Math 4032, Test # 3. Spring 2006 **Name:** ______________________

In problems (2)-(14) **circle the number of the problem you want counted and show your work.** Recall that the function $x \mapsto [x]$ is defined by $[x] = \sup\{n \in \mathbb{Z} \mid n \leq x\}$.

1[15P]) True (T) or false (F). **Explain your answer**:

a) If $f \in \mathcal{BV}[a,b]$ then f is continuous on $[a,b]$.
Solution: F. Let $p \in (a,b)$ and take the function $f(x) = 0$ for $a \leq x < p$ and $f(x) = 2$ for $p \leq x \leq b$.

b) If $f \in \mathcal{RS}([a,b],g)$ then $g \in \mathcal{RS}([a,b],f)$.
Solution: T. See Theorem 6.3.1 in the text, integration by parts.

c) Let $f,g : [a,b] \to \mathbb{R}$ and assume that f and g are **both** discontinuous at the point $p \in (a,b)$. Then $f \notin \mathcal{RS}([a,b],g)$.
Solution: N. Theorem 6.2.4 in the text.

d) Let $f,g,h : [a,b] \to \mathbb{R}$ and assume that $f \in \mathcal{RS}([a,b],g)$. Let $p \in (a,b)$ and assume that $h(x) = g(x)$ for all $p \in [a,b] \setminus \{p\}$. Then $f \in \mathcal{RS}([a,b],h)$ and $\int_a^b f\,dh = \int_a^b f\,dg$.
Solution: Not correct as stated, one has to assume that f is continuous. See the solution to problem 6.2-7.

2[48P]) Evaluate the following integrals:

a) $\int_{-1}^1 x\,d[x] = 1$
Solution: Recall that the function $[x]$ is given by:

$$[x] = \begin{cases} -1 & , \quad -1 \leq x < 0 \\ 0 & , \quad 0 \leq x < 1 \\ 1 & , \quad x = 1 \end{cases} .$$

We can solve this in two ways.
a) There are jumps by 1 at the points 0 and 1 and hence

$$\int_{-1}^1 x\,d[x] = 0 + 1 = 1\,.$$

b) We have

$$\begin{aligned}\int_{-1}^{1} x\,d[x] &= x[x]|_{-1}^{1} - \int_{-1}^{1} [x]\,dx \\ &= (-1)\cdot(-1) - 1\cdot 1 - (-1) \\ &= 1\,.\end{aligned}$$

b) $\int_0^1 x\,d(e^x) = 1$

Solution: We use the rule $\int_a^b f dg = \int_a^b f g' dx$ if g' exists and $fg' \in \mathbb{R}[a,b]$. Hence

$$\begin{aligned}\int_0^4 x\,d(e^x) &= \int_0^1 xe^x\,dx \\ &= xe^x|_0^1 - \int_0^1 e^x\,dx \\ &= e - e^x|_0^1 \\ &= e - e + 1 \\ &= 1\,.\end{aligned}$$

Note, that this is just the rule of integration by parts!

c) $\int_1^{2\pi} x\,d\cos(x) = 2\pi$.

Solution: We use again integration by parts:

$$\begin{aligned}\int_0^{2\pi} x\,d\cos(x) &= x\cos(x)|_0^{2\pi} - \int_0^{2\pi} \cos(x)\,dx \\ &= 2\pi + \sin(x)|_0^{2\pi} \\ &= 2\pi - 0 = 2\pi\,.\end{aligned}$$

d) $\int_{-1}^1 x\,d|x| = 1$

Solution: We use again integration by parts:

$$\begin{aligned}\int_{-1}^{1} x\, d|x| &= x|x|\,|_{-1}^{1} - \int_{-1}^{1} |x|\, dx \\ &= 1-(-1)-1 \\ &= 1\,.\end{aligned}$$

e) $\int_0^2 \cos(\pi x)\, dg = -1$ where $g(x) = \begin{cases} 0 & , \quad 0 \le x < 1 \\ 2 & , \quad x = 1 \\ 1 & , \quad 1 < x \le 2 \end{cases}$.

Solution: Use the solution to problem 6.2-1 to get:

$$\int_0^2 \cos(\pi x)\, dg = \cos(\pi)\cdot 1 = -1\,.$$

f) $\int_1^2 x\, d\log x$

Solution: We have

$$\int_1^2 x\, d\log x = \int_1^2 x \cdot \frac{1}{x}\, dx = 2 - 1 = 1\,.$$

Prove one of the following three statements:

3[17P]) Let $f_1, f_2 \in \mathcal{RS}([a,b], g)$ and $c \in \mathbb{R}$. Then $\int_a^b cf_1 + f_2\, dg = c\int_a^b f_1\, dg + \int_a^b f_2\, dg$.

This is a Theorem in the book. Find which!

4[17P]) Let $p \in [a,b]$ and define $T : C[a,b] \to \mathbb{R}$ by $T(f) = f(p)$. Then T is a bounded linear functional on $C[a,b]$ (with the $\|\cdot\|_\infty$ norm) and $\|T\| = 1$.

Solution: This is problem 6.3-3. The solution is on page 7 on the solution to problems from Section 6.

5[17P]) Suppose $f'(x)$ exists on $[a,b]$ and that $f' \in \mathcal{R}[a,b]$. Show that $f \in \mathcal{BV}[a,b]$ and that $V_a^b(f) \le \int_a^b |f'(x)|\, dx$.

Solution: This is problem 6.1-14. The solution is on page 4 on the solution to problems from Section 6.

Prove **one** of the following two statements:

5[20P]) If $f, g \in \mathcal{BV}[a, b]$ then $fg \in \mathcal{BV}[a, b]$.

Solution: This is problem 6.1-9. The solution is on page 3 on the solution to problems from Section 6.

6[20P]) Let $f, g : [a, b] \to \mathbb{R}$ and $c \in \mathbb{R}$. Then $V_a^b(cf + g) \leq |c| V_a^b(f) + V_a^b(g)$.
Solution: This is problem 6.1-6. The solution is on page 3 on the solution to problems from Section 6.