Math 2025, Problems (Fall 2003)

1) Determine, if the following set of vectors are linearly independent or not. If the vectors are linearly dependent, write one of them as a combination of the others:

a) $(1,0,1),(1,1,0),(1,1,1)$

b) $(1,1),(1,-1)$

c) $(1,-2,1),(1,2,1),(1,1,-1),(1,0,1)$.

d) $(1,2),(-2,-4)$.

e) $(1,1,2),(1,-1,0),(-2,0,1)$.

f) $(1,1,-1,1),(1,2,3,2),(1,-2,2,0)$.

2) Determine if the following sets of functions are linearly dependent on the interval $[0,1)$ or not.

a) $\chi_{[0,1/2)}, \chi_{[1/2,1)}$.

b) $\chi_{[0,1)}, \chi_{[0,1/2)}$.

c) $\cos(2\pi t), \sin(2\pi t)$.

d) $f(x)=x, g(x)=x^2, h(x)=1$.

f) $\chi_{[0,1/2)}, x^2$.

g) $te^t, t^2, t+1$.

3) Show that the vectors $(1,2)$ and $(-2,1)$ are a basis for $\mathbb{R}^2$. Then determine the constant a,b such that

$$(5,-9)=a(1,2)+b(-2,1)\,.$$

4) Show that the vectors $(1,-1,1),(1,1,0),(1,-1,-2)$ form an orthogonal basis for $\mathbb{R}^3$ and then determine the constants $a,b,c\in\mathbb{R}$ such that

$$(2,-4,3)=a(1,-1,1)+b(1,1,0)+c(1,-1,-2)\,.$$

5) Show that the vectors $(1,2,0),(1,1,1),(1,0,1)$ form a basis for $\mathbb{R}^3$. Then determine the constants $a,b,c\in\mathbb{R}$ such that

$$(4,-1,2)=a(1,2,0)+b(1,1,1)+c(1,0,1)\,.$$

6) Which of the following sets is generating for $\mathbb{R}^2$?

a) $(1, 1)$

b) $(-1, 1), (1, 2)$,

c) $(1, 1), (0, 1), (-1, 0)$.

d) $(5, 1), (-1, 5)$.

7) Which of the following sets is generating for $\mathbb{R}^3$?

a) $(1, -1, 1), (1, 1, 0)$,

b) $(1, 1, -3), (1, 2, 1), (1, -1, 1)$

c) $(1, 1, 0), (1, -1, 0), (1, 0, -1), (2, 1, 0)$.

8) Write the following vectors as a linear combination of $(1, 1)$ and $(1, -1)$:

a) $(2, 4)$,

b) $(3, 3)$,

c) $(5, -3)$.

9) Express the following vectors as a linear combination of $(2, 1, 0), (1, 0, 1)$ and $(0, 1, 0)$:

a) $(1, 0, 0)$,

b) $(3, 1, 0)$,

c) $(5, -1, 1)$.

d) $(2, 3, 2)$.

10) Which of the following sets is a basis for $\mathbb{R}^2$?

a) $(1, 1), (1, 0)$,

b) $(1, 0), (0, 1), (1, 2)$.

c) $(1, 1)$.

d) $(2, 1), (2, 2)$.

e) $(3, 6), (-7, 14)$.

11) Which of the following sets is a basis for $\mathbb{R}^3$?

a) $(1, 2, 1), (2, -1, 1)$.

b) $(1, 0, 1), (1, 1, -1), (2, 0, 1)$.

c) $(1, 2, 3), (1, -1, 2), (2, 0, 0), (2, -1, 2)$

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Starting from the given set of vectors, use the Gram-Schmidt orthogonalization to construct a set of orthogonal vectors with the same linear span.

1) $v_1 = (1,1), v_2 = (2,1)$
2) $v_1 = (1,0,1), v_2 = (1,1,1)$
3) $v_1 = (1,1,0), v_2 = (0,1,1), v_3 = (1,0,1)$
4) $v_1 = (1,2,0), v_2 = (1,-1,1)$.
5) $v_1(x) = 1, v_2(x) = x$, where we use the inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$.
6) $v_1(x) = 1, v_2(x) = x+1$.
7) $v_1(x) = 1, v_2(x) = x^2, v_3(x) = x-1$
8) $v_1(x) = \chi_{[0,1/2)}(x), v_2(x) = \chi_{[0,1)}(x)$
9) $v_1(x) = x, v_2(x) = e^x$.