An idea how to solve some of the problems

5.2-2. (a) Does not converge: By multiplying across we get

$$\frac{2k}{2k^2-1} \geq \frac{1/2}{k} \Leftrightarrow 2k^2 \geq k^2 - 1/2 \Leftrightarrow k^2 \geq -1/2$$

Hence

$$\frac{2k}{2k^2-1} \geq \frac{1/2}{k}.$$

As the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges the same must hold for the original series.
(b) Converges: We have $(k-1)/(k2^k) \leq 2^{-k}$ and the series $\sum_{k=1}^{\infty} 2^{-k}$ converges.
(c) Divergent: In this case $1/(2k-1) > 1/(2k)$ (multiply in cross) and the series $\sum_{k=1}^{\infty} 1/k$ diverges.

(d) Divergent:

5.2-4. Assume first that $p > 1$ and take $f(x) = x^{-p}$. Then f is monotonically decreasing to zero. Furthermore

$$\int_1^{\infty} f(t)dt = \lim_{T\to\infty} \int_1^T t^{-p}\, dt = \lim_{T\to\infty} \frac{1}{1-p} T^{1-p} + \frac{1}{p-1} = \frac{1}{p-1} < \infty.$$

The claim follows then from Theorem 5.2.2.
Let now $p = 1$. We have $\int_1^T x^{-1}\, dx = \log T \to \infty$ as $T \to \infty$. It follows that $\int_1^{\infty} 1/x\, dx$ does not exists and hence $\sum_{k=1}^{\infty} k^{-1}$ does not converge according to Theorem 5.2.2. If $0 \leq p \leq 1$ then $1/k^p \geq 1/k$ and hence $\sum_{k=1}^{\infty} k^{-p}$ diverges.

5.2-8. Suppose $x_k \geq 0$ for all $k \in \mathbb{N}$, and suppose that $\lim_{k\to\infty} \sqrt[k]{x_k} = L$ exists.

(a) If $L > 1$ then $\sum_{k=1}^{\infty} x_k$ diverges: Let $1 < r < L$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $r \leq \sqrt[k]{x_k}$. Hence $x_k \geq r^k$. The claim follows now because $\sum_{k=N}^{\infty} r^k$ does not exists.

(b) If $L < 1$ then $\sum_{k=1}^{\infty} x_k$ converges: Let $L < r < 1$. Then there exists $N \in \mathbb{N}$ such that $\sqrt[k]{x_k} \leq r$ for all $n \geq N$. This implies that $x_k \leq r^k$ and hence

$$\begin{aligned} \sum_{k=1}^{\infty} x_k &= x_1 + \ldots + x_{N-1} + \sum_{k=N}^{\infty} x_k \\ &\leq x_1 + \ldots + x_{N-1} + \sum_{k=N}^{\infty} r^k < \infty. \end{aligned}$$

Hence the series converges.

(c) If $L = 1$ there is no information: Let $x_k = 1$ for all k. Then $\sqrt[k]{x_k} = 1$ and the series $\sum_{k=1}^{\infty} x_k$ diverges. On the other hand, if $x_k = k^{-2}$ then $\lim_{k\to\infty} \sqrt[k]{x_k} = 1$ as we will see in a moment and this time the series $\sum_{k=1}^{\infty} x_k$ converges.

Let $n \in \mathbb{N}$ and consider the sequence $x_k = \sqrt[k]{k^n}$. Taking the log we see that (using L'Hospital)

$$\lim_{k\to\infty} \log x_k = \lim_{k\to\infty} \frac{n \log k}{k} = \lim_{k\to\infty} \frac{n}{k} = 0\,.$$

Hence

$$\lim_{k\to\infty} x_k = e^0 = 1\,.$$

5.2-11: Test for convergence:

(a) $\sum_{k=0}^{\infty} k!/k^k$: Convergent because with $x_k = k!/k^k$ we have

$$\frac{x_{k+1}}{x_k} = \frac{(k+1)!k^k}{k!(k+1)^{k+1}} = \frac{1}{(1+1/k)^k} \to 1/e < 1\,.$$

(b) $\sum_{k=0}^{\infty} k/e^{-k^2}$: Convergent because

$$\frac{(k+1)e^{k^2}}{ke^{(k+1)^2}} = (1+1/k)e^{-2k-1} \to 0$$

as $k \to \infty$.

(c) $\sum_{k=2}^{\infty} 1/(\log k)^k$: Convergent. Use the root test (fill in the details).

5.3-1. We have

$$\sum_{k=1}^{\infty} \frac{1}{3^k} - \frac{1}{4^k} = \sum_{k=1}^{\infty} \frac{1}{3^k} - \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

5.3-2. If the sequence $\{c_k\}$ is summable then it follows that c_k is bounded, i.e., there exists a $C > 0$ such that $|c_k| \le C$ for all k (use that $\lim c_k = 0$). Hence

$$\sum_{k=1}^{\infty} |c_k x^k| \le C \sum_{k=1}^{\infty} |x|^k < \infty$$

for $0 \leq x < 1$. If $x = 1$ then $c_k x^k = c_k$ is summable by our assumption on c_k.

5.3-6. We do (a) Let $\epsilon > 0$ be given. Let $N > 2/\epsilon$. Then, if $n > m \geq N$ there exists $\mu \in (1/n, 1/m)$ such that

$$f(1/n) - f(1/m) = f'(\mu)\left(\frac{1}{n} - \frac{1}{m}\right)$$

As $|f'(\mu)| < 1$ it follows that

$$|f(1/n) - f(1/m)| < \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{2}{N} < \epsilon\,.$$

It follows that $\{f(1/n)\}$ is a Cauchy sequence and hence

$$\lim_{n\to\infty} f(1/n) = L$$

exists.

(b) Let now $\{x_k\}$ be an arbitrary sequence $x_k \to 0$. Then, by the same argument as above it follows that $\{f(x_k)\}$ is a Cauchy sequence and hence $\lim f(x_k) = L_1$ exists. Define a new sequence $y_{2k} = 1/k$ and $y_{2k+1} = x_k$. Then $y_k \to 0$ and the above argument show that $\lim_k f(y_k)$ exists. Add the details to show that this implies that $L = L_1$ (use subsequence).

5.4-4. It was shown that all the limits exists, so we will not do it here (on an exam you would have to do the details). Let $v, w \in V$ and $c \in \mathbb{R}$. Then

$$T_n(cv + w) = cT_n(v) + T_n(w)$$

because T_n is linear. As all the limits exists we have:

$$\begin{aligned} T(cv+w) &= \lim_{n\to\infty}(T_n(cv+w)) \\ &= \lim_{n\to\infty}(cT_n(v) + T_n(w)) \\ &= c\lim_{n\to\infty} T_n(v) + \lim_{n\to\infty} T_n(w) \\ &= cT(v) + T(w) \end{aligned}$$

which shows that T is linear.

5.4-6. First we have to show that $\|\cdot\|_\infty$ is a norm on ℓ_∞. Let $x = \{x_k\}, y = \{y_k\} \in \ell_\infty$ and $c \in \mathbb{R}$. Note first that

$$|cx_k + y_k| \leq |cx_k| + |y_k| = |c||x_k| + |y_k|\,.$$

Hence

$$\begin{aligned}
\|cx+y\|_\infty &= \sup_k |cx_k + y_k| \\
&\leq \sup_k(|cx_k| + |y_k|) \\
&\leq |c| \sup_k |x_k| + \sup_k |y_k| \\
&= |c|\|x\|_\infty + \|y\|_\infty
\end{aligned}$$

Furthermore $\|x\| = 0$ if and only if all $x_k = 0$ which happen if and only if $x = 0$.
Next we have to show that ℓ_∞ is complete. Let $\{x^n\}$ be a Cauchy sequence in ℓ_∞. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have

$$\|x^n - x^m\|_\infty = \sup_k |x_k^n - x_k^m| < \epsilon/2\,.$$

It follows that the sequence $\{x_k^n\}_n$ is a Cauchy sequence in $\mathbb{R}$ and hence there exists a $x_k \in \mathbb{R}$ such that $x_k^n \to x_k$. Let $x = \{x_k\}$ we have to show that $x^n \to x$ and that $x \in \ell_\infty$. Let N be as above. Then

$$|x_k^n - x_k^m| < \epsilon/2\,.$$

Letting $m \to \infty$ this implies that

$$|x_k^n - x_k| \leq \epsilon/2 < \epsilon.$$

Thus

$$(\forall n \geq N) \qquad \|x^n - x\|_\infty < \epsilon$$

and

$$\|x\|_\infty = \|x - x^N + x^N\|_\infty \leq \|x - x^N\|_\infty + \|x^N\|_\infty < \epsilon + \|x^N\|_\infty < \infty\,.$$

This proves both statements.

5.4-7: Recall that the sequence $x^n \in \ell_1$ is defined by $x_k^n = (n+1)/(n2^k)$.
a) Show that $x^n \in \ell_1$: We have

$$\|x^n\|_1 = \sum_{k=1}^\infty \frac{n+1}{n} 2^{-k} \leq 2 \sum_{k=1}^\infty 2^{-k} < \infty$$

because

$$\frac{n+1}{n} \leq 1 + 1/n \leq 2\,.$$

b) By the above we have that

$$\lim_{n\to infty} x_k^n = 2^{-k} = x_k$$

exists and the sequence $x = \{x_k\}$ is in ℓ_1 because $\sum_{k=1}^\infty 2^{-k} < \infty$.
c) We have $|x_k^n - x_k| = \frac{1}{n2^k}$. Furthermore $\sum_{k=1}^\infty 2^{-k} = 1$. Hence

$$\|x^n - x\|_1 = \sum_{k=1}^\infty |x_k^n - x_k| = \frac{1}{n}\,.$$

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $N > 1/\epsilon$. Then, if $n \geq N$ we have

$$\|x^n - x\|_1 = \frac{1}{n} \leq \frac{1}{N} < \epsilon\,.$$

5.4-8 In this problem we define x^n by $x_k^n = 1$ if $k \leq n$ and $x_k^n = k^{-2}$ if $k > n$.
(a) We have

$$\|x^n\|_1 = \sum_{k=1}^\infty x_k^n = n + \sum_{k=n+1}^\infty k^{-2} < \infty\,. \tag{1}$$

Hence $x^n \in \ell_1$.
(b) Let $k \in \mathbb{N}$, then for all $n \geq k$ we have $x_k^n = 1$. Hence $x_k = \lim_{n\to\infty} x_k^n = 1$ for all k. In particular $x = \{x_k\} \notin \ell_1$.
(c) The sequence $\{x^n\}$ can not be a Cauchy sequence because otherwise $\lim x^n = x \in \ell_1$ would exists.

5.5-2. If $0 \leq \alpha < 1$ show that $\sum_{k=1}^\infty x^k$ conveges uniformly on $[0, \alpha]$.

Solution: We have $M_k = \sup_{x\in[0,\alpha]} |x^k| = \alpha^k$ and hence the series $\sum_{k=1}^\infty M_k$ converges. The claim follows by the Weierstrass M-test.

5.5-4: If $\sum_{k=1}^\infty f_k$ converges uniformly on D, prove that $\|f_n\| \to 0$ as $n \to \infty$. Is the converse true?

Solution: As $\sum_{k=1}^\infty f_k$ converges uniformly it follows that the sequence of partial sums $s_n = \sum_{k=1}^n f_k$ is a Cauchy sequence in the supremum norm. Let $\epsilon > 0$. Then there exists a $N \in \mathbb{N}$ such that

$$\forall n, m \geq N \qquad \|s_n - s_m\|_\infty < \epsilon\,.$$

In particular for $n > N$:

$$\|f_n\|_\infty = \|s_n - s_{n-1}\|_\infty < \epsilon\,.$$

The converse is not true. For that let $f_k(x) = \frac{1}{k}$ on $[0,1]$. Then $\|f_k\| = 1/k \to 0$, but $\sum_{k=1}^\infty f_k(x)$ does not even converge at $x = 1$.

5.5-5: (a) The sequence $\sum_{k=1}^\infty e^{-kx}$ converges uniformly on $[1,\infty)$. For that note that on this interval we have

$$e^{-kx} \le e^{-k} = (1/e)^k$$

and the series $\sum_{k=1}^\infty (1/e)^k$ converges. The claim follows then by the Weierstrass M-test.
(b) $\sum_{k=1}^\infty \frac{\sin(kx)}{k^3}$ converges uniformly on $\mathbb{R}$ because

$$\left|\frac{\sin(kx)}{k^3}\right| \le \frac{1}{k^3}$$

and the series $\sum_{k=1}^\infty 1/k^3$ converges. The claim follows then by the Weierstrass M-test.
(c) The series $\sum_{k=1}^\infty \sin^k(x)$ converges uniformly on $[0,\pi/4]$ because on this interval $|\sin^k(x)| \le (1/\sqrt{2})^k$ and the series $\sum_{k=1}^\infty (1/\sqrt{2})^k$ converges.
(d) No, the series $\sum_{k=1}^\infty \tan^k x$ does not even converge at $x = \pi/4$.

5.6-2: (a) We have to show that

$$\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k+1} = x \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k}$$

converges uniformly on $[-1,1]$. We note that for all $x \in [-1,1]$ the series $\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k}$ is alternating and $x_k = \frac{x^{2k}}{2k+1} \to 0$ monotonically. Hence $\sum_k x_k$ exists and by Theorem 5.1.2 we we have with $s_n(x) = \sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k}$ and $s(x) = \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k}$:

$$|xs_n(x) - xs(x)| = |x||s_n(x) - s(x)| \le x_{n+1} = x \cdot \frac{x^{2n}}{2n+2} \le \frac{1}{2(n+1)}\,.$$

Hence

$$\|\sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k+1} - s(x)\|_\infty \le \frac{1}{2(n+1)}$$

which proves the claim.

(b) Define $g(x) = \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k}$ then it follows by (a) and Theorem 5.5.1, part a, it follows that $g(x)$ is continuous on $[-1,1]$. As $g(x) = \tan^{-1}(x)$ for $x \in (-1,1)$ and $\tan^{-1} x$ is continuous, it follows that $g(\pm 1) = \tan^{-1}(\pm 1)$.

(c) We know that $\tan^{-1}(1) = \frac{\pi}{4}$. Hence

$$\pi = 4\tan^{-1}(1) = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\,.$$

5.6-4: We have $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ if $|t| < 1$. Hence, by Theorem 5.6.1 and Theorem 5.5.1:

$$\sum_{k=0}^{\infty} \frac{t^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{t^k}{k} = \int_0^t \frac{du}{1-u} = -\log(1-u)\,.$$

Taking $t = 1/2$ we get

$$\sum_{k=1}^{\infty} \frac{1}{k2^k} = -\log(1/2) = \log 2\,.$$

5.6-5: Find the interval of convergence of the series $\sum c_k x^k$. We use the ratio test: In case

$$\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = L$$

exists, then

$$R = \frac{1}{L}\,.$$

(In case $L = 0$ this reads $R = \infty$ and $L = \infty$ reads $R = 0$.)

(a) $c_k = 1/(k!)$. Then

$$\frac{c_{k+1}}{c_k} = \frac{1}{k+1} \to 0\,.$$

Hence the power series converges for all $x \in \mathbb{R}$.

(b) $a = -1$ and $c_k = (-1)^{k+1}/(k+1)$. Then

$$\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = 1$$

and hence $R = 1$. If $x = 0$, then we have a alternating series so the power series converges at $x = 0$. If $x = -2$ then we are looking at the series

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{k+1}$$

which does not converge. So the power series converges on $(-2,1]$.

(c) $c_k = k!/k^k$ so

$$c_{k+1}/c_k = \frac{(k+1)!k^k}{k!(k+1)^{k+1}} = \left(\frac{1}{1+1/k}\right)^k \to 1/e\,.$$

What about the endpoint?

(d) $c_k = 1/k^k$. Then

$$c_{k+1}/c_k = \frac{k^k}{(k+1)^{k+1}} = \frac{1}{k+1}\left(\frac{k}{(k+1)}\right)^k \leq \frac{1}{k+1} \to 0\,.$$

Hence the power series converges for all $x \in \mathbb{R}$, i.e, $R = \infty$.

5.7-2: The function e^x is analytic at 0 and so is $\tan^{-1}(x)$. It follows by Theorem 5.7.3 that $e^x \tan^{-1} x$ is analytic at 0. There are two ways to find the coefficient of x^4. First, just differentiate the function four times and use that $c_k = f^{(k)}(0)/k!$. The other way is to use that if $f(x) = \sum_{k=0}^\infty a_k x^k$ and $g(x) = \sum_{k=0}^\infty b_k x^k$ for $|x| \leq R$, then

$$fg(x) = \sum_{k=0}^\infty \sum_{j=0}^\infty a_k b_j x^{j+k} = \sum_{k=0}^\infty \left(\sum_{j=0}^k a_j b_{k-j}\right) x^k\,.$$

Hence the coefficient of x^k is

$$\sum_{j=0}^k a_j b_{k-j}\,.$$

It follows then from formula (5.2) p. 140 that the coefficient of x^4 is

$$\sum_{j=0}^4 \frac{1}{j!}\frac{(-1)^{4-j}}{2(4-j)+1} = \frac{1}{9} - \frac{1}{7} + \frac{1}{10} - \frac{1}{18} + \frac{1}{24} = \text{simplify}\,.$$

5.7-3: We have

$$\tan^{-1}(x) = \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k+1} = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} x^k\,.$$

Note, that the coefficients for the even powers of x are all zero. Hence $f^{(even)}(0) = 0$. In particular $f^{(100)}(0) = 0$. We have $101 = 2 \cdot 50 + 1$, so $k = 50$, and hence

$$f^{(101)}(0) = 101! \cdot \frac{1}{101} = 100!\,.$$

5.7-4: (a) The function $f(x) = |x|$ can not be analytic at zero, because it is not differentiable at zero (recall: analytic functions are smooth!).

(b) The function can not be analytic at zero because we have

$$f^{(k-1)}(x) = \begin{cases} k!x & , \quad x > 0 \\ 0 & , \quad x \leq \end{cases}$$

and this function is not differentiable at zero.

5.7-5: (a) True, the function is given by $f(x) = x^4$ on the interval $(0, 1)$.

(b) True, we have $f(x) = 0$ on the interval $(-1, 0)$.

(c) No (see problem 5.7-3 with $k = 4$.

5.7-6. Let

$$f(x) = \begin{cases} e^{-1/x^2} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} .$$

Note that f is ∞-times differentiable at all points $x \neq 0$ as that holds for the exponential function and the function $x \mapsto -1/x^2$.

(a) To see if $f'(0)$ we need to see if the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}$$

exists. Note that this limit is of the form $\frac{0}{0}$ so we can use L'Hospital. We set $u = 1/h$ and consider the limit $u \to \infty$:

$$\begin{aligned} \lim_{h \to 0} \frac{e^{-1/h^2}}{h} &= \lim_{u \to \infty} \frac{u}{e^{u^2}} \\ &= \lim_{u \to \infty} \frac{1}{2ue^{u^2}} \\ &= 0 \end{aligned}$$

Hence, $f'(0)$ exists and is equal to zero, $f'(0) = 0$.

Before we do the next parts let us note the following: Let $k \in \mathbb{N}$, then

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h^k} = \lim_{u \to \infty} \frac{u^k}{e^{u^2}}$$

$$\begin{aligned}
&= \lim_{u\to\infty} \frac{ku^{k-1}}{2ue^{u^2}} \\
&= \lim_{u\to\infty} \frac{k(k-1)u^{k-2}}{2e^{u^2}+4u^2e^{u^2}} \\
&= \lim_{u\to\infty} \frac{k!}{q(u)e^{u^2}} \\
&= 0
\end{aligned}$$

where $q(u) = 2^k u^k + \dots$ is a polynomial of degree k.
(b) We have

$$f'(x) = \begin{cases} 2e^{-1/x^2}/x^3 & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} .$$

Hence, by the above argument

$$\frac{f'(h) - f'(0)}{h} = \frac{2e^{-1/h^2}}{h^4} \to 0 \qquad h \to 0 \,.$$

Hence the derivative at zero exists and $f'(0) = 0$.
(c) Use induction to show that there exists an $n \in \mathbb{N}$ and constants c_j, $j = 0, \dots, n$ such that

$$f^{(k)}(x) = \begin{cases} \sum_{j=0}^{n} c_j \frac{e^{-1/x^2}}{x^j} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Hence, the above argument shows, that $f^{(k+1)}(x)$ exists for all $x \in \mathbb{R}$ and $f^{(k+1)}(0) = 0$.

5.7-8: We have

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \,.$$

Hence $\frac{\sin(x)}{x}$ is analytic and

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} \,.$$

(Fill in the details.)

5.8-1: The function $f(x) = 1/x$ is unbounded around 0, whereas every polynomial is bounded. Hence, assume that $p(x)$ is a polynomial. Then

$$\sum_{x\in(0,1)} |f(x) - p(x)| = \infty \,.$$

5.8-2: The function $f(x) = e^x$ is unbounded on $\mathbb{R}$. Even more holds. Let $p(x) = \sum_{j=0}^{n} a_j x^j$ be a polynomial with $a_n \neq 0$. Then for x big, we have

$$|e^x - p(x)| = |x|^n \left| \frac{e^x}{x^n} - a_n - a_{n-1}/x - \ldots - a_0/x^n \right| \to \infty$$

as $x \to \infty$.

5.8-3: (a) Assume that $f \in C([0,1])$ and that $\int_0^1 f(x)x^k\, dx = 0$ for all $k = 0, 1, \ldots$. Assume that $f \neq 0$, Then

$$\int_0^1 f(x)^2\, dx = A > 0\,.$$

Let $p(x)$ be a polynomial. Then

$$\int_0^1 f(x)p(x)\, dx = 0$$

and

$$\begin{aligned} \|f - p\|_\infty^2 &\geq \int_0^1 (f(x) - p(x))^2\, dx \\ &= \int_0^1 f(x)^2\, dx - 2\int_0^1 f(x)p(x)\, dx + \int_0^1 p(x)^2\, dx \\ &\geq A > 0\,. \end{aligned}$$

Let $0 < \epsilon < A$. Then, by Weierstrass Approximation Theorem, there exists a polynomial p such that

$$\|f - p\|_\infty < \epsilon < A$$

a contradiction.

(b) Define $T_k(f) = \int_0^1 f(x)x^k\, dx$, $k = 0, 1, \ldots$. Then $T_k(af + g) = aT_k(f) + T_k(g)$ because the Riemann integral is linear. Furthermore

$$\begin{aligned} |T_k(f)| &= \left| \int_0^1 f(x)x^k\, dx \right| \\ &\leq \int_0^1 |f(x)|x^k\, dx \\ &\leq \|f\|_\infty \int_0^1 x^k\, dx \\ &= \frac{\|f\|_\infty}{k+1}\,. \end{aligned}$$

Hence T_k is bounded.

(c) Assume that $f, g \in C([0,1])$ and that $T_k(g) = T_k(f)$. Then $T_k(g-f) = 0$ for all k and hence by (a) $g - f = 0$ or $g = f$.

5.8-4: Let $k(x) = \frac{1}{2}\chi_{[-1,1]}(x)$ where $\chi_{[-1,1]}$ denotes the indicator function of the interval $[-1,1]$. Then

$$k_n(x) = nk(nx) = \frac{n}{2}\chi_{[-1/n,1/n]}$$

(fill in the detail) and

$$\int_0^1 k_n(x)\,dx = \frac{n}{2}\int_{-1/n}^{1/n} dx = 1\,.$$

If $\delta > 0$, then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and hence $k_n(x) = 0$ for $\delta \leq |x| \leq 1$.