Math 4032, Test # 1. Spring 2006 **Name:** ______________________

1[15P]) True (T) or false (F):

a) If f is differentiable at x, then f is continuous at x. (T)

b) The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for all $p > 1$. (T)

c) The series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, k^2}{2k^3 - 1}$ converges absolutely. (F)

Answer 3 of the following 6 questions. **Circle the number of the problems you want counted.**

2[20P]) Use the Cauchy-Schwarz inequality to show that $\displaystyle\int_0^{\pi} \sqrt{x \sin(x)}\, dx \le \pi$

Solution: Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{\sin(x)}$. Then the left hand side is exactly $| < f, g > |$. According to the Cauchy-Schwarz inequality we know that $| < f, g > | \le \|f\|_2 \|g\|_2$. We have

$$\|f\|_2^2 = \int_0^{\pi} x\, dx = \frac{x^2}{2}|_0^{\pi} = \frac{\pi^2}{2}$$

and

$$\|g\|^2 = \int_0^{\pi} \sin(x)\, dx = -\cos(x)|_0^{\pi} = 2\,.$$

Hence

$$\int_0^{\pi} \sqrt{x \sin(x)}\, dx \le \frac{\pi}{\sqrt{2}} \cdot \sqrt{2} = \pi\,.$$

3[20P]) Let $f, g \in \mathcal{R}[a, b]$ and $\|g\|_2 > 0$. Find a constant $c \in \mathbb{R}$ such that $(f - cg) \perp g$.

Solution: By definition $f - cg \perp g$ if and only if $< f - cg, g > = < f, g > - c\|g\|^2 = 0$. Hence we take $c = < f, g > / \|g\|^2$ which is possible because $\|g\| > 0$.

4[20P]) Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{3k^4 + 2k^2 - k + 1}$.

Solution: (a) We have $\frac{1}{k(k+1)} \leq \frac{1}{k^2}$. Hence the series converges absolutely by the p-test.

(b) The series converges conditionally. The sequence $1/\sqrt{k}$ is monotonically decreasing to 0, and hence the alternating series converges. On the other hand

$$\frac{(-1)^{k+1}}{\sqrt{k}} = k^{-1/2}$$

and that series diverges by the p-test.

(c) First we note that there exists a constant $C > 0$ such that

$$\frac{2k+1}{3k^4 + 2k^2 - k + 1} \leq \frac{C}{k^3}\,.$$

To see that note that

$$\lim_{k\to\infty} \frac{2k^4 + k^3}{3k^4 + 2k^2 - k + 1} = 2/3\,.$$

Hence, there exists a N such that

$$\lim_{k\to\infty} \frac{2k^4 + k^3}{3k^4 + 2k^2 - k + 1} \leq 2$$

for all $k \geq N$. Then let

$$C := \max_{k=1,\ldots,N} \{\frac{2k^4 + k^3}{3k^4 + 2k^2 - k + 1}, 2\}\,.$$

The series $\sum_{k=1}^{\infty} \frac{C}{k^3}$ converges and it follows that the series in (c) converges absolutely.

5[20P]) Give an example of a sequence f_n such that $f_n' \neq 0$ and $f_n' \to 0$ uniformly on $\mathbb{R}$, yet $f_n(x)$ diverges for all $x \in \mathbb{R}$.

Solution: Let $f_n(x) = n\cos(x/n^2)$. If $n \to \infty$, then $x/n^2 \to 0$. Hence there exists a N such that for all $n \geq N$ we have $\cos(x/n^2) \geq 1/2$. Hence $f_n(x) \geq n/2 \to \infty$. Thus $\lim_n f_n(x)$ does not exists for any x. On the other hand, we have $f_n'(x) = -\sin(x/n^2)/n$. Hence $|f_n'(x)| \leq 1/n \to 0$ uniformly.

6[20P]) Let $f(x) = \begin{cases} x\sin(1/x) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$.

a) Show that f is continuous on $\mathbb{R}$.

b) Show that $f'(x)$ exists for all $x \neq 0$ and that $f'(0)$ does not exists.

Solution: (a) The function f is continuous at $x \neq 0$ because is a composition of continuous function in the domain $\mathbb{R} \setminus \{0\}$. For $x = 0$ we have

$$0 \leq |f(x)| \leq |x| \to 0, \quad x \to 0$$

as $|\sin(u)| \leq 1$. Hence f is continuous at $x = 0$.

(b) If $x \neq 0$ then f is differentiable at x because it is a composition of differentiable functions. For $x = 0$ we have to use the definition:

$$\frac{f(h) - f(0)}{h} = \sin(1/h)$$

and the limit $\lim_{h \to 0} \sin(1/h)$ does not exists. Hence f is not differentiable at $x = 0$.

7[20P]) Show that $\sin(x) < x$ for all $x > 0$.

Solution: This is clearly correct for $x > 1$ as $\sin x \leq 1$. Set $F(x) = x - \sin x$. Then $F'(x) = 1 - \cos x > 0$ for $0 < x < 2\pi$ and note that $1 < 2\pi$. Hence F is strictly increasing on the interval $(0, 2\pi)$. As $F(0) = 0$ it follows that $F(x) > 0$ for $x \in (0, 2\pi)$ and hence $\sin x < x$ for all $x > 0$.

Prove one of the following theorems. **Circle the one that you want graded**: For the solution look at the corresponding proofs in the book.

8[25P]) Suppose that f_n is defined on a finite interval I and that f_n' is continuous on I. Suppose f_n' converges uniformly on I to a function g. Suppose moreover that $f_n(a)$ converges for at least one point $a \in I$. Then, there exists a differentiable function f such that $f_n \to f$ uniformly on I and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in I$.

9[25P]) Suppose that $x_n \geq 0$ is a decreasing sequence with limit zero. Then the alternating sum $\sum_{k=1}^{\infty} (-1)^{k+1} x_k$ converges.