Solutions to some of the problems, part I.

3.4-1 Solution: Let $p \in (a, b)$ and define $f(x) = \begin{cases} 0 & x \neq p \\ 1 & x = p \end{cases}$. Then $f \neq 0$ but $\int_a^b f\,dx = 0$.

<u>3.4-2</u> <u>Solution:</u>

$$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f, f\rangle + \langle g, g\rangle + 2\langle f, g\rangle$$

$$= \|f\|^2 + \|g\|^2 + 2\langle f, g\rangle$$

But we have seen, that $2\langle f, g\rangle \leq 2\|f\| \cdot \|g\|$.
Hence

$$\|f+g\|^2 \leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|$$

$$= (\|f\| + \|g\|)^2.$$

Taking the square root on both sides, gives

$$\|f+g\| \leq \|f\| + \|g\|.$$

<u>3.4-4</u> <u>Solution</u>: Recall, that the Cauchy-Schwarz inequality states

$$|\langle f, g\rangle| \leq \|f\|_2 \cdot \|g\|_2.$$

a) $\int_0^\pi \sqrt{x\sin x}\,dx \le \pi$: Take $f(x)=\sqrt{x}$ and $g(x)=\sqrt{\sin x}$. Then

$$\|f\|_2^2 = \int_0^\pi x\,dx = \frac{x^2}{2}\Big]_0^\pi = \frac{\pi^2}{2}.$$

$$\|g\|_2^2 = \int_0^\pi \sin x\,dx = -\cos x\Big]_0^\pi = 2$$

Hence $\|f\|_2 \cdot \|g\|_2 = \frac{\pi}{\sqrt{2}} \cdot \sqrt{2} = \pi$ which implies

$$\int_0^\pi \sqrt{x\sin x}\,dx \le \pi.$$

b) $\int_0^{\pi/4} (1+\tan(x))\sqrt{x}\sec(x)\,dx$: Take $f(x)=\sqrt{x}$ and $g(x) = (1+\tan(x))\sec(x)$. Then

$$\|f\|_2^2 = \int_0^{\pi/4} x\,dx = \tfrac{1}{2}x^2\Big]_0^{\pi/4} = \frac{\pi^2}{32}$$

$$\|g\|_2^2 = \int_0^{\pi/4} (1+\tan(x))^2\sec^2(x)\,dx \quad ; \; u=\tan(x),\, du = \sec(x)^2\,dx$$

$$= \int_0^1 (1+u)^2\,du$$

$$= \frac{1}{3}(1+u)^3\Big]_0^1 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Hence $\int_0^{\pi/4} (1+\tan(x))\sqrt{x}\sec(x)\,dx \le \frac{\pi}{\sqrt{32}} \cdot \frac{\sqrt{7}}{\sqrt{3}} = \underline{\underline{\pi\sqrt{\frac{7}{96}}}}$

8) $f \perp g \iff \|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$.

__Solution__: We have

$$\|f+g\|_2^2 = \langle f+g, f+g\rangle$$
$$= \|f\|_2^2 + \|g\|_2^2 + 2\langle f,g\rangle.$$

Hence

$$\langle f,g\rangle = \frac{1}{2}\left(\|f+g\|_2^2 - \|f\|_2^2 - \|g\|_2^2\right)$$

which obviously implies the statement.

9) $f, g \in \mathcal{R}[a,b]$, $\|g\|_2 > 0$.

(a) we have $f - cg \perp g \iff 0 = \langle f - cg, g\rangle$
$$= \langle f, g\rangle - c\|g\|_2^2$$

Thus we take
$$c = \frac{\langle f, g\rangle}{\|g\|_2^2}.$$

(b) $f(x) = x$ and $g(x) = 1$ on $[0,1]$. Find c such that $f - cg \perp g$.

Solution: we have to take $c = \dfrac{\langle f, g\rangle}{\|g\|_2^2}$.

- $\|g\|_2^2 = \int_0^1 1\,dx = 1$
- $\langle f, g\rangle = \int_0^1 x\,dx = \frac{1}{2}$

Thus $c = \dfrac{1/2}{1} = \underline{\underline{\frac{1}{2}}}$.

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Exercises 4.1

1) $f(x) = |x|$. We do a little more than asked.

- $x_0 < 0$, then if $|h| < x_0$ we have
$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{-x_0 - h + x_0}{h} = -1$$

So $\lim\limits_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = -1$

- If $x_0 > 0$ and $|h| < x_0$ (⟸ why is that needed?)
$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{x_0 + h - x_0}{h} = 1 \to 1$$

• $x_0 = 0$: If $h > 0$ then $\frac{f(0+h)-f(0)}{h} = \frac{h}{h} = 1$. But, if $h < 0$, then $\frac{f(0+h)-f(0)}{h} = \frac{-h}{h} = -1$, so

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

does not exists.

4) (a) If $g'(x)$ exists and $g(x) \neq 0$, then - because g is continuous at x (why?) - there exists a $\delta > 0$ such that $g(y) \neq 0$ for all y s.t. $|x-y| < \delta$. Let $f(y) = \frac{1}{g(y)}$ and $0 < |h| < \delta$. Then

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \frac{g(x) - g(x+h)}{g(x+h)g(x)h}$$

$$= \frac{1}{g(x+h)g(x)} \cdot \frac{g(x) - g(x+h)}{h}.$$

As g is continuous at x it follows, that

$$\lim_{h\to 0} \frac{1}{g(x+h)g(x)} = \frac{1}{g(x)^2}.$$

As $g'(x)$ exist, it follows that

$$\lim_{h\to 0} \frac{g(x) - g(x+h)}{h} = -g'(x).$$

Hence

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = -\frac{g'(x)}{g(x)^2} = \left(\frac{1}{g}\right)'(x).$$

5) $f(x) = \begin{cases} x\sin(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

a) $f \in C([0,\pi])$.

Solution: If $x_0 \neq 0$, then it is clear that f is continuous at x_0 because

$$(*)\begin{cases} \bullet\ x \to x \text{ is continuous} \\ \bullet\ x \to \frac{\pi}{x} = u \text{ is continuous (at } x \neq 0) \\ \bullet\ u \mapsto \sin(u) \text{ is continuous} \end{cases}$$

and composition and product of continuous functions is continuous.

If $x_0 = 0$, then $|f(x) - f(0)| = |x\sin(\frac{\pi}{x})| \leq |x| \to 0$ as $x \to 0$.

b) If $x_0 \neq 0$ then, as all the functions in $(*)$ are differentiable, it follows that f is differentiable at x_0. If $x_0 = 0$, then

$$\frac{f(x_0+h) - f(x_0)}{h} = \sin(\frac{\pi}{h})$$

and the limit $h \to 0$ of $\sin(\frac{\pi}{h})$ does not exist.
(take $h_n = \frac{1}{n}$, then $\sin(\frac{\pi}{h_n}) = \sin(n\pi) = 0 \to 0$
Take $h_m = \frac{2}{2n+1} \to 0$, then $\sin(\pi/h_m) = 1 \to 1$.)

6) Let $f(x) = \begin{cases} x^2\sin(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$

(a) Same as in 5.

(b) If $x_0 \neq 0$, same as in #5. If $x_0 = 0$, then

$$\frac{f(x_0+h) - f(x_0)}{h} = h\sin(\frac{\pi}{h}) \to 0$$

Hence $f'(0) = 0$.

(c) If $x \neq 0$, then

$$f'(x) = 2x\sin\left(\frac{\pi}{x}\right) - \pi\cos\left(\frac{\pi}{x}\right).$$

If $x \to 0$, then $2x\sin\left(\frac{\pi}{x}\right) \to 0$, but $\lim_{x\to 0} \pi\cos\left(\frac{\pi}{x}\right)$ does not exists. Take

$$x_n = 1/n,\ \pi\cos\left(\frac{\pi}{x}\right) = \pi\cos(\pi n) = \begin{cases} -\pi & n \text{ odd} \\ \pi & n \text{ even.} \end{cases}$$

8) Use the chain rule!

Exercises 4.2

4) $f(x) = x^3$ on $[-1,1]$. f is strictly increasing but $f'(0) = 0$.

5) $f(x) = x$ for $x \in [-1,0)$ and $f(x) = x+1$ for $x \in [0,1]$.

6) $(1-x^2)^n \geq 1 - nx^2$ for all $x \in [0,1]$.

Solution: Let $f(x) = (1-x^2)^n - 1 + nx^2$. Then $f(0) = 0$ and

$$f'(x) = -2xn(1-x^2)^{n-1} + 2nx = 2nx\left(1 - (1-x^2)^{n-1}\right).$$

If $0 < |x| \leq 1$, then $0 \leq (1-x^2)^{n-1} \leq 1$ and hence $f'(x) > 0$. It follows that f is strictly increasing and hence for $x > 0$

$$0 = f(0) < f(x)$$

or $(1-x^2)^n > 1 - nx^2$.

7) Let $f(x) = x - \log(x+1)$. Then $f(0) = 0$. Furthermore

$$f'(x) = 1 - \frac{1}{x+1} = \frac{x}{x+1} > 0 \text{ if } x > 0.$$

Hence f is strictly increasing. Thus $f(x) > f(0) = 0$ if $x > 0$, which implies that $x > \log(x+1)$.

11) Suppose $|f'(x)| \le M$ for all $x \in I$.

(a) f is uniformly continuous on I.

a) Note first that $f(y) - f(x) = f'(\xi)(y-x)$, where ξ is some point between y and x. Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{M+1}$ (note, that we have not assumed that $M \neq 0$). Then, if $|y-x| < \delta$ we get

$$
\begin{aligned}
|f(y)-f(x)| &= |f'(\xi)||y-x| \\
&< M \cdot \delta \\
&= M \frac{\varepsilon}{M+1} < \varepsilon .
\end{aligned}
$$

(b) Assume that (a,b) is a finite interval, then $\lim_{x \to a^+} f(x)$ exists.

Solution: Let $x_n \to a$, then $|f(x_n) - f(x_m)| \le M \cdot |x_n - x_m|$ which shows that $\{f(x_n)\}$ is a Cauchy-sequence (do the details!). Hence $\lim_{n\to\infty} f(x_n) = A$ exists. We have to show that A does not depend on the sequence. So assume that $\{y_n\}$ is another sequence $y_n \to a$ and let $B = \lim_{n\to\infty} f(y_n)$. Let $\varepsilon > 0$. Then we can find $N \in \mathbb{N}$ such that for all $n \geq N$ we have

- $|a - y_n| < \frac{\varepsilon}{3(M+1)}$, $|a - x_n| < \frac{\varepsilon}{3(M+1)}$
- $|A - f(x_n)| < \frac{\varepsilon}{3}$, $|B - f(y_n)| < \frac{\varepsilon}{3}$.

Hence

$$
\begin{aligned}
|A-B| &= |A - f(x_n) + f(x_n) - f(y_n) + f(y_n) - B| \\
&\le |A - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - B| \\
&< \varepsilon \qquad (\leftarrow \text{do the details}).
\end{aligned}
$$

Exercises 4.3

2) Let $F(x) = \int_0^{x^2} e^{-t^2}\,dt$, find $F'(x)$.

Solution: Let $G(u) = \int_0^u e^{-t^2}\,dt$. Then - by Thm 4.3.2 - $G'(u)$ exists and $G'(u) = e^{-u^2}$. Use the chain rule, with $u = x^2$ to show that

$$F'(x) = \frac{dG(u)}{du} \cdot \frac{du}{dx}(x)$$

$$= \underline{\underline{2xe^{-x^4}}}$$

3) Suppose $f \in \mathcal{R}[a,b]$ and let $F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$. Then F is continuous.

<u>Proof</u>. Let $M \geq 0$ be such that $|f(t)| \leq M$ for all $x \in [a,b]$. Then (assume $a \leq y < x \leq b$)

$$(*)\quad \left\{ \begin{aligned} |F(x) - F(y)| &= \left|\int_y^x f(t)\,dt\right| \\ &\leq \int_y^x |f(t)|\,dt \\ &\leq M(x-y) \end{aligned} \right.$$

Hence, if $\varepsilon > 0$, we let $\delta = \frac{\varepsilon}{M+1}$. If $|x-y| < \delta$ then $(*)$ implies that

$$|F(x) - F(y)| < \varepsilon .$$

Thus F is continuous.

10) (a) f odd $\Rightarrow f'$ even

Proof: $$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$$

$$= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} \qquad (f \text{ odd}) \qquad (*)$$

$$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= f'(x)$$

So f' is odd.

(b) f even $\Rightarrow f'$ odd. The proof is the same as in (a) except in $(*)$ where we get

$$\lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$$

$$= -\lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} = -f'(x).$$

(d) f' even does not imply that f is odd.

Let $f(x) = x^3 + 1$, not odd, but $f'(x) = x^2$ is even.

(c) If f' is odd, then f is even.

Proof: Let $g(x) = f(x) - f(-x)$. Then

$$g'(x) = f'(x) + f'(-x) = 0 \qquad (f' \text{ odd})$$

Thus $g(x) \equiv c$ is constant. But $g(0) = 0$ and hence $g \equiv 0$ or $f(x) = f(-x)$ ∎

1) $|f_n(x)| \leq \frac{1}{n} \to 0$, but $f_n'(x) = 2n\cos(n^2 x)$. In particular $f_n'(0) = 2n \to \infty$.

3) Let $f_n(x) = n\cos(\frac{x}{n^2})$. If $n \to \infty$, then $\frac{x}{n^2} \to 0$ and hence

$$\cos(\tfrac{x}{n^2}) \to 1.$$

Let N be such that $\cos(\frac{x}{n^2}) \geq \frac{1}{2}$ for all $n \geq N$. Then $f_n(x) \geq \frac{n}{2}$ so $f_n(x)$ diverges. On the other hand we have

$$f_n'(x) = -\frac{\sin(x/n^2)}{n} \to 0$$

uniformly, because $|f_n'(x)| \leq \frac{1}{n}$.

Exercises 4.5

4) Find $\lim_{x\to 0^+} x^x$. Solution: Take log to get.

$$\lim_{x\to 0^+} x\log x = -\lim_{x\to 0^+} \frac{\log(1/x)}{1/x} = \lim_{u\to\infty} \frac{\log(u)}{u}$$

$$= \lim_{u\to\infty} \frac{1/u}{1} = \lim_{u\to\infty} \frac{1}{u} = 0$$

Thus - by L'Hospital's Rule

$$\lim_{x\to 0^+} x\log x = 0.$$

Using that exp is continuous we get

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x} = e^{\lim_{x\to 0^+} x\log x}$$

$$= e^0$$

$$= \underline{\underline{1}}$$

5) If $m \in \mathbb{N}$ and $p>0$, find $\lim\limits_{x\to\infty} \dfrac{(\log x)^m}{x^p}$.

<u>Solution</u>: Note $\dfrac{\frac{d}{dx}(\log x)^m}{\frac{d}{dx}x^p} = \dfrac{m}{p}\,\dfrac{(\log x)^{m-1}}{x^p}$. Doing the differentiation m-times, we get

$$\frac{\left(\frac{d}{dx}\right)^m (\log x)^m}{\left(\frac{d}{dx}\right)^m x^p} = \frac{n!}{p^m x^p} \to 0 \quad \text{as } x \to \infty$$

Hence, using L'Hospital's rule m-times, we get

$$\lim_{x\to\infty} \frac{(\log x)^m}{x^p} = 0 .$$

7) <u>Claim</u> $\lim\limits_{x\to 0} \dfrac{e^{-1/x^2}}{x^k} = 0$ for all $k \in \mathbb{N}$.

<u>Proof</u>. Let $u = \frac{1}{x} \to \infty$. Then

$$\frac{e^{-1/x^2}}{x^k} = \frac{1/x^k}{e^{1/x^2}} = \frac{u^k}{e^{u^2}} .$$

Now use L'Hospital k-times.

<u>Exercises 4.6</u>

1) Let f be a polynomial of degree n on $(a-r, a+r)$.
Then $f(x) \equiv P_n(x)$.
Proof. Note that $f^{(n+1)} \equiv 0$. Hence $R_n \equiv 0$.
(defined on p. 111). As $f(x) - P_n(x) = R_n(x) = 0$
it follows that $f(x) = P_n(x)$.

4) Let $p(x) = 8x^3 + 2x^2 - x + 1$. Then we can write

$$p(x) = \sum_{k=0}^{3} c_k (x-1)^k \quad (\text{recall } \#1)$$

and by Definition 4.6.1 we have

$$c_k = \frac{p^{(k)}(1)}{k!}$$

Thus

- $c_0 = p(1) = 3 + 2 - 1 + 1 = 5.$
- $p'(x) = 9x^2 + 4x - 1$

  $c_1 = p'(1)/1! = 9 + 4 - 1 = 12$
- $p''(x) = 18x + 4$ and

  $c_2 = p''(1)/2 = \frac{22}{2} = 11$
- $p'''(x) = 18$ and

  $c_3 = p'''(1)/6 = 3.$

It follows that

$$p(x) = 5 + 12(x-1) + 11(x-1)^2 + 3(x-1)^3.$$

Exercises 5.1

2) Test for absolute convergence, conditional convergence, or divergence:

(a) $\sum_{k=1}^{\infty} (-1)^{k+1}$, $c_k = (-1)^{k+1}$ does not converge to zero and hence divergent by Thm. 5.1.1.

(b) Let $c_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

Absolut convergent

$$s_m = \sum_{k=1}^{m} c_k = 1 - \frac{1}{n+1} \Rightarrow 0$$

(But we can now (on the final) also use the p-test $\frac{1}{k(k+1)} \leq \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.)

(d) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ convergent by Thm. 5.1.2

because $\frac{1}{\sqrt{k}} \searrow 0$. Not absolut convergent because

$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ is divergent.

(f) $\sum_{k=0}^{\infty} (-\frac{\pi}{3})^k$ divergent, $\pi > 3$ and hence $(-\frac{\pi}{3})^k$ can not converge to zero (or use the test for the geometric series).

(g) $\sum_{k=0}^{\infty} (\frac{e}{3})^k$ convergent/absolut convergent because $0 < \frac{e}{3} < 1$.

(4) Find a_k so that $s_n = \log(n)$.

Solution: Note that $a_n = s_n - s_{n-1} = \log(\frac{n}{n-1})$ if $n > 1$. For $n = 1$ we have $s_n = a_n = 0$.

7) Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction with corresponding constant $0 < r < 1$.

(a) f is uniformly continuous: Given $\varepsilon > 0$ let $\delta = \varepsilon$. Then if $|x-y| < \delta$ we get

$$|f(x) - f(y)| \leq r|x-y|$$
$$< \delta = \varepsilon.$$

(b) Let $x_0 \in \mathbb{R}$ be arbitrary and $x_n = f(x_{n-1})$, $n \geq 1$.
Then $|x_{n+1} - x_n| \leq r^n |x_1 - x_0|$.
We prove this by induction: If $m = 1$, then we get

$$|x_2 - x_1| = |f(x_1) - f(x_0)| \leq r|x_1 - x_0|$$

Assume, the statement is correct for n. Then

$$|x_{(n+1)+1} - x_{n+1}| = |f(x_{n+1}) - f(x_n)|$$

$$\leq r|x_{n+1} - x_n|$$
$$\leq r \cdot r^n |x_1 - x_0| \quad \text{(induction)}$$
$$= r^{n+1} |x_1 - x_0|$$

(c) Show that $\{x_n\}$ is a Cauchy sequence.
<u>Solution</u>: Let $m \geq 1$, $m \in \mathbb{N}$. Then, if $n \in \mathbb{N}$,

$$|x_{n+m} - x_n| \leq \sum_{j=0}^{m-1} |x_{n+j+1} - x_{n+j}| \qquad \text{(why?)}$$
$$\leq \sum_{j=0}^{m-1} r^{n+j} |x_1 - x_0|$$
$$= r^n |x_1 - x_0| \sum_{j=0}^{m-1} r^j$$
$$\leq r^n \frac{|x_1 - x_0|}{1 - r}$$

If $x_1 = x_0$, then $x_n = x_0$ for all n and we are done. Otherwise, let $\varepsilon > 0$. Then, as $r^n \to 0$, we can find $N \in \mathbb{N}$, such that for all $n \geq N$,

$$r^n < \varepsilon \frac{1 - r}{|x_1 - x_0|}$$

But then $|x_{n+m} - x_n| < \varepsilon$ for all $n \geq N$.

The claim follows now as m was arbitrary.

(d). Let $p = \lim x_n$. Then

$$
\begin{aligned}
f(p) &= f(\lim x_n) \\
&= \lim f(x_n) && (f \text{ is continuous}) \\
&= \lim_{n\to\infty} x_{n+1} && (\text{by the def. of } x_{n+1}) \\
&= p.
\end{aligned}
$$

(e). If $f(p) = p$ and $f(q) = q$ then $p = q$ (thus, there is one and only one fixed point).

<u>Solution</u>: We have

$$|p-q| = |f(p) - f(q)| \leq r\,|p-q|.$$

So, if $p \neq q$, we would have $1 \leq r$, a contradiction.

(f). Assume that $\|f'\|_\infty = r < 1$. Then f is a contraction.

<u>Solution</u>: Let $x, y \in \mathbb{R}$. Then there exists a ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x-y).$$

Thus

$$
\begin{aligned}
|f(x) - f(y)| &\leq \|f'\|_\infty |x-y| \\
&= r\,|x-y|
\end{aligned}
$$

and $r < 1$. ∎