Wavelets, Problems

Due, Friday March 7

1) Suppose that $f, \psi \in \mathcal{S}(\mathbb{R}^n)$. Show that for each $x \in \mathbb{R}^n$ that the following integrals exists and

$$
\begin{aligned}
f(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} S_\psi f(b,\omega)\psi_{b,\omega}(x)\, d\omega db \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} S_\psi f(b,\omega)\psi_{b,\omega}(x)\, db d\omega
\end{aligned}
$$

2) Assume that $n = 1$. Let $\psi(x) = e^{-\pi x^2/2}$. Find the reproducing kernel for the Hilbert space $\mathcal{H}_\psi = \mathrm{Im}(S_\psi)$. (Hint: The reproducing kernel will again have something to do with the Gaussian.)

3) Let $\psi \in L^2(\mathrm{R}^n)$. Show that $S_\psi : L^2(\mathrm{R}^n \times \mathrm{R}^n) \to L^2(\mathbb{R}^n)$ is given by

$$
S_\psi^*(F)(x) = \int_{\mathrm{R}^n} \mathcal{F}_2(F)(b,-x)\psi(x-b)\, db
$$

where $\mathcal{F}_2$ stands for the Fourier transform in the second variable. Thus, if $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then $\omega \mapsto F(b,\omega) =: F_b(\omega)$ is in $L^2(\mathbb{R}^n)$ for almost all b and hence $\mathcal{F}_2(F)(b,\omega) := \mathcal{F}(F_b)(\omega)$ is well defined for almost all $b \in \mathbb{R}^n$. (Hint: Try first ψ and F in $\mathcal{S}$ or even F of the form $F = f \otimes g : (x,y) \mapsto f(x)g(y)$ for $f, g \in \mathcal{S}$.)