Math 4032, Test # 1. Spring 2006 **Name:** ______________________

1[10P]) True or false:
a) Let $f : (a,b) \to \mathbb{R}$ be continuous, then f is differentiable. (F) [Take $f(x) = |x|$)

b) The function $f(x) = \begin{cases} x\sin(1/x) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ is continuous but not differentiable. (T)

Answer 3 of the following 6 questions. **Circle the number of the three problems you want graded.** Show your work, a correct argument is what counts.

2[10P]) Give an example of a function $f \in C(\mathbb{R})$ such that $f'(0)$ does not exists.
Solution: $f(x) = |x|$.

3[10P]) Give an example of a sequence of functions $\{f_n\}_n$ such that $\lim_{n\to\infty} f_n(t) = 0$ for all t, but $\lim_{n\to\infty} f'(t)$ does not exists for any t.
Solution: $f_n(x) = \frac{1}{n}(\cos(n^2 x) + \sin(n^2 x))$.

4[10P]) Expand the polynomial $p(t) = t^3 + 3t^2 - 2t + 1$ in powers of $x - 1$.
Solution: We have

$$p(t) = \sum_{n=0}^{3} \frac{p^{(n)}(1)}{n!}(x-1)^n .$$

Furthermore:

- $p(1) = 1 + 3 - 2 + 1 = 3$;
- $p'(t) = 3t^2 + 6t - 2$ and hence $p'(1) = 3 + 6 - 2 = 7$;
- $p''(t) = 6t + 6$ and hence $p''(1) = 12$;
- and finally $p'''(t) = 6$.

Hence

$$p(t) = 3 + 7(x-1) + \frac{12}{2}(x-1)^2 + \frac{6}{3}(x-1)^3 = 3 + 7(x-1) + 6(x-1)^2 + (x-1)^3 .$$

5[10P]) Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. Solution: Absolute convergent, compare it to the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ which converges.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. Conditional convergent, because $k^{-1/2}$ is monotonically decreasing with limit zero, but $\sum_k k^{-1/2}$ does not converge.

6[10P]) Find a formula for a_k such that $s_n = \sum_{k=1}^n a_k = \log(n)$.
Solution: Let $n \geq 2$, then

$$a_n = s_n - s_{n-1} = \log(n) - \log(n-1) = \log\left(\frac{n}{n-1}\right).$$

7[10P]) Use the Cauchy-Schwarz inequality to show that $\displaystyle\int_0^{\pi} \sqrt{x\sin(x)}\, dx \leq \pi$.
Solution: Use the Cauchy-Schwarz inequality (p. 87) with $f(x) = \sqrt{x}$ and $g(x) = \sqrt{\sin x}$. Then

$$\int_0^{\pi} f(x)^2\, dx = \int_0^{\pi} x\, dx = \frac{\pi^2}{2}$$

and

$$\int_0^{\pi} g(x)^2\, dx = \int_0^{\pi} \sin(x)\, dx = 2\,.$$

Hence

$$\int_0^{\pi} \sqrt{x\sin(x)}\, dx \leq \left(\int_0^{\pi} f(x)^2\, dx\right)^{1/2} \left(\int_0^{\pi} g(x)^2\, dx\right)^{1/2} = \pi\,.$$

Prove **3** of the following statements. **Circle the problems that you want graded.**

8[20P]) Suppose that $\|f\|_2\|g\|_2 > 0$, where $f, g \in \mathcal{R}[a,b]$. Show that $|(f,g)| \leq \|f\|_2\|g\|_2$ and that $|(f,g)| = \|f\|_2\|g\|_2$ if and only if there exists a $r \in \mathbb{R}$ such that $\int_a^b (f(x) + rg(x))^2\, dx = 0$.
Solution: For the first part see the proof of the Cauchy-Schwarz Theorem p. 87. Define

$$F(r) = \int_a^b (f(x) + rg(x))^2\, dx = \|f\|^2 + 2r(f,g) + r^2\|g\|^2 \geq 0\,.$$

For the minimal value we have

$$0 = F'(r) = 2(f,g) + 2r\|g\|^2$$

or $r = -(f,g)/\|g\|^2$. Insert this value of r into the first equation to get

$$0 \le \|f\|^2 - 2(f,g)^2/\|g\|^2 + (f,g)^2/\|g\|^2 = \|f\|^2 - (f,g)^2/\|g\|^2 \,.$$

Thus

$$|(f,g)| \le \|f\|\|g\| \,.$$

Finally $F(r) = 0$ if and only if $\|f\|^2 - (f,g)^2/\|g\|^2 = 0$ or $\|f\|\|g\| = |(f,g)$.

9[20]) Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Show that there exists exactly one point $p \in \mathbb{R}$ such that $f(p) = p$.
Solution: Let $x \in \mathbb{R}$ and define $x_1 = f(x)$ and then inductively $x_{n+1} = f(x_n)$. Then induction shows that

$$|x_{n+k} - x_n| \le |x_{n+1} - x_n| \sum_{j=1}^{k} r^j = r^n \sum_{j=1}^{k} r^j \le \frac{r^n}{1-r} \,.$$

It follows that $\{x_n\}$ is a Cauchy sequence and hence $\lim_{n\to\infty} x_n = p$ exists. We have (because f is continuous)

$$f(p) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \lim_{n\to\infty} x_{n+1} = p \,.$$

If $f(p) = p$ and $f(q) = q$, then

$$|p - q| = |f(p) - f(q)| \le r|p - q|$$

which is impossible (because $r < 1$) unless $p = q$.

10[20P]) Find the n^{th} Taylor polynomial for the function $f(x) = \sin(x)$ and then show that the reminder $R_n(x)$ goes to zero.
The Taylor series is

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} \,.$$

Compare to the solution to problem 6, p. 114.

11[20P]) Show that $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

Solution: Define a function $f(x) = \log(1+x)$ and $g(x) = x$. Then

$$\lim_{t\to 0} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f'(x)}{g'(x)} = \lim_{t\to 0} = 1\,.$$

It follows that

$$\lim_{n\to\infty} n \log\left(1 + \frac{1}{n}\right) \lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n}\right)}{1/n} = 1$$

and hence

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1 = e$$

because the exponential function is continuous.