§8 The real life inversion

Recall from the last lecture that for $f \in S(\mathbb{R}^d)$, in particular $f \in C_c^\infty(\mathbb{R}^d)$, i.e. a smooth function with compact support $J$, we have

$$f(x) = c \int \int_{J} \frac{\partial \hat{f}(\omega, p)}{x \cdot \omega - p} dp \, dw.$$  

There are two obvious problems with this inversion formula:

- The singularity of the kernel function $\frac{1}{x \cdot \omega - p}$.

But note that it is mainly located at $p = x \cdot \omega$ which is the place where the measurements are done.

- The differentiation $\frac{\partial}{\partial p} \hat{f}(\omega, p)$.

Note, that both of these problems come from the multiplication by $|x|$ in the frequency domain

$$\hat{g}(\omega, t) = c \int |\omega| \hat{f}(\omega, r) \, dt.$$  

The multiplication by $|x|$ can be
Factorized into two parts:
1. Multiplication by $\lambda \Leftrightarrow$ corresponds to the derivative $\partial_p$ in the inversion formula.
2. Multiplication by $\text{sign} \lambda$ which corresponds to the Hilbert transform.

The simplest idea would be to replace $\lambda \Leftrightarrow M$ by a smooth version, that also vanishes at $\infty$.

This would replace
\[
\int_{-\infty}^{\infty} \partial_p Rf(w,p) \frac{dp}{x \cdot w - p}
\]
by
\[
\int_{-\infty}^{\infty} (x \cdot w - p) Rf(w,p) dp
\]
which can "easily" be discretized using a finite sum approximation to the integral. Another way to look at this is the "filtered backprojection" which is based on the following lemma.
Lemma. Let \( f \in S(\mathbb{R}^d) \) and \( g \in S(\mathbb{R}) \).

Then
\[
(R^v g) * f(x) = R^v(g * _p Rf)(x)
\]

Remark: Here the subscript \( p \) in the convolution signs indicates convolution in the \( p \)-variable.

\[
g * _p f(w, \cdot) = \int_{-\infty}^{\infty} g(p-t) f(q) \, dt
\]

Proof: We have
\[
(R^v g) * f(x) = \int_{\mathbb{R}^d} R^v g(y) f(x-y) \, dy
\]

\[
= \int_{\mathbb{R}^d} \int_{S^{d-1}} g(w, y \cdot w) \, d\sigma(w) f(x-y) \, dy
\]

\[
= \int_{S^{d-1}} \int_{\mathbb{R}^d} g(w, y \cdot w) f(x-y) \, dy \, d\sigma(w)
\]

\[
= \int_{S^{d-1}} \int_{\mathbb{R}^d} g(w, y \cdot w) f(x-pw-z) \, dz dp \, d\sigma(w)
\]

\[
= \int_{S^{d-1}} \int_{\mathbb{R}^d} g(w, p) f((x \cdot w - p)w - z) \, dz dp \, d\sigma(w)
\]

\[
= \int_{S^{d-1}} \int_{\mathbb{R}^d} g(w, p) Rf(w, x \cdot w - p) \, dp \, d\sigma(w)
\]
This can now be used to approximate the inversion in parallel-beam scan.

Other algorithms should be used for other types of scanning.

- If \( R^v \) would be the \( \delta \)-distribution then we have

\[
R^v g \ast f = \delta \ast f = f = R^v (g \ast Rf)
\]

which would give an exact inversion.

- So we choose \( g \) such that

1. \( g(w \cdot p) = g(p) = g(c - p) \) does not depend on the direction \( w \)
2. \( R^v g \) is "close" to the \( \delta \)-distribution
Then we get
\[ f \sim R_y \ast p(x) = \int_{S^{d-1}} g \ast p \hat{f}(w, x, w) dw \]
approximate the inner integral by a Riemannian sum
\[ \approx \int h \sum_{j} g (x \cdot w - jh) \hat{f}(w, jh) dw \]
the inner integral by a finite sum
we would readily only take finite sum
\[ \approx \frac{h}{Nh} \sum_{k=0}^{N} \sum_{j} g (x \cdot w - k \cdot jh) \hat{f}(w, k \cdot jh) \]
\[ \text{"known measurements"} \]

We know this for a finite sequence of inputs \( \{ x, w, k \cdot jh \} \) so we can easily program this.

**Remark**

1. The starting point is to fix the "filter" \( g \). A priori knowledge about \( f \) (the material) can help here (low pass filter etc.)

This way the linear map
\[ S^{-1} g \ast p \]

can be chosen arbitrary close to
the inverse of the Radon transform on "good" functions, or so that it preserves, enhances, properties that we are interested in.

(2) The numbers $g(\mathbf{x} \cdot \mathbf{w} - h_j)$ can be stored and we do not have to redo them each time.

(3) We then get numbers that are close to $f(\mathbf{x}_k)$, but what does this "really" tell us about $f$, are there error estimates, ... see the book by Natterer.

§9 Sampling theory

The final question that we will now discuss is the following: We can only use finitely many $\mathbf{x}$, so how exactly do the numbers $2f(\mathbf{x}_k)$ determine the function $f$? How can we reconstruct $f$ from the discrete values?
Probably the best known theorem is the Whittaker-Shannon-Kotel'nikov (WSK) - sampling theorem for band-limited functions. From now on we will assume that $d = 1$.

**Definition.** The function $f \in L^2(\mathbb{R})$ is called band-limited if $\hat{f}$ has compact support, i.e., $\int_{|\lambda| > R} |\hat{f}(\lambda)|^2 d\lambda = 0$.

In this case,

$$f(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda$$

which implies that $f$ extends to a holomorphic function and hence $f$ is determined if we know $\{f(x_j)\}_j$, any sequence $(x_j)_j$ which has a limit point. But this does not tell us how to construct $f$ from $\{f(x_j)\}_j$.

Assume for simplicity that $R = \pi$, then the functions

$$\theta_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

form an orthonormal basis for $L^2([-\pi, \pi])$ and hence

$$\hat{f} = \sum (\hat{f}, e_n) e_n$$

But

$$(\hat{f}, e_n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\lambda) e^{-in\lambda} d\lambda = f(-n)$$
We also note that

\[
\int_{-\pi}^{\pi} e^{i(nx)} e^{i(n+x)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(n(n+x))} dx
\]

\[= \frac{1}{\sqrt{2\pi}} \frac{\sin(\pi(n+x))}{\pi(n+x)}\]

\[= \sqrt{2\pi} \sin c(\pi(x+n))\]

It follows that

**Theorem (WSK).** Assume that \(\text{supp} \hat{f} \subseteq [-\pi, \pi]\).
Then

\[f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(m) \sin c(\pi(x-n)).\]

In general, if \(\text{supp} \hat{f} \subseteq [-R, R]\), then

\[f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(\frac{n\pi}{R}) \sin c(R(x-\frac{n\pi}{R}))\]

The only problem with this formula is that

\[|\sin c(x)| \sim \frac{1}{1+x^2}, \quad |x| \gg 0\]

so the sum converges very slowly. One way is to replace \(X_{[-R, R]}\) by a smoother function

\[f(x) \sim 1 \text{ on } [-R, R].\]

Will need go more into that.
Now, if \( g \) has compact support as is the case in the Radon transform of "real" objects, then \( g \) is not analytic so \( g \) can not be bandlimited, thus for all \( R \):

\[
g(x) = \sum_{n=-\infty}^{\infty} g\left(\frac{n \pi}{R}\right) \text{sinc}\left(R(x - \frac{n \pi}{R})\right)
\]
as a function of \( x \).

**Definition:** Let \( \varepsilon > 0 \), \( R > 0 \). We say that \( f \in L^2(\mathbb{R}) \) is essentially \( R \)-bandlimited if

\[
\int_{|x| \geq R} |\hat{f}(x)| \, dx \leq \varepsilon
\]

**Theorem:** Assume that \( f \) essentially \( R \)-bandlimited

Then there exists a constant \( c > 0 \) so that

\[
\left| \sum_{n=-\infty}^{\infty} g\left(\frac{n \pi}{R}\right) \text{sinc}\left(R(x - \frac{n \pi}{R})\right) - g(x) \right| \leq c \cdot \varepsilon
\]

Note, that this is a bandlimited function that agrees with \( g \) at \( \frac{n \pi}{R} \), \( n \in \mathbb{Z} \), and is close to \( g \) in the \( \ell^2 \)-norm, whereas

\[
\hat{f} \mapsto \hat{f} \chi_{L-2R,2R} \mapsto \hat{f}^\prime \chi_{L-2R,2R}
\]
gives a bandlimited function which is close to \( f \) in the \( L^2 \)-norm.