1[8P] Apply the two dimensional Haar wavelet transform to the matrix \( \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \).

Answer: \( \begin{pmatrix} \frac{11}{4} & -\frac{5}{4} \\ -\frac{1}{4} & \frac{3}{2} \end{pmatrix} \)

2[12P] Apply the two dimensional Haar wavelet transform to the matrix \( \begin{pmatrix} 4 & -2 & 11 & -1 \\ 2 & 0 & 5 & -3 \\ 20 & -4 & 2 & -2 \\ 8 & 2 & -4 & -4 \end{pmatrix} \).

Answer: \( \begin{pmatrix} \frac{17}{8} & \frac{13}{8} & 2 & 5 \\ -\frac{1}{8} & -\frac{21}{8} & 15/2 & 1 \\ 0 & 2 & 1 & 1 \\ \frac{2}{3} & 2 & \frac{9}{2} & 1 \end{pmatrix} \).

3[8P] Let \( z = 2 + 3i \) and \( w = \frac{1}{2+i} \). Evaluate the following:

a) \( z \cdot w = \frac{7}{5} + \frac{4}{5}i \)

Notice first that for a complex number \( \frac{1}{x+iy} \), we have

\[
\frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i,
\]

Hence \( w = \frac{1}{5}(2 - i) \). Thus

\[
z \cdot w = \frac{1}{5}(2 + 3i) \cdot (2 - i)
\]

\[
= \frac{1}{5}(2 \cdot 2 - (3i) \cdot i) + \frac{1}{5}(3i \cdot 2 + 2 \cdot (-i))
\]

\[
= \frac{1}{5}(4 + 3) + \frac{i}{5}(6 - 2) = \frac{7}{5} + \frac{4}{5}i
\]

b) \( \bar{z} = 2 - 3i \): Recall that for a complex number \( z = x + iy \) we have \( \bar{z} + iy = x - iy \).

c) \( z^2 = z \cdot z = (2 + 3i) \cdot (2 + 3i) = (4 - 9) + 2 \cdot 3i = -5 + 6i \).

d) \( |w|^2 = \frac{1}{25}(4 + 1) = \frac{1}{5} \).

Recall that for any complex number \( z = x + iy \) the number \( |x + iy|^2 \) is a nonnegative real number give by

\[
|x + iy|^2 = (x + iy) \cdot (x + iy)
\]

\[
= (x + iy) \cdot (x - iy)
\]

\[
= x^2 + y^2
\]

4[8P] Evaluate the following multiplication of matrices:

a) \[
\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 10 \\ 1 & -16 & 5 \end{bmatrix}
\]

Recall first of all that we can only multiply \( m \times n \) matrix by an \( n \times q \) matrix and the outcome is always a \( m \times q \) matrix. Furthermore if \( A \cdot B = C \) then we have

\[
C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}
\]

Thus
\[
\begin{bmatrix}
1 & 2 \\
-1 & 3
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 4 \\
1 & -5 & 3
\end{bmatrix}
= \begin{bmatrix}
1 \cdot 2 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot (-5) & 1 \cdot 4 + 2 \cdot 3 \\
(1) \cdot 2 + 3 \cdot 1 & (-1) \cdot 1 + 3 \cdot (-5) & (-1) \cdot 4 + 3 \cdot 3
\end{bmatrix}
= \begin{bmatrix}
4 & -9 & 10 \\
1 & -16 & 5
\end{bmatrix}
\]

\(b) \ \begin{bmatrix}
2 & 2 & -1 & 1 \\
2 & 2 & 1 & 2 \\
-1 & 2 & 4 & 3
\end{bmatrix}
= \begin{bmatrix}1 & 9\end{bmatrix}.

First notice that this is a product of a \(1 \times 4\) matrix by a \(4 \times 2\) matrix. The outcome should therefore by a \(1 \times 2\) matrix (or row vector):

\[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
-1 & 2 \\
4 & 3
\end{bmatrix}
\begin{bmatrix}
2 & 2 & -1 & 1 \\
2 & 2 & 1 & 2 \\
-1 & 2 & 4 & 3
\end{bmatrix}
= \begin{bmatrix}2 & 1 + 2 \cdot 2 +(-1) \cdot (-1) + 1 \cdot 4, 2 \cdot 2 + 2 \cdot 2 + (-1) \cdot 2 + 1 \cdot 3 \\
= \begin{bmatrix}2 & 4 + 1 + 4, 4 + 4 - 2 + 3 \\
= \begin{bmatrix}11, 9\end{bmatrix}
\]

Before discussing the next problems let me recall few facts:

**Definition:** Let \(F\) be a field. A vector space \(V\) over \(F\) is a nonempty set with operations of vector addition, i.e., a map

\[V \times V \ni (u, v) \mapsto u + v \in V\]

and a scalar multiplication, i.e., a map

\[F \times V \ni (r, v) \mapsto r \cdot v \in V\]

satisfying the following properties:

A1 (Commutativity of addition) For all vectors \(u, v \in V\) we \(u + v = v + u\);

A2 (Associativity for addition) For all \(u, v, w \in V\) : \(u + (v + w) = (u + v) + w\);

A3) (Existence of additive identity) There exists an element, denote by \(0 \in V\), such that for all \(u \in V\) : \(u + 0 = u\);

A4) (Existence of additive inverse) For every \(u \in V\) there exists an element, denoted by \(-u\), such that \(u + (-u) = 0\);

A5) For all \(u \in V\) : \(1 \cdot u = u\);

A6) (Associativity of scalar multiplication) For all \(r, s \in F\) and \(u \in V\) we have \((rs) \cdot u = r \cdot (s \cdot u)\);

A7) (First distributive property) For all \(r \in F\) and \(u, v \in V\) we have \(r \cdot (u + v) = (r \cdot u) + (r \cdot v)\);

A8) (Second distributive property) For all \(r, s \in F\) and all \(u \in V\) : \((r + s) \cdot u = (r \cdot u) + (s \cdot u)\);

The first thing the check is therefore always: Is the addition and multiplication defined, and do those operations always give an element in \(V\)!

From the axioms A1-A8 it follows that:

1. We have \(0 \cdot u = 0\) for all \(u \in V\).

2. The additive inverse is \(-u = (-1) \cdot u\). That is, we take the vector \(u\) and multiply it by \(-1\). This follows from

\[
\begin{align*}
u + (-1) \cdot u &= 1 \cdot u + (-1) \cdot u \quad \text{(by A5)}
&= (1 + (-1)) \cdot u \quad \text{(by A8)}
&= 0 \cdot u
&= 0 \quad \text{(by the remark just made)}
\end{align*}
\]
Important examples of vector spaces:

1. The space \( \mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{F}\} \). Here the addition and scalar multiplication is given by

\[
(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)
\]

\[
r \cdot (x_1, \ldots, x_n) = (rx_1, \ldots, rx_n).
\]

Notice that we can actually view \( \mathbb{F}^n \) as the space of row vectors (\( 1 \times n \)-matrices) or as the space of column vectors (\( n \times 1 \)-matrices).

2. Let \( S \) be a set and let \( V = \mathbb{F}^S = \) the space of functions from \( S \) to \( \mathbb{F} \). Then we can define addition and scalar multiplication by

\[
(f + g)(s) = f(s) + g(s)
\]

\[
(r \cdot f)(s) = rf(s).
\]

We will not prove here that this gives us a vector space. Notice that in this example we can replace the target space \( \mathbb{F} \) by any vector space over \( \mathbb{F} \).

3. Let \( M(n \times m, \mathbb{F}) \) be the set of \( n \times m \) matrices with coefficients in \( \mathbb{F} \). Define addition and scalar multiplication by

\[
[a_{ij}] + [b_{ij}] := [a_{ij} + b_{ij}]
\]

\[
r[a_{ij}] = [ra_{ij}].
\]

Often we construct vector spaces in the following way:

1. We have given a vector space \( V \). In particular we know that all the axioms A1-A8 are valid for elements in \( V \).

2. Then we define a subset \( S \) of \( V \) by

\[ S = \{v \in V \mid \text{some conditions holds for } v \} \]

Thus \( S \) is in general not all of \( V \) but only those elements that satisfy the given condition. Here are some examples:

(a) Let \( \mathbb{F} = \mathbb{R} \), and let \( I \) be a nonempty interval in \( \mathbb{R} \). Let \( V \) be the vector space of functions on \( I \). According to above, we know that \( V \) is a vector space. Now let us consider the condition continuous. Thus we set

\[ C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous} \} \]

So a function on \( I \) is in the subset \( C(I) \) if and only if \( f \) is continuous. Let for example \( I = [0,1) \) for a moment, then the function \( f \) defined by \( f(x) = x^2 \) is in \( C([0,1)) \) but the function \( \varphi_0^2 \) is not in \( S \).

(b) If \( \mathbb{F} = \mathbb{C} \) then we write \( C(I, \mathbb{C}) \) for the set of functions \( f : I \to \mathbb{C} \) that are continuous.

(c) Let \( V = \mathbb{R}^2 \) and consider \( S = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0 \} \). Then only those elements in \( \mathbb{R}^3 \) that are solutions to the equation \( 3x - y + 2z = 0 \) belong to \( S \). As an example the point/vector \((1, 3, 0)\) is in \( S (3 \cdot 1 - 3 + 2 \cdot 0 = 0) \) whereas \((1,1,1)\) does not belong to \( S (3 \cdot 1 - 1 + 2 \cdot 1 = 4 \neq 0) \). One can show that \( S \) is the plane of points in \( \mathbb{R}^3 \) that are perpencular to \((3,-1,2)\).

(d) Let \( V = M(1 \times m, \mathbb{R}) \) and let \( A \) be a \( m \times k \) matrix. Consider \( S = \{x \in V \mid xA = 0 \} \). Thus \( S \) is the set of solutions of a system of \( k \)-equations with \( m \) unknowns.

(e) Let \( V = \mathbb{R}^I \), i.e., the space of functions on an interval \( I \). Define \( S = \{f \in V \mid f \text{ is piecewise continuous} \} \). Then all the continuous functions are in \( S \) as well as all the functions that are discontinuous at finitely many points. For example if \( I = [0,1) \) then all the functions \( \varphi_0^N \) and all the functions \( \psi_0^N \) are elements in \( S \). Here the condition that has to be satisfied is that \( f \) is piecewise continuous.
(f) Let $V$ be the space of all functions on the real line $\mathbb{R}$ (thus we are looking at the above example with $I = \mathbb{R}$). Let $S$ be the set be the set of polynomials of degree $\leq n$ where $n$ is some fixed nonnegative integer. Thus every element in $S$ can be written in the form $p(x) = \sum_{j=0}^{n} a_j x^j$ where $a_j$ are real numbers.

3. After defining a set $S$ in this way we often need to know if $S$ is a vector space or not. For that we again notice some simple facts:

(a) If $u$ and $v$ are two elements in $S$ then we can define $u + v \in V$ because both $u$ and $v$ are elements of the vector space $V$ and addition is defined in $V$;

(b) If $u$ is in $S$ and $r \in \mathbb{F}$ then - again because $V$ is a vector space - the vector $r \cdot u \in V$ is defined.

(c) As $V$ is a vector space it follows that all the axioms A1-A8 are valid.

(d) What is missing is the first part in the definition: Are the vectors $u + v$ and $r \cdot u$ again in $S$? If that is the case it follows that $S$ is in fact a vector space.

4. We collect this in the following:

**Definition:** Let $V$ be a vector space and $S$ a nonempty subset of $V$. Then $S$ is said to be a (vector) **subspace** (of $V$) if $S$ with the addition and scalar multiplication from $V$ is a vector space.

**Theorem:** Let $V$ be a vector space and $S$ a nonempty subset of $V$, then $S$ is a subspace of $V$ if for all $u, v \in S$ and $r \in \mathbb{F}$ we have

- (S is closed under addition): $u + v \in S$;
- (S is closed under scalar multiplication): $r \cdot u \in S$.

Notice that this implies that $0 \in S$ by taking $r = 0$ and using that $0 \cdot u = 0$ for all $u \in V$. As $S$ is supposed to be closed under scalar multiplication it follows that $0 \in S$. We can therefore conclude:

**Corollary:** Suppose that $S$ is a nonempty subset of $V$ and $0 \not\in S$, then $S$ is not a vector subspace.

Notice: This conclusion is only one way. From $0 \in S$ it does not follows that $S$ is a subspace. To show that a subset is a vector subspace, we have to show that it is closed under addition and scalar multiplication!

Notice: We can replace the two conditions $u + v \in S$ and $r \cdot u \in S$ by one condition: For all $u, v \in S$ and all $r, s \in \mathbb{F}$: $ru + sv \in V$.

5. It can now be shown that all the examples for (a)-(f) above are vector spaces.

Let now $V$ and $W$ be two vector spaces. Then we are mainly interested in special kind of maps from $V$ to $W$. Those are the functions that preserve the algebraic structure that we have.

**Definition:** Let $V$ and $W$ be vector spaces. A map $T : V \to W$ is said to be **linear** if

$$T(ru + sv) = rT(u) + sT(v)$$

for all $r, s \in \mathbb{F}$ and all $u, v \in V$.

Notice that this one condition can also be split up in two condition: $T(u + v) = T(u) + T(v)$ and $T(ru) = rT(u)$ for all $u, v \in V$ and all $r \in \mathbb{F}$.

**Lemma:** Let $T : V \to W$ be linear. Then $T(0_V) = 0_W$ where $0_V$ is the zero element in $V$ and $0_W$ is the zero element in $W$.

**Proof:** Let $u \in V$ and take $r = 0$. Then

$$T(0_V) = T(r \cdot u) \quad (\text{because } 0 \cdot u = 0_V)$$

$$= rT(u) \quad (\text{because } T \text{ is linear})$$

$$= 0_W.$$
Notice again, that this this is only a one way conclusion. \( T(0) = 0 \) does not imply that \( T \) is linear!

**Lemma:** Let \( T : V \to W \) be linear, then the set
\[
S = \{ u \in V \mid T(u) = 0 \}
\]
is a subspace of \( V \). This subspace is denoted by \( \text{Ker}(T) \).

**Proof:** Let \( u, v \in \text{Ker}(T) \) and \( r, s \in \mathbb{F} \). Then
\[
T(ru + sv) = rT(u) + sT(v) = 0 .
\]

Hence \( ru + sv \in \text{Ker}(T) \).

**Lemma:** Let \( T : V \to W \) be linear, then the set
\[
S = \{ w \in W \mid \exists v \in V : w = T(v) \}
\]
is a subspace of \( W \). This space is denoted by \( \text{Im}(T) \).

**Proof:** Let \( w, z \in \text{Im}(T) \) and \( r, s \in \mathbb{F} \). To show that \( rw + sz \in \text{Im}(T) \) we need to find a vector \( a \in V \) such that \( T(a) = rw + sz \). The only thing we know for sure is, that by definition there are vectors \( u, v \in V \) such that \( T(u) = w \) and \( T(v) = z \). Let \( a = ru + sv \in V \). Then
\[
T(a) = T(ru + sv) = rT(u) + sT(v) = rw + sz .
\]

**Lemma:** Let \( V, W \) be vector spaces, let \( S, T : V \to W \) be linear maps and let \( r, s \in \mathbb{F} \). Then the map \( rR + sS : V \to W \) \( u \mapsto rR(u) + sS(u) \), is linear.

**Proof:** Let \( a, b \in \mathbb{F} \) and \( u, v \in V \). Then the following holds:
\[
(rR + sS)(au + bv) = arR(u) + asS(u) + brR(v) + bsS(v) \quad (R \text{ and } S \text{ are linear})
\]
\[
= a(rR + sS)(u) + b(rR + sS)(v) .
\]

**Remark:** What we have in fact shown is that the space of linear maps from \( V \) to \( W \) is a vector space!

Let us now take few examples:

1. \( V = \mathbb{R}^n \) and \( W = \mathbb{R}^m \) (both viewed as row vectors). Let \( A = [a_{ij}] \) be a \( n \times m \) matrix and define a map \( T : V \to W \) by
\[
T([x_1, \ldots, x_n]) = [x_1, \ldots, x_n]A .
\]

Then \( T \) is linear. This follows from the rules of matrix multiplication: \( [rx + sy]A = r(xA) + s(yA) \).

2. If \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear map, then there exists a matrix \( A \) such that \( T(x) = xA \). To find \( A \) we let \( e_1 = [1, 0, \ldots, 0] \), \( e_2 = [0, 1, 0, \ldots, 0] \), \ldots, \( e_n = [0, \ldots, 0, 1] \). Let \( a_j = T(e_j) \).

Let
\[
A = \begin{bmatrix}
a_1 \\
\vdots \\
a_n \\
\end{bmatrix} .
\]

We leave it out as an exercise to show that \( T(x) = xA \).
Determine if the each of the following sets is a vector space or not, and state why:

a) The space of polynomials of degree \( \leq 5 \), i.e., \( V = \left\{ \sum_{j=0}^{5} a_j x^j \mid \forall j : a_j \in \mathbb{R} \right\} \); Answer: This is a vector space.

**Solution:** As this is a subset of the **vector space** of all functions on the real line, we only have to show that \( V \) is closed under addition and scalar multiplication.

Closed under addition: Let \( p(x) = \sum_{j=0}^{5} a_j x^j \) and \( q(x) = \sum_{j=0}^{5} b_j x^j \) be elements in \( V \). Then

\[
(p + q)(x) = \sum_{j=0}^{5} a_j x^j + \sum_{j=0}^{5} b_j x^j = \sum_{j=0}^{5} (a_j + b_j) x^j \in V
\]

Closed under scalar multiplication: Let \( r \in \mathbb{R} \), then

\[
(rp)(x) = r \sum_{j=0}^{5} a_j x^j = \sum_{j=0}^{5} (ra_j) x^j \in V.
\]

b) \( V = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + x^2 - y + 2z = 0\} \); Answer: This is not a vector space.

**Solution:** Notice that \((0, 0, 0) \in V \) because \( 2 \cdot 0 + 0^2 - 0 + 2 \cdot 0 = 0 \). We will now give different ways to show that this is not a vector space.

**Solution 1:** The vector \((2, 0, 0)\) is in \( V \). But \( 2 \cdot (2, 0, 0) = (4, 0, 0) \notin V \) because

\[
2 \cdot 4 - 4^2 = 8 - 16 = -8 \neq 0.
\]

Thus \( V \) is not closed under scalar multiplication. (This shows how one can use concrete examples to show that a set is not closed under scalar multiplication).

**Solution 2:** \( V \) is not closed under scalar multiplication. Let \((x, y, z) \in V \) and \( r \in \mathbb{R} \). Then \( 2x + x^2 - y + 2z = 0 \). On the other hand we have \( r \cdot (x, y, z) = (rx, ry, rz) \). We test now the condition:

\[
2(rx) + (rx)^2 - (ry) + 2(rz) = r(2x + rx^2 - y + 2z)
= r(2x + x^2 - y + 2z) + (r - 1)rx^2
= (r - 1)rx^2.
\]

Here I have used that \( 2x + x^2 - y + 2z = 0 \). I also added and substracted \( rx^2 \) to get it into the correct form. So we see that the right hand is only zero if \( r = 0, r = 1, \) or \( x = 0 \). By taking \((x, y, z) \) element in \( S \) with \( x \neq 0 \) and take \( r \neq 0, 1 \) we se that \( V \) is not closed under scalar multiplication.

**Solution 3:** \( V \) is not closed under addition. Let \((x, y, z), (r, s, t) \in V \). Then we have to test if \((x+r, y+s, z+t) \in V \). For that we calculate:

\[
2(x+r) + (x+r)^2 - (y+s) + 2(z+t) = 2(x+r) + x^2 + r^2 + 2xr - (y+s) + 2(z+t)
= (2x + x^2 - y + 2z) + (2r + r^2 - s + 2t) + 2xr
= 2xr.
\]

The right hand side is only zero if \( xr = 0 \). So we take two elements in \( V \) with the first coordinate not equal to zero, i.e., \((2, 0, 0)\) in both cases.

c) \( V = \left\{ f \in C([-1, 1]) \mid \int_{-1}^{1} f(t) \, dt = 0 \right\} \); Answer: This is a vector space.
Solution: Let \( f, g \in V \) and \( r, s \in F \) then
\[
\int_{-1}^{1} rf + sg \, dt = r \int_{-1}^{1} f \, dt + s \int_{-1}^{1} g \, dt = 0 + 0 = 0.
\]
Hence \( rf = sg \in V \).

d) The space \( V_3 \) of all functions on the interval \([0,1]\) of the form \( \sum_{j=0}^{7} a_j \psi_j^3 \), with arbitrary real numbers \( a_1, \ldots, a_7 \). Here \( \psi_j^3(t) = \psi(8t-j) \). Answer: This is a vector space.

Solution: Let \( f = \sum_{j=0}^{7} a_j \psi_j^3, g = \sum_{j=0}^{7} b_j \psi_j^3 \in V \) and \( r, s \in \mathbb{R} \). Then
\[
(r f + s g) = r \sum_{j=0}^{7} a_j \psi_j^3 + s \sum_{j=0}^{7} b_j \psi_j^3 = \sum_{j=0}^{7} (ra_j + sb_j) \psi_j^3 \in V.
\]

e) Let \( A \) be a \( n \times m \) matrices and \( V = \{ \mathbf{x} = [x_1, \ldots, x_n] \in \mathbb{R}^n \mid \mathbf{x} A = \mathbf{0} \} \). Answer: This is a vector space.

Solution: Let \( \mathbf{x}, \mathbf{y} \in V \) and \( r, s \in \mathbb{R} \). Then
\[
(r \mathbf{x} + r \mathbf{y}) A = r(x A) + s(y A)
= 0 + 0
= 0.
\]

f) \( V = \{ u \in U \mid T(u) = y \} \) where \( U \) and \( W \) are vector spaces, \( T : U \to W \) is linear and \( y \in W, y \neq 0 \). Answer: This is not a vector space.

Solution: If \( T \) is a linear map, then \( T(0_U) = 0_W \), so \( 0 \notin V \).

g) The space of functions on the real line \( \mathbb{R} \) that are solutions to the differential equation \( y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0 \), i.e. \( V = \{ y \in C^\infty(\mathbb{R}) \mid y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0 \} \). Answer: This is a vector space.

Solution: Let \( f, g \in V \) and \( r, s \in \mathbb{R} \). Then we have to show that
\[
(r f + s g)^{(n)} + a_{n-1} (r f + s g)^{(n-1)} + \ldots + a_0 (r f + s g) = 0.
\]
But
\[
(r f + s g)^{(n)} + a_{n-1} (r f + s g)^{(n-1)} + \ldots + a_0 (r f + s g) = r f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f + s g^{(n)} + a_{n-1} g^{(n-1)} + \ldots + a_0 g = r \left(f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f\right) + s \left(g^{(n)} + a_{n-1} g^{(n-1)} + \ldots + a_0 g\right) = 0.
\]

6[24P)] Determine if the following maps are linear or not, state why:

a) \( T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y, z) = (2x + y - z, xy) \). Answer: This map is not linear because of the factor \( xy \). (Do the details!)

b) \( V \) the space of polynomials of degree \( \leq 5 \) and \( W \) the space of polynomials of degree \( \leq 4 \), \( T(p)(x) = 2p'(x) + 3y''(x) \). Answer: This map is linear because differentiation is linear and linear combination of linear maps is linear. (You can also show this directly by plugging a linear combination in the definition of \( T \).

c) \( T : \mathbb{R}^4 \to \mathbb{R}; T(x_1, x_2, x_3, x_4) = 2x_1 + x_2 - 3x_3 + 4x_4 \). Answer: Linear.

Solution: Write \( T \) as
\[
T(x) = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix}
\]
and use that any map of this form is linear (see above or notes from class).

d) Let \( V_N = \{ \sum_{j=0}^{2^{N-1}} s_j \varphi_j^N \mid \forall j = 0, \ldots, 2^N - 1 : a_j \in \mathbb{R} \} \) and \( T : V_N \to V_{N-1} \) given by

\[
T \left( \sum_{j=0}^{2^{N-1}} s_j \varphi_j^N \right) = \sum_{j=0}^{2^{N-1}-1} \frac{s_{2j} + s_{2j+1}}{2} \varphi_j^{N-1}.
\]

**Answer:** This map is linear.

**Solution:** Let \( f = \sum_{j=0}^{2^{N-1}} s_j \varphi_j^N \) and \( g = \sum_{j=0}^{2^{N-1}} t_j \varphi_j^N \) be vectors in \( V_N \). Then for \( r, s \in \mathbb{R} \) we get:

\[
rf = \sum_{j=0}^{2^{N-1}} r s_j \varphi_j^N
\]

\[
sg = \sum_{j=0}^{2^{N-1}} s t_j \varphi_j^N
\]

and

\[
rf + sg = \sum_{j=0}^{2^{N-1}} (r s_j + s t_j) \varphi_j^N
\]

and hence

\[
T(rf + sg) = \sum_{j=0}^{2^{N-1}-1} \frac{r s_{2j} + s t_{2j} + s t_{2j+1}}{2} \varphi_j^{N-1}
\]

\[
= \sum_{j=0}^{2^{N-1}-1} \left( r \frac{s_{2j}}{2} + s \frac{t_{2j} + t_{2j+1}}{2} \right) \varphi_j^{N-1}
\]

\[
= r \sum_{j=0}^{2^{N-1}-1} \frac{s_{2j}}{2} \varphi_j^{N-1} + s \sum_{j=0}^{2^{N-1}-1} \frac{t_{2j} + t_{2j+1}}{2} \varphi_j^{N-1}
\]

\[
= r T(f) + s T(g)
\]

e) \( T: \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (2x + y - 3z, 3x + y + 2, x - 4y + z) \). **Answer:** Not linear.

**Solution:** By direct calculation we get

\[
T(0,0,0) = (0,2,0) \neq (0,0,0).
\]

f) \( T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), T(f) = f'' + f' \cdot f \). **Answer:** Not linear because of the factor \( f' \cdot f \).

7[12P] In the following problems, evaluate the given linear map \( T \) at the given point:

a) \( T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y, z) = (2x + 3y, -x + 4y), (x, y, z) = (2, -1, 4) \). **Answer:** (1, -6)

b) \( T: \mathbb{C}^3 \to \mathbb{C}^3, T(z_1, z_2, z_3) = ((1 + i)z_1 + 2z_2 - iz_3, z_1 + (1 - i)z_2, z_2 - \frac{1}{1 + i} z_3), (z_1, z_2, z_3) = (i, 1 + i, 2 + i). \) **Answer:** (2 + i, 2 + i, -\frac{1}{2} + \frac{3}{2} i)

c) \( T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), T(f) = f'' + 4f, f = 2 \cos(x) + \sin(x) + e^x \). **Answer:** 6\cos(x) + 3\sin(x) + 5e^x

d) \( T: C([-1, 1]) \to \mathbb{R}, T(f) = \int_{-1}^{1} f(t) \, dt, f(t) = t^2 + t + \cos(\pi t) \). **Answer:** \frac{2}{3}.