Wavelets, Problems due Fr. March 28

1) Suppose that $H$ is a finite dimensional Hilbert space. Show that a finite set $\{f_n\}$ in $H$ is a frame if and only if $\{f_n\}$ is generating.

2) Let $H$ be a separable Hilbert space. Let $\{f_n\}$ be a sequence in $H$. Then $\{f_n\}$ is called a Bessel sequence if there exists a $B > 0$ such that
\[ \sum_n |(x, f_n)|^2 \leq B\|x\|^2 \]
for all $x \in H$. Define the Gram matrix associated to $\{f_n\}$ by $G = ((f_k, f_j))_{j,k}$. Show that $\{f_n\}$ is a Bessel sequence with bound $B$ if and only if $G$ defines a bounded linear operator $(x_n) \mapsto (\sum_n x_n (f_n, f_j))$ on $\ell^2 = \{(c_n) \mid c_n \in \mathbb{C}, \sum_n |c_n|^2 < \infty\}$.

3) Assume that $\{f_n\}$ is a Bessel sequence with bound $B$. Prove that the following holds:
   a) $\|f_n\|^2 \leq B$ for all $n \in \mathbb{N}$.
   b) If $\|f_n\| = B$ for some $n \in \mathbb{N}$, then $(f_n, f_k) = 0$ for all $k \neq n$. 