1[42P]) Calculate the derivatives:

a) \( \frac{d}{dx} \sin \left( \frac{1}{x^2 + 1} \right) = \) ________________

b) \( \frac{d}{dx} \sqrt{6x + \sqrt{4x}} = \) ________________

c) \( \frac{d}{dx} \left( \frac{x(x + 2)}{(4x^2 + 1)(2x + 2)} \right) = \) ________________

d) \( \frac{d}{dx} \sin^{-1}(x^2 + x - 1) = \) ________________
2[8P]) Let \( h(x) = \sqrt{x} \). Find \( h''(1) = \) \underline{\hspace{2cm}} .

3[8P] Find the equation of the tangent line of \( x^2y + 2xy^2x + 2y \) at the point (1, 1).
Answer: The equation of the tangent line is \underline{\hspace{2cm}} .

4[9P]) A conical tank has height 3 \( m \) and radius 2 \( m \) at the top. Water flows in at a rate of 2 \( m^3/min \). How fast is the water level rising when it is 2 \( m \)? (The volume of conical tank is \( V = \frac{4}{3}\pi r^2h \))
Answer: The water level is rising \underline{\hspace{2cm}} .

5[8P]) Estimate the quantity \( \sqrt{26} - 5 \approx \) \underline{\hspace{2cm}} using the Linear Approximation. Show your work, calculator will give you the wrong answer!
For Partial Credit, show your Work. You may use that \[ \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6} \.

1[30P]) Suppose that
\[
f(x) = \frac{1}{x} + \frac{1}{x-1}.
\]
Then
\[
f'(x) = \frac{x^2 + (x-1)^2}{x^2(x-1)^2} \quad \text{and} \quad f''(x) = \frac{2(2x-1)(x^2-x+1)}{x^3(x-1)^3}.
\]

(A) Find all critical values of \( f(x) \). If there are no critical values, enter \textit{None}. If there are more than one, enter them separated by commas.
Critical value(s) =

(B) Use \textit{interval notation} to indicate where \( f(x) \) is increasing. If it is increasing on more than one interval, enter the union of all intervals where \( f(x) \) is increasing.
Increasing:

(C) Use \textit{interval notation} to indicate where \( f(x) \) is decreasing. If it is decreasing on more than one interval, enter the union of all intervals where \( f(x) \) is decreasing.
Decreasing:

(D) Find the \( x \)-coordinates of all local maxima of \( f(x) \). If there are no local maxima, enter \textit{None}. If there are more than one, enter them separated by commas.
Local maxima at \( x = \)
(E) Find the $x$-coordinates of all local minima of $f(x)$. If there are no local minima, enter $None$. If there are more than one, enter them separated by commas.
Local minima at $x =$

(F) Use interval notation to indicate where $f(x)$ is concave up.
Concave up:

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down:

(H) Find all inflection points of $f$. If there are no inflection points, enter $None$. If there are more than one, enter them separated by commas.
Inflection point(s) at $x =$

(I) Find all horizontal asymptotes of $f$. If there are no horizontal asymptotes, enter $None$. If there are more than one, enter them separated by commas.
Horizontal asymptote(s): $y =$

(J) Find all vertical asymptotes of $f$. If there are no vertical asymptotes, enter $None$. If there are more than one, enter them separated by commas.
Vertical asymptote(s): $x =$
(K) Use all of the preceding information to sketch a graph of $f$.

2[15P]) A landscape architect wished to enclose a rectangular garden on one side by a brick wall costing $60/ft and on the other three sides by a metal fence costing $10/ft. If the area of the garden is 42 square feet, find the dimensions of the garden that minimize the cost.
Length of side with bricks $x =$ ____________
Length of adjacent side $y =$ ____________

3[15P]) Use L'Hôpital's Rule to evaluate the following limits:

a) $\lim_{x \to 0} \frac{1 - \cos(2x)}{\sin(3x)} =$ ________
b) \[ \lim_{x \to 0^+} \sqrt{x} \ln(x) = \ \] 

\[ \lim_{x \to \infty} \sqrt{x^2 + 3x + 2} - x = \ \]

4[15P]) Use Newton’s Method with the function \( f(x) = x^2 - 2 \) and initial value \( x_0 = 1 \) to calculate \( x_1 \) and \( x_2 \).
\[ x_1 = \ \] \[ x_2 = \ \]

5[10P]) Evaluate the following two antiderivatives:

a) \[ \int x(x + 2x^3) \, dx = \ \]
b) \( \int e^{x^2} dx = \) 

6[15P]) Let \( f(x) = 2x^2 + x \).

a) Calculate \( R_4 \) on \([0, 1] \). \( R_4 = \) 

b) For \( N \) an integer calculate \( R_N \) on \([0, 1] \). \( R_N = \) 

b) Use (b) to find the area below the graph of \( y = x^2 \) and above the interval \([0, 1] \). The area is: \( \)
For Partial Credit, show your Work. You may use that \[ \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}. \]

1[30P]) Suppose that \[ f(x) = \frac{1}{x} + \frac{1}{x-1}. \]
Then \[ f'(x) = -\frac{x^2 + (x-1)^2}{x^2(x-1)^2} \quad \text{and} \quad f''(x) = \frac{2(2x-1)(x^2-x+1)}{x^3(x-1)^3}. \]

(A) Find all critical values of \( f(x) \). If there are no critical values, enter \textit{None}. If there are more than one, enter them separated by commas.
Critical value(s) = \textit{None}
\[ x^2 + (x-1)^2 > 0 \quad \text{for all } x \text{ where defined} \]

(B) Use \textit{interval notation} to indicate where \( f(x) \) is increasing. If it is increasing on more than one interval, enter the union of all intervals where \( f(x) \) is increasing.
Increasing: \textit{Not increasing}
\[ f'(x) < 0 \quad \text{for all } x \text{ where defined} \]

(C) Use \textit{interval notation} to indicate where \( f(x) \) is decreasing. If it is decreasing on more than one interval, enter the union of all intervals where \( f(x) \) is decreasing.
Decreasing: \( (-\infty) \cup (0,1) \cup (1,\infty) \)

(D) Find the \( x \)-coordinates of all local maxima of \( f(x) \). If there are no local maxima, enter \textit{None}. If there are more than one, enter them separated by commas.
Local maxima at \( x = \textit{None} \)
\[ \text{No local max or min as } f'(x) \text{ is never zero} \]
(E) Find the $x$-coordinates of all local minima of $f(x)$. If there are no local minima, enter None. If there are more than one, enter them separated by commas.
Local minima at $x = \text{None}$

(F) Use interval notation to indicate where $f(x)$ is concave up.
Concave up: $(0, \frac{1}{2}) \cup (1, \infty)$

<table>
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<th>$x$</th>
<th>$2x-1$</th>
<th>$x^2-x+1$</th>
<th>$x^3$</th>
<th>$(x-1)^3$</th>
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<td>$x \geq \frac{1}{2}$</td>
<td>TR</td>
<td>$x &gt; 0$</td>
<td>$x &gt; 1$</td>
</tr>
<tr>
<td>never</td>
<td>$x &lt; \frac{1}{2}$</td>
<td>never</td>
<td>$x &lt; 0$</td>
<td>$x &lt; 1$</td>
</tr>
</tbody>
</table>

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down: $(-\infty, 0) \cup (\frac{1}{2}, 1)$

(H) Find all inflection points of $f$. If there are no inflection points, enter None. If there are more than one, enter them separated by commas.
Inflection points at $x = \frac{1}{2}$

(I) Find all horizontal asymptotes of $f$. If there are no horizontal asymptotes, enter None. If there are more than one, enter them separated by commas.
Horizontal asymptote(s): $y = 0$

(J) Find all vertical asymptotes of $f$. If there are no vertical asymptotes, enter None. If there are more than one, enter them separated by commas.
Vertical asymptote(s): $x = 0, 1$
(K) Use all of the preceding information to sketch a graph of $f$.

2[15P]) A landscape architect wished to enclose a rectangular garden on one side by a brick wall costing $60/ft$ and on the other three sides by a metal fence costing $10/ft$. If the area of the garden is 42 square feet, find the dimensions of the garden that minimize the cost.

Length of side with bricks $y = \frac{\sqrt{12}}{2}$

Length of adjacent side $x = \frac{42/\sqrt{12}}{2}$

$$
\text{area} = xy = 42, \quad x = \frac{42}{y}
$$

$$
\text{cost} = 60y + 20x + 10y
$$

$$
\text{cost} = 70y + \frac{840}{y}
$$

$$
\text{cost}^1 = 70 - \frac{840}{y^2} = 0, \quad y^2 = \frac{840}{70} = 12
$$

$$
y = \sqrt{12}
$$

$$
x = \frac{42/\sqrt{12}}{2}
$$

3[15P]) Use L'Hopital's Rule to evaluate the following limits:

a) $\lim_{x \to 0} \frac{1 - \cos(2x)}{\sin(3x)} = 0$

$$
\lim_{x \to 0} \frac{2 \sin(2x)}{3 \cos(3x)} = 0
$$
b) \( \lim_{x \to 0^+} \sqrt{x} \ln(x) = 0 \)

\[
\lim_{x \to 0^+} \sqrt{x} \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-\frac{1}{2}}} = \lim_{x \to 0^+} \frac{-\frac{1}{x}}{x^{-\frac{3}{2}}} = \lim_{x \to 0^+} \frac{-2x^{\frac{3}{2}}}{x} = 0
\]

= \lim_{x \to 0^+} (-2\sqrt{x}) = 0

c) \( \lim_{x \to \infty} \sqrt{x^2 + 3x + 2 - x} = \frac{3}{2} \)

\[
\lim_{x \to \infty} \sqrt{x^2 + 3x + 2 - x} = \lim_{x \to \infty} \frac{\sqrt{\left(1 + \frac{3}{x} + \frac{2}{x^2}\right)} - 1}{x} = \lim_{x \to \infty} \frac{\sqrt{\left(1 + \frac{3}{x} + \frac{2}{x^2}\right)} - 1}{x} = \lim_{u \to 0} \frac{\sqrt{1+3u+2u^2} - 1}{u} = \lim_{u \to 0} \frac{3 + 4u}{2\sqrt{1+3u+2u^2}} = \frac{3}{2}
\]

4[15P]) Use Newton's Method with the function \( f(x) = x^2 - 2 \) and initial value \( x_0 = 1 \) to calculate \( x_1 \) and \( x_2 \).

\( x_1 = \frac{3}{2} \)

\[
x_{n+1} = x_n - \frac{f'(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}x_n + \frac{1}{x_n}
\]

\( x_0 = 1 \):

\( x_1 = \frac{3}{2} + 1 = \frac{3}{2} \)

\( x_2 = \frac{3}{4} + \frac{2}{3} = \frac{9 + 8}{12} = \frac{17}{12} \)

5[10P]) Evaluate the following two antiderivatives:

a) \( \int x(x + 2x^3) \, dx = \frac{1}{3}x^3 + \frac{2}{5}x^5 + C \)

\[
\int x^2 + 2x^4 \, dx = \frac{1}{3}x^3 + \frac{2}{5}x^5 + C
\]
1[14P]) Evaluate the integrals:

a) \[ \int_{-2}^{2} (1 + t^2 - t^3) \, dt = \frac{28}{3} \]

1 and \( t^2 \) are even, \( t^3 \) odd. The integral is therefore the same as

\[ 2 \int_0^2 1 + t^2 \, dt = 2 \left[ t + \frac{t^3}{3} \right]_0^2 = 2 \left[ 2 + \frac{8}{3} \right] = \frac{28}{3} \]

b) \[ \int_0^{\pi/4} \tan^2(x) \sec^2(x) \, dx = \frac{1}{2} \]

\( \text{Let } u = \tan^2(x). \) Then \( du = 2 \tan(x) \sec(x) \, dx \)

\[ \int_0^1 u \, du = \frac{u^3}{3} \bigg|_0^1 = \frac{1}{2} \]

2[7P]) Calculate the derivative \( \frac{d}{dx} \int_0^x \sqrt{t} \, dt = \frac{d}{dx} x^{3/2} = 2x^{1/2}x \).

Use the chain rule and the fundamental theorem of calculus.

3[21P]) Evaluate the integrals:

a) \[ \int x \sqrt{1 + x^2} \, dx = \frac{2}{3} \left( 1 + x^2 \right)^{3/2} + C \]

\( \text{Set } u = 1 + x^2, \quad du = \frac{1}{2} \, 2x \, dx, \quad x \, dx = \frac{1}{2} \, du \)

\[ \frac{1}{2} \int u^{1/2} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} \left( 1 + x^2 \right)^{3/2} + C \]
b) \[ \int x^3 \sqrt{1 + x^2} \, dx = \frac{1}{2} \left( \frac{5}{2} \left( 1 + x^2 \right)^{\frac{5}{2}} - \frac{1}{3} \left( 1 + x^2 \right)^{\frac{3}{2}} \right) + C \]

\[ u = 1 + x^2 \quad \text{and} \quad \frac{du}{dx} = 2x \]

\[ \frac{1}{2} \int (u - 1) u^{\frac{1}{2}} \, du = \frac{1}{2} \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du = \frac{1}{2} \left[ \frac{2}{5} u^{\frac{5}{2}} - \frac{3}{3} u^{\frac{3}{2}} \right] + C \]

\[ \int \frac{3}{9 + 4x^2} \, dx = \frac{1}{2} \arctan \left( \frac{2x}{3} \right) + C \]

\[ \int \frac{3}{9 + 4x^2} \, dx = \frac{1}{3} \int \frac{dx}{1 + \left( \frac{2}{3} x \right)^2} \]

\[ \text{Let } u = \frac{2x}{3}, \quad \frac{du}{dx} = \frac{2}{3} \]

\[ = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \arctan(u) + C = \frac{1}{2} \arctan \left( \frac{2x}{3} \right) + C \]

4(14P) a) Sketch the region enclosed by the curves \( y = \frac{1}{x}, \quad y = \frac{1}{x^2}, \quad x = 2. \)

b) Find the area of the region in part (a). Area = \( \frac{\ln(2)}{2} - \frac{1}{2} \)

\[ \int_{1}^{2} \frac{1}{x^2} \, dx = \ln x + \frac{1}{x} \bigg|_{1}^{2} = \ln(2) + \frac{1}{2} - 1 = \ln(2) - \frac{1}{2} \]
5(14P) Let \( V \) be the volume of a right circular cone of height \( h = 4 \) whose base is a circle of radius \( r = 2 \).

a) Find the area \( A(y) \) of the horizontal cross section at a height \( y \).
\[
A(y) = \frac{\pi}{r^2} \left( 2 - \frac{y}{2} \right)^2 = \frac{\pi}{4} (4 - y)^2
\]

\[
\frac{h}{r} = \frac{z}{x} = \frac{4-y}{x}, \quad dx = 4-y \cdot x = 2 - \frac{y}{2}
\]

b) Calculate \( V \) by integrating the cross-sectional areas. \( V = 8\pi \).
\[
\frac{\pi}{4} \int_0^4 (4-y)^2 \, dy = -\frac{\pi}{4} \int_0^4 u^2 \, du
\]
\[
u = 4-y \quad \Rightarrow \quad du = -dy
\]
\[
= \frac{\pi}{4} \int_0^4 u^2 \, du = \frac{\pi}{8} u^3 \bigg|_0^4 = \frac{\pi}{8} (64) = 8\pi
\]
In the following three problems set up, but do NOT evaluate, an integral needed to find the volume. Do not forget the limits of integration:

6[10P]) Set up an integral for the volume of the solid obtained by rotating the region under the graph of the function \( f(x) = 3x^2 - x \) over the interval \([1, 2]\) about the axis \( y = -1 \).

\[
V = \int_{1}^{2} \pi \left( (3x^2 - x + 1)^2 - 1 \right) \, dx \quad (\text{you can also simplify this})
\]

\[
R = f(x) + 1 = 3x^2 - x + 1
\]

\[
r = 1
\]

7[10P]) Set up an integral for the volume of the solid obtained by rotating the region enclosed by the graphs \( x = \sqrt{y} \) and \( x = y^2 \) about the \( y \)-axis.

\[
V = \pi \int_{0}^{1} y - y^2 \, dy
\]

\[
= \pi \left[ \frac{y^2}{2} - \frac{y^3}{3} \right]_{0}^{1}
\]

8[10P]) Set up an integral needed to compute the volume of the solid obtained by rotating the region enclosed by the graphs of the functions \( y = x^2 \), \( y = 8 - x^2 \) and \( x = 0 \) about the \( y \)-axis by using the Shell Method.

\[
V = \int_{0}^{2} 2\pi x \left( 8 - 2x^2 \right) \, dx
\]

\[
= 2\pi \int_{0}^{2} x \left( 8 - 2x^2 \right) \, dx
\]

\[
= 8 - x^2 \quad (8 - x^2 - x^2 = 8 - 2x^2)
\]

\[
8 - x^2 = x^2 \quad 8 - 2x^2, \quad x = 2
\]