1[15P]) True (T) or false (F):

a) If $f$ is differentiable at $x$, then $f$ is continuous at $x$. (T)

b) The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for all $p > 1$. (T)

c) The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{2k^3 - 1}$ converges absolutely. (F)

Answer 3 of the following 6 questions. **Circle the number of the problems you want counted.**

2[20P]) Use the Cauchy-Schwarz inequality to show that $\int_{0}^{\pi} \sqrt{x \sin(x)} \, dx \leq \pi$

**Solution:** Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{\sin(x)}$. Then the left hand side is exactly $| \langle f, g \rangle |$. According to the Cauchy-Schwarz inequality we know that $| \langle f, g \rangle | \leq \|f\|_2 \|g\|_2$. We have

$$\|f\|_2^2 = \int_{0}^{\pi} x \, dx = \frac{x^2}{2}\bigg|_{0}^{\pi} = \frac{\pi^2}{2}$$

and

$$\|g\|_2^2 = \int_{0}^{\pi} \sin(x) \, dx = -\cos(x)\bigg|_{0}^{\pi} = 2.$$

Hence

$$\int_{0}^{\pi} \sqrt{x \sin(x)} \, dx \leq \frac{\pi}{\sqrt{2}} \cdot \sqrt{2} = \pi.$$

3[20P]) Let $f, g \in \mathcal{R}[a, b]$ and $\|g\|_2 > 0$. Find a constant $c \in \mathbb{R}$ such that $(f - cg) \perp g$.

**Solution:** By definition $f - cg \perp g$ if and only if $\langle f - cg, g \rangle = \langle f, g \rangle - c\|g\|_2^2 = 0$. Hence we take $c = \langle f, g \rangle /\|g\|^2$ which is possible because $\|g\| > 0$.

4[20P]) Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$.

c) $\sum_{k=1}^{\infty} \frac{2k + 1}{3k^4 + 2k^2 - k + 1}$. 
Solution: (a) We have $\frac{1}{k(k+1)} \leq \frac{1}{k^2}$. Hence the series converges absolutely by the $p$-test.

(b) The series converges conditionally. The sequence $1/\sqrt{k}$ is monotonically decreasing to 0, and hence the alternating series converges. On the other hand

$$\frac{(-1)^{k+1}}{\sqrt{k}} = k^{-1/2}$$

and that series diverges by the $p$-test.

(c) First we note that there exists a constant $C > 0$ such that

$$\frac{2k+1}{3k^4 + 2k^2 - k + 1} \leq \frac{C}{k^3}.$$ 

To see that note that

$$\lim_{k \to \infty} \frac{2k^4 + k^3}{3k^4 + 2k^2 - k + 1} = 2/3.$$ 

Hence, there exists a $N$ such that

$$\lim_{k \to \infty} \frac{2k^4 + k^3}{3k^4 + 2k^2 - k + 1} \leq 2$$

for all $k \geq N$. Then let

$$C := \max_{k=1,\ldots,N} \left\{ \frac{2k^4 + k^3}{3k^4 + 2k^2 - k + 1}, 2 \right\}.$$ 

The series $\sum_{k=1}^{\infty} \frac{C}{k^3}$ converges and it follows that the series in (c) converges absolutely.

5[20P]) Give an example of a sequence $f_n$ such that $f_n' \neq 0$ and $f_n' \to 0$ uniformly on $\mathbb{R}$, yet $f_n(x)$ diverges for all $x \in \mathbb{R}$.

Solution: Let $f_n(x) = n \cos(x/n^2)$. If $n \to \infty$, then $x/n^2 \to 0$. Hence there exists a $N$ such that for all $n \geq N$ we have $\cos(x/n^2) \geq 1/2$. Hence $f_n(x) \geq n/2 \to \infty$. Thus $\lim_n f_n(x)$ does not exists for any $x$. On the other hand, we have $f_n'(x) = -\sin(x/n^2)/n$. Hence $|f_n'(x)| \leq 1/n \to 0$ uniformly.

6[20P]) Let $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$.

a) Show that $f$ is continuous on $\mathbb{R}$.

b) Show that $f'(x)$ exists for all $x \neq 0$ and that $f'(0)$ does not exists.

Solution: (a) The function $f$ is continuous at $x \neq 0$ because is a composition of continuous function in the domain $\mathbb{R} \setminus \{0\}$. For $x = 0$ we have

$$0 \leq |f(x)| \leq |x| \to 0, \quad x \to 0$$

as $|\sin(u)| \leq 1$. Hence $f$ is continuous at $x = 0$. 

(b) If \( x \neq 0 \) then \( f \) is differentiable at \( x \) because it is a composition of differentiable functions. For \( x = 0 \) we have to use the definition:

\[
\frac{f(h) - f(0)}{h} = \sin(1/h)
\]

and the limit \( \lim_{h \to 0} \sin(1/h) \) does not exists. Hence \( f \) is not differentiable at \( x = 0 \).

7[20P]) Show that \( \sin(x) < x \) for all \( x > 0 \).

**Solution:** This is clearly correct for \( x > 1 \) as \( \sin x \leq 1 \). Set \( F(x) = x - \sin x \). Then \( F'(x) = 1 - \cos x > 0 \) for \( 0 < x < 2\pi \) and note that \( 1 < 2\pi \). Hence \( F \) is strictly increasing on the interval \((0, 2\pi)\). As \( F(0) = 0 \) it follows that \( F(x) > 0 \) for \( x \in (0, 2\pi) \) and hence \( \sin x < x \) for all \( x > 0 \).

Prove one of the following theorems. **Circle the one that you want graded:** For the solution look at the corresponding proofs in the book.

8[25P]) Suppose that \( f_n \) is defined on a finite interval \( I \) and that \( f'_n \) is continuous on \( I \). Suppose \( f'_n \) converges uniformly on \( I \) to a function \( g \). Suppose moreover that \( f_n(a) \) converges for at least one point \( a \in I \). Then, there exists a differentiable function \( f \) such that \( f_n \to f \) uniformly on \( I \) and \( f'(x) = \lim_{n \to \infty} f'_n(x) \) for all \( x \in I \).

9[25P]) Suppose that \( x_n \geq 0 \) is a decreasing sequence with limit zero. Then the alternating sum \( \sum_{k=1}^{\infty} (-1)^{k+1} x_k \) converges.