Math 2025, Problems (Fall 2003)

1) Determine, if the following set of vectors are linearly independent or not. If the vectors are linearly dependent, write one of them as a combination of the others:
   a) \((1, 0, 1), (1, 1, 0), (1, 1, 1)\)
   b) \((1, 1), (1, -1)\)
   c) \((1, -2, 1), (1, 2, 1), (1, 1, -1), (1, 0, 1)\).
   d) \((1, 2), (-2, -4)\).
   e) \((1, 1, 2), (1, -1, 0), (-2, 0, 1)\).
   f) \((1, 1, -1, 1), (1, 2, 3, 2), (1, -2, 2, 0)\).

2) Determine if the following sets of functions are linearly dependent on the interval \([0, 1]\) or not.
   a) \(\chi_{[0,1/2]}, \chi_{[1/2,1]}\).
   b) \(\chi_{[0,1]}, \chi_{[0,1/2]}\).
   c) \(\cos(2\pi t), \sin(2\pi t)\).
   d) \(f(x) = x, g(x) = x^2, h(x) = 1\).
   f) \(\chi_{[0,1/2]}, x^2\).
   g) \(te^t, t^2, t + 1\).

3) Show that the vectors \((1, 2)\) and \((-2, 1)\) are a basis for \(\mathbb{R}^2\). Then determine the constant \(a, b\) such that
   \[(5, -9) = a(1, 2) + b(-2, 1)\].

4) Show that the vectors \((1, -1, 1), (1, 1, 0), (1, -1, -2)\) form an orthogonal basis for \(\mathbb{R}^3\) and then determine the constants \(a, b, c \in \mathbb{R}\) such that
   \[(2, -4, 3) = a(1, -1, 1) + b(1, 1, 0) + c(1, -1, -2)\].

5) Show that the vectors \((1, 2, 0), (1, 1, 1), (1, 0, 1)\) form a basis for \(\mathbb{R}^3\). Then determine the constants \(a, b, c \in \mathbb{R}\) such that
   \[(4, -1, 2) = a(1, 2, 0) + b(1, 1, 1) + c(1, 0, 1)\].
Which of the following sets is a basis for \( \mathbb{R}^2 \)?

\[ \{(1, 2, 3, 4), (2, -1, 0, 1), (2, 0, 0, 1)\} \]

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Express the following vectors as a linear combination of \((1, 0, 0, 1)\) and \((0, 1, 0, 1)\):

\[ \{(1, 2, 3, 4), (2, 0, 0, 1), (2, 0, 0, 1)\} \]

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Starting from the given set of vectors, use the Gram-Schmidt orthogonalization to construct a set of orthogonal vectors with the same linear span.

1) \( V_1 = (1, 1), V_2 = (2, 1) \)
2) \( V_1 = (1, 0, 1), V_2 = (1, 1, 1) \)
3) \( V_1 = (1, 1, 0), V_2 = (0, 1, 1), V_3 = (1, 0, 1) \)
4) \( V_1 = (1, 2, 0), V_2 = (1, -1, 1) \)
5) \( V_1(x) = 1, V_2(x) = x \), where we use the inner product \( \langle f, g \rangle = \int_a^b f(x)g(x)\,dx \).
6) \( V_1(x) = 1, V_2(x) = x + 1 \).
7) \( V_1(x) = 1, V_2(x) = x^2, V_3(x) = x - 1 \)
8) \( V_1(x) = x|_{[0, 1]}, V_2(x) = x|_{[0, 1]} \)
9) \( V_1(x) = x, V_2(x) = e^x \).