The first 14 problems count each 4 points, leading to 56 points. Each of the three "proof" problems at the end counts 15 points leading to 45 points for that part. To total numbers of points therefore 101. Your final grade is therefore \( \frac{\text{number of points}}{101} \times 100 \) rounded to the closest integer. In the first 14 give a short argument, a counter example, or refer to a Theorem/Lemma.

1) True or False: The set \( \{(x,y) \in \mathbb{R}^2 \mid x^2 = 1\} \) is connected.

2) True or False: The function defined on \( \mathbb{R}^2 \setminus \{0\} \) by \( f(x,y) = \frac{xy^2}{x^2 + y^2} \) is continuous.

3) True or False: \( \langle x, y \rangle = x_1y_1 - x_2y_2 \) defines an inner product on \( \mathbb{R}^2 \).

4) True or False: The dimension of \( L(\mathbb{R}^3, \mathbb{R}^4) \) is 12.

5) True or False: If \( E_1 \supseteq E_2 \supseteq \ldots \supseteq E_j \supseteq E_{j+1} \supseteq \ldots \) is a decreasing sequence of non-empty open sets in \( \mathbb{R}^n \) then \( \bigcap E_j = \emptyset \).

6) True or False: If \( D \subseteq \mathbb{R}^n \) is compact and \( f \in C(D, \mathbb{R}^m) \) then \( f(D) \) is compact.

7) True or False: Let \( D \subseteq \mathbb{R}^n \) be open and \( f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m \). Let \( p \in D \). Then \( f \) is differentiable at \( p \) if and only if the all partial derivatives \( \frac{\partial f_i}{\partial x_j}(p) \) exists.

One has to assume that \( \frac{\partial f_i}{\partial x_j} \) are continuous.

Thm 10.2.1
Prove three out of the following six statements. Here $D$ stands for a non-empty subset of $\mathbb{E}^n$

15) Suppose $f \in C(D, \mathbb{E}^m)$. If $D$ is connected then $f(D)$ is connected. Assume not. Let $u_1, u_2 \in D$ be open, $u_1 \cup u_2 \supseteq f(D)$, $f(D) \cap u_1 \cap u_2 \neq \emptyset$, and $f(D) \cap u_2 = \emptyset$. Then $f^{-1}(u_1)$ is open in $D$, $D = f^{-1}(u_1) \cup f^{-1}(u_2)$ and $f^{-1}(u_2) \neq \emptyset$, contradicting $D$ connected.

16) Suppose $D \subseteq \mathbb{E}^n$ is open and that $f : D \to \mathbb{E}^m$ is differentiable on $D$. Then $f$ is continuous on $D$. We have $\frac{f(x) - f(y)}{\|x - y\|} = \frac{Df(y)(x - y) + \varepsilon(x - y)}{\|x - y\|}$ when $\|x - y\| \to 0$ as $\|x - y\| \to 0$. Thus $\lim_{\|x - y\| \to 0} \frac{f(x) - f(y)}{\|x - y\|} = \lim_{\|x - y\| \to 0} \frac{Df(y)(x - y)}{\|x - y\|} = Df(y)(0)$.

17) Let $T \in L(\mathbb{E}^n, \mathbb{E}^n)$. Assume that $\|T\| < 1$. Then $T_K = \sum_{j=0}^{K} T^j$ is Cauchy and $T_K \to (I - T)^{-1}$.

18) Assume that $D$ is compact and $f \in C(D, \mathbb{E}^m)$. Then $f$ is uniformly continuous. Let $\varepsilon > 0$. If $x \in D$ then there exist $\delta > 0$ such that $\|y - x\| < \delta \Rightarrow \|f(y) - f(x)\| < \varepsilon/2$, we have $D \subseteq U_{\delta/2}(x)$. $D$ compact $\Rightarrow \exists x_1, \ldots, x_k \in D$ such that $D \subseteq \bigcup_{i=1}^{k} B_{\delta/2}(x_i)$. Assume $x \in D$. First, there is $j$ such that $\|x - x_j\| < \delta/2$. Then $\|f(x) - f(x_j)\| < \varepsilon/2$ and $\|y - x_j\| < \delta$.

19) Let $\cdot$ be on the vector space $V$ and $\|\cdot\| = \sqrt{x, x}$ the corresponding norm-function. Let $x, y \in V$. Then $\|x + y\| = \|x\| + \|y\|$ if and only if $<x, y> = 0$.

$\|x + y\|^2 = <x + y, x + y> = <x, x> + <y, y> + 2<x, y>$

$= \|x\|^2 + \|y\|^2 + 2<x, y>$ or $\frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2) = <x, y>$.

20) Let $U_j, j \in J$ be a collection of open sets in $\mathbb{E}^n$. Then the union $\bigcup_{j \in J} U_j$ is open.

Let $x \in U \subseteq U_j \Rightarrow \exists j : x \in U_j \subseteq \bigcup_{j \in J}$.
8) Given an example of a bounded set that is not compact.
\[(0, 1). \text{ Not closed.}\]

9) Let \(D = \{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\}\). Give an example of a set \(E \subseteq D\) that is open in \(D\) but closed in \(\mathbb{R}^n\).

10) State if the following sets are (a) open (b) closed (c) neither:
\[\begin{array}{l}
(1) \{(x, y) \mid x^2 + y^2 > 1\}. \quad \text{(a)} \\
(2) \{(x, y) \mid x + y > 1, x \geq 0\}. \quad \text{(c)} \\
(3) \{(x, y) \mid xy = 0\}. \quad \text{(b)} \\
(4) \{(x, y) \mid xy > 1\}. \quad \text{(a)} \\
(5) \{(x, y) \mid x + 2y = 1\}. \quad \text{(b)}
\end{array}\]

11) Define \(f : \mathbb{R}^2 \setminus \{0\} \) by \(f(x, y) = \frac{x^2y^3}{x^2 + y^2}\). Find \(\lim_{x \to 0} f(x) = 0\).
\[f(r(\cos \theta, \sin \theta)) = r^3 \cos^2 \theta \sin \theta \to 0\]

12) Let \(f : \mathbb{R}^3 \to \mathbb{R}^3\) be given by \(f(x_1, x_2, x_3) = (\cos(2\pi x_1), x_2 \sin(x_3), x_1 x_2)\). Find \(\frac{\partial f_2}{\partial x_1}(1, 1, 1) = 0\).
\(f_2\) does not depend on \(x_1\).

13) Assume that \(f : \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 < 2\} \to \mathbb{R}^3\) is differentiable and \(Df(1, 1, 0) =\)
\[
\begin{pmatrix}
1 & -1 & 2 \\
-3 & 4 & 0
\end{pmatrix}
\]. Find the directional derivative \(D_{(1, -1, 3)}f(1, 1, 0)\).
\[
\begin{pmatrix}
1 & -1 & 2 \\
-3 & 4 & 0
\end{pmatrix}
\begin{pmatrix}
-1 \\
3
\end{pmatrix}
= \begin{pmatrix}
1 + 3 + 6 \\
-3 - 4 + 0
\end{pmatrix}
= \begin{pmatrix}
8 \\
-7
\end{pmatrix}
\]

14) State the definition of a compact set in \(\mathbb{R}^n\).
\[\text{Any open covering has a finite subcover.}\]