MATH 1552-3, Summer 2011 Name: ____________________

Test 2, Wednesday, July 6, 2010. For partial credit, show all your work!

1[28P]) Determine which of the following series is convergent. Write **convergent** if the series is convergent, otherwise write **not convergent**. Give a short explanation for your answer:

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+2n+1}$ not convergent $\frac{n}{n^2+2n+1} \sim \frac{1}{n}$ p-test, $p=1$.

b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{n+1}$ convergent, alternating series $\frac{2}{n+1} \searrow 0$

c) $\sum_{n=0}^{\infty} n^2 e^{-n}$ convergent, $\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \frac{1}{e^n} = \left(1+\frac{1}{n}\right)\frac{1}{e} \to \frac{1}{e} < 1$

d) $\sum_{n=1}^{\infty} n! 5^{-n}$. not convergent, $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{5^n}{5^{n+1}} = \frac{n+1}{5} \to \infty > 1$.

2[24P]) Let $S = \sum_{n=1}^{\infty} a_n$ be an infinite series such that $S_N = \sum_{n=1}^{N} a_n = 5 - \frac{7}{N}$.

a) What is $S_{10} =$ $\frac{43}{10} = 4.3$, $S_{10} = 5 - \frac{7}{10} = \frac{50-7}{10} = \frac{43}{10}$

b) For $n > 1$ find a general formula for $a_n =$ $\frac{7}{n(n-1)}$

$a_n = S_n - S_{n-1} = \left(5 - \frac{7}{n}\right) - \left(5 - \frac{7}{n-1}\right) = 7\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{7}{n(n-1)}$

c) Evaluate the sum $\sum_{n=1}^{\infty} a_n =$ 5, $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = 5 - \lim_{N\to\infty} 7/N = 5$

3[7P]) Evaluate the infinite sums $\sum_{n=2}^{\infty} 4 \cdot 2^{-n} =$ 2

Geometric series starting at $n=2$ $\sum_{n=2}^{\infty} 4 \cdot 2^{-n} = 4 \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$

$= 4 \cdot \frac{\frac{1}{2^2}}{1 - \frac{1}{2}} = \underline{\underline{2}}$

4[18P] If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$ what can you say about the convergence of the following series. Write *convergent* if the series converges, otherwise *divergent*.

a) $\sum_{n=1}^{\infty} \underbrace{n^3 a_n}_{=b_n}$ convergent. $\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(n+1)^3}{n^3}\frac{a_{n+1}}{a_n}\right| = \left(1+\frac{1}{n}\right)^3\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{4} < 1$

b) $\sum_{n=1}^{\infty} \underbrace{5^n a_n}_{=c_n}$. divergent $\left|\frac{c_{n+1}}{c_n}\right| = \frac{5^{n+1}}{5^n}\left|\frac{a_{n+1}}{a_n}\right| = 5\cdot\left|\frac{a_{n+1}}{a_n}\right| \to 5\cdot\frac{1}{4} > 1$.

5[18P]) Determine the radius of convergence of the following power series:

a) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{12^n}$. $R = 12$, $a_n = 12^{-n}$, $\frac{a_{n+1}}{a_n} = \frac{12^n}{12^{n+1}} = \frac{1}{12} = L$, $R = \frac{1}{L} = 12$

b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(2n)!}$. $R = \infty$, $a_n = \frac{1}{(2n)!}$, $\frac{a_{n+1}}{a_n} = \frac{\cancel{(2n)!}}{(2(n+1))(2n+1)\cancel{(2n)!}} \to 0$

6[12P]) Find the interval of convergence of the power series $\sum_{n=1} \frac{4^n}{n^2}(x-1)^n$. Remember to consider the endpoints! $\left[\frac{3}{4}, \frac{5}{4}\right]$

$a_n = \frac{4^n}{n^2}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}}{(n+1)^2}\cdot\frac{n^2}{4^n} = \frac{4}{(1+\frac{1}{n})^2} \to 4$, $\boxed{R = \frac{1}{4}}$

$x -$ left endpoint $= \frac{3}{4}$, $\sum_1^{\infty} \frac{4^n}{n^2}\left(-\frac{1}{4}\right)^n = \sum_1^{\infty} \frac{(-1)^n}{n^2}$ convergent

$x =$ right endpoint $= \frac{1}{4}$, $\sum_1^{\infty} \frac{4^n}{n^2}\cdot\frac{1}{4^n} = \sum 1/n^2$ convergent.

7) Suppose that $\frac{2x}{16-x^2} = \sum_{n=0}^{\infty} c_n x^n$.

a [12P]) Find the first four coefficients:

$c_0 =$ 0

$\frac{2x}{16-x^2} = \frac{x}{8}\cdot\frac{1}{1-(\frac{x}{4})^2} =$

$= \frac{x}{8}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^{2n} = \sum_{n=0}^{\infty}\frac{1}{8\cdot 4^{2n}} x^{2n+1}$

$= \frac{1}{8}x + \frac{1}{8\cdot 16}x^3 + \frac{1}{8\cdot 16^2}x^{5}\ldots$

$c_1 = 1/8$

$c_2 = 0$

$c_3 = \frac{1}{128}$

b[9P]) Find the radius of convergence R of the power series. $R = 4$

$|\frac{x}{4}|^2 < 1, \ |x| < 4$

8[10P]) Find the Taylor polynomial $T_4(x)$ for $\cos(x)$ at $c = \pi/4$.

$T_4(x) = \frac{1}{\sqrt{2}}\left(1 - (x - \frac{\pi}{4}) - \frac{1}{2}(x - \frac{\pi}{4})^2 + \frac{1}{6}(x - \frac{\pi}{4})^3 + \frac{1}{24}(x - \frac{\pi}{4})^4\right)$

j	$\cos^{(j)}(x)$	$\cos^{(j)}(\pi/4)$
0	$\cos(x)$	$1/\sqrt{2}$
1	$-\sin(x)$	$-1/\sqrt{2} = 2^{-1/2}$
2	$-\cos(x)$	$-1/\sqrt{2}$
3	$\sin(x)$	$1/\sqrt{2}$
4	$\cos(x)$	$1/\sqrt{2}$

$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!}(x-c)^j$

9[12P]) Recall that the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$. Find the power series at $c = 0$ and the radius of convergence for

$f(x) = x\tan^{-1}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n-1}$; $\boxed{R = 1}$

$\tan^{-1}(x) = \int_0^x \frac{du}{1+u^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n u^{2n}\,du = \sum_{n=0}^{\infty} (-1)^n \int_0^x u^{2n}\,du$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$x\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(n+1)}}{2n+1}$

$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n-1}$.

The geometric series converges for $|x| < 1$, so $\underline{R = 1}$.