§8 The real life inversion

Recall from the ~~last~~ first lecture that for $f \in S(\mathbb{R}^2)$ [in particular $f \in C_c^\infty(\mathbb{R}^2)$, i.e. a smooth function with compact support] we have

$$f(x) = c \int_{S^1} \int_{-\infty}^{\infty} \frac{\partial_p Rf(\omega,p)}{x\cdot\omega - p}\, dp\, d\omega.$$

There are two obvious problems with this inversion formula

- The singularity of the kernel function

$$\frac{1}{x\cdot\omega - p}$$

But note that it is mainly located at $p = x\cdot\omega$ which is the place where the measurements are done.

- The differentiation $\frac{\partial}{\partial p} Rf(\omega,p)$.

Note, that both of those problem come from the multiplication by $|\lambda|$ in the frequency domain

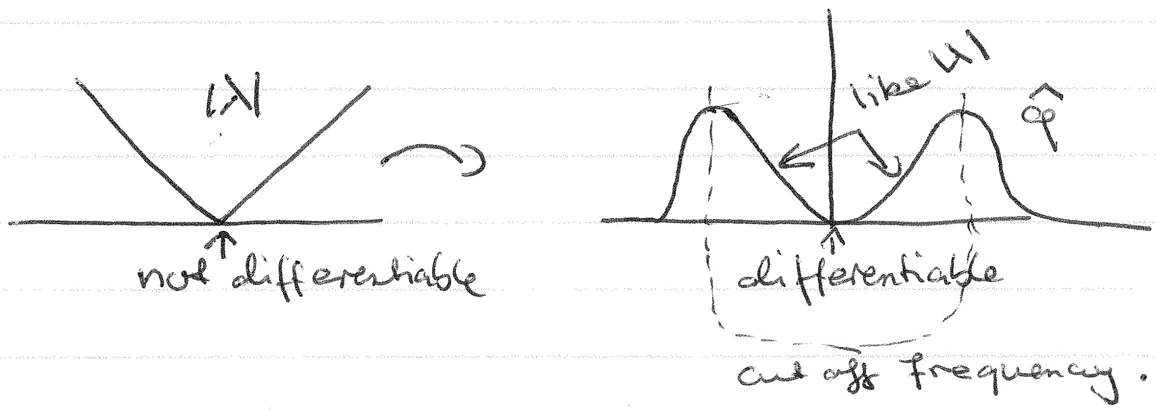
$$\mathcal{G}Rf(\omega,t) = c\, \mathcal{F}_1\left(|r|\,\mathcal{F}_1 Rf(\omega,r)\right)(t).$$

The multiplication by $|\lambda|$ can be factorized into two parts:

(1) Multiplication by $\lambda \Leftrightarrow$ correspands to the derivative ∂_p in the inversian formula.

(2) Multiplication by sign λ whic correspands to the Hilbert transform.

The simplest idle would by to replace $\lambda \to |\lambda|$ by a smooth version, that also vanishes at ∞.



This would replace

$$\int_{-\infty}^{\infty} \frac{\partial_p Rf(\omega,p)}{x\cdot\omega - p}\, dp$$

by

$$\int_{-\infty}^{\infty} \varphi(x\cdot\omega - p)\, Rf(\omega,p)\, dp$$

which can "easily" be discretizesed using a finite sum approximation to the integral. Another way to look at this is the "filtered backprojection" which is based on the following lemma.

Lemma Let $f \in S(\mathbb{R}^d)$ and $g \in S(\Xi)$.
Then
$$(R^\vee g) * f(x) = R^\vee(g *_p Rf)(x)$$

Remark: Here the subscript p in the convolution signs indicates convolution in the p-variable
$$g *_p \hat{f}(\omega,p) = \int_{-\infty}^{\infty} g(p-t)\hat{f}(\omega,t)\,dt$$
$$= \int_{-\infty}^{\infty} g(t)\,\hat{f}(\omega,p-t)\,dt \qquad \square$$

Proof: We have

$$(R^\vee g) * f(x) = \int_{\mathbb{R}^d} R^\vee g(y) f(x-y)\,dy$$

normalized measure on S^{d-1}.

$$= \int_{\mathbb{R}^d} \int_{S^{d-1}} g(\omega, y\cdot\omega)\,d\sigma(\omega) f(x-y)\,dy$$

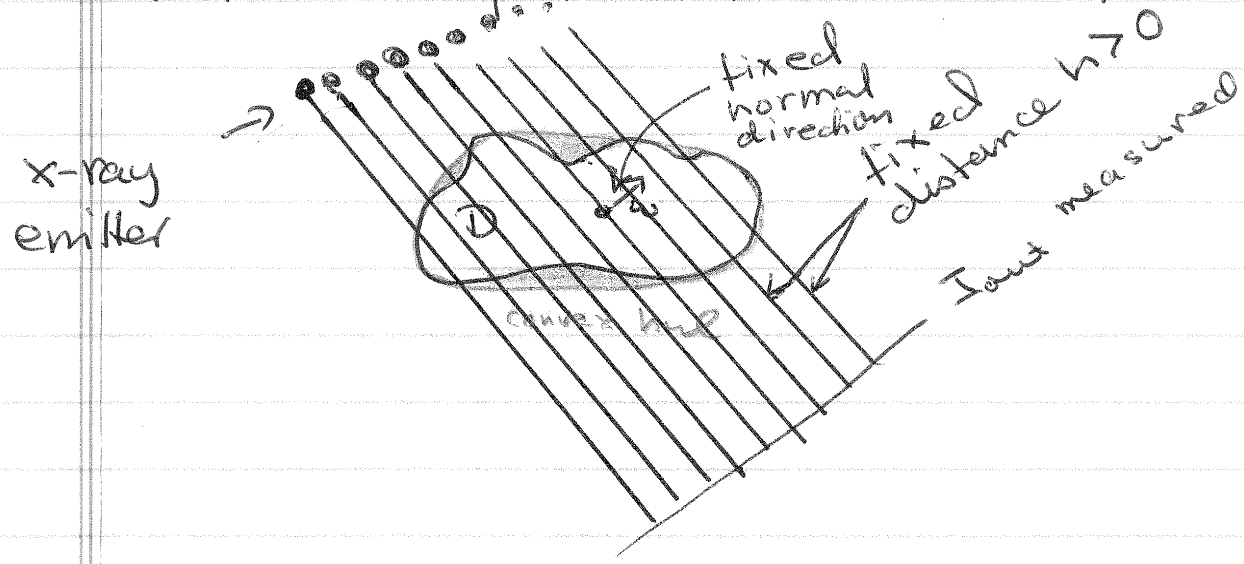
$$= \int_{S^{d-1}} \int_{\mathbb{R}^d} g(\omega, y\cdot\omega) f(x-y)\,dy\,d\sigma(\omega)$$

$$= \int_{S^{d-1}} \int_{-\infty}^{\infty} \int_{\omega^\perp} g(\omega,p) f(x-p\omega-z)\,dz\,dp\,d\sigma$$

$$= \int_{S^{d-1}} \int_{-\infty}^{\infty} \int_{\omega^\perp} g(\omega,p) f((x\cdot\omega-p)\omega-z)\,dz\,dp\,d\sigma$$

$$= \int_{S^{d-1}} \int_{-\infty}^{\infty} g(\omega,p) Rf(\omega, x\cdot\omega-p)\,dp\,d\sigma(\omega) \qquad \blacksquare$$

This can now be used to approximate the inversion in parallel-beam scan:



other algoritms should be used for other types of scanning.

- If $\overset{\vee}{R}g$ would be the δ-distribution then we have

$$\overset{\vee}{R}g * f = \delta * f = f$$
$$\cancel{= R^{\vee}(R\delta}$$
$$= R^{\vee}(g * Rf)$$

which would give an exact inversion.

- So we choose g such that

(1) $g(\omega, p) = g(p) = g(-p)$ does not depend on the direction ω

(2) $\overset{\vee}{R}g$ is "close" to the δ-distribution

Then we get

$$f \sim R^{\vee}g * f(x) = \int_{S^{d-1}} g *_p \hat{f}(\omega, x\cdot\omega)\, d\sigma(\omega)$$

approximate the inner integral by a Riemannian sum

$$\approx \int_{S^{d-1}} h \sum_j g(x\cdot\omega - jh)\, \hat{f}(\omega, jh)\, d\omega$$

we would realy only take finite sum

discretizese the outer integral by a finite sum

$$\approx \frac{h}{NH} \sum_{k=0}^{N} \sum_j g(x\cdot\omega_k - jh)\, \underbrace{\hat{f}(\omega_k, jh)}_{\text{"known measurements"}}$$

We know this for a finite sequence of inputs $\{x_e \cdot \omega_k - jh\}_{e,k,j}$ so we can easily program this.

Remark

(1) The starting point is to fix the "filter" g. A priory knowledge about f (the material) can help here (low pass filter etc.)
This way the linear map

$$\int_{S^1} g *_p$$

can be choosen arbitrary close to

the inverse of the Radon transform on "good" functions, or so that it preserves, enhances, properties that we are interested in.

(2) The numbers $g(x_k \cdot w_e - hj)$ can be stored and we do not have to redo them each time.

(3) We then get numbers that are close to $f(x_k)$, but what does this "really" tell us about f, are there error estimates, ... see the book by Natterer.

§9 Sampling theory

The final question that we will now discuss is the following: We can only use finitely many x, so how how exactly do the numbers $\{f(x_k)\}$ determine the function f? How can we reconstruct f from the discrete values?

Probably the best know theorem is the Whittaker-Shannon-Kotel'nikov (WSK)-sampling theorem for bandlimited functions. From now on we will assume that $d=1$

<u>Def</u> The function $f \in L^2(\mathbb{R})$ is called band limited if $\hat{f}$ has compact support, i.e. $\exists R\ \hat{f}(\lambda) = 0 \quad |\lambda| > R$.

In this case

$$f(x) = (2\pi)^{-1/2} \int_{-R}^{R} \hat{f}(\lambda) e^{i\lambda x}\, d\lambda$$

which implies that f extends to a holomorphic function and hence f is determined if we know $\{f(x_j)\}_j$ on <u>any</u> sequence $\{x_j\}$ which has a limit point. But this does not tell us how to construct f from $\{f(x_j)\}_j$!

Assume for simplicity that $R = \pi$, then the functions

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

form an orthonormal basis for $L^2([-\pi,\pi])$ and hence

$$\hat{f} = \sum (\hat{f}, e_n) e_n$$

But

$$(\hat{f}, e_n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\lambda) e^{-in\lambda}\, d\lambda = f(-n)$$

We also note that

$$\int_{-\pi}^{\pi} e_n(\lambda) e^{i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(n+x)\lambda}\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{i(n+x)\lambda}}{i(n+x)} \right]_{-\pi}^{\pi}$$

$$= \sqrt{2\pi}\, \frac{\sin(\pi(n+x))}{\pi(n+x)}$$

$$= \sqrt{2\pi}\, \operatorname{sinc}(\pi(x+n))$$

It follows that

Theorem (WSK). Assume that $\operatorname{supp}\hat{f} \subseteq [-\pi,\pi]$. Then

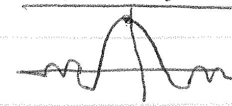
$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc}(\pi(x-n)).$$

[In general if $\operatorname{supp}\hat{f} \subseteq [-R,R]$, then

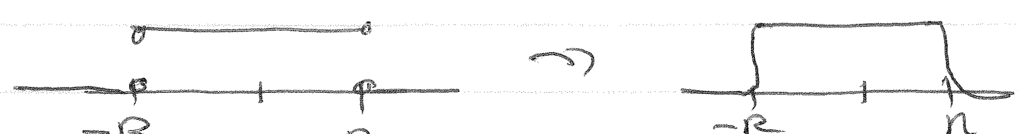
$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{R}\right) \operatorname{sinc}\left(R\left(x - \frac{n\pi}{R}\right)\right)]$$

The only problem with this formula is, that

localized wave

$$|\operatorname{sinc}(x)| \sim \frac{1}{|x|} \qquad |x| \gg 0$$ 

so the sum converges very slowly. One way is to replace $\chi_{[-R,R]}$ by a smoother function



$g \equiv 1$ on $[-R,R]$.

Will not go more into that.

Now, if g has compact support as is the case in the Radon transform of "real" objects. Then g is <u>not</u> analytic so g can <u>not</u> be bandlimited, thus for all R:

$$g(x) \neq \sum_{n=-\infty}^{\infty} g\left(\frac{n\pi}{R}\right) \operatorname{sinc}\left(R\left(x - \frac{n\pi}{R}\right)\right)$$

as a function of x.

"Def." Let $\varepsilon > 0$, $R > 0$. We say that $f \in L^2(\mathbb{R}^1)$ is essentially R-bandlimited if

$$\int_{|\lambda| \geq R} |\hat{f}(\lambda)| \, d\lambda \leq \varepsilon$$

Theorem Assume that f essentially R-bandlimited. Then there exists a constant $C > 0$ s.t.

$$\Big|\underbrace{\sum_{n=-\infty}^{\infty} g\left(\frac{n\pi}{R}\right) \operatorname{sinc}\left(R\left(x - \frac{n\pi}{R}\right)\right)} - g(x)\Big| \leq C \cdot \varepsilon.$$

Note, that this is a bandlimited function that agrees with g at $\frac{n\pi}{R}$, $n \in \mathbb{Z}$, and is close to g in the ∞-norm, whereas

$$f \rightsquigarrow \hat{f} \rightsquigarrow \hat{f}\chi_{[-R,R]} \rightsquigarrow \mathcal{F}^{-1}(\hat{f}\chi_{[-R,R]})$$

gives a bandlimited function which is close to f in the L^2-norm.