2-MODULAR MATRICES[*]

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Abstract. An integer matrix A is Δ-modular if the determinant of each $\text{rank}(A) \times \text{rank}(A)$ submatrix has absolute value at most Δ. The class of 1-modular, or unimodular, matrices is of fundamental significance in both integer programming theory and matroid theory. A 1957 result of Heller shows that the maximum number of nonzero, pairwise non-parallel columns of a rank-r unimodular matrix is $\binom{r+1}{2}$. We prove that, for each sufficiently large integer r, the maximum number of nonzero, pairwise non-parallel columns of a rank-r 2-modular matrix is $\binom{r+2}{2} - 2$.

Key words. integer programming, unimodular matrix, representable matroid, excluded minor

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1. Introduction. This paper is motivated by the theory of integer programming. Given a *constraint matrix* $A \in \mathbb{Z}^{m \times n}$ and vectors $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$, a fundamental problem of integer programming is to find an integer vector x in the polyhedron $\{x \in \mathbb{R}^n \colon Ax \leq b\}$ that maximizes the inner product $c^T x$. In general, one cannot expect to optimize an integer program efficiently, but better efficiency is possible if the constraint matrix has special structure.

Given a positive integer Δ, an integer matrix A is Δ-*modular* if the determinant of each $\text{rank}(A) \times \text{rank}(A)$ submatrix of A has absolute value at most Δ. A 1-modular matrix is *unimodular*, and integer programs with a unimodular matrix can be optimized in strongly polynomial time, because every vertex of the polytope $\{x \in \mathbb{R}^n \colon Ax \leq b\}$ is integral [9]. This fact has led to considerable interest in the complexity of integer programming with a Δ-modular constraint matrix (see, for example, [14] and [1]). This complexity is unknown when $\Delta \geq 3$, but a recent breakthrough of Artmann, Weismantel, and Zenklusen [2] shows that integer programs with a 2-modular, or *bimodular*, constraint matrix can be optimized in strongly polynomial time. In view of this result, the structural properties of 2-modular matrices are of significant interest. A well-known result of Heller [8] shows that a rank-r unimodular matrix has at most $\binom{r+1}{2}$ nonzero, pairwise non-parallel columns. We prove the analogous result for $\Delta = 2$.

THEOREM 1.1. *For each sufficiently large integer r, the maximum number of nonzero, pairwise non-parallel columns of a rank-r 2-modular matrix is $\binom{r+2}{2} - 2$.*

This was proven independently by Lee et al. [10], who showed that this bound holds for all r. The previous best upper bound was $\binom{r+1}{2} + C \cdot r$ for a large constant C, due to Geelen, Nelson, and Walsh [5]. The bound of Theorem 1.1 is tight, for example, for the matrices A_r and A_r' in Figure 1, where D_r is the $r \times \binom{r}{2}$ matrix whose columns consist of all r-tuples with exactly two nonzero entries, the first equal to 1 and the second equal to -1.

We prove Theorem 1.1 using matroids. For each positive integer Δ, let $\mathcal{M}_\Delta$ denote the class of matroids with a representation over $\mathbb{R}$ as a Δ-modular matrix. We say that such a matroid is Δ-*modular*. An integer matrix A is *totally Δ-modular* if the determinant of every submatrix of A has absolute value at most Δ. We define $\overline{\mathcal{M}}_\Delta$ to be the class of matroids with a representation over $\mathbb{R}$ as a totally Δ-modular matrix.

$$A_r = \begin{bmatrix} & I_r & & D_r & & \begin{array}{c} 1 \; \cdots \; 1 \\ \hline I_{r-1} \end{array} \end{bmatrix}$$

$$A'_r = \begin{bmatrix} & I_r & & D_r & & \begin{array}{c|c} \begin{array}{ccc} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{array} & \begin{array}{c} 1 \\ 1 \end{array} \\ \hline & -I_{r-2} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array} \end{bmatrix}$$

FIG. 1. A_r and A'_r.

Clearly $\overline{\mathcal{M}}_\Delta \subseteq \mathcal{M}_\Delta$ for each $\Delta \geq 1$, because every totally Δ-modular matrix is also Δ-modular. Since every unimodular matrix is row-equivalent to a totally unimodular matrix, the classes $\overline{\mathcal{M}}_1$ and $\mathcal{M}_1$ are in fact equal.

The class $\mathcal{M}_1$ is the class of *regular matroids*. Tutte [13] proved that this class coincides with the class of matroids representable over every field, and that there are exactly three minor-minimal matroids not in $\mathcal{M}_1$. For $\Delta \geq 2$, the class $\mathcal{M}_\Delta$ is not very well-studied. It is proved in [5] that $\mathcal{M}_\Delta$ is minor-closed, and that every matroid in $\mathcal{M}_\Delta$ is representable over every field with characteristic greater than Δ. In particular, every matroid in $\mathcal{M}_2$ is representable over GF(3) and GF(5); such a matroid is called *dyadic*.

In view of Tutte's results and the fundamental significance of unimodular matrices, there are several natural open questions concerning $\mathcal{M}_\Delta$.

PROBLEM 1.2. *Let Δ be a positive integer.*
  (i) *What is the maximum size of a simple rank-r matroid in $\mathcal{M}_\Delta$, and which matroids attain this maximum?*
 (ii) *What are the minor-minimal matroids that are not in $\mathcal{M}_\Delta$?*
(iii) *Is $\overline{\mathcal{M}}_\Delta$ equal to $\mathcal{M}_\Delta$?*
 (iv) *Is $\overline{\mathcal{M}}_\Delta$ closed under duality?*

Theorem 1.1 shows that the maximum size of a simple rank-r 2-modular matroid is $\binom{r+2}{2} - 2$, if r is sufficiently large, and the vector matroids of A_r and A'_r show that there are at least two matroids that attain this maximum. In order to prove Theorem 1.1, we show that (iii) and (iv) have affirmative answers for $\Delta = 2$, and we find several excluded minors for $\mathcal{M}_2$, including $U_{2,5}$, F_7, $U_{2,4} \oplus U_{2,4}$, and AG$(2,3)\backslash e$. In fact, we prove the following result, which is stronger than Theorem 1.1. A *line* in a matroid is a rank-2 flat, and a line is *long* if it contains at least three *points*, that is, rank-one flats. The matroid R_9 is the ternary Reid geometry [11, p. 654], which is the simple rank-3 matroid consisting of long lines L_1, L_2, L_3 with a common intersection point x so that $|L_1| = |L_2| = 4$, $|L_3| = 3$, and both elements in $L_3 - \{x\}$ are on four long lines.

THEOREM 1.3. *For each sufficiently large integer r, the maximum size of a simple rank-r matroid with no minor in $\{U_{2,5}, F_7, U_{2,4} \oplus U_{2,4}, R_9\}$ is $\binom{r+2}{2} - 2$.*

While we make progress on Problem 1.2 for $\Delta = 2$, it is surprisingly difficult for $\Delta \geq 3$. For fixed $\Delta \geq 3$, the current best upper bound on the maximum size of a simple rank-r matroid in $\mathcal{M}_\Delta$ is $\binom{r+1}{2} + C \cdot r$ for a large constant C, proved by Geelen,

Nelson, and Walsh [5]. Recently, Lee et al. [10] proved an upper bound of $\Delta^2\binom{r+1}{2}$, which is the first bound that is polynomial in both r and Δ. Both of these are recent additions to a long sequence of previous results; we direct the reader to [7] for more details. However, it is unclear what the correct bound should be for $\Delta \geq 3$. It is also unclear in general what the rank-2 uniform excluded minor is for $\mathcal{M}_\Delta$. On a more positive note, it follows from a result of D'Adderio and Moci [4, Theorem 2.2] that $\mathcal{M}_\Delta$ is closed under duality for all positive Δ. In section 4 we present an elementary proof of this result due to Marcel Celaya (private communication). However, for $\Delta \geq 3$ it is unknown whether $\overline{\mathcal{M}}_\Delta = \mathcal{M}_\Delta$, and whether $\overline{\mathcal{M}}_\Delta$ is closed under duality. Some of the difficulties of these open problems are discussed in section 4. After introducing some preliminaries in section 2, we show in section 3 that A_r and A'_r attain equality in Theorem 1.1. The proofs of Theorem 1.1 and Theorem 1.3 are given in section 9. A crucial case in the proof of the latter, when M has a spanning-clique restriction, is treated in section 8. Prior to that, section 5, section 6, and section 7 consider 1-, 2-, and k-element extensions of a clique.

2. Preliminaries. We follow the notation of [11], unless otherwise stated. For a matroid M, we write $|M|$ and $\varepsilon(M)$ for, respectively, $|E(M)|$ and the number of points of M. For an integer $\ell \geq 2$, we denote by $\mathcal{U}(\ell)$ the class of matroids with no $U_{2,\ell+2}$-minor.

We first describe a construction that shows how the vector matroids of A_r and A'_r are related to $M(K_{r+2})$. If a matroid M is obtained from a matroid N by deleting a non-empty subset T of $E(N)$, then N is an *extension* of M. If $|T| = 1$, then N is a *single-element extension* of M, and if $|T| = k$ for $k \geq 2$, then N is a *k-element extension* of M. The single-element extensions of a matroid M are in one-to-one correspondence with the *modular cuts* of M, which are the sets $\mathcal{F}$ of flats with the following two properties:

(i) If $F \in \mathcal{F}$ and F' is a flat of M containing F, then $F' \in \mathcal{F}$.
(ii) If $F_1, F_2 \in \mathcal{F}$ and $r_M(F_1) + r_M(F_2) = r_M(F_1 \cup F_2) + r_M(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{F}$.

A pair of flats that satisfies (ii) is a *modular pair*. A modular cut $\mathcal{F}$ is *proper* if it does not contain all flats of M. The modular cut of M corresponding to a single-element extension N of M by an element e is the set of flats of M that span e in N. Conversely, Crapo [3] showed that, for every modular cut $\mathcal{F}$ of M, there is a unique single-element extension N of M by an element e for which $\mathcal{F}$ is precisely the set of flats of M that span e in N (see [11, Theorem 7.2.3]). Given a set $\mathcal{F}$ of flats of M, the modular cut of M *generated* by $\mathcal{F}$ is the intersection of all modular cuts of M that contain every flat in $\mathcal{F}$. If N is a single-element extension of M by an element e corresponding to the modular cut generated by a single flat F, then we say that e has been *freely added* to F.

A matroid M' is an *elementary projection* or an *elementary quotient* of M if there is a single-element extension N of M by an element e so that $M' = N/e$. Thus, every elementary projection of M corresponds to a modular cut of M. We prove in the following section that the vector matroids of the matrices A_r and A'_r are both simplifications of elementary projections of $M(K_{r+2})$.

Two classes of matroids arise in the proof of Theorem 1.3: spikes and Reid geometries. A *spike* is a simple matroid S with an element t so that the simplification of S/t is a circuit, and each parallel class of S/t has size two. The element t is a *tip* of S. A spike S may have several elements that can serve as a tip. For example, the Fano plane is a spike, and every element can serve as a tip. However, a spike of rank

at least four has a unique tip. We will also make use of the fact that spikes are not graphic (see [11, p. 662]).

A *Reid geometry* is a simple rank-3 matroid N consisting of long lines L_1, L_2, L_3 with a common intersection point x so that $|L_1| = |L_2|$ and $|L_3| = 3$, while both elements in $L_3 - \{x\}$ are on $|L_1|$ long lines of N. If $|L_1| = 3$, then N is the Fano plane F_7. If $|L_1| = 4$, then N is the ternary Reid geometry R_9.

3. The Maximum-Sized Matroids. In this section, we describe two matroids that meet the bound of Theorem 1.3. We first show that the matrices A_r and A'_r are in fact 2-modular.

LEMMA 3.1. *For each integer $r \geq 2$, the matrix A_r is 2-modular.*

Proof. Let r be minimal so that the lemma is false. Then $r > 2$. Let A be an $r \times r$ submatrix of A_r so that $|\det(A)| > 2$. If A has a row with exactly one nonzero entry, then the determinant of A is ± 1 times the determinant of an $(r-1) \times (r-1)$ submatrix of A_{r-1}, and so $|\det(A)| \leq 2$, by the minimality of r. Then each row of A has at least two nonzero entries, and, since each column of A_r has at most two nonzero entries, this implies that A has precisely two nonzero entries in each row and column. By permuting rows and columns of A and multiplying some rows and columns of A by -1, we may obtain a matrix A' for which all but at most one column has nonzero entries 1 and -1. If A' has no column with two entries equal to 1, then $|\det(A')| \leq 1$. Otherwise, by expanding along a column with two ones, we see that the determinant of A is the sum or difference of the determinants of two matrices with determinant in $\{-1, 0, 1\}$. In either case, $|\det(A')| \leq 2$. But $|\det(A')| = |\det(A)|$, so this contradicts the choice of A.   □

LEMMA 3.2. *For each integer $r \geq 2$, the matrix A'_r is 2-modular.*

Proof. For each integer $r \geq 3$, let $H_r = [I_r|D_r|v]$, where $v = [1, -1, -1, 0, \ldots, 0]^T$. Then A'_r can be obtained from H_{r+1} by pivoting on the first entry of the last column, deleting the first row and last column, and then deleting two columns. This implies that each $r \times r$ submatrix of A'_r has the same determinant as an $(r+1) \times (r+1)$ submatrix of H_{r+1}, so it suffices to show that H_r is 2-modular for each $r \geq 3$.

Let $A = [A'|v]$ be an $r \times r$ submatrix of H_r. Let $A_1 = [A'|e_1 - e_2]$ and $A_2 = [A'|e_3]$, where e_i is the ith unit vector. Then $|\det(A_1)| \leq 1$ and $|\det(A_2)| \leq 1$, since each is a submatrix of $[I_r|D_r]$. Since the determinant is linear in the last column and $v = (e_1 - e_2) - e_3$, we have $\det(A) = \det(A_1) - \det(A_2)$, so $|\det(A)| \leq 2$.   □

The vector matroids of the matrices A_r and A'_r are both simplifications of elementary projections of $M(K_{r+2})$. For each integer $r \geq 2$, the matroid T_r is the simplification of the matroid obtained by projecting $M(K_{r+2})$ by an element freely added to a 3-point line. The matrix A_r is a well-known representation of T_r [12], so T_r is a 2-modular matroid by Lemma 3.1. Note that $|T_r| = |M(K_{r+2})| - 2 = \binom{r+2}{2} - 2$. We now define another single-element extension of $M(K_{r+2})$, by its corresponding modular cut.

PROPOSITION 3.3. *Let L_1 and L_2 be 2-point lines of $M(K_{r+2})$ so that $L_1 \cup L_2$ is a 4-element circuit. Then the set of flats of $M(K_{r+2})$ that contain L_1 or L_2 is a proper modular cut. Moreover, the corresponding single-element extension of $M(K_{r+2})$ is represented over both $\mathbb{R}$ and $\mathrm{GF}(3)$ by the matrix $[I_{r+1}|D_{r+1}|v]$, where $v = [1, -1, -1, 0, \ldots, 0]^T$.*

Proof. Note that the matrix $H_{r+1} = [I_{r+1}|D_{r+1}|v]$ is a submatrix of A'_{r+1}, and is thus 2-modular, by Lemma 3.2. Therefore, the vector matroids of H_{r+1} over $\mathbb{R}$ and

GF(3) are isomorphic; let M denote this matroid. In the matroid M, let e label the last column of H_{r+1}. Then $M\backslash e$ is $M(K_{r+2})$. Let $\mathcal{F}$ be the set of flats of $M\backslash e$ that span e in M. Note that $\mathcal{F}$ contains no 1-element flat, because M is simple. Let a, b, c, d be the labels of the columns $[1, -1, 0, \ldots, 0]^T$, $[0, 1, 0, \ldots, 0]^T$, $[0, 0, 1, 0, \ldots, 0]^T$, and $[1, 0, -1, 0, \ldots, 0]^T$, respectively, of H_{r+1}. Then $\{a, c, e\}$, $\{b, d, e\}$, and $\{a, b, c, d\}$ are circuits of M, and so $\{a, c\}$ and $\{b, d\}$ are flats of $M(K_{r+2})$ in $\mathcal{F}$. Let F_0 denote the 6-element flat of $M(K_{r+2})$ spanned by $\{a, b, c, d\}$; then $M(K_{r+2})|F_0 \cong M(K_4)$. It is straightforward to check that $\{a, c, e\}$ and $\{b, d, e\}$ are the only long lines of $M|(F_0 \cup \{e\})$ that contain e.

Suppose that there is a flat $F \in \mathcal{F}$ that does not contain $\{a, c\}$ or $\{b, d\}$. Since $F \in \mathcal{F}$ and $F_0 \in \mathcal{F}$, the flat $L = F \cap F_0$ is in $\mathcal{F}$ as well, by property (ii) of modular cuts. This flat L has rank less than three, since it does not contain $\{a, c\}$ or $\{b, d\}$. It has size at least two, since M is simple. But $\{a, c, e\}$ and $\{b, d, e\}$ are the only long lines of $M|(F_0 \cup \{e\})$ that contain e, which contradicts the existence of F. Thus, $\mathcal{F}$ is precisely the set of flats that contain $\{a, c\}$ or $\{b, d\}$. Since $\mathcal{F}$ is proper, the proposition holds. $\square$

For each integer $r \geq 2$, let T'_r denote the simplification of the matroid obtained by projecting $M(K_{r+2})$ by an element corresponding to the modular cut of Proposition 3.3. It is not hard to see that the matrix A'_r can be obtained from the real matrix $[I_{r+1}|D_{r+1}|v]$ by pivoting on the first entry of the last column, deleting the first row and last column, and then deleting one column from each of the two parallel pairs. Thus, A'_r represents T'_r over $\mathbb{R}$, and so T'_r is a 2-modular matroid by Lemma 3.2. Note that $|T'_r| = |M(K_{r+2})| - 2 = \binom{r+2}{2} - 2$. The matroids T_r and T'_r are likely the unique simple rank-r 2-modular matroids of size $\binom{r+2}{2} - 2$ for $r \geq 5$, but this seems difficult to prove.

4. Duality and Excluded Minors. In this section, we find several excluded minors for $\mathcal{M}_2$, show that $\overline{\mathcal{M}}_2 = \mathcal{M}_2$, and present a proof showing that $\mathcal{M}_\Delta$ is closed under duality for each positive integer Δ. We make use of the following result.

LEMMA 4.1. *Let A be a 2-modular representation of a rank-r matroid M, and let e be a non-loop of M that labels the first column of A. Then there is a 2-modular matrix A' of the form $[I_r|X]$ whose first column is labeled by e such that A' is obtained from A by elementary row operations, column swaps, and dividing rows by two.*

Proof. We may assume that e labels the first column of A, that A has r rows, and that $r \geq 2$. Let B be a basis of M containing e, and let A_B be the corresponding submatrix of A. By performing elementary row operations, we may transform A into another 2-modular matrix in which A_B is in Hermite normal form. If each entry on the diagonal of A_B is 1, then the lemma holds. Otherwise, since A is 2-modular, exactly one entry on the diagonal of A_B is equal to 2 and all the rest are equal to 1. We may assume that the row of A with the entry 2 on the diagonal of A_B has an entry in $\{-1, 1\}$, or else we may divide this row by 2. By pivoting on that entry and swapping columns, we obtain a 2-modular matrix of the form $[I_r|X]$, where the first column is labeled by e. $\square$

Since row operations do not change the vector matroid, Lemma 4.1 implies that $\mathcal{M}_2$ is closed under duality (see [11, Theorem 2.2.8]). Also, since every Δ-modular matrix of the form $[I_r|X]$ is in fact totally Δ-modular, Lemma 4.1 implies that $\mathcal{M}_2 = \overline{\mathcal{M}}_2$. Unfortunately, the analogue of Lemma 4.1 is false for Δ-modular matrices with

$\Delta \geq 3$. For example, the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

with the first column labeled by e is 3-modular, but is not row-equivalent to a matrix with an I_2-submatrix that uses the first column. This is the main difficulty in showing that $\overline{\mathcal{M}}_\Delta = \mathcal{M}_\Delta$ for $\Delta \geq 3$.

Next we present a proof showing that $\mathcal{M}_\Delta$ is closed under duality. This follows from a result of D'Adderio and Moci concerning arithmetic matroids [4, Theorem 2.2], but we present an elementary proof due to Marcel Celaya (private communication). We also thank the anonymous referee for informing us that this result is in fact known.

We first develop some notation. Let $A \in \mathbb{Z}^{m \times n}$ be a matrix with columns indexed by a set E. For every $X \subseteq E$, we write A_X for the submatrix of A consisting of the columns indexed by X. If A has full row-rank, then we write $\gcd(A)$ for the greatest common divisor of the determinants of the $m \times m$ submatrices of A. We write $[n]$ for $\{1, 2, \ldots, n\}$.

THEOREM 4.2 (D'Adderio, Moci). *For each positive integer Δ, the class $\mathcal{M}_\Delta$ is closed under duality.*

Proof. Let M be a Δ-modular matroid with ground set $[n]$, and let $A \in \mathbb{Z}^{m \times n}$ be a Δ-modular representation of M with full row rank. We first show that there is a matrix $C \in \mathbb{Z}^{(n-m) \times n}$ with $\gcd(C) = 1$ whose rows form a basis of $\ker(A)$. Let $C_1 \in \mathbb{Z}^{(n-m) \times n}$ be a matrix whose rows form a basis of $\ker(A)$. Let $P \in \mathbb{Z}^{(n-m) \times (n-m)}$ and $Q \in \mathbb{Z}^{n \times n}$ be unimodular matrices so that PC_1Q is the Smith normal form of C_1. Then $PC_1Q = [D|0]$, where $D \in \mathbb{Z}^{(n-m) \times (n-m)}$ is an invertible diagonal matrix. Then $D^{-1}PC_1Q = [I|0]$ is an integer matrix with $\gcd(D^{-1}PC_1Q) = 1$. Since Q is unimodular, so is Q^{-1}, and therefore $(D^{-1}PC_1Q)Q^{-1} = D^{-1}PC_1$ is an integer matrix. Clearly the Smith normal form of $D^{-1}PC_1$ is $D^{-1}PC_1Q = [I|0]$. Since a matrix B with full row-rank has $\gcd(B) = 1$ if and only if the Smith normal form of B is $[I|0]$, it follows that $\gcd(D^{-1}PC_1) = 1$. Therefore, $D^{-1}PC_1$ is an integer matrix row-equivalent to C_1 with $\gcd(D^{-1}PC_1) = 1$, so we may take $C = D^{-1}PC_1$.

We will show that C is a Δ-modular representation of M^*. For each set $X \subseteq [n]$, we write $\bar{X}$ for $[n] - X$. Note that A and C both have columns indexed by $[n]$, and that $B \subseteq [n]$ is a basis of M if and only if $\bar{B}$ is a basis of M^*. By rearranging columns, we may assume that $B = [m]$ is a basis of M. Then $A_B^{-1}A = [I_m | A_B^{-1}A_{\bar{B}}]$. Since $A_B^{-1}A$ is row-equivalent to A, it represents M over $\mathbb{Q}$. Let $C' = [(A_B^{-1}A_{\bar{B}})^T | -I_{n-m}]$. Then by Theorem 2.2.8 in [11], C' represents M^* over $\mathbb{Q}$. Also, the rows of C' form a basis of $\ker(A_B^{-1}A) = \ker(A)$, so C' is row-equivalent to C, and thus C represents M^* as well. Moreover, since C' and C are row-equivalent, there exists an invertible rational $(n-m) \times (n-m)$ matrix Q such that $C' = QC$. Then each set $B_1 \subseteq E(M)$ of size m satisfies $\det(C'_{\bar{B}_1}) = \det(Q)\det(C_{\bar{B}_1})$.

Let B_0 be a basis of M obtained from B by a basis exchange, so $B_0 = (B \cup \{j\}) - \{i\}$ for some $i \in B$ and $j \notin B$. Then the matrix $C'_{\bar{B}_0}$ is obtained from $-I_{n-m}$ by replacing the $(j-m)^{th}$ column with the ith column of $(A_B^{-1}A_{\bar{B}})^T$, so

(4.1) $$\det(C'_{\bar{B}_0}) = (-1)^{n-m-1}(A_B^{-1}A_{\bar{B}})^T_{j-m,i}$$

(4.2) $$= (-1)^{n-m-1}(A_B^{-1}A_{\bar{B}})_{i,j-m}$$

(4.3) $$= (-1)^{n-m-1}(A_B^{-1}A)_{i,j}.$$

Furthermore, the system $(A_B)x = A_{\{j\}}$ has unique solution $A_B^{-1}A_{\{j\}} = (A_B^{-1}A)_{\{j\}}$,

so by Cramer's rule, the ith entry of the solution $(A_B^{-1}A)_{\{j\}}$ is

$$(A_B^{-1}A)_{i,j} = \frac{\det(A_{B_0})}{\det(A_B)}.$$

Combined with (3), this implies that

$$(4.4) \qquad \det(C'_{\bar{B}_0}) = (-1)^{n-m-1}\frac{\det(A_{B_0})}{\det(A_B)}.$$

Since $C' = QC$ and $\det(C'_{\bar{B}}) = (-1)^{n-m}$, (4) implies that

$$(4.5) \qquad \frac{\det(C_{\bar{B}_0})}{\det(C_{\bar{B}})} = \frac{\det(C'_{\bar{B}_0})}{\det(C'_{\bar{B}})} = -\frac{\det(A_{B_0})}{\det(A_B)}.$$

Since every basis of M can be obtained from any other basis by a sequence of basis exchanges, (5) implies that each pair (B_1, B_2) of bases of M satisfies

$$(4.6) \qquad \frac{\det(C_{\bar{B}_1})}{\det(C_{\bar{B}_2})} = \pm\frac{\det(A_{B_1})}{\det(A_{B_2})}.$$

Suppose that some basis B_1 of M satisfies $\det(C_{\bar{B}_1}) = c \cdot \det(A_{B_1})$ for some $c \in \mathbb{Q}$ with $|c| > 1$. Then (6) implies that each other basis B_2 of M satisfies $\det(C_{\bar{B}_2}) = \pm c \cdot \det(A_{B_2})$. But then $c \cdot \gcd(A)$ is an integer with absolute value greater than 1 which divides $\gcd(C)$, a contradiction. Thus, each basis B of M satisfies $|\det(C_{\bar{B}})| \le |\det(A_B)| \le \Delta$, so C is Δ-modular. □

We now turn our attention back to $\mathcal{M}_2$. To find excluded minors for $\mathcal{M}_2$, we can use the fact that every matroid in $\mathcal{M}_2$ is dyadic. The excluded minors for the class of dyadic matroids include $U_{2,5}$, F_7, $\mathrm{AG}(2,3)\backslash e$, $\Delta(\mathrm{AG}(2,3)\backslash e)$, T_8, and their duals. Here $\Delta(\mathrm{AG}(2,3)\backslash e)$ is obtained from $\mathrm{AG}(2,3)\backslash e$ by a Δ-Y exchange, while T_8 is a self-dual ternary spike with the tip deleted [11, p. 649]. In fact, each of these matroids is an excluded minor for $\mathcal{M}_2$. The proof, which we omit, amounts to finding a 2-modular representation for each single-element deletion or contraction of each of these matroids. Also, since the ternary Reid geometry R_9 has $\mathrm{AG}(2,3)\backslash e$ as a restriction, R_9 is not in $\mathcal{M}_2$. There are two other known excluded minors for the class of dyadic matroids, namely N_1 and N_2, and the question of whether this list is complete is a long-standing open problem (see [11, Problem 14.7.11]).

PROBLEM 4.3. *Determine if*

$$\{U_{2,5}, U_{3,5}, F_7, F_7^*, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*, \Delta(\mathrm{AG}(2,3)\backslash e), T_8, N_1, N_2\}$$

is the complete list of excluded minors for the class of dyadic matroids.

The matroid N_2 may be an excluded minor for $\mathcal{M}_2$, but N_1 is not, as we shall see shortly. To find excluded minors for $\mathcal{M}_2$ that are dyadic, we use Lemma 4.1.

PROPOSITION 4.4. $U_{2,4} \oplus U_{2,4}$ *is an excluded minor for* $\mathcal{M}_2$.

Proof. Suppose A is a 2-modular representation of $U_{2,4} \oplus U_{2,4}$ with four rows. By Lemma 4.1, we may assume that A has an I_4-submatrix. Since $U_{2,4}$ is not regular, this implies that A has a 4×4 block-diagonal submatrix where the determinant of each nonzero block has absolute value at least two. But then the determinant of this submatrix has absolute value at least four, a contradiction, so $U_{2,4} \oplus U_{2,4} \notin \mathcal{M}_2$. Each

single-element contraction of $U_{2,4} \oplus U_{2,4}$ is isomorphic to $U_{1,3} \oplus U_{2,4}$, and it is easy to see that this matroid is in $\mathcal{M}_2$. Since $U_{2,4} \oplus U_{2,4}$ is self-dual, this implies that every minor of $U_{2,4} \oplus U_{2,4}$ is in $\mathcal{M}_2$.   □

The matroid $U_{2,4} \oplus U_{2,4}$ is signed-graphic, which means that it has a representation over $\mathrm{GF}(3)$ so that each column has at most two nonzero entries. In fact, $\mathcal{M}_2$ has at least two other signed-graphic excluded minors, represented over $\mathbb{R}$ by the following matrices:

$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0
\end{bmatrix}
$$

We write U_8 for the vector matroid of the first matrix, and U_8' for the vector matroid of the second matrix. Both matroids are self-dual. One can prove that they are excluded minors for $\mathcal{M}_2$ using a similar proof to that of Proposition 4.4. The excluded minor N_1 for the dyadic matroids has a U_8-minor, but N_2 does not have a minor isomorphic to $U_{2,4} \oplus U_{2,4}$, U_8, or U_8', and thus it may be an excluded minor for $\mathcal{M}_2$.

Finding the excluded minors for $\mathcal{M}_\Delta$ likely becomes more difficult as Δ increases. Indeed, we have not yet solved the following.

PROBLEM 4.5. *Find the unique rank-2 excluded minor for $\mathcal{M}_\Delta$.*

Since $\mathcal{M}_\Delta$ is closed under parallel extension and adding loops, each excluded minor for $\mathcal{M}_\Delta$ is simple. Since every rank-2 simple matroid is uniform, this implies that $\mathcal{M}_\Delta$ has a unique rank-2 excluded minor, which must be uniform. If p is the smallest prime greater than Δ, then $U_{2,p+2} \notin \mathcal{M}_\Delta$, because every matroid in $\mathcal{M}_\Delta$ is $\mathrm{GF}(p)$-representable. Also, it is not hard to see that $U_{2,\Delta+2} \in \mathcal{M}_\Delta$ for each $\Delta \geq 1$, by taking the representation with two rows, two unit columns, and all columns of the form $[1, m]^T$, where $1 \leq m \leq \Delta$. This implies that if $\Delta + 1$ is prime, then $U_{2,\Delta+3}$ is an excluded minor for $\mathcal{M}_\Delta$. It is natural to conjecture that $U_{2,p+2}$ is always an excluded minor for $\mathcal{M}_\Delta$, where p is the smallest prime greater than Δ. However, it appears (through Sage code) that the rank-2 excluded minor for $\mathcal{M}_7$ is $U_{2,11}$, even though nine is not prime. Thus, it is unclear what the correct answer to Problem 4.5 may be as a function of Δ.

5. Single-Element Extensions. The remainder of this paper is devoted to the proof of Theorem 1.3. In this section, we show that, up to isomorphism, there are only two rank-r non-trivial single-element extensions of a clique that have no minor isomorphic to $U_{2,5}$, F_7, or R_9. We need the following straightforward lemma to identify R_9.

LEMMA 5.1. *Let M be a simple rank-3 matroid on nine elements consisting of three long lines through a common point. Then either M has a $U_{2,5}$-minor, or $M \cong R_9$.*

Proof. Suppose that M has no $U_{2,5}$-minor, and let x be a point that is on three long lines. Since $|M| = 9$, two of these lines have four points, and one has three points. Let L_1 and L_2 be the 4-point lines through x, and let $\{x, y, z\}$ be the 3-point line through x. Since M has no $U_{2,5}$-minor, each element of $L_1 - \{x\}$ is on a long line with y and a long line with z. Thus, y and z are both on four long lines of M, so $M \cong R_9$.   □

We now show that, up to isomorphism, there are only two rank-r non-trivial

single-element extensions of a clique that have no minor isomorphic to $U_{2,5}$, F_7, or R_9.

PROPOSITION 5.2. *Let M be a simple matroid of rank at least three with a set X so that $M|X \cong M(K_{r(M)+1})$, and let $e \in E(M) - X$. If M has no minor isomorphic to $U_{2,5}$, F_7, or R_9, then either*
(a) *the extension of $M|X$ by e corresponds to the modular cut generated by a 3-point line, or*
(b) *the extension of $M|X$ by e corresponds to the modular cut generated by a pair of 2-point lines whose union is a circuit.*

Proof. Let $\mathcal{L}$ denote the set of lines of $M|X$ that span e. Since $M|X \cong M(K_{r(M)+1})$, each line in $\mathcal{L}$ has size at most three. The 3-point lines in $\mathcal{L}$ correspond to triangles of $K_{r(M)+1}$, and the 2-point lines in $\mathcal{L}$ correspond to 2-edge matchings of $K_{r(M)+1}$.

CLAIM 5.3. *$\mathcal{L}$ is non-empty.*

Proof. Assume that $\mathcal{L}$ is empty. Then $(M/e)|X$ is simple. Let M_1 be a minimal simple minor of M of rank at least three so that
- $e \in E(M_1)$,
- there is a set $X_1 \subseteq X$ so that $M_1|X_1 \cong M(K_{r(M_1)+1})$, and
- $(M_1/e)|X_1$ is simple.

Since M has no $U_{2,5}$-minor and $(M_1/e)|X_1$ is simple, each plane of $M_1|X_1$ that spans e has size at most four. Thus, $r(M_1) \geq 4$. By minimality, each element of X_1 is in a plane of $M_1|X_1$ that spans e. Each plane of $M_1|X_1$ has size 3 or 4, corresponding to a 3-edge matching, or a triangle and disjoint edge. Since each plane of $M_1|X_1$ is spanned by an $M(K_m)$-restriction of $M_1|X_1$ for some $m \leq 6$, the minimality of M_1 implies that $r(M_1) \leq 5$.

Suppose $r(M_1) = 4$. Then each element of X_1 is in a 4-element plane of $M_1|X_1$ that spans e. Since $|X_1| = 10$, there is some $x \in X_1$ that is in at least two such planes. Each element of X_1 is on at least three long lines of $M_1|X_1$, and is thus on at least three long lines of $(M_1/e)|X_1$, or else M_1/e has a $U_{2,5}$-restriction, since $(M_1/e)|X_1$ is simple. But x is on two 4-point lines in M_1/e, so Lemma 5.1 implies that M_1 has a minor isomorphic to $U_{2,5}$ or R_9, a contradiction. Thus, $r(M_1) = 5$.

By the minimality of M_1, each plane of $M_1|X_1$ that spans e has size three. Moreover, each element $x \in X_1$ is in at least two planes of $M_1|X_1$ that span e; otherwise e is spanned by a unique line of $(M_1/x)|(X_1 - \{x\})$, and this line has two points, so $M_1/\{x, e\}$ has a $U_{2,5}$-restriction. We may assume that the ground set of $M_1|X_1$ is the edge set of the complete graph with vertex set $\{0, 1, 2, 3, 4, 5\}$. Let $x = 01$ and choose $y = 02$. We will show that $M_1/\{e, y\}$ has at least three long lines through x, two of which have four points, and obtain a contradiction using Lemma 5.1. Let Z be the union of the three 3-point lines of $M_1|X_1$ that contain x but not y.

Note that Z does not span e or y in M_1. Suppose $(M_1/\{e, y\})|Z$ is not simple. Then there is a circuit $\{x_1, x_2, e, y\}$ of M_1 where x_1 and x_2 are in $Z - \{x\}$. Therefore, $\{y, x_1, x_2\}$ is not a 3-edge matching in the K_6 corresponding to $M_1|X_1$, contradicting the fact that each plane of $M_1|X_1$ spanning e has size three. Thus, $(M_1/\{e, y\})|Z$ is simple, so there are at least three long lines of $M_1/\{e, y\}$ that contain x.

There is no plane of $M_1|X_1$ that spans e and contains both x and y, since such a plane would have at least four elements. Since x is in at least two planes of $M_1|X_1$ that span e, this implies that x is in at least two planes of $(M_1/y)|(X_1 - \{y\})$ that span e in M_1/y. Moreover, these planes each have at least four points, since $\text{si}((M_1/y)|(X_1 -$

$\{y\})) \cong M(K_5)$ and so has no 3-point plane. Thus, in $M_1/\{y,e\}$, the element x is on at least two 4-point lines. Since x is on at least three long lines, Lemma 5.1 implies that $M_1/\{e,y\}$ has an R_9-restriction or a $U_{2,5}$-minor, a contradiction.   □

Since $\mathcal{L}$ is non-empty, the sets in $\mathcal{L}$ are pairwise disjoint and pairwise coplanar. To finish the proof, we will use the following two facts:
  (i) If (F, F') is a modular pair of flats of $M|X$ such that F and F' both span e, then the flat $F \cap F'$ spans e as well.
 (ii) If F and F' are flats of $M|X$, and $M|F'$ is isomorphic to a clique, then (F, F') is a modular pair.
First assume that $\mathcal{L}$ contains a 3-point line L. If F is a flat of $M|X$ that spans e, then $F \cap L$ is a flat of $M|X$ that spans e, by (i) and (ii). Since M is simple, $r(F \cap L) \neq 1$, so F contains L. Thus, (a) holds, so we may assume that each line in $\mathcal{L}$ has size two. Then each such line corresponds to a 2-edge matching in $K_{r(M)+1}$. Since the lines in $\mathcal{L}$ are pairwise coplanar, the union of two such lines corresponds to a 4-cycle of $K_{r(M)+1}$. Thus, $|\mathcal{L}| \leq 3$. Moreover, if $\mathcal{L} = \{L_1, L_2, L_3\}$, then $M|(L_1 \cup L_2 \cup L_3) \cong M(K_4)$. As every element of $M|(L_1 \cup L_2 \cup L_3 \cup \{e\})$ is on three long lines, $M|(L_1 \cup L_2 \cup L_3 \cup \{e\}) \cong F_7$, a contradiction. Thus, $|\mathcal{L}| \leq 2$.

Let $Z \subseteq X$ so that $M|Z \cong M(K_4)$ and Z contains each line in $\mathcal{L}$. If $|\mathcal{L}| = 1$, then $(M/e)|Z \cong U_{2,5}$, a contradiction. Therefore $|\mathcal{L}| = 2$. Let L_1 and L_2 be the two lines in $\mathcal{L}$. If F is a flat of $M|X$ that spans e, then $F \cap Z$ is a flat of $M|X$ that spans e, by (i) and (ii). Since M is simple, the flat $F \cap Z$ has rank at least two. If F does not contain L_1 or L_2, this implies that $F \cap Z$ is a line of $M|X$ other than L_1 or L_2. This gives a contradiction as either $\mathcal{L}$ contains a 3-point line, or $|\mathcal{L}| > 2$. We conclude that F contains L_1 or L_2, and so (b) holds.   □

Let M be a matroid with a set X so that $M|X \cong M(K_{r(M)+1})$. We say that an element $e \in E(M) - X$ is *type-(a)* if e is freely added to a 3-point line of $M|X$, and is *type-(b)* if the extension of $M|X$ by e corresponds to the modular cut generated by a pair of 2-point lines of $M|X$ whose union is a circuit. Proposition 5.2 says that if M is simple and has no minor isomorphic to $U_{2,5}$, F_7, or R_9, then every element in $E(M) - X$ is type-(a) or type-(b). Also note that e is type-(a) if and only if $\mathrm{si}((M/e)|X) \cong T_{r(M)-1}$, and e is type-(b) if and only if $\mathrm{si}((M/e)|X) \cong T'_{r(M)-1}$.

6. Two-Element Extensions. We now analyze the structure of a 2-element extension of a clique, under the additional assumption that $U_{2,4} \oplus U_{2,4}$ is excluded as a minor. In a matroid M, for sets $A, B \subseteq E(M)$, we write $\sqcap(A, B)$ for $r_M(A) + r_M(B) - r_M(A \cup B)$.

LEMMA 6.1. *Let M be a simple matroid of rank at least five with no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$, and with a set X so that $M|X \cong M(K_{r(M)+1})$. If $e, f \in E(M) - X$ so that e is type-(a) and f is type-(b), then there is a set $Z \subseteq X$ so that $M|Z \cong M(K_4)$ and $e, f \in \mathrm{cl}_M(Z)$.*

Proof. Since e is a type-(a) element, there is a 3-point line L_1 of $M|X$ so that e is freely added to L_1. By Proposition 3.3, since f is a type-(b) element, there are 2-point lines L_2 and L'_2 of $M|X$ so that $L_2 \cup L'_2$ is a 4-element circuit, and a flat F of $M|X$ spans f if and only if F contains L_2 or L'_2.

We may assume that $\mathrm{cl}_M(L_2 \cup L'_2)$ does not contain e otherwise the lemma holds. Thus $\sqcap(L_2 \cup L'_2, L_1) \neq 2$. If $\sqcap(L_2 \cup L'_2, L_1) = 0$, then $L_1 \cup \{e\}$ and $\mathrm{cl}_{M|X}(L_2 \cup L'_2)$ are skew rank-2 sets in M/f each with four points, so M has $U_{2,4} \oplus U_{2,4}$ as a minor, a contradiction. We deduce that $\sqcap(L_2 \cup L'_2, L_1) = 1$. Hence there is some $Z \subseteq X$ such that $L_1 \cup L_2 \cup L'_2 \subseteq Z$ and $M|Z \cong M(K_5)$. Let $Z \subseteq Y \subseteq X$ so that $M|Y \cong M(K_6)$.

We may assume that the ground set of $M|Y$ is the edge set of the complete graph with vertex set $\{0,1,2,3,4,5\}$, and that the ground set of $M|Z$ is the edge set of the complete graph with vertex set $\{0,1,2,3,4\}$. As $\sqcap(L_2 \cup L_2', L_1) = 1$, we may assume that $L_1 = \{01, 02, 12\}$, and $L_2 = \{13, 24\}$.

Suppose that $\{z, e, f\}$ is a circuit of M for some $z \in Z$. Then either z meets precisely one of 0, 1, or 2, or $z = 34$. In the first case, $L_1 \cup \{z\}$ spans an $M(K_4)$-restriction of $M|X$, so f is spanned by a plane of $M|X$ that does not contain L_2 or L_2', a contradiction. If $z = 34$, then $e \in \mathrm{cl}_M(L_2 \cup L_2')$, a contradiction. Thus, no such element z exists.

Assume that $L_2' = \{14, 23\}$. As $e \notin \mathrm{cl}_M(L_2 \cup L_2')$, the elements $13, 14, 12, 34$ are on distinct lines of M/e through f. Moreover, each element of $(M/e)|(Z \cup \{f\})$ is on one of these four lines, or else $M/\{e, f\}$ has a $U_{2,5}$-restriction. The sets $\{f, 13, 24\}$ and $\{f, 14, 23\}$ have rank two in M/e, and do not span any element in $\{03, 04, 34\}$, otherwise e is spanned by a plane of $M|Z$ that does not contain L_1, a contradiction. This means that each element in $\{03, 04, 34\}$ is spanned in M/e by either $\{f, 12\}$ or $\{f, 34\}$. Since $r_{M/e}(\{03, 04, 34\}) = 2$, this implies that $\{03, 04, 34\}$ is spanned in M/e by $\{f, 34\}$, so $\{03, 04, 34, f\}$ is a $U_{2,4}$-restriction of $(M/e)|(Z \cup \{f\})$. But $\{01, 03, 04, 05\}$ is a basis of $(M/e)|Y$, and $\{01, 05, 15, 25\}$ is a $U_{2,4}$-restriction of M/e, since it spans L_1 in M. Thus, M/e has a $(U_{2,4} \oplus U_{2,4})$-restriction, a contradiction.

Now assume that $L_2' = \{12, 34\}$. Let $N = (M/e)|(Z \cup \{f\})$, and let $N' = N \setminus \{01, 02\}$. Then N' is a simple rank-3 matroid on nine elements. The sets $\{12, 03, 13, 23\}$ and $\{12, 04, 14, 24\}$ are lines of N' because they span e in M, and the set $\{12, 34, f\}$ is a line of N' because it is a line of M that does not span e. But then N' has a $U_{2,5}$-minor or $N' \cong R_9$, by Lemma 5.1, a contradiction. $\qquad\square$

The following lemma imposes structure on a different 2-element extension of a clique.

LEMMA 6.2. *Let M be a simple matroid of rank at least six with no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$ and with a set X so that $M|X \cong M(K_{r(M)+1})$. Let C_1 and C_2 be distinct 4-element circuits of $M|X$ so that, for each $i \in \{1, 2\}$, there is an element $e_i \in E(M) - X$ spanned by both 2-point lines of $M|X$ contained in C_i. Then $|C_1 \cap C_2| = 2$, and there is a set $Z \subseteq X$ so that $M|Z \cong M(K_5)$ and $C_1 \cup C_2 \subseteq Z$.*

Proof. We may assume that $E(M) = X \cup \{e_1, e_2\}$. Note that $e_1 \neq e_2$, and that e_1 and e_2 are both type-(b). Let x and y be distinct elements in $\mathrm{cl}_{M|X}(C_1) - C_1$. Then either x or y is not in $\mathrm{cl}_{M|X}(C_2)$, or C_1 and C_2 are contained in an $M(K_4)$-restriction of $M|X$ and the lemma follows. Assume that $x \notin \mathrm{cl}_{M|X}(C_2)$. The matroid $(M/x)|(X - \{x\})$ has a spanning-clique restriction, and e_2 is spanned by both 2-point lines of $(M/x)|(X - \{x\})$ contained in C_2. Moreover, the simplification of $(M/x)|(C_1 \cup \{y\})$ is a 3-point line, L, that spans e_1 in M/x. Then $L \cup C_2$ is contained in an $M(K_4)$-restriction of M/x, by Lemma 6.1. Since C_1 spans x in M, this implies that there is a set $Z \subseteq X$ so that $M|Z \cong M(K_5)$ and $C_1 \cup C_2 \subseteq Z$.

We now prove that $|C_1 \cap C_2| = 2$. We may assume that the ground set of $M|Z$ is the edge set of the complete graph with vertex set $\{0, 1, 2, 3, 4\}$. Assume that $|C_1 \cap C_2| \neq 2$. Then $|C_1 \cap C_2| = 1$ since K_5 has no edge-disjoint 4-cycle pairs. Up to re-labeling vertices, we may assume that $C_1 = \{01, 12, 23, 03\}$ and $C_2 = \{01, 14, 42, 02\}$.

Suppose there is an element $z \in Z$ for which $\{z, e_1, e_2\}$ is a circuit of M. Assume z is in $\mathrm{cl}_M(C_1)$ or $\mathrm{cl}_M(C_2)$. Then C_1 or C_2 spans both e_1 and e_2. Since $(\mathrm{cl}_M(C_1), \mathrm{cl}_M(C_2))$ is a modular pair of flats of M whose intersection contains $\{01, 12, 02\}$ and e_1 or e_2, we deduce that e_1 or e_2 is spanned by $\{01, 12, 02\}$, a contradiction. We conclude that $z \notin \mathrm{cl}_M(C_1) \cup \mathrm{cl}_M(C_2)$, and so $z = 34$. The set

$\mathrm{cl}_M(\{12, 03, z\})$ contains neither $\{01, 24\}$ nor $\{02, 14\}$, and thus $e_2 \notin \mathrm{cl}_M(\{12, 03, z\})$ because e_2 is type-(b). This implies that $e_2 \notin \mathrm{cl}_{M/z}(\{12, 03\})$. But $e_1 \in \mathrm{cl}_{M/z}(\{12, 03\})$ because $e_1 \in \mathrm{cl}_M(\{12, 03\})$, and so e_1 and e_2 are not parallel in M/z, a contradiction.

The set $\{01, 02, 03, 13\}$ is a $U_{2,4}$-restriction of M/e_1. Since 01 and 23 are parallel in M/e_1 and $\{e_2, 01, 24\}$ is a line of M and $\{e_1, e_2\}$ spans no element of X, the set $\{01, 24, 34, e_2\}$ is a $U_{2,4}$-restriction of M/e_1. Since $\{01, 04, 14\}$ is a $U_{2,3}$-restriction of M/e_1 and $M/e_1 \setminus \{12, 23\}$ is simple, Lemma 5.1 implies that M/e_1 restricted to the union of the three sets $\{01, 02, 03, 13\}$, $\{01, 24, 34, e_2\}$, and $\{01, 04, 14\}$ is either isomorphic to R_9, or has a $U_{2,5}$-minor, a contradiction. □

7. k-Element Extensions. In this section, we place structure on k-element extensions of a clique with $k \geq 3$, in three cases. We say that a point x of a simple matroid M is *special* if it is on at least two 4-point lines, is a tip of at least two spike restrictions of M, or is a tip of a spike restriction of M and is on a 4-point line.

LEMMA 7.1. *Let M be a simple matroid with no minor in $\{U_{2,5}, F_7, R_9\}$, and with a set X so that $M|X \cong M(K_{r(M)+1})$. If $|E(M) - X| \leq 3$, then M has at most* 21 *special points.*

Proof. If $r(M) \leq 2$, then $|E(M)| \leq |X| + |E(M) - X| \leq 6$ and therefore M has at most 6 special points, so we may assume that $r(M) \geq 3$. Since $|E(M) - X| \leq 3$ and M has no $U_{2,5}$-restriction, the points in $E(M) - X$ span at most three lines, each of which spans at most two points in X. By Proposition 5.2, there are at most 12 elements in X that are on a line of $M|X$ that spans an element of $E(M) - X$. Thus, there are at most 18 elements in X that are on a long line of M that is not a long line of $M|X$. Because no spike is graphic, $M|X$ has no spike restriction. Therefore, at most 18 elements in X are a tip of a spike or on a 4-point line. Since $|E(M) - X| \leq 3$, it follows that M has at most 21 special points. □

We must work harder when $|E(M) - X|$ is not bounded.

LEMMA 7.2. *Let M be a simple matroid with no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$, and with a set X so that $M|X \cong M(K_{r(M)+1})$. If $|E(M) - X| \geq 4$ and each element in $E(M) - X$ is type-(a), then M has exactly one special point. Moreover, this point is in every 3-point line of $M|X$ which spans an element in $E(M) - X$.*

Proof. We may assume that the ground set of $M|X$ is the edge set of the complete graph with vertex set $\{0, 1, 2, \ldots, r(M)\}$. Since each point in $E(M) - X$ is type-(a), each is freely added to a 3-point line of $M|X$. Let $\mathcal{L}$ denote the set of 3-point lines of $M|X$ that span an element of $E(M) - X$.

CLAIM 7.3. *The lines in $\mathcal{L}$ intersect in a common point.*

Proof. Suppose that the lines in $\mathcal{L}$ do not intersect in a common point. Since M has no $(U_{2,4} \oplus U_{2,4})$-minor, each pair of lines in $\mathcal{L}$ has a common element. Then since $|E(M) - X| \geq 4$, M has a restriction N that can be obtained from $M(K_4)$ by freely adding a point to each long line. Let z be an element of this $M(K_4)$-restriction. Then z is on at least two 4-point lines of N. Suppose z is on three or more long lines of N. Then by Lemma 5.1, N has a minor isomorphic to R_9 or $U_{2,5}$, a contradiction. Therefore z is on exactly two long lines of N. But since $|N| = 10$ and N has no $U_{2,5}$-restriction, this implies that $N/z \cong U_{2,5}$, a contradiction. □

Without loss of generality, we may assume that the lines in $\mathcal{L}$ intersect in the point $x = 01$. Since $|\mathcal{L}| \geq 4$, x is special. We will show that x is the unique special

point of M. Note that the points of $(M/x)|(X - \{x\})$ of size at least two form a basis of $\mathrm{si}((M/x)|(X - \{x\}))$; this implies that $E(M) - X$ is independent in M.

CLAIM 7.4. *Let $e_1, e_2 \in E(M) - X$, and, for each $i \in \{1, 2\}$, let L_i be a line in $\mathcal{L}$ that spans e_i. If $z \in X$ so that $\{z, e_1, e_2\}$ is a circuit of M, then z is the unique element in $\mathrm{cl}_{M|X}(L_1 \cup L_2) - (L_1 \cup L_2)$. Moreover, every 4-point line of M contains exactly three elements of X including x.*

Proof. Certainly $z \in \mathrm{cl}_M(L_1 \cup L_2)$ since $\{z, e_1, e_2\}$ is a circuit. If $z \in L_1$, then L_1 spans e_2. But then L_1 is a flat of $M|X$ that spans e_2 but does not contain L_2, which contradicts Proposition 5.2(i). Thus, $z \notin L_1$, and, by symmetry, $z \notin L_2$. Since $|\mathrm{cl}_{M|X}(L_1 \cup L_2)| = 6$ and $|L_1 \cup L_2| = 5$, the element z is unique. As $E(M) - X$ is independent, it follows that each 4-point line of M contains exactly three elements of X including x. □

By Claim 7.4, if an element y of X is on a line in $\mathcal{L}$, then each long line of M through y is spanned by a long line of $M|X$ through y. As no spike is graphic, y is not a tip of a spike restriction of M. Moreover, if $y \neq x$, then y is on exactly one 4-point line of M, so X contains at most one special point of M that is on a line in $\mathcal{L}$.

If an element y of X is not on a line in $\mathcal{L}$, then y is not on a 4-point line, since $E(M) - X$ is independent. By Claim 7.4, the element y is on at most one more long line in M than in $M|X$, which implies that y is a tip of at most one spike restriction of M. Thus, y is not a special point. Therefore, M has at most one special point in X, namely x.

Now, let $e \in E(M) - X$. We will show that e is not special. We may assume that e is freely added to the 3-point line $L' = \{01, 02, 12\}$ of $M|X$. By Claim 7.4, each long line of M through e contains an element of X that uses the vertex 2. Let T be a transversal of the points of M/e so that each element in T is in X and uses the vertex 2; then T is independent in $M|X$. Also, T contains at most one of 02 and 12, and so T does not span L' in $M|X$. Since e is freely added to L', this implies that T does not span e in M. Since T is independent in M and does not span e, the element e is not a tip of a spike restriction of M. Since e is on exactly one 4-point line of M, the element e is not special. Thus, x is the unique special point of M, so the lemma holds. □

We need an analogous lemma for a different case.

LEMMA 7.5. *Let M be a simple matroid with no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$, and with a set X so that $M|X \cong M(K_{r(M)+1})$. If there is a 3-point line $\{x, y, z\}$ of $M|X$ so that each element in $E(M) - X$ is either freely added to $\{x, y, z\}$, or is spanned by both 2-point lines of $M|X$ in some 4-element circuit of $M|X$ containing x and y, then M has at most two special points.*

Proof. Let $L = \{x, y, z\}$. We may assume that the ground set of $M|X$ is the edge set of the complete graph with vertex set $\{0, 1, 2, \ldots, r(M)\}$, and that $(x, y, z) = (01, 02, 12)$. By Proposition 5.2, each element in $E(M) - X$ that is not freely added to L is type-(b). Let $\mathcal{C}$ denote the set of circuits C of $M|X$ for which both 2-point lines of $M|X$ in C span a common element of $E(M) - X$. By hypothesis, each circuit in $\mathcal{C}$ contains x and y.

The points of $(M/x)|(X - \{x\})$ of size at least two form a basis of $\mathrm{si}((M/x)|(X - \{x\}))$ because they all use the composite vertex obtained by identifying 0 and 1. This implies that $E(M) - X$ is independent in M. Since $E(M) - X$ is independent, there is at most one 4-point line of M, namely $\mathrm{cl}_M(L)$.

CLAIM 7.6. *Let $e_1, e_2 \in E(M) - X$ so that neither is freely added to L, and, for each $i \in \{1, 2\}$, let C_i be a circuit in $\mathcal{C}$ that spans e_i. If $w \in X$ so that $\{w, e_1, e_2\}$ is a circuit of M, then w is the unique element in $\mathrm{cl}_{M|X}(C_1 \cup C_2) - (\mathrm{cl}_{M|X}(C_1) \cup \mathrm{cl}_{M|X}(C_2))$.*

Proof. Certainly $w \in \mathrm{cl}_M(C_1 \cup C_2)$, since $\{w, e_1, e_2\}$ is a circuit. If $w \in \mathrm{cl}_{M|X}(C_1)$, then $\mathrm{cl}_{M|X}(C_1)$ is a flat of $M|X$ that spans e_2, but does not contain either 2-point line of $M|X$ contained in C_2, which contradicts that e_2 is of type-(b). Thus, $w \notin \mathrm{cl}_M(C_1)$, and, by symmetry, $w \notin \mathrm{cl}_M(C_2)$. As $C_1 \cap C_2 = \{x, y\}$, it follows that $|\mathrm{cl}_{M|X}(C_1 \cup C_2)| = 10$ and $|\mathrm{cl}_{M|X}(C_1) \cup \mathrm{cl}_{M|X}(C_2)| = 9$. Thus, the element w is unique. ☐

The following deals with a slightly different case.

CLAIM 7.7. *Let $e \in E(M) - X$ be freely added to L, and let $f \in E(M) - (X \cup e)$ be spanned by a circuit $C \in \mathcal{C}$. If $w \in X$ so that $\{w, e, f\}$ is a circuit of M, then w is the unique element in $\mathrm{cl}_{M|X}(C) - (L \cup C)$.*

Proof. Certainly $w \in \mathrm{cl}_{M|X}(C)$, since C spans L and $\{w, e, f\}$ is a circuit. If $w \in L$, then $\mathrm{cl}_M(L)$ contains $\{x, y, z, e, f\}$, a contradiction. If $w \in C$, then there is a 2-point line of $M|X$ that contains w and spans e but does not contain L, which contradicts the fact that e is freely added to L. Since $|\mathrm{cl}_{M|X}(C)| = 6$ and $|L \cup C| = 5$, the element w is unique. ☐

If $w \in X - L$ is in a circuit in $\mathcal{C}$, then w is incident to one of the vertices 1 and 2. This implies that there is no pair (L, C) or (C_1, C_2) with $C, C_1, C_2 \in \mathcal{C}$ so that w is the unique element in $\mathrm{cl}_{M|X}(C) - (L \cup C)$ or $\mathrm{cl}_{M|X}(C_1 \cup C_2) - (\mathrm{cl}_{M|X}(C_1) \cup \mathrm{cl}_{M|X}(C_2))$. So, by Claim 7.6 and Claim 7.7, there is no pair of elements in $E(M) - X$ that span w. Since $w \notin L$ and each circuit in $\mathcal{C}$ contains x and y, the element w is in exactly one circuit in $\mathcal{C}$. Therefore, w is on exactly one long line, L', of M that is not a long line of $M|X$. Thus, w is a tip of at most one spike in M, otherwise M has a spike not using the line L', so the graphic matroid $M|X$ has a spike, a contradiction. Since $w \notin L$, it is not on a 4-point line, so it is not special.

Since $z = 12$, we see that $z \in L$ and $z \in \mathrm{cl}_{M|X}(C)$ for all $C \in \mathcal{C}$. So, by Claim 7.6 and Claim 7.7, there is no pair of elements in $E(M) - X$ that span z. This implies that each long line of M through z is spanned by a long line of $M|X$ through z, so z is not a tip of a spike. Since z is on at most one 4-point line, it is not special.

If $w \in X - L$ is not in a circuit in $\mathcal{C}$, then there is at most one pair (L, C) or (C_1, C_2) with $C, C_1, C_2 \in \mathcal{C}$ so that w is the unique element in $\mathrm{cl}_{M|X}(C) - (L \cup C)$ or $\mathrm{cl}_{M|X}(C_1 \cup C_2) - (\mathrm{cl}_{M|X}(C_1) \cup \mathrm{cl}_{M|X}(C_2))$. So, by Claim 7.6 and Claim 7.7, there is at most one pair of elements in $E(M) - X$ that span w. Then, since w is not in a circuit in $\mathcal{C}$ and is not in L, it is on at most one long line of M that is not a long line of $M|X$. Thus, it is a tip of at most one spike. Since $w \notin L$, it is not on a 4-point line, so it is not special. Therefore, M has at most two special points in X.

Now, let $e \in E(M) - X$. We will show that e is not special. We consider two cases. First assume that e is freely added to L. As $E(M) - X$ is independent, by Claim 7.7, each long line of M through e contains an element of X that uses the vertex 0. Let T be a transversal of the non-trivial parallel classes of M/e so that each element in T is in X and uses the vertex 0. Then T is independent in $M|X$. Also, T contains at most one of 01 and 02, and so T does not span L in $M|X$, which implies that T does not span e in M. Since T is independent in M and does not span e, the element e is not a tip of a spike restriction of M. Since e is on at most one 4-point line of M, the element e is not special.

Now assume that e is not freely added to L. Then e is a type-(b) element and is on no 4-point lines of M. Let C be a circuit in $\mathcal{C}$ so that both 2-point lines of $M|X$ in C span e. We may assume that $C = \{01, 13, 23, 02\}$. Then $\{01, 23\}$ and $\{02, 13\}$ both span e. By Claim 7.7 and Claim 7.6, each long line of M through e contains an element of X that uses the vertex 3. Let T be a transversal of the non-trivial parallel classes of M/e so that each element in T is in X and uses the vertex 3. Then T is independent in $M|X$. Note that T contains 13 and 23, and that $T - \{13\}$ does not span $\{01, 23\}$ or $\{02, 13\}$ in M. Since e is of type-(b), this implies that $T - \{13\}$ does not span e. Then $T - \{13\}$ is an independent set that does not span e, and is a transversal of all but one long line of M through e. Thus, e is a tip of at most one spike. Since e is not on a 4-point line of M, this implies that e is not special. Thus, M has no special point in $E(M) - X$, and at most two special points in X, so the lemma holds. □

8. The Spanning-Clique Case. We prove the following result for matroids with a spanning-clique restriction. Recall that a point x of a simple matroid M is *special* if it is on at least two 4-point lines, is a tip of at least two spike restrictions of M, or is a tip of a spike restriction of M and is on a 4-point line.

PROPOSITION 8.1. *Let M be a simple matroid of rank at least six with a spanning-clique restriction. If M has no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$, then*

(i) $|M| \le \binom{r(M)+2}{2} - 2$, *and*

(ii) M *has at most* 21 *special points.*

Proof. Let $M|X \cong M(K_{r(M)+1})$. Note that $|M| \le \binom{r(M)+2}{2} - 2$ if and only if $|E(M) - X| \le r(M) - 1$. By Proposition 5.2, each element of $E(M) - X$ is either freely added to a 3-point line of $M|X$, or corresponds to the modular cut generated by a pair of 2-point lines of $M|X$ whose union is a circuit. Let $\mathcal{L}$ be the set of 3-point lines of $M|X$ that span an element in $E(M) - X$. Let $\mathcal{C}$ be the set of 4-element circuits C of $M|X$ for which both 2-point lines of $M|X$ contained in C span a common element of $E(M) - X$.

Clearly each line in $\mathcal{L}$ spans exactly one element of $E(M) - X$, or else M has a $U_{2,5}$-restriction. If there is a pair $\{L, L'\}$ of 2-point lines L and L' of $M|X$ whose union is a circuit so that L and L' each span elements e and f in $E(M) - X$, then $L \cup \{e, f\}$ and $L' \cup \{e, f\}$ are distinct lines of M that intersect in two points, a contradiction. Thus, $|E(M) - X| = |\mathcal{L}| + |\mathcal{C}|$.

We first deal with the case when two circuits in $\mathcal{C}$ span each other.

CLAIM 8.2. *If there are distinct circuits C_1 and C_2 in $\mathcal{C}$ whose union is contained in an $M(K_4)$-restriction of $M|X$, then $|E(M) - X| \le 3$, and (i) and (ii) hold.*

Proof. For each $i \in \{1, 2\}$, let e_i be the element of M that is spanned by each 2-point line of $M|X$ contained in C_i. Since $C_1 \cup C_2$ is contained in an $M(K_4)$-restriction of $M|X$, there is a line $\{x, y\}$ of $M|X$ contained in both C_1 and C_2.

Suppose that $\mathcal{L} \ne \varnothing$, and let $L \in \mathcal{L}$. Then $L \subseteq C_1 \cup C_2$, or else either (L, C_1) or (L, C_2) violates Lemma 6.1. Either x or y is in L, since every triangle of K_4 intersects every size-2 matching; assume without loss of generality that $x \in L$. But then x is on the 4-point lines $\mathrm{cl}_M(L)$ and $\{x, y, e_1, e_2\}$, and also a 3-point line other than L in $C_1 \cup C_2$, a contradiction to Lemma 5.1. Thus, $\mathcal{L} = \varnothing$.

If there is some circuit $C_3 \in \mathcal{C}$ that is not contained in $C_1 \cup C_2$, then there are at most two elements of C_3 in $C_1 \cup C_2$. Since $|C_1 \cap C_3| = 2$ and $|C_2 \cap C_3| = 2$ by Lemma 6.2, this implies that C_3 contains $\{x, y\}$. But the only 4-element circuits of $M|X$ that contain $\{x, y\}$ are C_1 and C_2. Thus, each circuit in $\mathcal{C}$ is contained in

$C_1 \cup C_2$. Since $C_1 \cup C_2$ is contained in an $M(K_4)$-restriction of $M|X$, it follows that $|\mathcal{C}| \leq 3$. Then $|E(M) - X| \leq 3$ so (i) holds, and (ii) holds by Lemma 7.1. $\qquad \square$

Next we treat the case when the lines in $\mathcal{L}$ do not share a common element.

CLAIM 8.3. *If $\mathcal{L}$ contains a set of three lines that do not intersect in a common point, then $|E(M) - X| = 3$, and (i) and (ii) hold.*

Proof. Since M has no $(U_{2,4} \oplus U_{2,4})$-minor, each pair of lines in $\mathcal{L}$ share a common element. Since the lines in $\mathcal{L}$ have no common point of intersection, there is a set $Z \subseteq X$ so that $M|Z \cong M(K_4)$ and Z contains each line in $\mathcal{L}$. Suppose $|\mathcal{L}| \geq 4$. Then Lemma 7.2 shows that the lines in $\mathcal{L}$ intersect in a common element, a contradiction. Thus $|\mathcal{L}| = 3$, and so there are three elements $x, y, z \in Z$ that are each on two lines in $\mathcal{L}$.

Suppose $|\mathcal{C}| \geq 1$. By Lemma 6.1, each circuit in $\mathcal{C}$ is contained in Z, and so each circuit in $\mathcal{C}$ contains one of x, y, z, since $|Z| = 6$. But then one of x, y, z is on two 4-point lines and a 3-point line of $M|\operatorname{cl}_M(Z)$, so M has an R_9-restriction or a $U_{2,5}$-minor by Lemma 5.1. This is a contradiction, so $\mathcal{C} = \varnothing$, and so $|E(M) - X| = 3$. Then $|E(M) - X| \leq 3$ so (i) holds, and (ii) holds by Lemma 7.1. $\qquad \square$

We now consider four cases, depending on the size of $\mathcal{L}$. Suppose first that $|\mathcal{L}| \geq 3$. By Claim 8.3, the lines in $\mathcal{L}$ share a common element x; thus, $|\mathcal{L}| \leq r(M) - 1$. Since $|\mathcal{L}| \geq 3$, the set $\mathcal{C}$ is empty, or else there is a line $L \in \mathcal{L}$ and a circuit $C \in \mathcal{C}$ so that (L, C) violates Lemma 6.1. Then $|E(M) - X| = |\mathcal{L}| \leq r(M) - 1$, and (i) holds. Also, (ii) holds by Lemma 7.2 if $|\mathcal{L}| \geq 4$ and by Lemma 7.1 if $|\mathcal{L}| = 3$.

Next assume that $|\mathcal{L}| = 2$. The union of the two lines in $\mathcal{L}$ is contained in an $M(K_4)$-restriction $M|Z$ of $M|X$, and by Lemma 6.1, each circuit in $\mathcal{C}$ is contained in Z. By Lemma 5.1, the element x that is on both lines in $\mathcal{L}$ is in no circuit in $\mathcal{C}$. This implies that $|\mathcal{C}| \leq 1$, and so $|E(M) - X| \leq 3 \leq r(M) - 1$, and (ii) holds by Lemma 7.1.

Now suppose that $|\mathcal{L}| = 1$. Let $L \in \mathcal{L}$, and let e be the element of $E(M) - X$ that is spanned by L. By Claim 8.2, we may assume that each circuit in $\mathcal{C}$ spans at most one element of $E(M) - X$ other than e. By Lemma 6.1, each circuit in $\mathcal{C}$ contains two elements of L. If, for two different circuits C_1 and C_2 in $\mathcal{C}$, we have $C_1 \cap L \neq C_2 \cap L$, then, as $r(C_1 \cup C_2) = 4$, we see that $|C_1 \cap C_2| = 1$, which contradicts Lemma 6.2. Thus, there are elements x and y in L that are contained in each circuit in $\mathcal{C}$. This implies that $|\mathcal{C}| \leq r(M) - 2$, and so $|E(M) - X| \leq r(M) - 1$. Also, (ii) holds by Lemma 7.5.

Finally, suppose that $|\mathcal{L}| = 0$. By Claim 8.2, we may assume that each $M(K_4)$-restriction of $M|X$ spans at most one element of $E(M) - X$. By Lemma 6.2, each pair of these $M(K_4)$-restrictions has a common triangle. If there is no shared triangle among all of these restrictions, then there are exactly three $M(K_4)$-restrictions of $M|X$ that span an element of $E(M) - X$, and they are contained in an $M(K_5)$-restriction of $M|X$. Then $|E(M) - X| = 3$, so (i) and (ii) hold, using Lemma 7.1. Otherwise, there is a common triangle L contained in each $M(K_4)$-restriction that spans an element of $E(M) - X$. By the same reasoning at the end of the previous case, there are elements x and y in L that are contained in each circuit in $\mathcal{C}$, and so (i) and (ii) hold. $\qquad \square$

9. The Proofs of the Main Results. In this section, we prove Theorem 1.3 and Theorem 1.1. We use two results of Geelen, Nelson, and Walsh from [5]. To state these results, we need a notion of matroid connectivity. A *vertical j-separation* of a matroid M is a partition (X, Y) of $E(M)$ so that $r(X) + r(Y) - r(M) < j$ and

$\min(r(X), r(Y)) \geq j$. A matroid M is *vertically k-connected* if it has no vertical j-separation with $j < k$. The first result [5, Theorem 6.1.3] finds structure in a minimal counterexample. The second, which is a consequence of Theorem 7.6.1 in [5], allows us to move from a clique minor to a spanning-clique restriction.

THEOREM 9.1. *There is a function $r_{9.1} \colon \mathbb{R}^6 \to \mathbb{Z}$ so that, for all integers $\ell \geq 2$ and $t, s \geq 1$, and any real polynomial $p(x) = ax^2 + bx + c$ with $a > 0$, if $M \in \mathcal{U}(\ell)$ satisfies $r(M) \geq r_{9.1}(a, b, c, \ell, t, s)$ and $\varepsilon(M) > p(r(M))$, then M has a minor N with $\varepsilon(N) > p(r(N))$ and $r(N) \geq t$ such that either*

(1) N has a spanning-clique restriction, or

(2) N is vertically s-connected and has an s-element independent set S so that $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N) - 1)$ for each $e \in S$.

THEOREM 9.2. *There are functions $s_{9.2} \colon \mathbb{Z} \to \mathbb{Z}$ and $r_{9.2} \colon \mathbb{Z}^3 \to \mathbb{Z}$ so that, for all integers $\ell \geq 2$, $k \geq 1$, and $m \geq 3$, if M is a vertically $s_{9.2}(k)$-connected matroid with no $U_{2,\ell+2}$-minor and no Reid geometry minor, and with an $M(K_{r_{9.2}(\ell,m,k)+1})$-minor and $2k$ special points, then M has a simple minor N of rank at least m with an $M(K_{r(N)+1})$-restriction and k special points.*

In order to apply Theorem 9.2, we need the following result of Geelen and Whittle [6].

THEOREM 9.3. *There is a function $\alpha_{9.3} \colon \mathbb{Z}^2 \to \mathbb{Z}$ so that, for all integers $\ell, t \geq 2$, if $M \in \mathcal{U}(\ell)$ satisfies $\varepsilon(M) > \alpha_{9.3}(\ell, t) \cdot r(M)$, then M has an $M(K_{t+1})$-minor.*

We now prove a restatement of Theorem 1.3 and then Theorem 1.1.

THEOREM 9.4. *There is a constant n_1 so that if M is a simple matroid of rank at least n_1 with no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$, then $|M| \leq \binom{r(M)+2}{2} - 2$.*

Proof. Let $\mathcal{M}$ be the class of matroids with no minor in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$. Note that $\binom{x+2}{2} - 2 = \frac{1}{2}x^2 + \frac{3}{2}x - 1$ for any positive number x. Let n_0 be an integer so that $\binom{x+2}{2} - 2 > \alpha_{9.3}(3, r_{9.2}(3, 6, 22)) \cdot x$ for all $x \geq n_0$. Define

$$n_1 = r_{9.1}\left(\frac{1}{2}, \frac{3}{2}, -1, 3, n_0, \max(s_{9.2}(22), 44)\right).$$

Assume that there is a simple matroid $M \in \mathcal{M}$ with $r(M) \geq n_1$ and with $|M| > \binom{r(M)+2}{2} - 2$. By Theorem 9.1 with $p(x) = \binom{x+2}{2} - 2$ and $t = n_0$ and $s = \max\left(s_{9.2}(22), 44\right)$, there is a minor N of M so that $r(N) \geq n_0$ and $|N| > \binom{r(N)+2}{2} - 2$, and either

(a) N has a spanning-clique restriction, or

(b) N is vertically $s_{9.2}(22)$-connected, and has a 44-element independent set S so that each $e \in S$ satisfies $\varepsilon(N) - \varepsilon(N/e) \geq r(N) + 2$.

By Proposition 8.1(i), outcome (a) does not hold, so outcome (b) holds.

CLAIM 9.5. *Each element in S is special.*

Proof. For $e \in S$, let $\mathcal{L}_N(e)$ denote the set of long lines of N that contain e. Then

(9.1) $$\varepsilon(N) - \varepsilon(N/e) = 1 + \sum_{L \in \mathcal{L}_N(e)} \left(|L| - 2\right).$$

Let $N_1 = N|(\cup_{L \in \mathcal{L}_N(e)} L)$, and note that $|\operatorname{si}(N_1/e)| = |\mathcal{L}_N(e)|$. If e is on at least two 4-point lines of N, then e is special. If e is on no 4-point line of N, then, since $\varepsilon(N) - \varepsilon(N/e) \geq r(N) + 2$, it follows from (1) that $|\mathcal{L}_N(e)| \geq r(N) + 1$. Thus,

$|\operatorname{si}(N_1/e)| \geq r(N) + 1$. Since $\operatorname{si}(N_1/e)$ has rank at most $r(N) - 1$, this implies that $\operatorname{si}(N_1/e)$ has corank at least two, and so it contains at least two distinct circuits. Then e is a tip of two spike restrictions of N, and is thus special. If e is on exactly one 4-point line of N, then (1) implies that $\operatorname{si}(N_1/e)$ has corank at least one and thus contains a circuit, so e is a tip of a spike restriction of N.   □

Since $r(N) \geq n_0$ and $|N| > \binom{r(N)+2}{2} - 2$, Theorem 9.3 and the definition of n_0 imply that N has an $M(K_{r_{9.2}(3,6,22)+1})$-minor. Then, since N has no $U_{2,5}$- or R_9-minor, Theorem 9.2 implies that N has a simple minor of rank at least six with a spanning-clique restriction and at least 22 special points, which contradicts Proposition 8.1(*ii*).   □

Proof of Theorem 1.1. Let n_1 be the constant from Theorem 9.4, and let A be a 2-modular matrix of rank $r \geq n_1$ with no zero-column and no parallel pair of columns. Let M be the vector matroid of A. Then M is a simple rank-r matroid with no minor isomorphic to a matroid in $\{U_{2,5}, F_7, R_9, U_{2,4} \oplus U_{2,4}\}$. By Theorem 9.4, $|M| \leq \binom{r(M)+2}{2} - 2$, so A has at most $\binom{r(M)+2}{2} - 2$ columns.   □

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