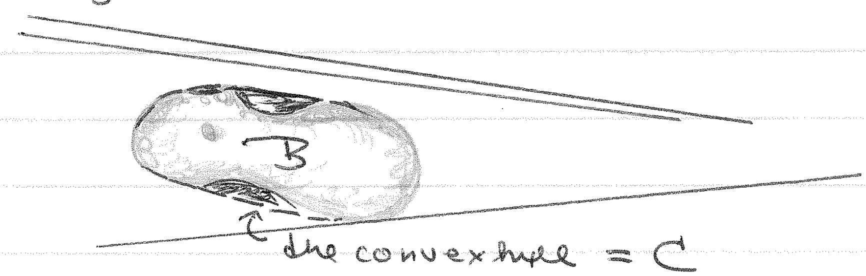
§6 Support Theorems

X-ray tomography is the 2-D Radon transform and we have seen that the reconstruction formula involves global information on $\hat{f}$. The following shows, that under the assumption that μ (or f) vanishes rapidly at ∞, we only need information on $\hat{f}$ on lines, that actually go through the body.

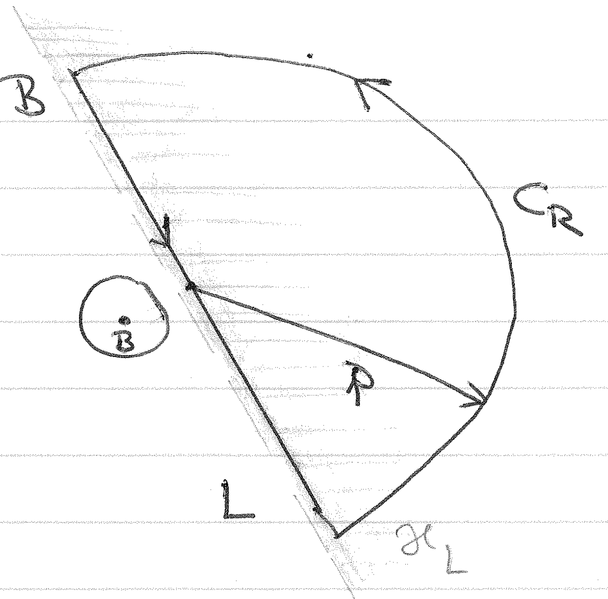


Theorem Let $f \in S(\mathbb{R}^d)$ and let $C \subseteq \mathbb{R}^d$ be convex and compact. If $\hat{f}(L) = 0$ for all L s.t. $L \cap C = \emptyset$, then $f(x) = 0$ for all $x \notin C$.

It is surprisingly hard to prove this theorem, and it is fact necessary to assume that f vanishes rapidly at infinity. Take

$$f(x,y) = (x+iy)^{-5}$$

outside a small ball $B_R \ni 0$ and such that f is smooth on $\mathbb{R}^2$. Let $L \subseteq \mathbb{R}^2$ be a line disjunct from B



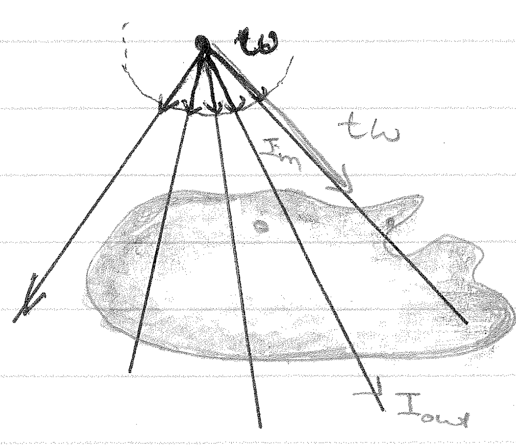
Then there exists a half-plane $\mathcal{H}_L$ such that $\mathcal{H}_L \cap B = \emptyset$ and $L \subseteq \mathcal{H}_L$. As f is holomorphic in $\mathcal{H}_L$ it follows that

$$\int_L f = -\int_{C_R} f \xrightarrow{R\to\infty} 0$$

Hence $\int_L f = 0$, but obviously $f \neq 0$ outside B.

A related statement involves to socalled fan-beam (or divergent beam) transform. For $x \in \mathbb{R}^d$ and $\omega \in S^{d-1}$ define

$$Df(x,\omega) = \int_0^\infty f(x+t\omega)\,dt$$

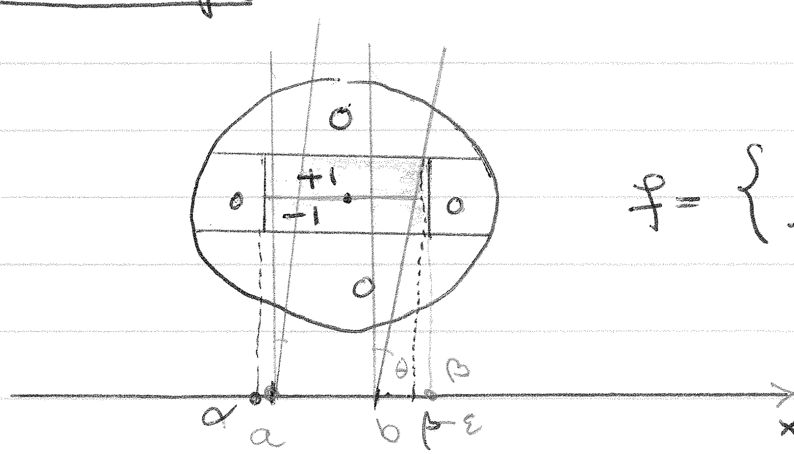


<u>Theorem</u> Let $\mathcal{U} \subseteq S^{d-1}$ be open and let $C \subseteq \mathbb{R}^d$ be a continuously differentiable curve. Let $\Omega \subseteq \mathbb{R}^d$ be bounded and open. Assume that for each $\omega \in \mathcal{U}$ there is a $x \in C$ such that the half line

$$L_+(x, \omega) = \{ x + t\omega \mid t \geq 0 \} = x + \mathbb{R}^+ \omega$$

does not intersect Ω. Then, if f is smooth, $\operatorname{Supp} f \subseteq \Omega$ and $\mathcal{D}f(x, \omega) = 0$ for all $x \in C$, $\omega \in$, then $f = 0$ in $\{x + t\omega \mid x \in C, \omega \in \ , t \geq 0\}$.

<u>Counterexample</u>: Let $\Omega = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}$



$$f = \begin{cases} +1 & \text{upper part} \\ -1 & \text{lower part} \end{cases}$$

$\alpha < a < b < \beta - \varepsilon$

we take $C = \{ a < x < b \}$ and $\mathcal{U}$ is the set of directions making an angle less than θ with the y-axis. Then obviously

$$\int_{L_+(x, \omega)} f = 0$$

for all such half-lines. Now f is not smooth. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} \varphi \subseteq B_\delta(0)$, $\delta << \varepsilon$. Then

Then
$$\int_0^\infty \varphi * f(x+t\omega)\,dt = \int_{|s|<\delta} \varphi(s) \int_0^\infty f(a-s+t\omega)\,dt\,ds$$
$$= 0$$
if δ is small enough.

Proof. One shows that (by induction)
$$D_k f(x,y) = \int_0^\infty t^k f(x+ty)\,dt = 0$$
for all $x \in C$ and $y \in \mathcal{U}$. As $t \to f(x+ty)$ is compactly supported and the polynomial hence dense in $L^2(\operatorname{Supp} f)$ it follows that $f(x+ty)=0$ for all $t \geq 0$.

Theorem Let C be a set of directions such that no non-trivial homogeneous polynomial vanishes on C. If $f \in C_c^\infty(\mathbb{R}^d)$ and $r \mapsto Rf(\omega, r) = 0$ for all $\omega \in C$, then $f = 0$.

Proof As f has compact support it follows that $\mathcal{F}f$ has a holomorphic extension to $\mathbb{C}^d$. Thus
$$\mathcal{F}f(\lambda) = \sum_{k=0}^\infty p_k(\lambda)$$
where p_k is a homogeneous polynomial of degree k. It follows that $\exists c \neq 0$
$$c\mathcal{F}_1 \hat{f}(\omega, r) = \mathcal{F}f(r\omega)$$
$$= \sum_{k=0}^\infty r^k p_k(\omega)$$
$$= 0 \text{ for all } r \text{ and } \omega \in A.$$

It follows that $a_k(\omega) = 0$ for all $\omega \in \mathbb{C}$ and hence $a_k \equiv 0$. $\square$

For similar statements about P and D, see the book by Natterer, p. 33 ff.

§ 7 Further properties of the Radon transform.

<u>a-differentiation:</u> For $f \in S(\mathbb{R}^d)$, $a \in \mathbb{R}^d$ and $t \in \mathbb{R}$ let

$$f_{t,a}(x) = f(x+ta).$$

Hence $D_a f(x) = \frac{d}{dt} f_{t,a}(x)\Big|_{t=0}$. Then

$$\hat{f}_{t,a}(\omega,p) = \int_{x\cdot\omega=p} f_{t,a}(x)\,dx$$

$$= \int_{x\cdot\omega=p} f(x+ta)\,dx$$

$$= \int_{x\cdot\omega=p+t\omega\cdot a} f(x)\,dx$$

$$= \hat{f}(\omega, p+t\omega\cdot a).$$

It follows that:

<u>Lemma</u> Let $f \in S(\mathbb{R}^d)$, $a \in \mathbb{R}^d$. Then

$$R(D_a f)(\omega,p) = \omega\cdot a \frac{\partial}{\partial p}\hat{f}(\omega,p).$$

In particular

$$R(\Delta f)(\omega,p) = \frac{\partial^2}{\partial p^2} Rf(\omega,p).$$