Enclosed are the tests in math 1552 for the fall 2004. The first part includes the tests without the solutions. The second part includes the tests with solution. In few cases the questions might differ from those actually given in class. Different problems are sometimes also discussed in the solution part. There are no solutions to the make up tests. Please let me know if you find any typos in the solutions.

Math 1552, Test # 1. Fall 2004 Name: ________________________________

SHOW YOUR WORK!

1[18P]) Evaluate the integrals:

a) $\int x \sin(2\, x^2)\, dx =$ ________________

b) $\int e^t \cos(t)\, dt =$ ________________

2[18P]) Evaluate the trigonometric integrals:

a) $\int \cos^4(x) \sin^3(x)\, dx =$ ________________

b) $\int \cos^2(t)\, dt =$ ________________

3[10P]) Find the antiderivative of $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\, dx =$ ________________

4[12P]) Write out the form of the following partial fraction decomposition. **Do not determine the numerical value of the constants.**

a) $\displaystyle\frac{2x}{(x+1)(x+3)^3} =$ __________________________

b) $\displaystyle\frac{1}{(x+2)(x^2+1)} =$ __________________________

6[12P]) Evaluate $\displaystyle\int \frac{x^3-x-2}{x^3-x^2+x-1}\, dx =$ ____

7[12P] Use the Simpson's Rule, with $n = 5$ to approximate the integral $\int_1^2 \sqrt{x^2-1}\, dx \approx$ ________________

8[20P] Determine whether each integral is convergent on divergent.

	Integral	Convergent/divergent		Integral	Convergent/divergent
a)	$\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$		b)	$\displaystyle\int_1^{2} \frac{1}{2-x}\, dx$	
c)	$\displaystyle\int_0^{2} \frac{1}{x-1}\, dx$		d)	$\displaystyle\int_0^{\infty} \frac{x}{x^3+2x+4}\, dx$	

Bonus question [10P]) How large do we need to take n in order to guarantee, that the Trapezoidal Rule approximation for $\displaystyle\int_1^2 \frac{1}{1+x^3}\, dx$ are accurate within 10^{-6}?

Math 1552, Test # 2. Fall 2004 Name: ________________________

1[8P]) Evaluate the following dot products:

a) $< 1, -2, 1 > \cdot < 2, 1, 0 >=$ ______

b) $(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) =$ ______

2[7P]) Evaluate the cross product $< 1, -1, 1 > \times < 1, 0, 2 >=$ ____________

3[10P]) Find the **radius** and **center** of the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z - 2 = 0$.

Radius: ____________

Center: ____________

4[10P]) Circle the correct answer:

a) The points $A(2, 1, -1)$, $B(1, 2, 1)$, and $C(-1, 2, 2)$ **lie** / **do not lie** on a line.
b) The points $K(2, 1, 0)$, $L(3, 2, 1)$, and $M(1, 0, -1)$ **lie** / **do not lie** on a line.

5[10P]) Find the **angel** between the following vectors:

a) $\mathbf{a} =< 1, -1, 1 >$, $\mathbf{b} =< 1, 1, 0 >$, $\theta =$ ______

b) $\mathbf{a} =< 1, 2, -2 >$, $\mathbf{b} =< 2, 2, 1 >$, $\theta =$ ______

6[14P]) Find the **scalar** and **vector** projection of $\mathbf{b} =< 1, 2, 1 >$ onto $\mathbf{a} =< 1, 2, -2 >$,

$\text{comp}_{\mathbf{a}}\mathbf{b} =$ ________ $\qquad$ $\text{proj}_{\mathbf{a}}\mathbf{b} =$ ____________

7[9P]) Find a **parametric** and a **symmetric** equation for the line through the points $(2, 1, -1)$ and $(1, -1, 2)$.
8[14P]) a) Find a parametric equation for the line that passes through the point $(5, 1, 0)$ and is parallel to the line with parametric equation $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$.
b) Find the point where the above line intersects the plane given by the equation $3x - 2y + z = 1$.
9[9P]) Find the equation of the plane through the point $P(1, 1, -1)$ and parallel to the plane $2x + y - 5z = 5$.
10[9P]) Find the equation of the plane containing the point $P(2, 1, -1)$ and the line given by the parametric equation $x = 3 + 2t, y = t$, and $z = 8 - t$.

Math 1552, Test # 3. Fall 2004 Name: ________________________

1[10P]) Find the Cartesian equation for the curves:

a) $x = \sqrt{t}$, $y = 1 - t$; **Solution:**
b) $r = 9\cos(\theta)$. **Solution:**

2[10P]) Find dy/dx and d^2y/dx^2 for $x = t^3 - 6t^2$ and $y = t^2 - 1$.

$$\frac{dy}{dx} = \qquad \frac{d^2y}{dx^2} =$$

3[14P]) Find the equation of the tangent line to the curve at the point corresponding to the given value of the parameter:

a) $x = t^4 + 1$, $y = t^3 + t$, $t = -1$. **The equation is:**
b) $x = \cos(\theta) + \sin(2\theta)$, $y = \sin(\theta) + \cos(2\theta)$; $\theta = \pi$. **The equation is:**

4[14Pt]) Find the points on the curve $x = 2t^3 + 3t^2 - 12t + 1$, $y = 3t^4 + 16t^3 + 24t^2 - 1$, where the tangent is horizontal or vertical.

Horizontal: **Vertical:**

5[8P]) At what point of the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with equation $x = -7t$, $y = 12t - 5$?

6[8P]) Find the length of the curve $x = t^2\cos(t)$, $y = t^2\sin(t)$, $0 \le t \le 1$.
7[8Pt]) Set up, **but do not evaluate**, an integral that represents the length of the curve $x = t\sin t$, $y = t\cos t$, $0 \le t \le \pi$;
[In most cases cases] 7[8P]) Set up an integral that gives the area that inside the curve $r = 4\sin(\theta)$ and outside the curve $r = 2$.
8[8P]) Find the equation for the parabola with vertex at $(1, 2)$ and focus at $(2, 2)$.
9[12P]) Find the vertices and foci of the ellipse $25x^2 + 9y^2 - 50x + 36y = 164$, and then sketch it graph indicating the vertices and foci.
10[8P]) This is problem from the book

Math 1552, Test # 3 make up. Fall 2004 Name: ____________________

1[10P]) Find the Cartesian equation for the curves:

a) $x = 1 - t^2$, $y = t$; **Solution:** ____________________

b) $r = 4\sin(\theta)$. **Solution:** ____________________

2[10P]) Find dy/dx and d^2y/dx^2 for $x = t^2 - 2t$ and $y = t^3 - 1$.

$$\frac{dy}{dx} = \underline{\qquad\qquad\qquad} \qquad\qquad \frac{d^2y}{dx^2} = \underline{\qquad\qquad\qquad}$$

3[14P]) Find the equation of the tangent line to the curve at the point corresponding to the given value of the parameter:

a) $x = t^4 + t$, $y = t^2$, $t = 1$. **The equation is:** ____________________

b) $x = \cos(\theta) + \sin(\theta)$, $y = \sin(\theta)$; $\theta = \pi$. **The equation is:** = ____________________

4[14P]) Find the points on the curve $x = 2t^3 + 3t^2 - 12t + 1$, $y = 3t^4 + 16t^3 + 24t^2 - 1$, where the tangent is horizontal or vertical.

Horizontal: ____________________

Vertical: ____________________

5[8P]) At what point of the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with equation $x = -7t$, $y = 12t - 5$?
6[8P]) Find the length of the curve $x = e^x\cos t$, $y = e^x\sin t$, $0 \le t \le 1$.
7[8P]) Set up an integral that gives the area that is inside the curve $r = 8\sin(\theta)$ and outside the curve $r = 4$.
8[8P]) Find the equation for the parabola with vertex at $(2, 2)$ and focus at $(2, 0)$. **9[12P])** Find the vertices and foci of the ellipse $25x^2 + 9y^2 - 50x + 36y = 164$. Then sketch its graph indicating clearly the vertices and foci.
10[9P]) Ellipse. This problem included a picture that I can not include here.

Math 1552, Test # 4. Fall 2004 Name: __________________________________

1[6P]) Find a formula for the general term a_n of the sequence $\{a_n\}_{n=1}^{\infty}$ if the first terms are given by $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$ **Solution:** $a_n =$ __________________________________

2[24p]) Determine if the sequence converges or diverges. If it converges find the limit.

a) $a_n = \dfrac{1+2n+n^2}{1+n}$;

b) $a_n = \dfrac{2^n}{3^{2n+1}}$

c) $a_n = n^2 e^{-n}$

d) $a_n = n\sin(1/n)$

3[18p]) Evaluate the series:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} =$ ________________

b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{5}{2^n} =$ ________________

c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{10^n} =$ ________________

4[24p]) Determine if the series **converges**, **converges absolutly** or **converges conditionally**:

a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

5[24p]) Find the radius of convergence of the following series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

b) $\displaystyle\sum_{n=1}^{\infty} 2^n (x-1)^n$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+3)^n}{n^2}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

6[7p]) Find the power series for the function $\dfrac{1}{(1-x)^2}$. What is the radius of convergence?

Math 1552, Test # 4. Make up Fall 2004 Name: __________________________________

1[6P]) Find a formula for the general term a_n of the sequence $\{a_n\}_{n=1}^{\infty}$ if the first terms are given by $\{\frac{5}{2}, \frac{5}{4}, \frac{5}{6}, \frac{5}{8}, \ldots\}$ **Solution:** $a_n =$ ____________________

2[24p]) Determine if the sequence converges or diverges. If it converges find the limit.

a) $a_n = \dfrac{1+2n+5n^2}{1+3n+4n^2}$;

b) $a_n = \dfrac{1+n+n^4}{3^{2n+1}}$

c) $a_n = \dfrac{\ln(n)}{2n}$

d) $a_n = n(\cos(1/n) - 1)$

3[18p]) Evaluate the series:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{4}{2^n} =$ ____________

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n} =$ ____________

c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{10^n} =$ ____________

4[24p]) Determine if the series **converges**, **converges absolutly** or **converges conditionally**, or diverges:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

5[24p]) Find the radius of convergence of the following series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n x^n}{n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(5x+2)^n}{2^n}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$

6[7p]) Find the power series for the function $\ln(1+x)$. What is the radius of convergence?

Math 1552, Test # 1. Fall 2004 Name: ____________________

SHOW YOUR WORK!

1[18P]) Evaluate the integrals:

a) $\int x \sin(2x^2)\, dx = -\frac{1}{4}\cos(2x^2) + C$

Solution: Use substitution $u = 2x^2$, $du = 4xdx$. Hence

$$\begin{aligned}\int x\sin(2\,x^2)dx &= \frac{1}{4}\sin\sin(u)du \\ &= -\frac{1}{4}\cos(u) + C \\ &= -\frac{1}{4}\cos(2x^2) + C\end{aligned}$$

If the question was $\int x\sin(x)\,dx$ then you use partial integration:

$$\int x\sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + C\,.$$

b) $\int e^t\cos(t)\,dt = \frac{1}{2}(e^t\cos(t) + e^t\sin(t)) + C$.
Solution: Use partial integration twice with $dv = e^t$ or $v = e^t$. Then we get

$$\begin{aligned}\int e^t\cos(t)dt &= e^t\cos(t) + \int e^t\sin(t)dt \\ &= e^t\cos(t) + e^t\sin(t) - \int e^t\cos(t)dt\end{aligned}$$

Moving $-\int e^t\cos(x)dx$ over to the left hand side and then divide by 2 gives the final answer
2[18P]) Evaluate the trigonometric integrals:

a) $\int\cos^4(x)\sin^3(x)\,dx = -\frac{1}{5}\cos^5(x) + \frac{1}{7}\sin^7(x) + C$
Solution: Write $\sin^2(x) = 1 - \cos^2(x)$. Then the integral becomes:

$$\int(\cos^4(x) - \cos^6(x))\sin(x)dx$$

Now set $u = \cos(x)$. Then $du = -\sin(x)dx$ and we get

$$\int -u^4 + u^6 du = -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C = -\frac{1}{5}\cos^5(x) + \frac{1}{7}\sin^7(x) + C$$

b) $\int\cos^2(t)\,dt = \frac{t}{2} + \frac{1}{4}\sin(2t) + C$
Solution: Write $\cos^2(t) = \frac{1}{2}(1+\cos(2t)$.

3[10P]) Find the antiderivative of $\displaystyle\int\frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right)$

Solution: Set $x = 3\sin\theta$. Then $\theta = \sin^{-1}(x/3)$. Furthermore $dx = 3\cos(\theta)d\theta$ and $9 - x^2 = 9 - 9\sin^2(\theta) = 9\cos^2(\theta)$. Hence the integral becomes:

$$\int\frac{3\cos(\theta)}{3\cos(\theta)}\,d\theta = \int d\theta = \theta + C = \sin^{-1}(x/3) + C\,.$$

4[12P]) Write out the form of the following partial fraction decomposition. **Do not determine the numerical value of the constants.**

a) $$\frac{2x}{(x+1)(x+3)^3} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{(x+2)^2} + \frac{D}{(x+3)^3}$$

b) $$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

6[12P]) Evaluate $\displaystyle\int \frac{x^3-x-2}{x^3-x^2+x-1}\,dx = x - \ln|x-1| + \ln(x^2+1) + C.$

We first have to divide to get
$$\frac{x^3-x-2}{x^3-x^2+x-1} = 1 + x^2 - 2x - 1\,.$$
Then we note that $x^3 - x^2 + x + 1 = (x-1)(x^2+1)$. The rest is then partial fractions.

7[12P] Use the Simpson's Rule, with $n = 5$ to approximate the integral $\int_1^2 \sqrt{x^2-1}\,dx \approx$ ______________
There will be no problem like this on the final.

8[20P] Determine whether each integral is convergent on divergent.

	Integral	Convergent/divergent		Integral	Convergent/divergent
a)	$\int_1^\infty \frac{1}{x^2}\,dx$	Convergent. Use the p-test	b)	$\int_1^2 \frac{1}{2-x}\,dx$	Divergent
c)	$\int_0^2 \frac{1}{x-1}\,dx$	Divergent	d)	$\int_0^\infty \frac{x}{x^3+2x+4}\,dx$	Convergent, use the p-test

Note, that in (c) the singularity is between the endpoints!

Bonus question [10P]) How large do we need to take n in order to guarantee, that the Trapezoidal Rule approximation for $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx$ are accurate within 10^{-6}? **Solution:** There will be no problem like this on the final.

Math 1552, Test # 2. Fall 2004 Name: ____________________________

1[8P]) Evaluate the following dot products:

a) $< 1, -2, 1 > \cdot < 2, 1, 0 >= 2 - 2 + 0 = 0$

b) $(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 2 - 1 + 4 = 5$

2[7P]) Evaluate the cross product $< 1, -1, 1 > \times < 1, 0, 2 >=< -2, -1, 1 >$

3[10P]) Find the **radius** and **center** of the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z - 2 = 0.$

Radius $= 4$ and **Center** $=(-2, 1, 3)$
Solution: Write

$$0 = x^2 + y^2 + z^2 + 4x - 2y - 6z - 2 = x^2 + 4x + 4 - 4 + y^2 - 2y + 1 - 1 + z^2 - 6z + 9 - 9 - 2$$

or

$$(x + 2)^2 + (y - 1)^2 + (z + 3)^2 = 16\,.$$

4[10P]) Circle the correct answer:

a) The points $A(2, 1, -1)$, $B(1, 2, 1)$, and $C(-1, 2, 2)$ (Wrong) **lie** / (Correct) **do not lie** on a line.
b) The points $K(2, 1, 0)$, $L(3, 2, 1)$, and $M(1, 0, -1)$ (Correct) **lie** / (Wrong) **do not lie** on a line.

5[10P]) Find the **angel** between the following vectors:
a) $\mathbf{a} =< 1, -1, 1 >$, $\mathbf{b} =< 1, 1, 0 >$, $\theta = \frac{\pi}{2}$
b) $\mathbf{a} =< 1, 2, -2 >$, $\mathbf{b} =< 2, 2, 1 >$, $\theta = \cos^{-1}(4/9)$
Solution: Recall that the angel is given by

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\,.$$

In (a) we have $\mathbf{a} \cdot \mathbf{b} = 0$. In (b) we have $\mathbf{a} \cdot \mathbf{b} = 2 + 4 - 2 = 4$ and $|\mathbf{a}| = \sqrt{1 + 4 + 4} = \sqrt{9}$ and $|\mathbf{b}| = \sqrt{4 + 4 + 1} = \sqrt{9}$.

6[14P]) Find the **scalar** and **vector** projection of $\mathbf{b} =< 1, 2, 1 >$ onto $\mathbf{a} =< 1, 2, -2 >$,

$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = 1$ $\quad\quad$ $\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{1}{3} < 1, 2, -2 >$

Solution: Recall first that

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

and

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}\,.$$

Here we have

$$\mathbf{a} \cdot \mathbf{b} = 1 + 4 - 2 = 3$$

and

$$|\mathbf{a}| = \sqrt{1 + 4 + 4} = 3\,.$$

Inserting this into the above formula gives the correct answer.

7[9P]) Find a **parametric** and a **symmetric** equation for the line through the points $(2, 1, -1)$ and $(1, -1, 2)$.

Solution: We need a point on the line and the direction of the line. For the direction take

$$< 2, 1, -1 > - < 1, -1, 2 >=< 1, 2, -3 > \,.$$

Then the line is given by

$$\{< 1, -1, 2 > +t < 1, 2, -3 >|\, t \in \mathbb{R}\}\,.$$

The parametric equation is therefore:

$$x = 1 + t\,, \qquad y = -1 + 2t\,, \qquad z = 2 - 3t\,.$$

Solving for t we get the symmetric equation:

$$\frac{1+t}{1} = \frac{y+1}{2} = \frac{z-2}{-3}$$

which can also be written as

$$1 + t = \frac{y+1}{2} = \frac{2-z}{3}\,.$$

Note, that there is more than one correct answer to this problem.

8[14P]) a) Find a parametric equation for the line that passes through the point $(5, 1, 0)$ and is parallel to the line with parametric equation $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$.

Solution: The equation is

$$x = 5 + 2t \qquad y = 1 + 3t \qquad z = -7t\,.$$

b) Find the point where the above line intersects the plane given by the equation $3x - 2y + z = 1$.

Solution: The point is

$$\left(\frac{59}{7}, \frac{43}{7}, -12\right)\,.$$

To solve the problem, we insert our parametric form for x, y and z into the equation for the plane to get:

$$3(5 + 2t) - 2(1 + 3t) - 7t = (15 - 2) + (6 - 6 - 7)t = 13 - 7t = 1\,.$$

Hence

$$t = \frac{12}{7}\,.$$

We then isert this t into the parametric equation and get

$$x = 5 + 2 \cdot (12/7) = 59/7, \quad y = 1 + 3 \cdot (12/7) = 43/7 \quad z = -7 \cdot (12/7) = -12\,.$$

9[9P]) Find the equation of the plane through the point $P(1, 1, -1)$ and parallel to the plane $2x + y - 5z = 5$.

Solution: The equation is$2x + y - 5z = 8$:

To find the equation we need a point on the plane and a normal vector. As our plane is parallel to the plane $2x + y - 5z = 5$ we know that $< 2, 1, -5 >$ is a normal vector. To find find the number on the right hand side, we insert the point $(1, 1, -1)$ which we know is on the plane:

$$2 \cdot 1 + 1 \cdot 1 + (-5) \cdot (-1) = 2 + 1 + 5 = 8\,.$$

Thus the equation is $2x + y - 5z = 8$.

10[9P]) Find the equation of the plane containing the point $P(2, 1, -1)$ and the line given by the parametric equation $x = 3 + 2t, y = t$, and $z = 8 - t$.

Solution: The equation is

$$8x + 19y + 3z = 48\,.$$

For the solution we need need a point on the plane and a normal vector. For that we need **3** points on the plane not all on a single line. For that we can take any two points on the given line. Take $t = 0$ and $t = 1$. Then we get the two points $(3, 0, 8)$ and $(5, 1, 7)$. Then form the vectors:

$$(3, 0, 8) - (2, 1, -1) = (1, -1, 9)$$

and

$$(5,1,7)-(2,1,-1)=(3,0,8)\,.$$

To find a normal vector we take the cross product

$$(3,0,8)\times(1,-1,9)=(8,19,3)$$

and then insert one of our points into the equation $8x+19y+3z=d$. Taking the point $(3,0,8)$ we get $8\cdot 3+3\cdot 8=48$.

Math 1552, Test # 3. Fall 2004 Name: ______________________________

1[10P]) Find the Cartesian equation for the curves:

a) $x=\sqrt{t}$, $y=1-t$; **Solution:** $y=1-x^2$.

b) $r=9\cos(\theta)$. **Solution:** $(x-9/2)^2+y^2=\dfrac{81}{4}$.

Solution: Note that $x=9\cos^2(\theta)$ and $y=9\cos(\theta)\sin(\theta)$. Hence

$$\begin{aligned} x^2+y^2 &= 81\cos^4(\theta)+81\cos^2(\theta)+81\sin^2(\theta) \\ &= 81\cos^2(\theta)[\cos^2(\theta)+\sin^2(\theta)] \\ &= 81\cos^2(\theta) \\ &= 9x \end{aligned}$$

Thus

$$x^2-9x+y^2=0$$

or

$$(x-9/2)^2+y^2=\frac{81}{4}\,.$$

2[10P]) Find dy/dx and d^2y/dx^2 for $x=t^3-6t^2$ and $y=t^2-1$.

$$\frac{dy}{dx}=frac23(t-4) \qquad \frac{d^2y}{dx^2}=frac{-}29t(t-4)^3$$

Solution: Recall that

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

and

$$\frac{d^2y}{dx^2}=\frac{d(dy/dx)}{dt}/(dx/dt)\,.$$

We have $\frac{dy}{dt}=2t$ and $\frac{dx}{dt}=3t^2-12t=3t(t-4)$. Hence $(dy/dt)/(dx/dt)=\frac{2}{3(t-4)}$.
We have

$$\frac{d}{dt}\left(\frac{2}{3(t-4)}\right)=\frac{2}{3}\frac{-1}{(t-4)^2}$$

and hence

$$\frac{d^2y}{dx}=\frac{-2/3}{(t-4)^2}/(3t(t-4))=\frac{-2}{9t(t-4)^3}\,.$$

3[14P]) Find the equation of the tangent line to the curve at the point corresponding to the given value of the parameter:

a) $x = t^4 + 1$, $y = t^3 + t$, $t = -1$. **The equation is:** $y = x$.
b) $x = \cos(\theta) + \sin(2\theta)$, $y = \sin(\theta) + \cos(2\theta)$; $\theta = \pi$. **The equation is:** $y = \frac{1}{2}(-x + 1)$.
Solution: Recall that the tangent line at the point (x_0, y_0) is given by

$$y - y_0 = k(x - x_0)$$

where $k = dy/dx$ is the slope at the point (x_0, y_0).
For (a) we have $x(1) = 1^4 + 1 = 2$, $y(1) = 1^3 + 1 = 2$, $dx/dt|_{t=1} = 4t^3|_{t=1} = 4$ and $dy/dt|_{t=1} = 3t^2 + 1|_{t=1} = 3 + 1 = 4$. Hence $k = 4/4 = 1$. It follows that the equation is

$$y - 2 = x - 2$$

or $y = x$.
For (b) we have $x(\pi) = \cos(\pi) + \sin(2\pi) = -1$ and $y(\pi) = \sin(\pi) + \cos(2\pi) = 1$. Furthermore

$$\frac{dy}{d\theta}|_{\theta=\pi} = \cos(\pi) - 2\sin(2\pi) = -1$$

and

$$\frac{dx}{d\theta}|_{\theta=\pi} = -\sin(\pi) + 2\cos(2\pi) = 2$$

. Hence the equation is

$$y - 1 = \frac{-1}{2}(x + 1)$$

or

$$y = \frac{-1}{2}x + \frac{1}{2}.$$

4[14Pt]) Find the points on the curve $x = 2t^3 + 3t^2 - 12t + 1$, $y = 3t^4 + 16t^3 + 24t^2 - 1$, where the tangent is horizontal or vertical.

Horizontal: $(1, -1), (21, 15)$ **Vertical:** $(-6, 42)$.

Solution: We have to solve the equations $dy/dt = 0$ and $dx/dt = 0$. We have to consider also the case $dy/dt = dx/dt = 0$. Here

$$\frac{dy}{dt} = 12t^3 + 48t^2 + 48t = 12t(t^2 + 4t + 4) = 12t(t + 2)^2.$$

Hence the two solutions $t = 0$ and $t = -2$. The corresponding points on the curve are

$$t = 0 : (1, -1), \qquad t = -2 : (21, 15).$$

We also have

$$\frac{dx}{dt} = 6t^2 + 6t - 12 = 6(t + t - 2) = 6(t + 2)(t - 1).$$

Hence $t = -2$ or $t = 1$. For $t = 1$ the point on the curve is:

$$t = 1 : (-6, 42).$$

Note that

$$\frac{dy/dt}{dx/dt} = \frac{2t(t+2)}{t-1}\,.$$

Hence $t=-2$ corresponds to a horizontal slope.

5[8P]) At what point of the curve $x = t^3+4t$, $y = 6t^2$ is the tangent parallel to the line with equation $x=-7t$, $y = 12t-5$?

Solution: The points are $(-5, 6)$ and $(-112/9, 32/3)$.
The tangent of the given line is

$$\frac{dy/dt}{dx/dt} = \frac{12t}{3t+4}\,.$$

As the tangent line is supposed to be parallel to the given line, which has slope $-12/7$ we must have: $\frac{12t}{3t^2+4} = -12/7$ or (after multiplying a cross and simplifying)

$$3t^2+7t+4 = 0\,.$$

The solutions to this equation are:

$$t = -\frac{7}{6} \pm \frac{1}{6}\sqrt{49-48} = -\frac{7}{6} \pm \frac{1}{6} = \begin{cases} -1 \\ -4/3 \end{cases}\,.$$

Insert those two values of t into the equation of the curve to get the final answer.

6[8P]) Find the length of the curve $x = t^2\cos(t)$, $y = t^2\sin(t)$, $0 \le t \le 1$.

Solution: The lenth of the curve is $\frac{1}{3}[5^{3/2}-8]$
Recall that the lenth of a curve is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

in general, and

$$s = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

for a curve $r = r(\theta)$. In our case $r = t^2$ and hence

$$s = \int_0^1 \sqrt{t^4+4t^2}dt = \int_0^1 t\sqrt{4+t^2}\, dt\,.$$

Set $u = 4+t^2$. Then $du = 2tdt$ and the endpoints are $u=4$ and $u=5$. Hence

$$s = \frac{1}{2}\int_4^5 \sqrt{u}\, du = \frac{1}{3}\left[u^{3/2}\right]_4^5 = \frac{1}{3}[5^{3/2}-8]\,.$$

7[8Pt]) Set up, **but do not evaluate**, an integral that represents the length of the curve $x = t\sin t$, $y = t\cos t$, $0 \le t \le \pi$;

Solution: In this problem we have (we can in fact interchange the role of x and y) $r = t$ and $dr/dt = 1$. Hence the integral becomes

$$\int_0^\pi \sqrt{t^2+1}\, dt\,.$$

[In most cases cases] 7[8P]) Set up an integral that gives the area that inside the curve $r = 4\sin(\theta)$ and outside the curve $r = 2$.
Solution: The curve intersects at $\theta = \pi/6$ and $\theta = 5\pi/6$. The integral becomes then (using that $A = 1/2\int_a^b r^2 d\theta$)

$$\frac{1}{2}\int_{\pi/5}^{5\pi/6} 16\sin^2\theta - 4\, d\theta = 2\int_{\pi/6}^{5\pi/6} 4\sin^2(\theta) - 1\, d\theta\,.$$

8[8P]) Find the equation for the parabola with vertex at $(1,2)$ and focus at $(2,2)$.
Solution: $4(x-1) = (y-2)^2$.
9[12P]) Find the vertices and foci of the ellipse $25x^2 + 9y^2 - 50x + 36y = 164$, and then sketch it graph indicating the vertices and foci.
Solution: (No picture) Completing the square, we get

$$\begin{aligned} 164 &= 25x^2 + 9y^2 - 50x + 36y \\ &= 25(x^2 - 2x) + 9(y^2 + 4y) \\ &= 25(x^2 - 2x + 1) + 9(y^2 + 4y + 4) - 25 - 36 \\ &= 25(x-1)^2 + 9(y+2)^2 - 61 \end{aligned}$$

or

$$25(x-1)^2 + 9(y+2)^2 = 225\,.$$

The equation in standard form is therefore

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1\,.$$

Hence $a = 5$ and $b = 3$ and $c^2 = 25-9 = 16$ or $c = 4$. Thus we have the vertices: $(4,-2), (-2,-2), (1,3), (1,-7)$. The focal points are at $(1,2), (1,-6)$.
10[8P]) This is problem from the book. See the solution manual

Math 1552, Test # 3 make up. Fall 2004 Name: ________________________

Math 1552, Test # 4. Fall 2004 Name: ________________________

1[6P]) Find a formula for the general term a_n of the sequence $\{a_n\}_{n=1}^{\infty}$ if the first terms are given by $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$ **Solution:** $a_n = 2^{-n}$
2[24p]) Determine if the sequence converges or diverges. If it converges find the limit.

a) $a_n = \dfrac{1+2n+n^2}{1+n}$; **Solution:** $frac 1 + 2n + n^2 1 + n = 1 + n$ so the sequence is divergent.

b) $a_n = \dfrac{2^n}{3^{2n+1}}$.
Solution: $\lim_{n\to\infty a_n} = 0$.
c) $a_n = n^2 e^{-n}$
Solution: We have $\lim_{n\to\infty} a_n = 0$. To see that recall that

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0\,.$$

d) $a_n = n\sin(1/n)$
Solution: We have $\lim a_n = 1$. For that use

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1\,.$$

3[18p]) Evaluate the series:

a) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{3}{2}$

b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{5}{2^n} = \frac{10}{3}$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n+1}}{10^n} = 9/7$

Solution: To solve this note that all of those series are **geometric series**. To evaluate the above sums we use:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}\,.$$

4[24p]) Determine if the series **converges**, **converges absolutly** or **converges conditionally**:

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$

Solution: This is an alternating series. We have (with $b_n = n/(n^2+1)$) that b_n is decreasing with limit 0 (why?). Hence the series is covergent. Taking the absolut value we are looking at the series $\sum_{n=0}^{\infty} n/(n^2+1)$. But this series has the same bahaviour as the series $\sum_{n=1}^{\infty} 1/n$ which is divergent. Hence the series is not absolut convergent. The final answer is, that the series is **conditionally convergent**.

b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln(n)}}$

Solution: Use the integral test with $f(x) = \frac{1}{x\sqrt{\ln(x)}}$. We have, using the substitution $u = \ln(x)$,

$$\int_2^{\infty} f(x)\,dx = \int_{\ln(2)}^{\infty}\frac{1}{\sqrt{u}}\,du = \infty\,.$$

Hence the series is divergent.

c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^n}$

Solution: We have

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty} 3/n = 0\,.$$

Hence the series is convergent/

5[24p]) Find the radius of convergence of the following series:

a) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$

Solution: $R = 1$.

b) $\sum_{n=1}^{\infty} 2^n (x-1)^n$

Solution: $R = 1/2$.

c) $\sum_{n=1}^{\infty} \frac{(4x+3)^n}{n^2}$

Solution: $R = 1/4$.

d) $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

Solution: $R = 3$.

6[7p]) Find the power series for the function $\frac{1}{(1-x)^2}$. What is the radius of convergence?

Solution: Differentiating the geometric series

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

we get

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (n+1)x^n .$$

The radius of convergence is $R = 1$.