Math 7390-1 Harmonic Analysis I

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Prerequisite: Math 7311 (Real Analysis I) or equivalent.

Text: Lecture notes by R. Fabec and G. Olafsson

Harmonic analysis has its origin in the work of Fourier on the heat equation, which led him to consider the expansion of an "arbitrary" 2π-periodic function into superposition of trigonometric functions:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt\, .$$

One can interpret this expansion either as the spectral decomposition of the differential operators with constant coefficients, or as decomposition of $L^2([0, 2\pi])$ into irreducible representations of the compact Lie group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

The subject proper of *harmonic analysis* is to study functions or function spaces by decomposing the functions into *simpler functions*, where "simpler" depends on the problem we are dealing with. In the theory of differential equations that means to write an arbitrary functions as a sum or integral of eigenfunctions. If we have a symmetry group acting on the system, then we would like to write an arbitrary function as a sum of functions that transforms in a simple and controllable way under the symmetry group. The simplest example is the use of polar coordinates and radial functions for rotation symmetric equations. Harmonic analysis includes spectral decomposition of differential operators, theory of special functions, integral transforms related to special functions, atomic decomposition of function spaces, and study of functions defined on a Lie group (or a homogeneous space) by decomposing them into pieces associated with the unitary irreducible representations of the group.

The basic example of noncompact Lie group is the real line $\mathbb{R}$ considered as an additive group. In this case all the unitary irreducible representations are one dimensional and given by the exponential functions $t \mapsto e^{i\lambda t}$ with $\lambda \in \mathbb{R}$. To each (sufficiently regular) function $f : \mathbb{R} \to \mathbb{C}$ we associate its Fourier transform

$$(\mathcal{F}f)(\lambda) \sim \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\lambda x}\, dx.$$

Here the Fourier inversion

$$f(x) \sim \int_{\mathbb{R}} (\mathcal{F}f)(\lambda) e^{i\lambda x} d\lambda$$

provides the required decomposition with respect to the irreducible representations or the spectral decomposition of the differential operators on $\mathbb{R}$ with constant coefficients. Harmonic analysis also studies in which sense this decomposition has to be considered. For instance, the symbol "∼" means convergence in L^2 norm if $f \in L^2(\mathbb{R})$, and even pointwise convergence if f is smooth and compactly supported.

The classical one-dimensional Fourier analysis on $\mathbb{R}$ has several generalizations. The real line can be replaced by a higher-dimensional Euclidean space, by a locally compact Hausdorff topological group, by a Lie group, or by a homogeneous space. Again, the simplest example is $\mathbb{R}^n$. We can consider $\mathbb{R}^n$ as a homogeneous space under the *group* $\mathbb{R}^n$ or the group of isometric affine transformations $\mathrm{SO}(n) \times_s \mathbb{R}^n$ (the semi-direct product of $\mathbb{R}^n$ and the group of translations). The use of those different groups leads to different decompositions of $L^2(\mathbb{R}^n)$. We can also replace $\mathrm{SO}(n) \times_s \mathbb{R}^n$ with other groups, resulting the wavelet transform.
The course is an introduction into the basic theory of classical Fourier analysis. Next semester, R. Fabec will give the second part, which will deal with more advanced part of modern harmonic analysis. The simplest examples of non-abelian harmonic analysis are harmonic analysis on the $(ax+b)$-group, which is the basic tool in the wavelet theory, the Heisenberg group, classical linear Lie groups.The second part discusses the basic properties of topological groups and homogeneous spaces. Then representations of topological groups will be discussed and several examples worked out.
The main topics this semester will be:

- Basic properties of Fourier series;
- Convergence of Fourier series;
- Function spaces on $\mathbb{R}^n$, in particular the space of rapidly decreasing functions, L^p-spaces, and the space of compactly supported functions;
- Basic properties of the Fourier transform on $\mathbb{R}^n$, inversion formula for rapidly decreasing functions;
- The L^2-theory for the Fourier transform, in particular the Plancherel theorem;
- We include some application, which will be chosen depending on time available and the interests of the students. Possible topics include: distribution theory and Fourier transforms of distributions, differential equations, Sobolev spaces, the Fock-space and Hermite polynomials, introduction to harmonic analysis on the Heisenberg group, the short time Fourier transform, and the continuous wavelet transform.
- If time allows, then we will also introduce some basic facts from representation theory.

You will have to turn in the solution to **one** problem a week. Solutions submitted **after the deadline** will not be considered **except you talked with me before the deadline**. You need 80% to get an A and 60% to get an B. There will be a test at the end of the semester. The grade criteria is the same as for the problems.
We will use lecture notes written my R. Fabec and G. Ólafsson. Those lecture notes and other material related to this class and harmonic analysis in general can be found on the webpage http://math.lsu.edu/harmonic. You can log in by using the same username and password as you use for your own account at LSU.