Math 2025, Quiz #3 (Fall 2003) Name: Solutions

Recall that an inner product on $\mathbb{R}^n$ is given by $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1y_1 + \ldots + x_ny_n$. The length or norm of a vector $\mathbf{u} = (x_1, \ldots, x_n)$ is the real number

$$||(x_1, \ldots, x_n)|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{(\mathbf{u}, \mathbf{u})}$$

Inner product on functions on $[a, b]$ (piecewise continuous, continous, etc.) is given by $(f, g) = \int_a^b f(t)g(t)\,dt$. The norm is

$$||f|| = \sqrt{(f, f)} = \sqrt{\int_a^b f(t)^2\,dt}\,.$$

1) Ecaluate the following inner products:

a) $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (-1, 1, 1)$, $(\mathbf{u}, \mathbf{v}) =$ $\underline{\underline{0}}$

$(u,v) = 1\cdot(-1) + 2\cdot 1 + (-1)\cdot 1 = -1 + 2 - 1 = 0$

b) $a = 0, b = 1$, $f(t) = t^2 + 1$, and $g(t) = -t + 2$. $(f, g) =$ $23/12$

$f(t)\cdot g(t) = (t^2+1)(2-t) = 2t^2 + 2 - t^3 - t$, $(f,g) = \int_0^1 2t^2 + 2 - t^3 - t\,dt =$
$= \frac{2}{3}t^3 + 2t - \frac{1}{4}t^4 - \frac{t^2}{2}\Big]_0^1 = \frac{2}{3} + 2 - \frac{1}{4} - \frac{1}{2} = \frac{8 + 24 - 3 - 6}{12} = 23/12$

2) Evaluate the norm of the following vectors:

a) $\mathbf{u} = (1, -2, 2, 1)$, $||\mathbf{u}|| =$ $\sqrt{10}$

$||u|| = \sqrt{(u,u)} = \sqrt{1+4+4+1} = \sqrt{10}$

b) $a = 0, b = 1$, $f(t) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, $||f|| =$ 1

$f(t)^2 = \chi_{[0,1/2)}(t)^2 - 2\chi_{[0,1/2)}(t)\chi_{[\frac{1}{2},1)}(t) + \chi_{[\frac{1}{2},1)}(t)^2 = \chi_{[0,1)}(t)$

$\int_0^1 \chi_{[0,1)}(t) = \int_0^1 1\,dt = t\Big]_0^1 = 1$

3) Which of the following pair of vectors are orthogonal?

a) $\mathbf{u} = (1, -1)$, $\mathbf{v} = (1, 1)$. $\boxed{\text{Yes}}$

$(u,v) = 1 - 1 = 0$

b) $a = 0, b = 1$, $f(t) = t$ and $g(t) = 2 - 3t$. $\boxed{\text{Yes}}$

$f(t)g(t) = 2t - 3t^2$

$(f,g) = \int_0^1 2t - 3t^2\,dt = t^2 - t^3\Big]_0^1 = 1 - 1 = 0$