Math 2057-5 Quiz #3 (Fall 2005) **Solutions**

Recall that $D = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)$ if f is differentiable.

1) Let $f(x,y) = \ln(x^2 + y^2)$.

a) Find the directional derivative of the function f at the point $(2,1)$, in the direction of the vector $\mathbf{v} =< -1, 2 >$.

Solution: The directional derivative of the function f at the point $(2,1)$, in the direction of the vector $\mathbf{v} =< -1, 2 >$ is 0.

Recall that the directional derivative of a function $f(x,y)$ at the point (a,b) in the direction of $\mathbf{u}$, $|\mathbf{u}| = 1$ is given by $\nabla f(a,b) \cdot \mathbf{u}$.

The gradient: We have $f_x(x,y) = \dfrac{2x}{x^2+y^2}$ and $f_y(x,y) = \dfrac{2y}{x^2+y^2}$. Inserting the point $(2,1)$ we get:

$$\nabla f(2,1) =< \frac{4}{5}, \frac{2}{5} > .$$

The direction: The direction is given by the vector $\mathbf{v} =< -1, 2 >$. The length of this vector is $|\mathbf{v}| = \sqrt{5}$. Hence $\mathbf{u} = \frac{1}{\sqrt{5}} < -1, 2 >$.

Finally, the directional derivative is:

$$< \frac{4}{5}, \frac{2}{5} > \cdot < \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} >= 0 .$$

Note, that could also have argued from the beginning that the directional vector $< -1, 2 >$ is perpenticular to the gradient, and hence there is no rate of change in that direction.

b) What is the maximum rate of change of f at the point $(2,1)$?

Solution: What is the maximum rate of change of f at the point $(2,1)$ is $\dfrac{2}{\sqrt{5}}$.

The maximal rate of change is always in the direction of the gradient, and the rate of change in that direction is $|\nabla f|$. We have already found the gradient at the point $(2,1)$. What we get it therefore: The maximum rate of change at the point $(2,1)$ is

$$|\nabla f(2,1)| = |< \frac{4}{5}, \frac{2}{5} >| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{2}{\sqrt{5}} .$$

2) Find the local maximum and minimum **values** and saddle point(s) of the function

$f(x,y) = x^4 + y^4 - 4xy + 2$.

Solution: The minimum value is 0. There is a saddle point at $(0,0)$ [or on the graph $(0,0,2)$], and there is no maximum.

The function is differentiable at all points. Hence, to find the critical points we have to solve $\nabla f(a,b) = 0$. This gives:

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 4x^3 - 4y = 0 \qquad \text{or} \qquad y = x^3 \\ \frac{\partial f}{\partial y}(x,y) &= 4y^3 - 4x = 0 \qquad \text{or} \qquad x = y^3 \,. \end{aligned}$$

Taking those two equations together, we get $x^9 = x$ or $x(x^8 - 1) = 0$. It follows, that $x = 0$, $x = 1$ or $x = -1$. Now we use the first equation to find y for each of those possible values of x:

$x = 0$, then $y = 0$. Gives the point $(0, 0)$.
$x = 1$, then $y = 1$. Gives the point $(1, 1)$.
$x = -1$, then $y = -1$. Gives the point $(-1, -1)$.

To find the nature of those points, we use the second derivative test. $f_{xx}(x, y) = 12x^2$. $f_{yy}(x, y) = 12y^2$ and $f_{xy} = -4$. Hence

$$D = 144x^2y^2 - 16\,.$$

At $(0, 0)$, $D = -16$ saddle point.
At $(1, 1)$, $D = 144 - 16 > 0$ and $f_{xx} = 12 > 0$. Hence, local minimum
At $(-1, -1)$, $D = 144 - 16 > 0$ and $f_{xx} = 12 > 0$. Hence, local minimum.

But this is **not** the final answer. The question is: "Find the local maximum and minimum **values** ...". We have to incert the points into f to find the values: $f(1,1) = 1+1-4+2 = 0$ and $f(-1,-1) = 1 + 1 - 4 + 2 = 0$. Hence the minimum value is 0. The final answer is therefore:

The minimum value is 0. There is a saddle point at $(0, 0)$ [or on the graph $(0, 0, 2)$], and there is no maximum.