Wavelets, Problems

Due, Friday March 7

1) Suppose that $f, \psi \in \mathcal{S}(\mathbb{R}^n)$. Show that for each $x \in \mathbb{R}^n$ that the following integrals exists and

$$f(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} S_\psi f(b, \omega) \psi_{b, \omega}(x) \, d\omega \, db$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} S_\psi f(b, \omega) \psi_{b, \omega}(x) \, db \, d\omega$$

2) Assume that $n = 1$. Let $\psi(x) = e^{-\pi x^2/2}$. Find the reproducing kernel for the Hilbert space $\mathcal{H}_\psi = \text{Im}(S_\psi)$. (Hint: The reproducing kernel will again have something to do with the Gaussian.)

3) Let $\psi \in L^2(\mathbb{R}^n)$. Show that $S_\psi : L^2(\mathbb{R}^n \times \mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is given by

$$S_\psi^*(F)(x) = \int_{\mathbb{R}^n} \mathcal{F}_2(F)(b, -x) \psi(x - b) \, db$$

where $\mathcal{F}_2$ stands for the Fourier transform in the second variable. Thus, if $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then $\omega \mapsto F(b, \omega) =: F_b(\omega)$ is in $L^2(\mathbb{R}^n)$ for almost all $b$ and hence $\mathcal{F}_2(F)(b, \omega) := F(F_b)(\omega)$ is well defined for almost all $b \in \mathbb{R}^n$. (Hint: Try first $\psi$ and $F$ in $\mathcal{S}$ or even $F$ of the form $F = f \otimes g : (x, y) \mapsto f(x)g(y)$ for $f, g \in \mathcal{S}$.)