^{\infty} E_j\right) Y\right) \quad \text{(definition)}$$

$$= \mu\left(\bigcup_{j=1}^{\infty} (E_j \cap Y)\right)$$

$$= \sum_{j=1}^{\infty} \mu(E_j \cap Y) \quad (\mu \text{ a measure})$$

$$= \sum_{j=1}^{\infty} \mu^X(E_j).$$

4) It is clear that $\mu(A) \geq 0$ because $a_j \geq 0$.
Let $\{E_j\}$ be a disjoint sequence in X. Then, because each $a_j \geq 0$, so the order of summation does not matter:

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{n \in \bigcup_{j=1}^{\infty} E_j} a_n = \sum_{j=1}^{\infty} \sum_{n \in E_j} a_n$$

$$= \sum_{j=1}^{\infty} \mu(E_j) \quad \square$$

The measure is σ-finite because

$$\mathbb{N} = \bigcup_{n=1}^{\infty} \{n\}$$

and $\mu(\{n\}) = a_n < \infty$. ∎

5) Prove that every infinite σ-finite countably additive measure μ is approximately finite. Thus we have to show: If $A \in \mathcal{A}$, $\mu(A) = \infty$ and $M > 0$, then there exists $B \subset A$ such that $M < \mu(B) < \infty$.

<u>Solution</u>: Let $\{X_j\}$ be a sequence in $\mathcal{A}$ such that $X = \bigcup_{j=1}^{\infty} X_j$ and $\mu(X_j)$. We can (as pointed out in class) use that the sets X_j are disjoint. Let $Y_N = \bigcup_{j=1}^{N} X_j \in \mathcal{A}$. Then $\{Y_N\}$ is an increasing sequence and hence $\{\mu(Y_N)\}$ is an increasing sequence of positive numbers with $\mu(Y_N) \nearrow \infty$. Now let A and M be as above. Then

$$\mu(A) = \sum_{j=1}^{\infty} \mu(A \cap X_j) = \lim_{N \to \infty} \mu(A \cap Y_N).$$

It follows that $\{\mu(A \cap Y_N)\}$ is an increasing sequence with $\mu(A \cap Y_N) \nearrow \infty$. In particular there exists N_0 s.t. for all $N \geq N_0$ we have

$$\mu(A \cap Y_N) > M$$

Let $B = A \cap Y_{N_0}$. Then $B \subseteq A$ and

$$M < \mu(B) \leq \sum_{j=1}^{N_0} \mu(X_j) < \infty$$

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Homework set 3, due Monday, Sept 10, 2012, at 11:30

1) Number 2.19 from the book.

2) Let $\mathcal{A}$ be an algebra on a set X and let μ be a finitely additive measure on $\mathcal{A}$. Assume that $\mu(X) < \infty$. Then the following are equivalent:

1. μ is countably additive.
2. For all decreasing sequences $\{A_j\}$ such that $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$ we have
$$\mu(\bigcap A_j) = \lim_{j\to\infty} \mu(A_j)\,.$$
3. For all increasing sequences $\{B_j\}$ in $\mathcal{A}$ such that $\bigcup_{j=1}^{\infty} B_j \in \mathcal{A}$ we have
$$\mu(\bigcup B_j) = \lim_{j\to\infty} \mu(B_j)\,.$$

(Explaine why those limits exists.)

3) Suppose $\{A_n\}$ is a sequence of sets. Define

$$\lim\sup(A_n) = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$$

and

$$\lim\inf(A_n) = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m.$$

If A$=$ $\lim\sup(A_n) = \lim\inf(A_n)$ we say the sequence $< A_n >$ converges to A and write $\lim_{n\to\infty} A_n = A$. In the following $(X, \mathcal{A}, \mu)$ will stand for a measure space and $\{A_n\}$ will always stand for a sequence of sets in $\mathcal{A}$.

1. Show that
$$\lim\sup(A_n) = \{x \mid x \in A_n \text{ for infinitely many } n\}$$
and
$$\lim\inf(A_n) = \{x \in A_n \text{ for all but finitely many } n\}$$
2. Show that $\lim\sup(A_n), \lim\inf(A_n) \in \mathcal{A}$.
3. Suppose $A_n \subset A_{n+1}$ for each n. Show that A_n converges and $\lim \mu(A_n) = \mu(\lim A_n)$.
4. If $\mu(\cup_{n=1}^{\infty}(A_n)) < \infty$ then $\mu(\lim\sup(A_n)) \geq \lim\sup \mu(A_n)$.
5. If $< A_n >$ converges and $\mu(\cup_{n=1}^{\infty} A_n) < \infty$, then $\mu(\lim A_n) = \lim \mu(A_n)$.

Solutions # 3

(1) Let μ be a finitely additive measure on the algebra $\mathcal{A}$ such that $\mu(X) < \infty$. Show that μ is countably additive if and only if for all decreasing sequences $\{A_j\}$ in $\mathcal{A}$ such that $\bigcap_{j=1}^{\infty} A_j = \emptyset$ we have $\lim_{j\to\infty} \mu(A_j) = 0$.

Solution We first note the following: As $X = A \dot{\cup} (X \setminus A)$ and $\mu(X) < \infty$ it follows that

$$\mu(X) = \mu(A) + \mu(X \setminus A)$$

or

$$\mu(X \setminus A) = \mu(X) - \mu(A).$$

Similarly $\mu(B \setminus A) = \mu(B) - \mu(A)$ for all $A \subseteq B$, $A, B \in \mathcal{A}$. Assume that μ is countably additive. Let $\{E_j\}$ be a decreasin sequence. Define $A_j = E_j \setminus E_{j+1}$. Then $E_1 = \dot{\bigcup}_{j=1}^{\infty} A_j$. It follows that

$$\begin{aligned}
\mu(E_1) &= \sum_{j=1}^{\infty} \mu(A_j) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \mu(E_j - E_{j+1}) \\
&= \lim_{N\to\infty} \left(\mu(E_1) - \mu(E_{N+1})\right) \\
&= \mu(E_1) - \lim_{N\to\infty} \mu(E_{N+1})
\end{aligned}$$

Hence (as $\mu(E_1) < \infty$) we must have

$$\lim_{N\to\infty} \mu(E_N) = 0.$$

Now assume that $\lim \mu(E_j) = 0$ for all decreasing sequences with $\bigcap E_j = \emptyset$. Let $\{A_j\}$ be a disjoint sequence of sets in $\mathcal{A}$ such that $\bigcup_j A_j = E \in \mathcal{A}$. Let

$$E_j = E \setminus \bigcup_{\nu=1}^{j} A_j$$

Then $\{E_j\}$ is a decreasin sequence with $\bigcap_{j=1}^{\infty} E_j = \emptyset$. Furthermore, as μ is finitely additive and $\mu(E) < \infty$,

$$0 = \lim_{N\to\infty} \mu(E_N) = \lim_{N\to\infty} \left(\mu(E) - \mu\left(\bigcup_{\nu=1}^{N} A_\nu\right)\right)$$

$$= \mu(E) - \lim_{N\to\infty} \sum_{\nu=1}^{N} \mu(A_\nu))$$

Thus

$$\mu(E) = \mu\left(\bigcup_{\nu=1}^{\infty} A_\nu\right) = \lim_{N\to\infty} \sum_{\nu=1}^{N} \mu(A_\nu) = \sum_{\nu=1}^{\infty} \mu(A_\nu)$$

So μ is countably additive.

2) We show that "(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)".

"(1) $\Rightarrow$ (2)". Let $\{A_j\}$ be a decrasing sequense in $\mathcal{A}$ such that $A = \bigcap_\nu A_\nu \in \mathcal{A}$. Let

$$E_j = A_j \setminus A_{j+1}.$$

Then $\{E_j\}$ is a disjoint sequence with

$$\bigcup_{j=1}^{\infty} E_j = A_1 \setminus A.$$

Hence

$$\mu(A_1) - \mu(A) = \sum_{j=1}^{\infty} \mu(A_j) - \mu(A_{j+1})$$

$$= \lim_{N\to\infty} (\mu(A_1) - \mu(A_{N+1}))$$

$$= \mu(A_1) - \lim_{N\to\infty} \mu(A_{N+1})$$

or $\mu(A) = \lim_{N\to\infty} \mu(A_N)$.

[Note that the limit exists because $\{\mu(A_j)\}$ is a decreasing sequence and bounded ≥ 0.]

"(2) $\Rightarrow$ (3)" Let $\{B_j\}$ be an increasing sequence with $B = \cup B_j \in \mathcal{A}$. Note that $0 \le \mu(B_1) \le \mu(B_2) \le \cdots \le \mu(X) < \infty$. Hence $\lim \mu(B_j)$ exists and is $\le \mu(B)$ because $B_j \subseteq B$ for all j. Let $A_j = X \setminus B_j$. Then $\{A_j\}$ is a decreasing sequence with

$$\bigcap_{j=1}^{\infty} A_j = X \setminus \bigcup_{j=1}^{\infty} B_j .$$

Hence

$$\begin{aligned}\mu(X-B) &= \mu(X) - \mu(B)\\ &= \lim_{N\to\infty} \mu(A_N)\\ &= \lim_{N\to\infty} \mu(X-B_N)\\ &= \lim_{N\to\infty} [\mu(X) - \mu(B_N)]\\ &= \mu(X) - \lim_{N\to\infty} \mu(B_N).\end{aligned}$$

Hence $\mu(B) = \lim_{N\to\infty} \mu(B_N)$.

"(3) $\Rightarrow$ (1)" Let $\{E_j\}$ be a disjoint sequence in $\mathcal{A}$ such that $E = \dot{\cup} E_j \in \mathcal{A}$. Let $B_N = \bigcup_{j=1}^{N} E_j$. Then $\{B_N\}$ is an increasing sequence with $\cup B_N = \cup E_j$. By (3) and the fact that μ is finitely additive, we get

$$\begin{aligned}\mu(\cup E_j) = \lim_{N\to\infty} \mu(B_N) &= \lim_{N\to\infty} \sum_{j=1}^{N} \mu(E_j)\\ &= \sum_{j=1}^{\infty} \mu(E_j). \quad \blacksquare\end{aligned}$$

8) Let $B_n = \bigcup_{m=n}^{\infty} A_m$. Then $\{B_n\}$ is a decreasing sequence and

$$\limsup A_n = \overline{\lim}\, A_n = \bigcap_{j=1}^{\infty} B_j .$$

Let $C_n = \bigcap_{m=n}^{\infty} A_m$. Then $\{C_n\}$ is increasing and

$$\liminf C_n = \underline{\lim}\, A_n = \bigcup_{\nu=1}^{\infty} C_\nu .$$

Let $\{a_n\}$ be a sequence of real numbers. Construct two new sequences

$$b_n = \sup_{m \geq n} \{a_m\}, \quad c_n = \inf_{m \geq n} \{a_n\}$$

Then $\{b_n\}$ is decreasing and $\{c_n\}$ is increasing. Hence the following two limits exist in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

$$\limsup a_n = \overline{\lim}\, a_n = \lim b_n$$
$$\liminf a_n = \underline{\lim}\, a_n = \lim c_n .$$

Now let $\{A_n\}$ be as in Exercise 3.

(i) Let $x \in \overline{\lim}\, A_n$. Then $x \in B_n$ for all n. Hence, for all $n \in \mathbb{N}$ there exists $m \geq n$ s.t. $x \in A_m$ hence $x \in \{y \mid y \in A_m$ for infinitely many $m\}$.

Similarly, let $x \in \{y \mid y \in A_m$ for infinitely many $m\}$ Let $n \in \mathbb{N}$. Then there exist $m \geq n$ such that $x \in A_m$. Hence $x \in B_n$. Thus $x \in \bigcap B_n = \overline{\lim}\, A_n$.

Solution #3

Let $B = \{x \mid x \in A_n$ for all but finitely many $n\}$.
Take $x \in X$. Then

$$x \in \underline{\lim}\, A_n \iff \exists m : x \in B_m$$
$$\iff \forall n \geq m : x \in A_n$$
$$\iff x \in B. \quad \blacksquare$$

(2) We have $B_m, C_m \in \mathcal{A}$ for all m, because $\mathcal{A}$ is closed under countable union and intersection. The same argument shows that

$$\overline{\lim}\, A_n = \bigcap_{j=1}^{\infty} B_j, \quad \underline{\lim}\, A_n = \bigcup_{j=1}^{\infty} C_j \in \mathcal{A}.$$

(3) If $\{A_n\}$ in increasing then

- $B_n = \bigcup_{m=n}^{\infty} A_m = \bigcup_{m=1}^{\infty} A_m$ for all n
- $C_n = \bigcap_{m=n}^{\infty} A_m = A_n$. Thus $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$

It follows that $\bigcap_{n=1}^{\infty} B_n = \bigcup_{m=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} C_n$. Or

$$\overline{\lim}\, A_n = \underline{\lim}\, A_n$$

As $\{A_n\}$ is increasing it follows that $\{\mu(A_n)\}$ is an increasing sequence and hence converges in $\overline{\mathbb{R}}$. Furthermore, as $A_n \subseteq \bigcup_{n=1}^{\infty} A_m$ for all n, $\mu(A_n) \leq \mu(\bigcup_{m=1}^{\infty} A_m)$
It follows that

$$\lim \mu(A_n) \leq \mu\left(\bigcup_{m=1}^{\infty} A_m\right).$$

In particular, if $\lim \mu(A_n) = \infty$, then $\mu(\cup A_n) = \infty$.
Assume now that $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$. Let $Y = \bigcup_{n=1}^{\infty} A_n$, and consider the σ-algebra (on Y)

$$\mathcal{A}_Y = \{B \cap Y \mid B \in \mathcal{A}\}$$

with the measure $\mu_Y(B\cap Y) = \mu(B \cap Y)$. Then μ_Y is finite and $\mu_Y(A_n) = \mu(A_n)$ for all n.
The claim that $\lim \mu(A_n) = \mu(\cup A_n)$ follows now from exercise 2, part 2.

(4) If $m \geq n$ then $A_m \subseteq B_n$ and hence

$$a_m = \mu(A_m) \leq \mu(B_n)$$

Hence $$\sup_{m \geq n} a_m = \sup_{m \geq n} \mu(A_m) \leq \mu(B_n).$$

As in (3) let $Y = \bigcup_{n=1}^{\infty} A_n$ and consider the σ-algebra $\mathcal{A}_Y$ with the finite measure μ_Y.
Then $\{B_n\}$ is a decreasing sequence in Y.
By exercise 2, part 2 we have

$$\mu(\overline{\lim} A_n) = \mu(\cap B_n) = \lim \mu(B_n)$$
$$\geq \lim_m a_m$$
$$= \overline{\lim}\, \mu(A_m).$$

(5) By replacing X by $Y = \cup A_n$ we can as in (3) & (4) assume that $\mu(X) < \infty$.

We have $C_n \subseteq A_m$ for all $m \geq n$. Thus

$$\mu(\cap C_n) \leq \mu(C_n) \leq \inf_{m \geq n} \mu(A_m)$$

It follows that

$$\mu(\underline{\lim}\, A_n) \leq \liminf \mu(A_n)$$
$$\leq \limsup \mu(A_n)$$
$$\leq \mu(\overline{\lim}\, A_n) \text{ (by 4)}.$$

Thus $\mu(\lim A_n) = \underline{\lim}\, \mu(A_n) = \overline{\lim}\, \mu(A_n)$.
In particular $\{\mu(A_n)\}$ is convergent with $\lim \mu(A_n) = \mu(\lim A_n)$. ∎

Math 7311, Analysis 1, Fourth Home Work Set, Due Monday, Sept 17, at 11:30 in Class

1) (See the textbook with hint on page 27) Recall: Let $\mathcal{A}$ be an algebra in $\mathcal{P}(X)$. An outer measure μ on $\mathcal{P}(X)$ is *outer regular* with respect to $\mathcal{A}$ if for all $S \subset X$ there exists $B \in \sigma(A)$ such that $\mu(S) = \mu(B)$. Show the following: Let $\mathcal{A}$ be an algebra on a non-empty set X and let μ be a countably additive measure on $\mathcal{A}$. Show that the outer measure μ^* constructed in class is outer regular.

2) Problem 3.2, p. 35

3) Problem 3.3, p. 35

Solutions #4

1) Let $S \in \mathcal{P}(X)$. If $\mu^*(S) = \infty$ then $\mu(X) = \infty$ and we can take $B = X$. Assume therefore that $\mu^*(S) < \infty$. Denote by μ (as usually) the restriction of μ^* to $\sigma(\mathcal{A})$. Let $\{E_j\}$ be a sequence in $\mathcal{A}$ such that $S \subseteq \bigcup E_j$. Let $F_1 = E_1$ and $F_{j+1} = E_{j+1} \setminus \bigcup_{\nu=1}^{j} E_\nu$. Then $\{F_j\}$ is a disjoint seq. such that

- $S \subseteq \bigcup_j F_j = \bigcup_j E_j \in \sigma(\mathcal{A})$
- $\mu(F_j) \leq \mu(E_j)$
- $\mu(\bigcup F_j) = \sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$

It follows that

$$\mu^*(S) = \inf\{\mu(F) \mid F \in \sigma(\mathcal{A}), S \subseteq F\}.$$

As $\mu^*(S) < \infty$ we can for each $n \in \mathbb{N}$ find $B_n \in \sigma(\mathcal{A})$ such that

- $S \subseteq B_n$
- $\mu(B_m) \leq \mu^*(S) + \frac{1}{m} < \infty$.

By repacing B_n by $C_n = \bigcap_{\nu=1}^{m} B_\nu \supseteq S$ we get a decreasing sequence in $\sigma(\mathcal{A})$ s.t. $S \subseteq \bigcap_{n=1}^{\infty} C_n$. Furthermore

$$\mu^*(S) \leq \lim_{n\to\infty} \mu(C_n) \leq \lim_{n\to\infty} \left(\mu^*(S) + \frac{1}{n}\right) = \mu^*(S)$$

Solution #4

Note that $\lim_{n\to\infty} \mu(C_n) = \mu(\bigcap_{n=1}^{\infty} C_n) = \mu^*(S)$.
As $\bigcap_{n=1}^{\infty} C_n = B \in \sigma(\mathcal{A})$ we can use this B.

2) Show that every open/closed sets in $[-N,N)$ is in the σ-algebra generated by the half open intervals $[a,b) \subseteq [-N,N)$.

<u>Solution</u> As the complement of a closed set is open it is enough to show that this holds for the open sets. So let $G \subseteq [-N,N)$ be open. Then there exists $U \subseteq \mathbb{R}$ open s.t.

$$G = [-N,N) \cap U.$$

As U is open in $\mathbb{R}$ we can write $U = \bigcup_{j=1}^{M} (a_j, b_j)$ where N is finite or ∞. Note, if $-N \notin G$ then $G \subseteq (-N,N)$ so G is open in $\mathbb{R}$ and in that case we can take $U = G$. If $-N \in G$, then $G \setminus \{-N\}$ is open and hence an union of open intervals. Let $a_0 < -N$. Then there exist $b_0 > -N$ s.t. $(a_0, b_0) \cap [-N,N) = [-N,b)$ is contained in G (because G is open in $[-N,N)$) So we can take $U = (a_0, b_0) \cup \bigcup (a_j, b_j)$ with $-N < a_j < b_j \leq N$. As $(a_0,b_0) \cap [-N,N) = [-N,b_0) \in \mathcal{A}$ we only have to show that each $(a,b) \subset (-N,N)$ is in the σ-algebra generated by $\mathcal{A}$. But this follows by Homework 1 (because

$$(a,b) = \bigcup_{m > \frac{1}{b-a}} [a + \tfrac{1}{m}, b) \in \sigma(\mathcal{A}).$$

Solutions #5

3) Show that $S \subseteq [-N,N)$ is measurable if and only if for every $\varepsilon>0$ there exists $F \subseteq S \subseteq G$, F closed, G open, and $\mu(G\setminus F)<\varepsilon$.

<u>proof:</u> "$\Rightarrow$" For $n\in\mathbb{N}$ let F_n, G_n be such that $F_n, G_n \subseteq [-N,N)$, F_n closed, G_n open, $F_n \subseteq S \subseteq G_n$, and $\mu(G_n\setminus F_n) < \frac{1}{n}$. We can assume that $\{G_n\}$ is decreasing and $\{F_n\}$ is increasing (otherwise replace G_n by $\bigcap_{j=1}^{n} G_n$ and F_n by $\bigcup_{j=1}^{n} F_j$). Let $F = \bigcup_{j=1}^{\infty} F_j$ and

$G = \bigcap_{j=1}^{\infty} G_j$. Then $F, G \in \mathbb{B}$ = the Borel σ-algebra, $F \subseteq S \subseteq G$ and $\mu(G\setminus F) = \lim_{n\to\infty} \mu(G_n\setminus F_n) = 0$, where we use that the sequence $\{G_n\setminus F_n\}$ is decreasing. Now Theorem 3.2.1 implies that S is measurable.

"$\Leftarrow$" Let $S\in\mathcal{P}(X)$ and let $\varepsilon>0$. Then (according to previous homework) there is a finite or countably infinite index set J ($J \subsetneq \mathbb{N}$ or $J=\mathbb{N}$) and $-N \le a_j < b_j \le N$, increasing sequences, s.t.

$S \subseteq \bigcup_{j\in J} [a_j, b_j)$ and $\mu^*(S) + \frac{\varepsilon}{4} > \mu\left(\bigcup_{j\in J}[a_j,b_j)\right) = \sum_{j\in J} b_j - a_j$. Let

$$G = \bigcup_{j\in J} \left(a_j - \frac{\varepsilon}{2^{j+2}}, b_j\right) \cap [-N,N).$$

$$\mu(G) \leq \sum_{j \in J} (b_j - a_j) + \sum_{j \in J} \frac{\varepsilon}{2^{j+2}}$$

$$\leq \mu(\cup [a_j, b_j)) + \frac{\varepsilon}{4} < \mu^*(S) + \frac{\varepsilon}{2}$$

It follows that we can find an open set G, $S \subseteq G$, s.t.

$$\mu(G) < \mu^*(S) + \frac{\varepsilon}{2}$$

In the same way we can find an open set G_1 such that $S^c \subseteq G_1$ and $\mu(G_1) < \mu^*(S^c) + \frac{\varepsilon}{2}$.

Assume now that S is measurable. Set $F = G_1^c \subseteq S$. Then F is closed. As S (and S^c) are measurable we get

$$\begin{aligned} \mu(F) &= \mu(X) - \mu(G_1) \\ &> \mu(X) - (\mu^*(S^c) + \tfrac{\varepsilon}{2}) \\ &= \mu(X) - (\mu(X) - \mu(S^c) + \tfrac{\varepsilon}{2}) \quad (S \text{ measurable}) \\ &= \mu(S^c) - \tfrac{\varepsilon}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \mu(G \setminus F) &= \mu(G) - \mu(F) \\ &= (\mu(G) - \mu(S)) - (\mu(F) - \mu(S)) \\ &< \varepsilon \end{aligned}$$

Solutions #5

1) Let $\{E_j\}$ be a countable sequence in $\mathcal{B}$. Then $f^{-1}(\bigcup_j E_j) = \bigcup_{j=1}^{\infty} f^{-1}(E_j) \in \mathcal{A}$ because each $f^{-1}(E_j)$ is in $\mathcal{A}$ and $\mathcal{A}$ is a σ-algebra. If $E \in \mathcal{B}$, then $f^{-1}(E^c) = f^{-1}(E)^c \in \mathcal{A}$. Hence $\bigcup E_j \in \mathcal{B}$, and $E^c \in \mathcal{B}$.

2) The set $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable, therefore we can number it as

$$\mathbb{Q}^n = \{q_j \mid j \in \mathbb{N}\}$$

For $\varepsilon > 0$ and $x \in \mathbb{R}^n$ let

$$Q_\varepsilon(x) = \{y \in \mathbb{R}^n \mid x_j - \frac{\varepsilon^{1/n}}{2} < y_j < x_j + \frac{\varepsilon^{\frac{1}{n}}}{2}\}$$

Then $Q_\varepsilon(x)$ is open and

$$\lambda(Q_\varepsilon(x)) = \varepsilon.$$

we set

$$U_\varepsilon = \bigcup_{j=1}^{\infty} Q_{\frac{\varepsilon}{2^j}}(q_j).$$

Then U_ε is open and dense ($\mathbb{Q}^n$ is already dense) and

$$\lambda(U_\varepsilon) \leq \sum_{j=1}^{\infty} \lambda(Q_{\frac{\varepsilon}{2^j}}(q_j))$$
$$= \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$