GENERALIZING COGRAPHS TO 2-COGRAPHS

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Abstract. In this paper, we consider 2-cographs, a natural generalization of the well-known class of cographs. We show that, as with cographs, 2-cographs can be recursively defined. However, unlike cographs, 2-cographs are closed under induced minors. We consider the class of non-2-cographs for which every proper induced minor is a 2-cograph and show that this class is infinite. Our main result finds the finitely many members of this class whose complements are also induced-minor-minimal non-2-cographs.

1. Introduction

In this paper, we only consider simple graphs. Except where indicated otherwise, our notation and terminology will follow [5]. A cograph is a graph in which every connected induced subgraph has a disconnected complement. By convention, the graph $K_1$ is assumed to be a cograph. Replacing connectedness by 2-connectedness, we define a graph $G$ to be a 2-cograph if $G$ has no induced subgraph $H$ such that both $H$ and its complement, $\overline{H}$, are 2-connected. Note that $K_1$ is a 2-cograph.

Cographs have been extensively studied over the last fifty years (see, for example, [6, 12, 4]). They are also called $P_4$-free graphs due to following characterization [3].

Theorem 1.1. A graph $G$ is a cograph if and only if $G$ does not contain the path $P_4$ on four vertices as an induced subgraph.

In other words, $P_4$ is the unique non-cograph with the property that every proper induced subgraph of $P_4$ is a cograph. The main result of the paper identifies non-2-cographs that are minimal in a certain natural sense.

In Section 2, we show that 2-cographs can be recursively defined and are closed under taking induced minors. We also generalize cographs to $k$-cographs for $k$ exceeding two and note that, in that case, the class of $k$-cographs is not closed under contraction. In addition, we correct a result of Akiyama and Harary [1] that had claimed to characterize when the complement of a 2-connected graph is 2-connected.

In Section 3, we show that, in contrast to Theorem 1.1, the class of non-2-cographs for which every proper induced minor is a 2-cograph is infinite. This prompts us to determine those non-2-cographs whose complements

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are also induced-minor-minimal non-2-cographs. The following is the main result of the paper. Its proof occupies most of Section 3.

**Theorem 1.2.** Let \( G \) be a graph. Suppose that \( G \) is not a 2-cograph but every proper induced minor of each of \( G \) and \( \overline{G} \) is a 2-cograph. Then \( 5 \leq |V(G)| \leq 10 \).

The unique 5-vertex graph satisfying the hypotheses of the last theorem is \( C_5 \), the 5-vertex cycle. In the appendix, we list all of the other graphs that satisfy these hypotheses.

2. Preliminaries

Let \( G \) be a graph. A vertex \( u \) of \( G \) is a **neighbour** of a vertex \( v \) of \( G \) if \( uv \) is an edge of \( G \). The **neighbourhood** \( N_G(v) \) of \( v \) in \( G \) is the set of all neighbours of \( v \) in \( G \). A **t-cut** of \( G \) is set \( X_t \) of vertices of \( G \) such that \( |X_t| = t \) and \( G - X_t \) is disconnected. A graph that has no \( t \)-cut for all \( t \) less than \( k \) is **\( k \)-connected**.

Viewing \( G \) as a subgraph of \( K_n \) where \( n = |V(G)| \), we colour the edges of \( G \) green while assigning the colour red to the non-edges of \( G \). In this paper, we use the terms **green graph** and **red graph** for \( G \) and its complementary graph \( \overline{G} \), respectively. An edge of \( G \) is called a **green edge** while a **red edge** refers to an edge of \( \overline{G} \). The **green degree** of a vertex \( v \) of \( G \) is the number of **green neighbours** of \( v \) while the **red degree** of \( v \) is its number of **red neighbours**. Similarly, a **green t-cut** refers to a \( t \)-cut of the graph \( G \) while a **red t-cut** is a \( t \)-cut of the graph \( \overline{G} \).

For graphs \( G_1 \) and \( G_2 \) having disjoint vertex sets, the **0-sum** \( G_1 \oplus G_2 \) of \( G_1 \) and \( G_2 \) is their disjoint union. A **1-sum** \( G_1 \oplus_1 G_2 \) of \( G_1 \) and \( G_2 \) is obtained by identifying a vertex of \( G_1 \) with a vertex of \( G_2 \).

We omit the straightforward proofs of the next three results.

**Lemma 2.1.** Let \( G \) be a 2-cograph.

(i) Every induced subgraph of \( G \) is a 2-cograph.

(ii) \( \overline{G} \) is a 2-cograph.

**Lemma 2.2.** Let \( G_1 \) and \( G_2 \) be 2-cographs and let \( G \) be a 0-sum or 1-sum of \( G_1 \) and \( G_2 \). Then \( G \) is a 2-cograph.

**Lemma 2.3.** Let \( G \) be a graph and let \( uv \) be an edge \( e \) of \( G \). Then the complement of \( G/e \) is the graph obtained by adding a vertex \( w \) with neighbourhood \( N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \) to the graph \( \overline{G} - \{u, v\} \).

A consequence of the fact that we consider only simple graphs here is that, when we write \( G/e \) for an edge \( e \) of a graph \( G \), we mean the graph that we get from the multigraph obtained by contracting the edge \( e \) and then deleting all but one edge from each class of parallel edges.

**Lemma 2.4.** Let \( G \) be a 2-cograph and \( e \) be an edge of \( G \). Then \( G/e \) is a 2-cograph.
Proof. Assume to the contrary that $G/e$ is not a 2-cograph. Then there is an induced subgraph $H$ of $G/e$ such that both $H$ and $\overline{H}$ are 2-connected. Let $e = uv$ and let $w$ denote the vertex in $G/e$ obtained by identifying $u$ and $v$. We may assume that $w$ is a vertex of $H$, otherwise $H$ is an induced subgraph of $G$, a contradiction. We assert that the subgraph $H'$ of $G$ induced on the vertex set $(V(H) \cup \{u,v\}) - w$ is 2-connected as is its complement $\overline{H'}$. Since $H$ is 2-connected, $H'$ is 2-connected unless it has $e$ as a cut-edge. In the exceptional case, we may assume that $H' - u \cong H$. Thus $G$ has an induced subgraph so that both it and its complement are 2-connected, a contradiction. We deduce that $H'$ is 2-connected.

Observe that the neighbours of $w$ in $\overline{H}$ are the common neighbours of $u$ and $v$ in $H'$. Thus the degrees of $u$ and $v$ in $\overline{H}$ each equal at least the degree of $w$ in $H$. Moreover, $\overline{H'} - u$ has a subgraph isomorphic to $\overline{H}$ and is therefore 2-connected. Since $u$ has degree at least two in $\overline{H}$, it follows that $\overline{H'}$ is 2-connected, a contradiction. \(\square\)

Note that Lemmas 2.1 and 2.4 imply that 2-cographs are closed under taking induced minors. Cographs are also called complement-reducible graphs due to the following recursive-generation result [3]. The operation of taking the complement of a graph is called complementation.

Lemma 2.5. A graph $G$ is a cograph if and only if it can be generated from $K_1$ using complementation and 0-sum.

Next, we show that if we include the operation of 1-sum in the above operations, we get the class of 2-cographs.

Lemma 2.6. A graph $G$ is a 2-cograph if and only if it can be generated from $K_1$ using complementation, 0-sum and 1-sum.

Proof. Let $G$ be a 2-cograph. If $|G| = 2$, the result holds. We proceed via induction on the number of vertices of $G$ and assume that the result holds for all 2-cographs of order less than $|G|$. Since $G$ is a 2-cograph, either $G$ or $\overline{G}$ is not 2-connected. Without loss of generality, we may assume that $G$ is not 2-connected and can therefore be written as a 0-sum or 1-sum of two induced subgraphs $G_1$ and $G_2$ of $G$. By Lemma 2.1, $G_1$ and $G_2$ are 2-cographs and the result follows by induction.

Conversely, let $G$ be a graph that can be generated from $K_1$ using complementation, 0-sum and 1-sum. Since $K_1$ is a 2-cograph, the result follows by Lemmas 2.1 and 2.2. \(\square\)

The following recursive-generation result for cographs is due to Royle [10]. It uses the concept of join of two disjoint graphs $G$ and $H$, which is the graph $G \vee H$ obtained by joining every vertex of $G$ to every vertex of $H$.

Lemma 2.7. Let $\mathcal{C}$ be the class of graphs defined as follows:

(i) $K_1$ is in $\mathcal{C}$;

(ii) if $G$ and $H$ are in $\mathcal{C}$, then so is their 0-sum; and
(iii) if $G$ and $H$ are in $\mathcal{C}$, then so is their join.
Then $\mathcal{C}$ is the class of cographs.

We generalise the join operation to 1-join $G \nabla_1 H$ of graphs $G$ and $H$ obtained by joining every vertex of $V(G) - V(H)$ to every vertex of $V(H) - V(G)$ where $|V(G) \cap V(H)|$ is allowed to be non-zero but is at most one. Note that $G \nabla_1 H$ is the graph $G \oplus H$ if $G$ and $H$ are disjoint. If $G$ and $H$ have a common vertex $v$, then $G \nabla_1 H = G \oplus_1 H$ where the 1-sum of $G$ and $H$ is obtained by identifying the common vertex $v$.

The class of 2-cographs can be recursively generated in a way similar to Lemma 2.7.

**Lemma 2.8.** Let $\mathcal{C}$ be the class of graphs defined as follows:

(i) $K_1$ is in $\mathcal{C}$;
(ii) if $G$ and $H$ are in $\mathcal{C}$, then so is their 0-sum and 1-sum; and
(iii) if $G$ and $H$ are in $\mathcal{C}$, then so is their 1-join.

Then $\mathcal{C}$ is the class of 2-cographs.

**Proof.** Since the operation of 1-join can be written in terms of 0-sum, 1-sum and complementation, every graph in $\mathcal{C}$ is a 2-cograph.

Conversely, let $G$ be a 2-cograph. If $|G| = 1$, then $G \in \mathcal{C}$. We proceed by induction on $|G|$ and assume that $H \in \mathcal{C}$ if $|H| < |G|$ for a 2-cograph $H$. Let $|G| = n \geq 2$. By Lemma 2.6, either $G$ or $\overline{G}$ is a 0-sum or 1-sum of two smaller 2-cographs. If $G$ is the graph that can be decomposed as above, the result follows by induction. Therefore we may assume that $G$ is 2-connected and $\overline{G}$ is either $G_1 \oplus G_2$ or $G_1 \oplus_1 G_2$ for two smaller 2-cographs $G_1$ and $G_2$. Observe that $G = \overline{G_1} \nabla_1 G_2$. By Lemma 2.1, $\overline{G_1}$ and $\overline{G_2}$ are 2-cographs and so are in $\mathcal{C}$ by induction. Therefore $G$ is in $\mathcal{C}$. \qed

For $k > 2$, call a graph $G$ $k$-cograph if, for every induced subgraph $H$ of $G$, either $H$ or $\overline{H}$ is not $k$-connected. Clearly, every $(k - 1)$-cograph is also a $k$-cograph. It is easy to observe that the class of $k$-cographs is closed under complementation and taking induced subgraphs. However, we note that, for $k$ exceeding two, the class of $k$-cographs is not closed under contraction.

![Figure 1. $k$-cographs are not closed under contraction.](image)

We can check that the graph $G$ in Figure 1 is a 3-cograph and is therefore a $k$-cograph for all $k > 2$. Since both $G/e$ and its complement are 3-connected, $G/e$ is not a 3-cograph and so is not a $k$-cograph for all $k > 2$.

The following lemma is easy to check.
Lemma 2.9. All graphs having at most four vertices are 2-cographs.

Akiyama and Harary [1, Corollary 1a] claimed that a 2-connected graph $G$ has a 2-connected complement if and only if the red and green degrees of every vertex of $G$ are at least two and $G$ has no spanning complete bipartite subgraph. However, this result is not true. The graphs in Figure 2 are complements of each other. The first graph in the figure satisfies the hypotheses of [1, Corollary 1a] but its complement is not 2-connected.

Figure 2. Counterexample to a result of Akiyama and Harary.

The Akiyama and Harary result can be easily repaired as follows.

Proposition 2.10. If $G$ is a 2-connected graph, then $\overline{G}$ is a 2-connected graph if and only if $G$ has no complete bipartite subgraph using at least $|V(G)| - 1$ vertices.

Proof. Note that if $\overline{G}$ is not 2-connected, then $G$ has a spanning complete bipartite subgraph or a complete bipartite subgraph on $|V(G)| - 1$ vertices. The converse is immediate. \hfill \Box

3. Induced-minor-minimal non-2-cographs

We begin this section by showing that the class of non-2-cographs for which every proper induced subgraph is a 2-cograph is infinite.

Lemma 3.1. Let $G$ be the complement of a cycle of length exceeding four. Then every proper induced subgraph of $G$ is a 2-cograph but $G$ is not.

Proof. It is easy to observe that $G$ is not a 2-cograph since both $G$ and its complementary cycle are 2-connected. Since every proper induced subgraph of $G$ is an induced subgraph of $G - v$ for some vertex $v$ of $G$, by Lemma 2.1, it is enough to show that $G - v$ is a 2-cograph for all vertices $v$ of $G$. Observe that $\overline{G} - v$ is a path and is therefore, a 2-cograph by Lemma 2.6. Since 2-cographs are closed under complementation, $G - v$ is a 2-cograph and the result follows. \hfill \Box

We noted in Section 2 that 2-cographs are closed under induced minors. In this section, we consider those non-2-cographs for which every proper induced minor is a 2-cograph. We call these graphs induced-minor-minimal non-2-cographs.

Next we show that the complements of cycles of length exceeding four are induced-minor-minimal non-2-cographs as well.
Lemma 3.2. Let $G$ be the complement of a cycle $C$ of length exceeding four. Then $G$ is an induced-minor-minimal non-2-cograph.

Proof. By Lemma 3.1, it suffices to show that $G/e$ is a 2-cograph for all edges $e$ of $G$. Let $e$ be an edge of $G$. By Lemma 2.3, the complement of $G/e$ is either a 0-sum of two paths and an isolated vertex, or a 0-sum of a path and $K_2$. This implies that the complement of $G/e$ is a 2-cograph and, by Lemma 2.1, the result follows. □

Note that cycles are not the only induced-minor-minimal non-2-cographs. For example, let $C_6^+$ be the graph obtained from a 6-cycle by adding a chord to create two 4-cycles. It can be checked that both $C_6^+$ and its complement are induced-minor-minimal non-2-cographs. These two graphs and $C_5$ are the members of $\mathcal{G}$, the class of induced-minor-minimal non-2-cographs $G$ such that $\overline{G}$ is also an induced-minor-minimal non-2-cograph. In the rest of the section, we show that all graphs in $\mathcal{G}$ have at most ten vertices. We give an exhaustive list of all these graphs in the appendix.

The following lemma is obtained by applying [9, Lemma 2.3] (see also [8, Lemma 4.3.10]) to the bond matroid of a 2-connected graph.

Lemma 3.3. Let $G$ be a 2-connected graph other than $K_3$ and let $v$ be an arbitrary vertex of $G$. Then $G$ has at least two edges $e$ incident to $v$ such that $G/e$ is 2-connected.

An edge $e$ of a 2-connected graph is contractible if $G/e$ is 2-connected. We omit the elementary proof of the next result.

Lemma 3.4. Let $G$ be an induced-minor-minimal non-2-cograph. Then both $G$ and $\overline{G}$ are 2-connected.

In the rest of the section, we use the following results of Chan about contractible edges in 2-connected graphs [2, Theorems 3.1 and 3.2].

Theorem 3.5. Let $G$ be a 2-connected graph non-isomorphic to $K_3$. Suppose all the contractible edges of $G$ meet a 3-element subset $S$ of $V(G)$. Then either $G - S$ has no edges, or $G - S$ has exactly one non-trivial component and this component has at most three vertices.

Theorem 3.6. Let $G$ be a 2-connected graph non-isomorphic to $K_3$. Suppose all the contractible edges of $G$ meet a 4-element subset $S$ of $V(G)$. Then $G - S$ has at most two non-trivial components and, between them, these components have at most four vertices.

We will also frequently use the following straightforward result.

Lemma 3.7. Let $G$ be a 2-connected graph. If $G$ has a 2-cut $\{g_1, g_2\}$ such that each of $g_1$ and $g_2$ has red degree at least two and the components of $G - \{g_1, g_2\}$ can be partitioned into two sets each of which contains at least two vertices, then the red graph $\overline{G}$ is 2-connected.
Lemma 3.8. Let $G$ be an induced-minor-minimal non-$2$-cograph such that $|V(G)| \geq 6$ and let $a$ and $d$ be a path $P$ of $G$ such that both $b$ and $c$ have degree two in $G$. Then $a$ and $d$ are adjacent.

Proof. Assume that $a$ and $d$ are not adjacent. By Lemma 3.4, $G$ is $2$-connected, so there is a path $P'$ joining $a$ and $d$ such that $P$ and $P'$ are internally disjoint. This implies that $G$ has $C_5$ as a proper induced minor. As $C_5 \in \mathcal{G}$, this is a contradiction. \qed

Lemma 3.9. Let $G$ be an induced-minor-minimal non-$2$-cograph. If $G$ has two adjacent vertices of degree two, then $|V(G)| \leq 10$.

Proof. Assume $|V(G)| \geq 11$. Let $a$ and $b$ be two vertices of $G$ of degree two such that $ab$ is a green edge. Let $c$ be the green neighbour of $a$ distinct from $b$, and let $d$ be the green neighbour of $b$ distinct from $a$. Then $c \neq d$, otherwise $G$ is not $2$-connected, contradicting Lemma 3.4. By Lemma 3.8, $cd$ is a green edge. Observe that every vertex of $V(G) - \{a, b, c, d\}$ has red edges joining it to each of $a$ and $b$. Thus $\overline{G} - \{c, d\}$ is $2$-connected.

Suppose that both $c$ and $d$ have red degree at least three. Let $w$ be a red neighbour of $d$. It follows by Lemma 3.3 that $w$ has a contractible green edge incident to it, say $e$, such that the other endpoint of $e$ is not $c$. Then $\overline{G}/e$ is $2$-connected, a contradiction.

Next suppose that both $c$ and $d$ have red degree two. First, we assume that $c$ and $d$ have the same red neighbour, say $v$, in $G - \{a, b\}$. Since $v$ has green degree at least two, we have two green neighbours of $v$, say $x$ and $y$. Note that $x$ and $y$ are in $V(G) - \{a, b, c, d\}$. Since $x$ and $y$ are adjacent to both $c$ and $d$ in the green graph, both the red and the green graph induced on $\{a, b, c, d, v, x, y\}$ are $2$-connected. This implies $|V(G)| \leq 7$, a contradiction. We may now assume that $c$ and $d$ have distinct red neighbours in $G - \{a, b\}$. Let $v$ and $w$ be the red neighbours of $c$ and $d$ respectively in $G - \{a, b\}$. Note that $vdcw$ is a green $vw$-path.

3.9.1. $G - \{a, b\}$ has no $vw$-path $P$ internally disjoint from the path $vdcw$.

Assume that $G - \{a, b\}$ has such a path. Observe that the red graph and the green graph induced on the vertex set $V(P) \cup \{a, b, c, d\}$ are $2$-connected and therefore, $V(G) = V(P) \cup \{a, b, c, d\}$. Now $|V(P)| \geq 7$ since $|V(G)| \geq 11$. Let $e$ be an edge in the path $P$ such that neither of the endpoints of $e$ is in $\{v, w\}$. Note that $G/e$ and $\overline{G}/e$ are both $2$-connected, a contradiction. Thus 3.9.1 holds.

Let $P_1$ and $P_2$ be shortest $vw$-paths in $G - \{a, b, d\}$ and $G - \{a, b, c\}$ respectively. By 3.9.1, we may assume that $P_1$ contains the vertex $c$ and $P_2$ contains $d$. Note that $V(G) = V(P_1) \cup V(P_2) \cup \{a, b\}$. As $|V(G)| \geq 11$, we may assume that $P_1 - w$ has length at least three. Let $e$ be an edge in $P_1 - w$ such that the endpoints of $e$ are not in $\{c, v\}$. Note that $G/e$ and $\overline{G}/e$ are both $2$-connected, a contradiction.

Finally, without loss of generality, we may assume that $c$ has red degree two and $d$ has red degree at least three. Let $v$ be the red neighbour of $c$
distinct from $b$. Suppose that $dv$ is red. Let $x$ and $y$ be two green neighbours of $v$ and let $P$ be a shortest path from $d$ to $\{v, x, y\}$ in $G - \{a, b, c\}$. Then, for $V' = \{a, b, c, d, v, x, y\} \cup V(P)$, the red and green graphs induced by $V'$ are 2-connected, so $V' = V(G)$. As $|V(G)| \geq 11$, we may assume that $P$ has length at least three. Let $e$ be an edge in $P$ such that the endpoints of $e$ are not in $\{d, v, x, y\}$. Note that $G/e$ and $\overline{G}/e$ are both 2-connected, a contradiction. Therefore, $dv$ is green. Let $w$ be a red neighbour of $d$ in $G - \{a, b\}$. Let $u$ be a green neighbour of $v$ distinct from $d$. Observe that $u \neq w$, otherwise $|V(G)| \leq 6$ since both $G[\{a, b, c, d, v, w\}]$ and $\overline{G}[\{a, b, c, d, v, w\}]$ are 2-connected. Let $P$ be a shortest path from $w$ to $\{d, u, v\}$ in $G - \{a, b, c\}$. Then $V(G) = \{a, b, c, d, u, v, w\} \cup V(P)$, so we may assume that $P$ has length at least three. Then for an edge $e$ of $P$ having neither endpoint in $\{d, u, v, w\}$, both $G/e$ and $\overline{G}/e$ are 2-connected, a contradiction.

In the following lemma, we note that if a path of a graph $G$ in $\mathcal{G}$ has three consecutive vertices of degree two, then $G \cong C_5$.

**Lemma 3.10.** For $G$ in $\mathcal{G}$, if $G$ has a path $P$ of length exceeding three such that all the internal vertices of $P$ are of degree two, then $G \cong C_5$.

**Proof.** Let $u$ and $v$ be vertices of $P$ such that the subpath $P_{uv}$ of $P$ joining $u$ and $v$ has length four. Since $G$ is 2-connected, there is a $uv$-path $P'$ such that $P_{uv}$ and $P'$ are internally disjoint. Assume that $P'$ is a shortest such path. Then contracting all but one edge in $P'$ and deleting all the vertices not in $V(P_{uv})$, we obtain $C_5$. This is a contradiction since $G$ cannot have $C_5$ as a proper induced minor. Thus $P'$ has length one and $V(P_{uv}) = V(G)$, so $G \cong C_5$. □

A 2-connected graph $H$ is **critically 2-connected** if $H - v$ is not 2-connected for all vertices $v$ of $H$.

**Lemma 3.11.** If $G \in \mathcal{G}$, then either $G$ or $\overline{G}$ is critically 2-connected, or both $G$ and $\overline{G}$ have vertex connectivity two.

**Proof.** Certainly, $G$ and $\overline{G}$ are 2-connected and, for all vertices $v$ of $G$, either $G - v$ or $\overline{G} - v$ is not 2-connected. Observe that if neither $G$ nor $\overline{G}$ is critically 2-connected, then $G$ has vertices $v$ and $v_c$ such that $G - v$ and $\overline{G} - v_c$ are 2-connected. It follows that $G - v_c$ and $\overline{G} - v$ are not 2-connected so both $G$ and $\overline{G}$ have vertex connectivity two. □

Next we show that $\mathcal{G}$ contains only two critically 2-connected graphs. We use the following result of Nebesky [7] for the proof.

**Lemma 3.12.** Let $G$ be a critically 2-connected graph such that $|V(G)| \geq 6$. Then $G$ has at least two distinct paths of length exceeding two such that the internal vertices of the paths have degree two in $G$.

**Proposition 3.13.** Let $G$ be a critically 2-connected graph in $\mathcal{G}$. Then $G$ is isomorphic to $C_5$ or $C_6^+$. 
Proof. Since $C_5$ is the unique member of $\mathcal{G}$ with at most five vertices, we may assume that $|V(G)| \geq 6$. Thus, by Lemma 3.12, $G$ has two paths $P_1$ and $P_2$ such that the internal vertices of $P_1$ and $P_2$ are of degree two. Since $G$ is not isomorphic to $C_5$, by Lemma 3.10, we may assume that both $P_1$ and $P_2$ have length three. Lemma 3.8 implies that the endpoints of these paths are adjacent and therefore, $G$ has $C_6^+$ as an induced minor. As $C_6^+ \in \mathcal{G}$, we deduce that $G \cong C_6^+$. □

Observe that the critically 2-connected graphs found in the last result and their complements have vertex connectivity two. Therefore, we have the following strengthening of Lemma 3.11.

**Corollary 3.14.** If $G \in \mathcal{G}$, then both $G$ and $\overline{G}$ have vertex connectivity two.

The following lemmas show that the number of vertices of the graphs in $\mathcal{G}$ are bounded given some conditions on the sizes of components after the removal of a green 2-cut and on the red degrees of the vertices in that cut.

**Lemma 3.15.** Let $\{g_1, g_2\}$ be a 2-cut of a graph $G$ in $\mathcal{G}$ such that each of $g_1$ and $g_2$ has red degree exceeding two and the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, $A$ and $B$, each having at least two vertices. Then $|V(G)| \leq 8$.

**Proof.** Assume that $|V(G)| > 8$. Without loss of generality, let $|V(A)| \geq 4$. Suppose $A$ has no red neighbour of $g_1$ or $g_2$. Then all vertices in $A$ are incident to both $g_1$ and $g_2$ via a green edge. Let $v$ be any vertex in $A$. Note that both $G - v$ and $\overline{G} - v$ are 2-connected, a contradiction. Therefore, we may assume that $A$ has a red neighbour, say $a_1$, of $g_1$. Lemma 3.3 implies that we can find a contractible green edge, say $e$, of $G$ incident to $a_1$ such that the other endpoint of $e$ is in $A$. By Lemma 3.7, $G / e$ is 2-connected, a contradiction. □

**Lemma 3.16.** Let $\{g_1, g_2\}$ be a 2-cut of a graph $G$ in $\mathcal{G}$ such that the red degree of $g_1$ is two and that of $g_2$ is greater than two. Suppose that the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs $A$ and $B$ such that $|V(A)| \geq |V(B)| \geq 2$ and $A$ contains exactly one red neighbour $v$ of $g_1$. If all contractible edges of $G$ having both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to a vertex in $\{g_1, g_2, v\}$, then $|V(A)| \leq 4$.

**Proof.** Assume that $|V(A)| > 4$. Let $G_A$ be the graph induced on $V(A) \cup \{g_1, g_2\}$ and let $Q$ denote the vertex set $\{g_1, g_2, v\}$. By colouring the edge $g_1g_2$ green if necessary, we may assume that $G_A$ is 2-connected. Since the contractible edges of $G_A$ must meet $Q$, by Theorem 3.5, either $G_A - Q$ has no edges, or $G_A - Q$ has one non-trivial component and this component has at most three vertices. First suppose that $G_A - Q$ is edgeless. Let $\Gamma = V(G_A) - Q$. Next we show the following.

**3.16.1.** There is no vertex $\gamma$ in $\Gamma$ such that $G_A - \gamma$ is 2-connected.
Lemma 3.7, both $V$ and $\overline{\gamma}$ is 2-connected, a contradiction.

Suppose $v$ and $g_2$ are adjacent in $G$. Let $\alpha$ be a neighbour of $v$ in $\Gamma$. Then $g_1\alpha v g_2 g_1$ is a cycle of $G_A$. Because $G_A - Q$ is edgeless and $G_A$ is 2-connected, every vertex in $\Gamma - \alpha$ is adjacent to at least two members of $\{g_1, g_2, v\}$. Thus $G_A - \gamma$ is 2-connected for all $\gamma$ in $\Gamma - \alpha$, a contradiction to Lemma 3.15. Therefore $v$ and $g_2$ are not adjacent in $G_A$.

Observe that $v$ and $g_2$ have a common neighbour $\beta$ in $\Gamma$, otherwise $G_A$ would be a cut vertex. Let $\alpha$ be a neighbour of $v$ in $\Gamma - \beta$. Since $g_1\alpha v \beta g_2 g_1$ is a cycle and all vertices in $\Gamma - \{\alpha, \beta\}$ are adjacent to at least two vertices in $\{g_1, g_2, v\}$, we deduce that $G_A - \gamma$ is 2-connected for all $\gamma$ in $\Gamma - \{\alpha, \beta\}$, a contradiction.

We may now assume that $G_A - Q$ has one non-trivial component, say $C_A$, and a set $I_A$ of isolated vertices. Moreover, $|V(C_A)| \leq 3$. Then $I_A$ is non-empty. Let $\alpha \beta$ be an edge in $C_A$. Note that $\alpha \beta$ is not contractible in $G_A$, so $\{\alpha, \beta\}$ is a 2-cut of $G_A$ and therefore, of $G$. Since $|V(B)| \geq 2$ and $I_A$ is non-empty, each of $\alpha$ and $\beta$ has red degree at least three in $G$. Therefore, by Lemma 3.15, as $|V(G)| = |V(A)| + 2 + |V(B)| > 8$, there is a vertex $t$ of $G$ whose only green neighbours are $\alpha$ and $\beta$. Since $g_1$ is adjacent to all vertices in $I_A \cup V(C_A)$, it follows that $t = v$. This implies that all vertices in $I_A$ are adjacent only to $g_1$ and $g_2$. Taking $w$ in $I_A$, we see, by Lemma 3.7, that $G - w$ and $\overline{G} - w$ are both 2-connected, a contradiction. \qed

**Lemma 3.17.** Let $\{g_1, g_2\}$ be a 2-cut of a graph $G$ in $\mathcal{G}$ such that the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs, $A$ and $B$, each having at least two vertices. If the red degree of $g_1$ is two and that of $g_2$ is greater than two such that one red neighbour of $g_1$ is in $A$ and the other is in $B$, then $|V(G)| \leq 10$.

**Proof.** Without loss of generality, assume $|V(A)| \geq |V(B)|$. Let $G_A$ be the graph induced on $V(A) \cup \{g_1, g_2\}$. Note that $G_A$ is 2-connected since $g_1 g_2$ is green. Denote the red neighbour of $g_1$ in $A$ by $v$ and let $Q = \{g_1, g_2, v\}$.

Observe that if we have a contractible edge $e$ of $G$ having both endpoints in $V(A) \cup \{g_1, g_2\}$ such that neither of the endpoints of $e$ is in $Q$, then, by Lemma 3.7, both $G/e$ and $\overline{G}/e$ are 2-connected, a contradiction. Therefore, we may assume that all contractible edges of $G$ that have both endpoints in $V(A) \cup \{g_1, g_2\}$ meet $Q$. Thus, by Lemma 3.16, $|V(A)| \leq 4$, so $|V(G)| \leq 10$. \qed

**Lemma 3.18.** Let $\{g_1, g_2\}$ be a 2-cut of a graph $G$ in $\mathcal{G}$ such that $g_1$ and $g_2$ are not adjacent in $G$ and the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, $A$ and $B$, each having at least two vertices. Then $|V(G)| \leq 10$.

**Proof.** Since the edge $g_1 g_2$ is red, by Lemma 3.9, we may assume that the red degree of $g_2$ exceeds two. By Lemma 3.15, we may further assume that the red degree of $g_1$ is two.
Let $v$ be a red neighbour of $g_1$ and, without loss of generality, assume that $v$ is in $A$. First, we show that $|V(B)| = 2$. Assume to the contrary that $|V(B)| \geq 3$. If $B$ contains no red neighbour of $g_2$, then all vertices in $B$ are incident to both $g_1$ and $g_2$ via green edges. This implies that both $G - u$ and $\overline{G} - u$ are 2-connected for each $u$ in $V(B)$, a contradiction. Therefore $B$ has a red neighbour, say $b$, of $g_2$. Now, by Lemma 3.3, we can find a contractible edge of $G$ incident to $b$, say $e$, such that both endpoints of $e$ are in $B$. Note that, by Lemma 3.7, $G/e$ is 2-connected, a contradiction. Thus $|V(B)| = 2$. Observe that, if we can find a contractible edge $e$ of $G$ having both endpoints in $V(A) - v$, then, by Lemma 3.7, $G/e$ is 2-connected, a contradiction. This implies that all the contractible edges of $G$ that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, v\}$. The result now follows by Lemma 3.16. 

Lemma 3.15 can be modified as follows.

**Proposition 3.19.** Let $\{g_1, g_2\}$ be a 2-cut of a graph $G$ in $G$ such that the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, $A$ and $B$, each having at least two vertices. If $g_1$ has red degree greater than two, then $|V(G)| \leq 10$.

**Proof.** Assume that $|V(G)| \geq 11$. Then, by Lemma 3.15, the red degree of $g_2$ is two. Let $x$ and $y$ be the two red neighbours of $g_2$. Note that if $x$ is in $A$ and $y$ is in $B$, then the result follows by Lemma 3.17. By Lemma 3.18, we may suppose that both $x$ and $y$ are in $A$. Observe that the edge $g_1g_2$ is green and the graph induced on $V(B) \cup \{g_1, g_2\}$ is 2-connected. We next show that

3.19.1. $|V(B)| = 2$.

Suppose $|V(B)| \geq 3$. If all vertices in $B$ are green neighbours of both $g_1$ and $g_2$, then $G - z$ is 2-connected for all $z$ in $V(B)$. But, by Lemma 3.7, $\overline{G} - z$ is also 2-connected, a contradiction. Thus $B$ has a red neighbour, say $b$, of $g_1$. Now, by Lemma 3.3, we can find a contractible edge, say $e$, of $G$ incident to $b$ such that both endpoints of $e$ are in $V(B)$. By Lemma 3.7, $G/e$ is 2-connected, a contradiction. Thus 3.19.1 holds.

The graph $G_A$ induced on $V(A) \cup \{g_1, g_2\}$ is 2-connected. Let $Q = \{g_1, g_2, x, y\}$. Then every contractible edge $e$ of $G_A$ must meet $Q$ otherwise, by Lemma 3.7, we obtain the contradiction that both $G/e$ and $G/e$ are 2-connected. By Theorem 3.6, $G_A - Q$ has at most two non-trivial components and, between them, these components have at most four vertices.

Let $I_A$ and $N_A$ be the sets of isolated and non-isolated vertices of $G_A - Q$ respectively. We note the following.

3.19.2. If two vertices $i_1$ and $i_2$ in $I_A$ have the same green neighbourhood, then $\{i_1, i_2\}$ is a green 2-cut.
It is clear that $G - i_1$ is 2-connected. Therefore, we may assume that $G - i_1$ is not 2-connected. It follows that $i_2$ is a cut-vertex of $G - i_1$ and so \{i_1, i_2\} is a green 2-cut.

First, suppose that $N_A$ is empty. As $|V(G)| \geq 11$, we see that $|I_A| \geq 5$. Assume that there is no red edge connecting $\{x, y\}$ to $I_A$. It follows that there are two vertices, say $i_1$ and $i_2$, in $I_A$ that have the same green neighbourhood. By 3.19.2, \{i_1, i_2\} is a green 2-cut. Since there is no vertex of $G$ that has green neighbourhood \{i_1, i_2\}, by Lemma 3.18, $|V(G)| \leq 10$. Therefore there is a red edge connecting $\{x, y\}$ to $I_A$. It follows that, for some $b$ in $V(B)$, both $G - b$ and $\overline{G} - b$ are 2-connected, a contradiction.

We may now assume that $G_A - Q$ has at least one non-trivial component. Let $C$ be such a component and let $\alpha \beta$ be an edge in $C$. Since $\alpha \beta$ is a non-contractible edge of $G_A$, we see that $\{\alpha, \beta\}$ is a green 2-cut of $G_A$ and thus of $G$. Then $G_A - Q \neq C$ otherwise, by Theorem 3.6, $|V(G)| \leq 10$, a contradiction. Thus both $\alpha$ and $\beta$ have red degree at least three in $G$.

Therefore, by Lemma 3.15, $G$ has a vertex $t$ that has green neighbourhood $\{\alpha, \beta\}$. Since all vertices in $G_A$ except $x$ and $y$ are adjacent to $g_2$ via a green edge, $t$ is either $x$ or $y$. As $\alpha \beta$ is an arbitrary green edge in $G_A - Q$, it follows that $G_A - Q$ has at most two edges and therefore has either one non-trivial component with at most three vertices or has two non-trivial components each with two vertices. Observe that if $G_A - Q$ has only one edge, then we may assume that $|I_A| \geq 3$ and green neighbourhood of every vertex in $I_A$ is contained in \{g_1, g_2, y\}. It is clear that if a vertex $w$ has green neighbourhood \{g_1, g_2\}, then $G - w$ and $\overline{G} - w$ are 2-connected. It follows that the green neighbourhood of a vertex in $I_A$ is either \{g_2, y\} or \{g_1, g_2, y\}. Let $i_1$ and $i_2$ be two vertices in $I_A$ that have the same green neighbourhood. By 3.19.2, \{i_1, i_2\} is a green 2-cut and so by Lemma 3.18, $|V(G)| \leq 10$. Therefore, we may assume that $3 \leq |N_A| \leq 4$. Observe that $I_A \neq \emptyset$ and all vertices in $I_A$ have green neighbourhood equal to \{g_1, g_2\} since $x$ and $y$ have their green neighbourhoods contained in $N_A$. Thus, for $w \in I_A$, both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction.

For a graph $G$ in $\mathcal{G}$, let $V_g$ and $V_r$ be, respectively, its sets of vertices of green-degree two and red-degree two. The following is an immediate consequence of Lemma 3.9.

**Corollary 3.20.** Let $G$ be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then the graph induced on the vertex set $V_g$ is a complete red graph and the graph induced on $V_r$ is a complete green graph.

The next five lemmas show that if the number of vertices of a graph $G$ in $\mathcal{G}$ exceeds ten, then both $V(G) - (V_g \cup V_r)$ and $V_g \cup V_r$ are non-empty, and either $|V_g|$ or $|V_r|$ is at most four.

**Lemma 3.21.** Let $G$ be a graph in $\mathcal{G}$. If $|V(G)| > 10$, then $V(G) \neq V_g \cup V_r$. 
Proof. Assume that \( V(G) = V_g \cup V_r \). By replacing \( G \) by \( \overline{G} \) if necessary, we may also assume that \( |V_g| \leq |V_r| \). There are \( 2|V_g| \) green edges and \( 2|V_r| \) edges joining a vertex in \( V_g \) to vertex in \( V_r \). Thus

\[ 3.21.1. \quad 2|V_g| + 2|V_r| = |V_g||V_r|. \]

If \( |V_g| = |V_r| \), then \( 4|V_r| = |V_r|^2 \), so \( |V_r| = 4 \), a contradiction. Therefore \( |V_g| \leq |V_r| - 1 \) so, by 3.21.1, \( |V_g||V_r| \leq 4|V_r| - 2 \). Thus \( |V_g| \leq 3 \). If \( |V_g| = 3 \), then, by 3.21.1, \( |V_r| = 6 \), so \( |V(G)| = 9 \), a contradiction. If \( |V_g| \leq 2 \), then we contradict 3.21.1.

Next we note two useful observations about the vertices in \( V(G) - (V_g \cup V_r) \) in the following lemmas.

Lemma 3.22. Let \( G \) be a graph in \( \mathcal{G} \) such that \( |V(G)| > 10 \). Then every vertex \( x \) in \( V(G) - (V_g \cup V_r) \) either has a green neighbour in \( V_g \) or a red neighbour in \( V_r \).

Proof. Since every vertex of \( G \) is either in a red 2-cut or a green 2-cut, the lemma follows by Proposition 3.19.

Lemma 3.23. Let \( G \) be a graph in \( \mathcal{G} \) such that \( |V(G)| > 10 \). Then the number of vertices in \( V(G) - (V_g \cup V_r) \) is at most \( 2|V_g \cup V_r| - |V_g||V_r| \). Thus \( 11 + |V_g||V_r| \leq 3|V_g| + 3|V_r| \).

Proof. There are \( |V_g||V_r| \) red or green edges joining a vertex in \( V_g \) to a vertex in \( V_r \). There are at most \( 2|V_g| \) green such edges and at most \( 2|V_r| \) red such edges. Thus at most \( 2|V_g \cup V_r| - |V_g||V_r| \) of the green edges meeting \( V_g \) and the red edges meeting \( V_r \) have an endpoint in \( V(G) - (V_g \cup V_r) \). Therefore, by Lemma 3.22, \( |V(G)| - (V_g \cup V_r) | \leq 2|V_g \cup V_r| - |V_g||V_r| \). Since \( |V(G)| \geq 11 \), we deduce that \( 11 + |V_g||V_r| \leq 3|V_g| + 3|V_r| \).

Lemma 3.24. Let \( G \) be a graph in \( \mathcal{G} \) such that \( |V(G)| > 10 \). Then neither \( V_g \) nor \( V_r \) is empty.

Proof. Assume that \( V_r \) is empty. By Lemma 3.22, every vertex outside \( V_g \) has a green neighbour in \( V_g \). Thus, by Lemma 3.23, \( |V(G)| \leq 3|V_g| \). Since \( |V(G)| \geq 11 \), we have \( |V_g| \geq 4 \). Let \( \{r_1, r_2\} \) be a red 2-cut \( T \). Since \( V_r \) is empty, applying Proposition 3.19 to \( \overline{G} \) gives that \( T \) is contained in \( V_g \). Let \( v \) be a vertex in \( V_g - T \) and let \( \alpha \) and \( \beta \) be the two green neighbours of \( v \). Consider the graph \( \overline{G} - T \). Note that \( \overline{G} - T \) is disconnected and \( v \) is incident to all the vertices in this graph except \( \alpha \) and \( \beta \). Let \( X \) be the component of \( \overline{G} - T \) containing \( v \). Since the red graph \( \overline{G} \) has no degree-two vertices, \( \overline{G} - T \) has exactly two components. The second component must have \( \{\alpha, \beta\} \) as its vertex set.

Let \( w \) be a vertex in \( V_g - T - v \). As \( w \) is in a different component of \( \overline{G} - T \) from \( \alpha \) and \( \beta \), both \( w\alpha \) and \( w\beta \) are green edges. Since \( w \) has green degree two, it follows that \( \{\alpha, \beta\} \) is the green neighbourhood of each vertex in \( V_g - T \). By Lemma 3.22, each vertex in \( V(G) - V_g - \{\alpha, \beta\} \) has a green
neighbour in \( V_g \). This neighbour is not in \( V_g - T \), so it is in \( T \). Thus |\( V(G) - V_g - \{\alpha, \beta\} \) ≤ 4. Hence |\( V_g - T \) ≥ 3. Therefore \( G - v \) and \( \overline{G} - v \) are both 2-connected, a contradiction. We conclude that \( V_r \) is non-empty. To see that \( V_g \) is non-empty, we modify the above argument taking \( T \) to be a green 2-cut, which must be contained in \( V_r \). The rest of the argument is completed by interchanging “green” and “red” above. \( \square \)

**Lemma 3.25.** Let \( G \) be a graph in \( \mathcal{G} \) such that |\( V(G) \) > 10. Then either |\( V_g \) or |\( V_r \) is at most four.

**Proof.** Assume that both |\( V_g \) and |\( V_r \) exceed four. Let \( \{a, b, c, d, e\} \subseteq V_g \) and \( \{\alpha, \beta, \gamma, \delta, \epsilon\} \subseteq V_r \). By Lemma 3.21, we have a vertex \( v \in V(G) - (V_g \cup V_r) \) and, by Lemma 3.22, without loss of generality \( v \) has a green neighbour, say \( a \), in \( V_g \). There are at most three green neighbours of \( \{a, b\} \) in \( V_r \). Thus there are at least two vertices, say \( \delta \) and \( \epsilon \), in \( V_r \) that have both \( a \) and \( b \) as their red neighbours. This implies that all the edges in \( \{c\delta, c\epsilon, d\delta, d\epsilon, e\delta, e\epsilon\} \) are green. Therefore the edges \( \alpha c, a d \) and \( a e \) are red, a contradiction to the fact that the red degree of \( \alpha \) is two. \( \square \)

Next we prove the main result of the paper, Theorem 1.2, which we restate for convenience.

**Theorem 3.26.** Let \( G \) be a graph in \( \mathcal{G} \). Then |\( V(G) \) ≤ 10.

**Proof.** Assume that |\( V(G) \) > 10. Without loss of generality, let |\( V_g \) ≤ |\( V_r \)\|. By Lemmas 3.24 and 3.25, 1 ≤ |\( V_g \) ≤ 4. Suppose |\( V_g \) = 4. Since every vertex in \( V_r \) has red degree two, every vertex in \( V_r \) is a green neighbour of at least two vertices in \( V_g \). This implies that |\( V_r \) ≤ 4, so |\( V_r \) = 4. Lemma 3.23 implies that \( V(G) - (V_g \cup V_r) \) is empty. Therefore |\( V(G) \) = 8, a contradiction.

Next we assume that |\( V_g \) = 3. Then every vertex in \( V_r \) is a green neighbour of at least one vertex in \( V_g \). Thus |\( V_r \) ≤ 6 as there are exactly six green edges incident to vertices in \( V_g \). Then, by Lemma 3.23, as |\( V_g \) = 3, we deduce that 11 ≤ 3|\( V_g \), a contradiction.

Now suppose that |\( V_g \) = 2. Then, by Lemma 3.23, |\( V_r \) ≥ 5. Let \( V_g = \{u, v\} \). Since there are only four green edges meeting \( V_1 \), there is a vertex \( w \) in \( V_r \) whose red neighbours are \( u \) and \( v \). Thus \( \{u, v\} \) is a red 2-cut. Suppose that \( w, x \) and \( y \) are vertices in \( V_r \) that are joined to both \( u \) and \( v \) by red edges. Then both \( G - w \) and \( \overline{G} - w \) are 2-connected, a contradiction. Thus \( V_r \) has at most two vertices that are joined to both \( u \) and \( v \) by red edges. Therefore |\( V_r \) ≤ 6 since \( V_g \) meets only four green edges. Assume that |\( V_r \) = 6. Then all the green neighbours of \( u \) and \( v \) are in \( V_r \) and are distinct. Since |\( V(G) \) ≥ 11, we see that |\( V(G) - (V_g \cup V_r) \) ≥ 3. Let \( \{w, x\} \) be the vertices in \( V_r \) having both \( u \) and \( v \) as their red neighbours. All the vertices in \( V_r - \{w, x\} \) have one red neighbour in \( V_g \). Since |\( V(G) - (V_g \cup V_r) \) ≥ 3, Lemma 3.22 implies that each vertex in \( V(G) - (V_g \cup V_r) \) has at most two red neighbours in \( V_r - \{w, x\} \) and thus has at least two green neighbours in
we obtain a contradiction by applying Proposition 3. Assume that \( G \) has two components, say \( X \) and \( Y \). Let \( v \) be a vertex in \( X \) such that \( v \) is not a red neighbour of \( \alpha \) or \( \beta \) and \( X - \{x\} \) contains at least two vertices of \( V_r - \{\alpha, \beta\} \). By Lemma 3.7, \( G - x \) is 2-connected. Moreover, each vertex of \( V_r - \{\alpha, \beta\} \) has its two red neighbours in \( Y \) and so is adjacent in \( G \) to every vertex of \( X \). Thus \( G - x \) is 2-connected, a contradiction. We conclude that \( V_r \) does not have a green 2-cut containing \( \alpha \).

Next, we show that no green 2-cut contains \( \alpha \). Assume that \( \{\alpha, z\} \) is a green 2-cut. We just showed that \( z \notin V_r \). By Proposition 3.19, \( G - \{\alpha, z\} \) has two components, \( X \) and \( Y \), where \( |Y| = 1 \). Since the vertex in \( Y \) has green degree two, \( Y = \{v\} \). Thus \( \alpha v \) is green, a contradiction. We conclude that deleting any red neighbour of \( v \) from \( V_r \) leaves a green graph that is still 2-connected.

To complete the proof of the theorem, we show that \( v \) has a red neighbour in \( V_r \) whose deletion from \( G \) leaves a 2-connected graph, thus arriving at a contradiction. Let \( \alpha \) and \( \beta \) be two red neighbours of \( v \) in \( V_r \). If \( \alpha \) and \( \beta \) have the same red neighbourhood, say \( \{x, v\} \), then \( \{x, v\} \) is a red 2-cut and we obtain a contradiction by applying Proposition 3.19 to \( G \). Thus \( \alpha \) and \( \beta \) have distinct red neighbourhoods, \( \{x, v\} \) and \( \{y, v\} \), respectively. Note that if \( xv \) is red, then \( G - \alpha \) is 2-connected. Thus we may assume that both \( xv \) and \( yv \) are green. This implies \( \gamma v \) is red for each \( \gamma \in V_r - \{\alpha, \beta\} \) since \( v \) has green degree two. Then the other red neighbour \( z \) of \( \gamma \) is distinct from \( x \) and \( y \). Since \( vz \) is red, we see that \( G - \gamma \) is 2-connected, a contradiction.

4. Appendix

We implement the algorithm described in this section using SageMath [11] and provide a list of all graphs in \( \mathcal{G} \) up to complementation. The graphs in this section are drawn using SageMath.

Graphs on six vertices. There are two graphs on six vertices in \( \mathcal{G} \), the graph in Figure 3 and its complement.
Graphs on seven vertices. There are sixteen graphs on seven vertices in $\mathcal{G}$, the graphs in Figure 4 and their complements.

Graphs on eight vertices. There are 87 graphs on eight vertices in $\mathcal{G}$, of which five are self-complementary. Figure 5 shows these self-complementary graphs. Figure 6 shows 41 non-self-complementary graphs that, with their complements, are the remaining 8-vertex graphs in $\mathcal{G}$.

Graphs on nine vertices. There are 86 graphs on nine vertices in $\mathcal{G}$. These are the 43 graphs in Figure 7 and their complements.

Graphs on ten vertices. There are two graphs on ten vertices in $\mathcal{G}$, the graph in Figure 8 and its complement.
Figure 6. Graphs on eight vertices in $\mathcal{G}$. 
Figure 7. Graphs on nine vertices in $\mathcal{G}$.
Algorithm Finding graphs in $G$ of order at most ten

Require: $n = 6, 7, 8, 9$ or 10.

Set FinalList ← $\emptyset$, $i ← 0$, $j ← 0$

Generate all two connected graphs of order $n$ using nauty geng and store in an iterator $L$

for $g$ in $L$ such that vertex connectivity of $g$ and $\overline{g}$ is 2 do
  Set $i ← 0$, $j ← 0$
  Create a list $T$ of induced subgraphs of $g$
  for $h$ in $T$ and non-empty subset $X$ of edges of $h$ do
    $i ← i + 1$
    if vertex connectivity of $h/X$ or $\overline{h}/X$ is less than two then
      $j ← j + 1$
    if $i$ equals $j$ then
      Add the graph $g$ to FinalList

for $g$ in FinalList do
  if FinalList does not contain $\overline{g}$ then
    remove $g$ from FinalList

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