CONSTRUCTING A 3-TREE FOR A 3-CONNECTED MATROID

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For our friend Geoff Whittle with thanks for many years of enjoyable collaboration

Abstract. In an earlier paper with Whittle, we showed that there is a tree that displays, up to a natural equivalence, all non-trivial 3-separations of a 3-connected matroid M. The purpose of this paper is to give a polynomial-time algorithm for constructing such a tree for M.

1. Introduction

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined for all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. For a positive integer k, a subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X|, |E - X| \geq k$. A k-separating set X, or a k-separating partition $(X, E - X)$, or a k-separation $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$.

We shall denote the set $\{1, 2, \ldots, n\}$ by $[n]$. Let X be an exactly 3-separating set of a matroid M. If there is an ordering $(x_1, x_2, \ldots, x_n)$ of X such that, for all i in $[n]$, the set $\{x_1, x_2, \ldots, x_i\}$ is 3-separating, then X is *sequential* and the ordering $(x_1, x_2, \ldots, x_n)$ is called a *sequential ordering* of X. An exactly 3-separating partition (X, Y) of M is *sequential* if either X or Y is a sequential 3-separating set. For a set X of M, we say that X is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The *full closure* of X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$.

The main theorem of [6], Theorem 9.1, shows that every 3-connected matroid M with at least nine elements has a tree decomposition that displays, up to equivalence, all non-sequential 3-separations. While the proof of that theorem does yield an algorithm for finding such a tree decomposition, that algorithm does not appear to be polynomial in $|E(M)|$. In this paper, we will describe such a polynomial algorithm. The proof that this algorithm works gives an alternative proof of [6, Theorem 9.1]. This paper will make repeated reference to the results of [6].

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2. Main Result

In this section, we state the main theorem of the paper together with the main result of [6]. The section begins by introducing the concepts and terminology needed to make these statements meaningful. Our terminology will follow Oxley [5]. We write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$.

Let $(P_1, P_2, \dots, P_n)$ be a *flower* Φ in a 3-connected matroid M, that is, $(P_1, P_2, \dots, P_n)$ is an ordered partition of $E(M)$ such that $\lambda_M(P_i) = 2 = \lambda_M(P_i \cup P_{i+1})$ for all i in $[n]$, where all subscripts are interpreted modulo n. The sets $P_1, P_2, \dots, P_n$ are the *petals* of Φ. Each must have at least two elements. It is shown in [6, Theorem 4.1] that every flower in a 3-connected matroid is either an *anemone* or a *daisy*. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \dots, P_n)$. A 3-separation (X, Y) is *displayed* by a flower if X is a union of petals of the flower.

Let Φ_1 and Φ_2 be flowers in a matroid M. A natural quasi ordering on the set of flowers of M is obtained by setting $\Phi_1 \preceq \Phi_2$ if every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. If $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$, then Φ_1 and Φ_2 are *equivalent* flowers. Such flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of M. Let Φ be a flower of M. The *order* of Φ is the minimum number of petals in a flower equivalent to Φ. An element e of M is *loose* in Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ; otherwise e is *tight*. A petal P_i is *loose* if all its elements are loose; and P_i is *tight* otherwise. A flower of order at least 3 is *tight* if all of its petals are tight. A flower of order 2 or 1 is *tight* if it has two petals or one petal, respectively. A flower Φ is *maximal* if Φ is equivalent to Φ' for every flower Φ' such that $\Phi \preceq \Phi'$.

The classes of anemones and daisies can be further refined using a useful companion function to the connectivity function. The *local connectivity*, $\sqcap(X, Y)$, is defined for all sets X and Y in a matroid M by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Let $(P_1, P_2, \dots, P_n)$ be a flower Φ with $n \geq 3$. If Φ is an anemone, then $\sqcap(P_i, P_j)$ takes a fixed value k in $\{0, 1, 2\}$ for all distinct i, j in $[n]$. We call Φ a *paddle* if $k = 2$, a *copaddle* if $k = 0$, and a *spike-like* flower if $k = 1$ and $n \geq 4$. Similarly, if Φ is a daisy, then $\sqcap(P_i, P_j) = 1$ for all consecutive i and j. We say Φ is *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and Φ is *Vámos-like* if $n = 4$ and $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, and *swirl-like* otherwise. Every flower with at least three petals is of one of these six *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous [6].

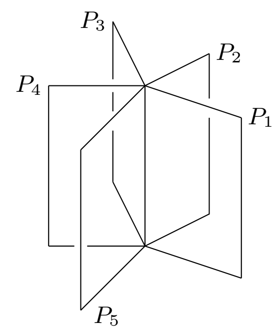

FIGURE 1. A representation of a rank-7 paddle.

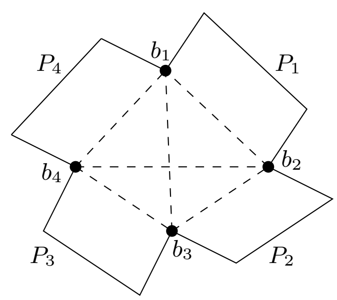

FIGURE 2. A representation of a rank-8 swirl-like flower.

To visualize a flower geometrically, it is helpful to think of a collection of lines in projective space along which the petals of the flower are attached. For example, we can obtain a paddle by gluing the petals along a single common line. Fig. 1 represents a 5-petal paddle in which each petal is a plane with enough structure to make the matroid 3-connected. This matroid has rank 7. Furthermore, Fig. 2 represents a 4-petal swirl-like flower. Again each petal is a plane. In that figure, the lines of attachment are the lines spanned by $\{b_1, b_2\}$, $\{b_2, b_3\}$, $\{b_3, b_4\}$, and $\{b_4, b_1\}$, where $\{b_1, b_2, b_3, b_4\}$ is an independent set and each of the elements in this set may or may not be in the matroid. The rank of this matroid is 8.

Flowers provide a way of representing 3-separations in a 3-connected matroid M. It was shown in [6] that, by using a certain type of tree, one can simultaneously display a representative of each equivalence class of non-sequential 3-separations of M. We now describe the type of tree that is used. Let π be a partition of a finite set E. Let T be a tree such that every member of π labels a vertex of T; some vertices may be unlabelled but no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*. If B is a bag vertex of T, then $\pi(B)$ denotes the subset of E that labels it. If the degree of B is at most one, then B is a *terminal* bag vertex; otherwise B is *non-terminal*.

Let G be a subgraph of T with components $G_1, G_2, \ldots, G_m$. Let X_i be the union of those bags that label vertices of G_i. Then *the subsets of E displayed by G* are $X_1, X_2, \ldots, X_m$. In particular, if $V(G) = V(T)$, then $\{X_1, X_2, \ldots, X_m\}$ is the *partition of E displayed by G*. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by $T\backslash e$. If $e = v_1v_2$ for vertices v_1 and v_2, then (Y_1, Y_2) is the *(ordered) partition of $E(M)$ displayed by v_1v_2* if Y_1 is the union of the bags in the component of $T\backslash v_1v_2$ containing v_1. Let v be a vertex of T that is not a bag vertex. The *partition of E displayed by v* is the partition displayed by $T - v$. The edges incident with v correspond to the components of $T-v$, and hence to the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \ldots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a 3-connected matroid with ground set E. Let T be a π-labelled tree for M, where π is a partition of E such that:

(I) For each edge e of T, the partition (X, Y) of E displayed by e is 3-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential 3-separation.
(II) Every non-bag vertex v is labelled either D or A; if v is labelled D, then there is a cyclic ordering on the edges incident with v.
(III) If a vertex v is labelled A, then the partition of E displayed by v is an anemone of order at least 3.
(IV) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a daisy of order at least 3.

By conditions (III) and (IV), a vertex v labelled D or A corresponds to a flower of M. The 3-separations displayed by this flower are the 3-separations *displayed by v*. A vertex of T is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. A 3-separation is *displayed by T* if it is displayed by some edge or some flower vertex of T. A 3-separation (R, G) of M *conforms* with T if either (R, G) is equivalent to a 3-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a 3-separation (R', G') with the property that either R' or G' is contained in a bag of T.

A π-labelled tree T for M satisfying (I)–(IV) is a *conforming tree* for M if every non-sequential 3-separation of M conforms with T. A conforming tree T is a *partial 3-tree* if, for every flower vertex v of T, the partition of E displayed by v is a tight maximal flower of M.

We now define a quasi order on the set of partial 3-trees for M clarifying the corresponding definition in [6, 7]. Let T_1 and T_2 be partial 3-trees for M. Define $T_1 \preceq T_2$ if every non-sequential 3-separation displayed by T_1 is equivalent to one displayed by T_2. If $T_1 \preceq T_2$ and $T_2 \preceq T_1$, then T_1 and T_2 are *equivalent* partial 3-trees. A partial 3-tree is *maximal* if it is maximal with respect to this quasi order. We shall call a maximal partial 3-tree a *3-tree*. Note that this terminology differs

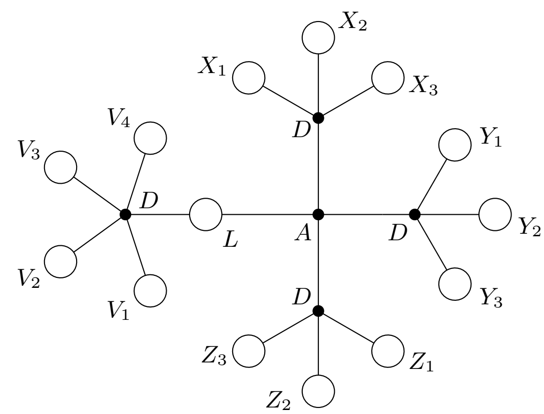

FIGURE 3. The 3-tree T.

from that used in [7] where we use the term '3-tree' for a particular type of maximal 3-tree defined in that paper.

As an example, for $n \geq 3$ and $k \geq 2$, the *free* (n,k)*-swirl* is the matroid that is obtained by beginning with a basis $\{1, 2, \ldots, n\}$, adding k points freely on each of the n lines spanned by $\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}$, and then deleting $\{1, 2, \ldots, n\}$. The usual free n-swirl coincides with the free $(n, 2)$-swirl. We observe that, when $n + k \geq 5$, the free (n,k)-swirl can be viewed as a swirl-like flower whose n petals consist of the sets of k points that were freely placed on the n lines above. The *spine* of a paddle $(P_1, P_2, \ldots, P_n)$ is the set $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \cdots \cap \mathrm{cl}(P_n)$, which coincides with each of the sets $\mathrm{cl}(P_i) \cap \mathrm{cl}(P_j)$ with $1 \leq i < j \leq n$.

Now, beginning with a free $(5, 4)$-swirl $S = (V_1, V_2, V_3, V_4, L)$, where each of V_1, V_2, V_3, V_4, and L is a line of S, use L as the spine of a paddle to which we attach three $(4, 4)$-swirls (X_1, X_2, X_3, L), (Y_1, Y_2, Y_3, L), and (Z_1, Z_2, Z_3, L). A possible 3-tree T for this matroid M is shown in Fig. 3, where large open circles represent bag vertices. At the end of Section 5, we will use this example, which is taken from [7], to illustrate our polynomial-time algorithm for finding a 3-tree. The 3-tree for M is not unique. Indeed, we can move the bag vertex labelled by L so that it occurs on one of the other edges incident with the anemone vertex of T to obtain another 3-tree for M.

The following theorem is the main result of [6, Theorem 9.1].

Theorem 2.1. *Let M be a 3-connected matroid with $|E(M)| \geq 9$. Then M has a 3-tree T. Moreover, every non-sequential 3-separation of M is equivalent to a 3-separation displayed by T.*

Throughout, we shall assume that each matroid M that we deal with is specified by a *rank oracle*, that is, a subroutine that, in unit time, gives the rank of any specified subset X of $E(M)$. The following is the main result of this paper.

Theorem 2.2. *Let M be a 3-connected matroid specified by a rank oracle and suppose that $|E(M)| \geq 9$. Then there is a polynomial-time algorithm for finding a 3-tree for M.*

The next section contains a number of preliminaries that we use to prove the last theorem. In Section 4, we use a result of Cunningham and Edmonds to show that, for a 3-connected matroid M with n elements, there is a polynomial $p(n)$ such that by making at most $p(n)$ calls to a rank oracle, we can either find a non-sequential 3-separation in M or show that no such 3-separation exists. Section 5 presents our algorithm for finding a 3-tree for M. In Section 6, we prove the correctness of the algorithm and thereby prove Theorem 2.2. Finally, Section 7 discusses why the proof of Theorem 2.1 in [6] does not appear to yield the desired polynomial-time algorithm for finding a 3-tree.

3. Preliminaries

In this section, we prove a number of lemmas needed to establish the main result. The first lemma is routine and often freely used.

Lemma 3.1. *Let (X, Y) be an exactly 3-separating partition of a matroid M.*

(i) *For $e \in E(M)$, the partition $(X \cup e, Y - e)$ is 3-separating if and only if $e \in \mathrm{cl}^{(*)}(X)$.*
(ii) *For $e \in Y$, the partition $(X \cup e, Y - e)$ is exactly 3-separating if and only if e is in exactly one of $\mathrm{cl}(X) \cap \mathrm{cl}(Y - e)$ and $\mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y - e)$.*
(iii) *The elements of $\mathrm{fcl}(X) - X$ can be ordered $(x_1, x_2, \ldots, x_n)$ so that $X \cup \{x_1, x_2, \ldots, x_i\}$ is 3-separating for all i in $[n]$.*

The connectivity function λ_M of a matroid M has many attractive properties. Clearly $\lambda_M(X) = \lambda_M(E - X)$. Moreover, one easily checks that $\lambda_M(X) = r(X) + r^*(X) - |X|$ for all subsets X of $E(M)$. Hence $\lambda_M(X) = \lambda_{M^*}(X)$. We often abbreviate λ_M as λ. This function is submodular, that is, $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E(M)$. The next lemma is a consequence of this. We make frequent use of it here and write *by uncrossing* to mean "by an application of Lemma 3.2."

Lemma 3.2. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

The next two lemmas were established in [8, Lemma 2.7] and [6, Lemma 5.9].

Lemma 3.3. *Let (X, Y) be a 3-separation in a 3-connected matroid M and let Y' be a non-sequential 3-separating set in M. If $Y' \subseteq Y$, then Y is non-sequential.*

Lemma 3.4. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight flower of order at least 3.*

(i) *If* $1 \le j \le n-2$, *then*

$$\mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_j) - (P_1 \cup P_2 \cup \cdots \cup P_j) \subseteq (\mathrm{fcl}(P_1) - P_1) \cup (\mathrm{fcl}(P_j) - P_j)$$

and every element of $(\mathrm{fcl}(P_1) - P_1) \cup (\mathrm{fcl}(P_j) - P_j)$ *is loose.*

(ii) *If* $2 \le j \le n-1$, *then* $P_1 \cup P_2 \cup \cdots \cup P_j$ *is a non-sequential* 3*-separating set. If, in addition,* $j \le n-2$, *then* $(P_1 \cup P_2 \cup \cdots \cup P_j, P_{j+1} \cup P_{j+2} \cup \cdots \cup P_n)$ *is a non-sequential* 3*-separation.*

The next result is a consequence of the last lemma.

Corollary 3.5. *Let* Φ *be a tight flower in a* 3*-connected matroid and* (U, V) *be a non-sequential* 3*-separation such that* U *is a union of petals of* Φ*. Then no petal of* Φ *is in the full closure of both* U *and* V*.*

Proof. Let P be a petal of Φ such that $P \subseteq U$ and $P \subseteq \mathrm{fcl}(V)$. Then P is a proper subset of U as (U, V) is non-sequential. Hence Φ has at least three petals. Therefore, by [6, Corollary 5.10], Φ has order at least three. Thus, by Lemma 3.4(i), P is loose; a contradiction. □

The next lemma was proved in [8, Lemma 3.1].

Lemma 3.6. *Let* $(P_1, P_2, \dots, P_k)$ *be a flower in a* 3*-connected matroid. If* P_2 *is loose and* P_1 *is tight, then* $P_2 \subseteq \mathrm{fcl}(P_1)$*.*

An ordered partition $(Z_1, Z_2, \dots, Z_k)$ of the elements of a 3-connected matroid is a 3-*sequence* if, for all i in $[k-1]$, the set $\cup_{j=1}^{i} Z_j$ is 3-separating. When a set Z_i consists of a single element z_i, we shall write z_i rather than $\{z_i\}$ in the 3-sequence.

Lemma 3.7. *Let* U *and* Y *be disjoint subsets of the ground set* E *of a* 3*-connected matroid* M*. Suppose that* U *and* $U \cup Y$ *are* 3*-separating and* $Y \subseteq \mathrm{fcl}(U)$*. If* $\mathrm{fcl}(U) \neq E$*, then there is an ordering* $(y_1, y_2, \dots, y_k)$ *of the elements of* Y *such that* $(U, y_1, y_2, \dots, y_k, E - (U \cup Y))$ *is a* 3*-sequence.*

Proof. Let $(u_1, u_2, \dots, u_l)$ be an ordering of $\mathrm{fcl}(U) - U$ such that $U \cup \{u_1, u_2, \dots, u_i\}$ is 3-separating for all i in $[l]$. Let $(y'_1, y'_2, \dots, y'_k)$ be the ordering of the elements of Y induced by this ordering of $\mathrm{fcl}(U) - U$. As $\mathrm{fcl}(U) \neq E$, we have $|E - \mathrm{fcl}(U)| \ge 4$ so, by uncrossing, $U \cup \{y'_1, y'_2, \dots, y'_j\}$ is 3-separating for all j in $[k]$. In particular, $(U, y'_1, y'_2, \dots, y'_k, E - (U \cup Y))$ is a 3-sequence in M. □

In [6], our approach to finding a 3-tree for a 3-connected matroid M relied on first constructing a maximal flower in M. As we shall see in Section 7, it is not clear how this approach can be used to produce a 3-tree for M in polynomial time. The basis of the algorithm that we shall introduce here will be to first find, if possible, a non-sequential 3-separation (X, Y) in M. Next we determine whether X has a partition (X', X'') so that $(X', X'' \cup Y)$ is a non-sequential 3-separation that is not equivalent to (X, Y). To facilitate our discussion of this process, we next introduce the notion of a 3-path. After formally defining this concept, we devote the rest of this section to proving various properties of 3-paths that we shall need.

Let M be a 3-connected matroid with ground set E. A 3-*path* in M is an ordered partition $(X_1, X_2, \dots, X_m)$ of E into non-empty sets, called *parts*, such that

(i) $(\cup_{j=1}^{i} X_j, \cup_{j=i+1}^{m} X_j)$ is a non-sequential 3-separation of M for all i in $[m-1]$; and
(ii) for all i in $\{2, 3, \dots, m-1\}$, the set X_i is not in the full closure of either $\cup_{j=1}^{i-1} X_j$ or of $\cup_{j=i+1}^{m} X_j$.

Condition (ii) is equivalent to the assertion that the non-sequential 3-separations $(\cup_{j=1}^{i} X_j, \cup_{j=i+1}^{m} X_j)$ and $(\cup_{j=1}^{i+1} X_j, \cup_{j=i+2}^{m} X_j)$ are inequivalent for all i in $[m-2]$. For a subset X_0 of E, an X_0-*rooted* 3-*path* is a 3-path of the form $(X_0 \cup X_1, X_2, \dots, X_m)$ where $X_0 \cap X_1 = \emptyset$. Thus a 3-path is just a $\emptyset$-rooted 3-path. An X_0-rooted 3-path is *maximal* if

(i) none of the sets X_i with $i \geq 2$ can be partitioned into sets $X_{i,1}, X_{i,2}, \dots, X_{i,k}$ for some $k \geq 2$ such that $(X_0 \cup X_1, X_2, \dots, X_{i-1}, X_{i,1}, X_{i,2}, \dots, X_{i,k}, X_{i+1}, \dots, X_m)$ is a 3-path; and
(ii) X_1 cannot be partitioned into sets $X_{1,1}, X_{1,2}, \dots, X_{1,k}$ for some $k \geq 2$ such that $(X_0 \cup X_{1,1}, X_{1,2}, \dots, X_{1,k}, X_2, \dots, X_m)$ is a 3-path.

Observe that, in (ii), the set $X_{1,1}$ may be empty when X_0 is non-empty although all of $X_{1,2}, X_{1,3}, \dots, X_{1,k}$ must be non-empty.

An X_0-rooted 3-path is *left-justified* if, for all i in $\{2, 3, \dots, m\}$, no element of X_i is in the full closure of $\cup_{j=0}^{i-1} X_j$. In a 3-path $(X_1, X_2, \dots, X_m)$, for each i in $[m]$, we denote the sets $\cup_{j=1}^{i-1} X_j$ and $\cup_{j=i+1}^{m} X_j$ by X_i^- and X_i^+, respectively. In particular, $X_1^- = \emptyset = X_m^+$. Observe that, in a 3-path $(X_1, X_2, \dots, X_m)$, each of X_1 and X_m has at least four elements as neither set is sequential, and each of $X_2, X_3, \dots, X_{m-1}$ has at least two elements by (ii).

In what follows, we shall frequently be referring to a 3-separation (R, G) of a 3-connected matroid M. In general, we shall view (R, G) as a colouring of the elements of $E(M)$, the elements in R and G being *red* and *green*, respectively. A non-empty subset X of E is *bichromatic* if it meets both R and G; otherwise it is *monochromatic*. We shall view the empty set as being monochromatic. In the lemmas that follow, we shall make repeated use of the fact [6, Lemma 3.3] that if (R, G) is non-sequential and (R', G') is a partition of $E(M)$ such that $\text{fcl}(R') = \text{fcl}(R)$ or $\text{fcl}(G') = \text{fcl}(G)$, then (R', G') is a non-sequential 3-separation of M.

Lemma 3.8. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted 3-path in a 3-connected matroid M. Let (R, G) be a non-sequential 3-separation in M. If, for some i in $\{2, 3, \dots, m-1\}$, both X_i^- and X_i^+ contain at least two red and at least two green elements, then X_i is monochromatic.*

Proof. Assume that X_i is bichromatic. Now $|X_i^+ \cap G| \geq 2$. Thus, by uncrossing, as R and $X_i^- \cup X_i$ are both 3-separating, so is their intersection, $(X_i^- \cup X_i) \cap R$. Again, by uncrossing, the union of the last set with X_i^-, which equals $X_i^- \cup (X_i \cap R)$, is 3-separating. By maximality, $(X_0 \cup X_1, X_2, \dots, X_{i-1}, X_i \cap R, X_i \cap G, X_{i+1}, \dots, X_m)$

is not a 3-path. But the original 3-path is left-justified, so $X_i \cap G \subseteq \mathrm{fcl}(X_i^+)$. By symmetry, $X_i^- \cup (X_i \cap G)$, is 3-separating, yet $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap G, X_i \cap R, X_{i+1}, \ldots, X_m)$ is not a 3-path, so $X_i \cap R \subseteq \mathrm{fcl}(X_i^+)$. We conclude that $X_i \subseteq \mathrm{fcl}(X_i^+)$; a contradiction. □

Lemma 3.9. *Let $(X_1, X_2, \ldots, X_m)$ be a 3-path in a 3-connected matroid M. Let X_0 be a subset of X_1, and (R, G) be a non-sequential 3-separation in M for which X_0 is monochromatic and no equivalent 3-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that, for some i in $[m]$, the set X_i is bichromatic. If, for some Z in $\{X_i^-, X_i^+\}$, there is at least one red element in Z, then there are at least two red elements in Z.*

Proof. Suppose first that $|X_i^+ \cap R| = 1$. As $(E - X_i^+, X_i^+)$ and (R, G) are non-sequential, $|X_i^+| \geq 4$ and $|R \cap (E - X_i^+)| \geq 3$. Thus, by uncrossing, $G \cap X_i^+$ is 3-separating. Since X_i^+ is also 3-separating, the one red element in X_i^+ can be recoloured green producing a 3-separation equivalent to (R, G) with fewer bichromatic parts; a contradiction. Hence $|X_i^+ \cap R| \geq 2$. A symmetric argument establishes that if $|X_i^- \cap R| \geq 1$, then $|X_i^- \cap R| \geq 2$. We note here that if $|X_i^- \cap R| = 1$ and the unique element of this set is in X_0, then $|X_0| = 1$ as X_0 is monochromatic. Thus X_0 stays monochromatic when the element of $X_i^- \cap R$ is recoloured and, as $X_0 \subseteq X_1$, we produce a 3-separation equivalent to (R, G) with fewer bichromatic parts. □

Lemma 3.10. *Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be a left-justified maximal X_0-rooted 3-path in a 3-connected matroid M. Let (R, G) be a non-sequential 3-separation in M for which X_0 is monochromatic and no equivalent 3-separation in which X_0 is monochromatic has fewer bichromatic parts. If, for some i in $\{2, 3, \ldots, m-1\}$, the set X_i is bichromatic, then either X_i is not 3-separating, or $X_i^- \cup X_i^+$ is monochromatic.*

Proof. Assume that X_i is 3-separating and that $X_i^- \cup X_i^+$ is not monochromatic. By Lemmas 3.8 and 3.9, X_i^- or X_i^+ is monochromatic and is green, say. Then, by Lemma 3.9, X_i^+ or X_i^-, respectively, contains at least two red elements. If X_i contains a single red element x, then x is the unique red element of some Y in $\{X_i^- \cup X_i, X_i^+ \cup X_i\}$. By uncrossing Y and G, we see that x can be recoloured green to produce a 3-separation equivalent to (R, G) with fewer bichromatic parts. If X_i contains a single green element, g, but more than one red element, then, by uncrossing, $X_i - g$ is 3-separating, so $g \in \mathrm{cl}^{(*)}(X_i - g)$ and we can recolour g red to reduce the number of bichromatic parts. We conclude that both $X_i \cap R$ and $X_i \cap G$ contain at least two elements. Now either X_i^- or X_i^+ is green. In the first case, by uncrossing, $X_i^- \cup (X_i \cap G)$ is 3-separating. As $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap G, X_i \cap R, X_{i+1}, \ldots, X_m)$ is not a 3-path, but the original 3-path is left-justified, it follows that $X_i \cap R \subseteq \mathrm{fcl}(X_i^+)$. It is straightforward to check that if X_i^+ is green, then $X_i \cap G \subseteq \mathrm{fcl}(X_i^+)$. Thus some Z in $\{X_i \cap R, X_i \cap G\}$ is a subset of $\mathrm{fcl}(X_i^- \cup X_i^+)$.

Suppose that $X_i \not\subseteq \mathrm{fcl}(X_i^- \cup X_i^+)$. Then, by Lemma 3.7, there is an ordering $(z_1, z_2, \ldots, z_k)$ of the elements of Z such that $(X_i^- \cup X_i^+, z_1, z_2, \ldots, z_k, X_i - Z)$ is a 3-sequence. Therefore $Z \subseteq \mathrm{fcl}(X_i - Z)$, so we can change the colour of all the elements of Z to give a 3-separation that is equivalent to (R, G) but has fewer bichromatic parts; a contradiction. We may now assume that $X_i \subseteq \mathrm{fcl}(X_i^- \cup X_i^+)$. Then there is an ordering $(z_1, z_2, \ldots, z_k)$ of the elements of X_i such that $(X_i^- \cup X_i^+, z_1, z_2, \ldots, z_k)$ is a 3-sequence. We can reorder the last three elements of this 3-sequence if necessary to obtain a 3-sequence whose last two elements are the same colour. Then we can recolour all of the elements of X_i this colour to get a 3-separation that is equivalent to (R, G) but has fewer bichromatic parts, again getting a contradiction. □

Lemma 3.11. *Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be a left-justified maximal X_0-rooted 3-path in a 3-connected matroid M. Let (R, G) be a non-sequential 3-separation in M for which X_0 is monochromatic and no equivalent 3-separation in which X_0 is monochromatic has fewer bichromatic parts. If, for some i in $\{2, 3, \ldots, m-1\}$, the set X_i^- is monochromatic but X_i is bichromatic, then $X_i^- \cup X_i^+$ is monochromatic.*

Proof. Assume that X_i^- is green but $X_i^- \cup X_i^+$ is bichromatic. Then, by Lemma 3.9, X_i^+ contains at least two red elements. Thus, by uncrossing, $X_i^- \cup (X_i \cap G)$ is 3-separating. As the 3-path $(X_0 \cup X_1, X_2, \ldots, X_m)$ is maximal and left-justified, it follows that $X_i \cap R \subseteq \mathrm{fcl}(X_i^- \cup (X_i \cap G))$, so $X_i \cap R \subseteq \mathrm{fcl}(G)$. Hence we can recolour all the elements in $X_i \cap R$ green thereby reducing the number of bichromatic parts; a contradiction. □

Lemma 3.12. *Let $(Z_0, Z_1, Z_2, \ldots, Z_m)$ be a 3-path in a 3-connected matroid M where $m \geq 2$. Let (R, G) be a non-sequential 3-separation of M such that*

(i) *each of $Z_1, Z_2, \ldots, Z_m$ is monochromatic;*
(ii) *$Z_{m-1} \cup Z_m$ is bichromatic;*
(iii) *either*
  (a) *Z_0 is monochromatic but $Z_0 \cup Z_1$ is not; or*
  (b) *Z_0 is bichromatic and $\min\{|Z_0 \cap R|, |Z_0 \cap G|\} \geq 2$.*

Then M has a flower $(Z_0, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ where each of $Z_{i,1} \cup Z_{i,2} \cup \cdots \cup Z_{i,s}$ and $Z_{j,t} \cup Z_{j,t-1} \cup \cdots \cup Z_{j,1}$ is monochromatic; each of $(Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s})$ and $(Z_{j,1}, Z_{j,2}, \ldots, Z_{j,t})$ is a subsequence of $(Z_1, Z_2, \ldots, Z_{m-1})$; and $\{Z_1, Z_2, \ldots, Z_{m-1}\} = \{Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}\} \cup \{Z_{j,1}, Z_{j,2}, \ldots, Z_{j,t}\}$. Moreover, when Z_0 is bichromatic, this flower can be refined so that $(Z_0', Z_0'', Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ is a flower where $\{Z_0', Z_0''\} = \{Z_0 \cap R, Z_0 \cap G\}$ and $Z_0'' \cup Z_{i,1}$ and $Z_0' \cup Z_{j,1}$ are monochromatic.

Proof. Without loss of generality, we may assume that $Z_m \subseteq G$ and $Z_{m-1} \subseteq R$. By assumption, $Z_0 \cup Z_1$ is bichromatic containing at least two red elements and at least two green elements. Let the subsequence of $(Z_2, Z_3, \ldots, Z_m)$ consisting of red sets be $(Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k})$. Then $p_k = m - 1$. By repeated applications of uncrossing, we get that $Z_{p_a} \cup Z_{p_{a+1}} \cup \cdots \cup Z_{p_k}$ is 3-separating for all a in $[k]$. As $Z_0 \cup Z_1 \cup \cdots \cup Z_b$ is 3-separating for all b in $[m-1]$, we deduce, by uncrossing, that each of $Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k}, Z_{p_1} \cup Z_{p_2}, Z_{p_2} \cup Z_{p_3}, \ldots, Z_{p_{k-1}} \cup Z_{p_k}$ is 3-separating. Moreover, $Z_{p_k} \cup Z_m = Z_{m-1} \cup Z_m$, so it is 3-separating.

Now let the subsequence of $(Z_2, Z_3, \ldots, Z_m)$ consisting of green sets be $(Z_{q_1}, Z_{q_2}, \ldots, Z_{q_l})$. Then $q_l = m$, so Z_{q_l} is 3-separating and, by uncrossing again, we deduce that each of $Z_{q_1}, Z_{q_2}, \ldots, Z_{q_{l-1}}, Z_{q_1} \cup Z_{q_2}, Z_{q_2} \cup Z_{q_3}, \ldots, Z_{q_{l-1}} \cup Z_{q_l}$ is 3-separating.

As each of $Z_{p_1} \cup Z_{p_2}, Z_{p_2} \cup Z_{p_3}, \ldots, Z_{p_{k-1}} \cup Z_{p_k}, Z_{p_k} \cup Z_{q_l}, Z_{q_l} \cup Z_{q_{l-1}}, \ldots, Z_{q_2} \cup Z_{q_1}$ is 3-separating, the union of all but the last of these sets is 3-separating and hence so is its complement, $Z_0 \cup Z_1 \cup Z_{q_1}$. Similarly, $Z_0 \cup Z_1 \cup Z_{p_1}$ is 3-separating. We deduce that $(Z_0 \cup Z_1, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k}, Z_m, Z_{q_{l-1}}, \ldots, Z_{q_1})$ is a flower. If Z_1 is red, then, by uncrossing, $Z_1 \cup Z_{p_1} \cup \cdots \cup Z_{p_k}$ is 3-separating, as are $Z_0 \cup Z_1$ and $Z_0 \cup Z_1 \cup Z_{p_1}$, so Z_1 and $Z_1 \cup Z_{p_1}$ are 3-separating. Also, $E - (Z_1 \cup Z_{p_1} \cup \cdots \cup Z_{p_k})$ is 3-separating and, by uncrossing, so too is $Z_0 \cup Z_{q_1}$. Hence $(Z_0, Z_1, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k}, Z_m, Z_{q_{l-1}}, \ldots, Z_{q_1})$ is a flower. If Z_1 is green, then, as Z_{m-1} is red, a similar argument gives that $(Z_0, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k}, Z_m, Z_{q_{l-1}}, \ldots, Z_{q_1}, Z_1)$ is a flower. We conclude, using the notation in statement of the lemma, that $(Z_0, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ is a flower.

Finally, assume that Z_0 is bichromatic.. Then, by uncrossing, $Z_0 \cap R$ and $Z_0 \cap G$ are both 3-separating and the argument at the end of the last paragraph implies that $(Z_0 \cap G, Z_0 \cap R, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ is a flower. □

In our algorithm, we shall construct maximal flowers from 3-paths. The next lemma is designed to cope with the fact that, whereas each 3-separation displayed by a 3-path is non-sequential, a maximal flower may have sequential petals.

Lemma 3.13. *Let M be a 3-connected matroid with at least nine elements and X be a non-sequential 3-separating set in M. Let (R, G) be a non-sequential 3-separation such that both $R \cap X$ and $G \cap X$ are sequential 3-separating sets. Let (U, V) be a non-sequential 3-separation with $\min\{|U - X|, |V - X|\} \geq 2$ such that $U \cap X \not\subseteq \mathrm{fcl}(U - X)$ and $V \cap X \not\subseteq \mathrm{fcl}(V - X)$. Then some of the elements of X can be recoloured to give a 3-separation (R', G') equivalent to (R, G) such that both $U \cap X$ and $V \cap X$ are monochromatic.*

Proof. Since X is non-sequential, $|X| \geq 4$. As $U \cap X \not\subseteq \mathrm{fcl}(U - X)$ and $V \cap X \not\subseteq \mathrm{fcl}(V - X)$, it follows that neither $U \cap X$ nor $V \cap X$ is empty. If $|U \cap X| = 1$, then $|V \cap X| \geq 2$ so, by uncrossing, $U - X$ is 3-separating and then $U \cap X \subseteq \mathrm{fcl}(U - X)$; a contradiction. Hence $|U \cap X| \geq 2$ and, by symmetry, $|V \cap X| \geq 2$.

Let $(r_1, r_2, \ldots, r_k)$ and $(g_1, g_2, \ldots, g_l)$ be sequential orderings of $R \cap X$ and $G \cap X$, respectively. Observe that the lemma trivially holds if either $k = 0$ or $l = 0$. Thus we may assume that $k, l \geq 1$. If $k = 1$, then, as $G \cap X$ is 3-separating, it follows by Lemma 3.1 that $r_1 \in \mathrm{cl}^{(*)}(G \cap X)$. Thus the lemma holds by recolouring r_1 green. Similarly, the lemma holds if $l = 1$, so we may assume that $k, l \geq 2$.

Suppose that $|R \cap X| \geq 3$. Then we may assume that $|\{r_1, r_2, r_3\} \cap U| \geq 2$. As $|U - X| \geq 2$, it follows by uncrossing that $X \cap V$ is 3-separating. Since $|R \cap (E - (X \cap V))| \geq 2$, it follows by another application of uncrossing that $G \cap (X \cap V)$ is 3-separating. As $|\{r_1, r_2, r_3\} \cap U| \geq 2$, it follows that $R \cap (X \cap V) \subseteq \mathrm{fcl}(U)$ and so, by Lemma 3.7, there is an ordering $(r'_1, r'_2, \ldots, r'_{k'})$ of the elements in $R \cap (X \cap V)$

such that
$$(U \cup (V - X), r'_1, r'_2, \dots, r'_{k'}, G \cap (X \cap V))$$
is a 3-sequence in M. Hence $R \cap (X \cap V) \subseteq \mathrm{fcl}(G \cap (X \cap V))$ so we can recolour the elements of $R \cap (X \cap V)$ green to obtain an equivalent 3-separation (R', G') in which $X \cap V$ is green, that is, $X \cap V \subseteq G'$. If $X \cap U \subseteq R'$, then the required result holds, so we may assume that $X \cap U \cap G' \neq \emptyset$. Thus $|X \cap G'| \geq 3$. Since $G \cap X$ is sequential, $G' \cap X$ is sequential. Take a sequential ordering $(g'_1, g'_2, \dots, g'_{l'})$ of $G' \cap X$. If at least two of g'_1, g'_2, and g'_3 are in U, then there is a 3-sequence in M of the form $(U, e_1, e_2, \dots, e_t, V - X)$ where $\{e_1, e_2, \dots, e_t\} = V \cap X$. Hence $V \cap X \subseteq \mathrm{fcl}(V - X)$; a contradiction. We deduce that at least two of g'_1, g'_2, and g'_3 are in V. Then there is a sequential ordering of $G' \cap X$ that first uses all of the elements of $V \cap X$. Let this ordering be $(v_1, v_2, \dots, v_a, u_1, u_2, \dots, u_b)$ where $\{v_1, v_2, \dots, v_a\} \subseteq V$ and $\{u_1, u_2, \dots, u_b\} \subseteq U$. Then, by uncrossing, $(X \cap U \cap R') \cup \{u_b, u_{b-1}, \dots, u_i\}$ is 3-separating for all i in $[b]$. Thus we can recolour the elements of $\{u_b, u_{b-1}, \dots, u_1\}$ red to get that $U \cap X$ is red and $V \cap X$ is green as required.

We may now assume that $|R \cap X| = 2$ and, by symmetry, that $|G \cap X| = 2$. The required result follows unless $U \cap X = \{r_1, g_1\}$ and $V \cap X = \{r_2, g_2\}$ where $\{r_1, r_2\} = R \cap X$ and $\{g_1, g_2\} = G \cap X$.

Since $|R|, |G| \geq 4$ and $|E(M)| \geq 9$, we may assume that $|R - X| \geq 3$. Then, without loss of generality, we may suppose that $U - X$ contains at least two red elements. Assume that $V - X$ contains at least one green element. Then, by uncrossing, $U \cap R$ is 3-separating and so $(U - X) \cup r_1$ is 3-separating. As $(U - X) \cup r_1 \cup g_1$ is 3-separating, it follows that $U \cap X \subseteq \mathrm{fcl}(U - X)$; a contradiction. We deduce that $(V - X) \cap G = \emptyset$, so $|(V - X) \cap R| \geq 2$. Then, by arguing as above, we get that $(U - X) \cap G = \emptyset$. Hence $E(M) - X \subseteq R$, so $|G| = 2$; a contradiction. □

Lemma 3.14. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted 3-path in a 3-connected matroid M. Let (R, G) be a non-sequential 3-separation in M for which X_0 is monochromatic and no equivalent 3-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that $m \geq 2$ and that X_m and X_m^- are bichromatic. Then both $R \cap X_m$ and $G \cap X_m$ are sequential 3-separating sets.*

Proof. By Lemma 3.9, $|R \cap X_m^-|, |G \cap X_m^-| \geq 2$. Therefore, as R and X_m are 3-separating and $|E(M) - (R \cup X_m)| \geq 2$, we have $R \cap X_m$ is 3-separating. Similarly, $G \cap X_m$ is 3-separating. If $(E(M) - (R \cap X_m), R \cap X_m)$ is non-sequential, then, as $(X_0 \cup X_1, X_2, \dots, X_m)$ is left-justified and maximal, $\mathrm{fcl}(R \cap X_m) = \mathrm{fcl}(X_m)$. In particular, by Lemma 3.7, we can recolour all the elements in $G \cap X_m$ red to give a 3-separation equivalent to (R, G) with fewer bichromatic parts; a contradiction. Thus $(E(M) - (R \cap X_m), R \cap X_m)$ is sequential, in particular, by Lemma 3.3, $R \cap X_m$ is sequential. Similarly, $G \cap X_m$ is sequential. □

Lemma 3.15. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted 3-path in a 3-connected matroid M. Let (R, G) be a non-sequential 3-separation in M for which X_0 is monochromatic and no equivalent 3-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that $\{2, 3, \dots, m-1\}$ contains an element j such that X_j and X_j^- are bichromatic, but X_j^+ is red. Then $R \cap X_j \subseteq$*

$\mathrm{fcl}(X_j^+)$. *Furthermore, there is a 3-separation* (R', G') *equivalent to* (R, G) *such that* $R' \cap X_j = X_j \cap \mathrm{fcl}(X_j^+)$ *while* $R' \cap X_i = R \cap X_i$ *and* $G' \cap X_i = G \cap X_i$ *for all* $i \neq j$.

Proof. By Lemma 3.9, $|G \cap X_j^-| \geq 2$ as $G \cap X_j^-$ is non-empty. Therefore, as R and $X_j \cup X_j^+$ are both 3-separating and avoid $G \cap X_j^-$, it follows by uncrossing that $(X_j^- \cup (G \cap X_j), R \cap (X_j \cup X_j^+))$ is a 3-separation. By Lemma 3.3, this 3-separation is non-sequential. But $(X_0 \cup X_1, X_2, \ldots, X_m)$ is maximal and left-justified, so $(X_j^- \cup (G \cap X_j), R \cap (X_j \cup X_j^+))$ is equivalent to $(X_j^- \cup X_j, X_j^+)$. Therefore $R \cap X_j \subseteq \mathrm{fcl}(X_j^+)$. Furthermore, by Lemma 3.7, recolouring all the elements in $(G \cap X_j) \cap \mathrm{fcl}(X_j^+)$ red, we have a 3-separation (R', G') equivalent to (R, G) with the desired properties. □

4. Finding a Non-Sequential 3-Separation

Finding a 3-tree for a 3-connected matroid M depends crucially on being able to find a non-sequential 3-separation for M or showing that M has no such 3-separation. We rely heavily on a polynomial-time algorithm of Cunningham and Edmonds (in Cunningham 1973) that, for any fixed positive integer k, will either find a k-separation in a matroid or will show that no such k-separation exists. Underlying this algorithm is the following result of Edmonds [4], which specifies the size of a largest common independent set of two matroids that share a common ground set.

Theorem 4.1. *Let* M_1 *and* M_2 *be matroids with rank functions* r_1 *and* r_2 *and a common ground set* E. *Then*

$$\max\{|I| : I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)\} = \min\{r_1(T) + r_2(E - T) : T \subseteq E\}.$$

The next result (see, for example, [5, Proposition 13.4.7]) provides the link between the existence of a certain k-separation and a common independent set of two matroids.

Proposition 4.2. *Let* M *be a matroid and* k *be a positive integer. If* X_1 *and* X_2 *are disjoint subsets of* $E(M)$ *each having at least* k *elements, then* M *has a* k*-separation* (Y_1, Y_2) *with* $X_1 \subseteq Y_1$ *and* $X_2 \subseteq Y_2$ *if and only if* $M/X_1 \backslash X_2$ *and* $M/X_2 \backslash X_1$ *do not have a common* t*-element independent set where* $t = r(M) + k - r(X_1) - r(X_2)$.

The matroid intersection algorithm finds, in polynomial time, not only a maximum-sized common independent set I of two matroids M_1 and M_2 on the same set E, but also a subset X of E that minimizes $r_1(X) + r_2(E - X)$, where r_i is the rank function of M_i. By Theorem 4.1, each of I and X verifies that the other has the specified property. By applying this algorithm to all pairs $M/X_1 \backslash X_2$ and $M/X_2 \backslash X_1$ for which X_1 and X_2 are disjoint 3-element subsets of $E(M)$, we get a polynomial-time algorithm for either finding a 3-separation in M or showing that no 3-separation exists. The difficulty with this process is that it may produce a sequential 3-separation and we want a non-sequential 3-separation. We show below how a minor modification of the algorithm will find a non-sequential 3-separation if one exists. First, we note that the basic idea in the matroid intersection algorithm is similar to that used in the algorithm for finding a maximum-sized matching in a bipartite graph: construction of an augmenting path. For a detailed description of the matroid intersection algorithm, the reader is referred to Cook, Cunningham, Pulleyblank, and Schrijver [1].

In order to find a non-sequential 3-separation in M if one exists, we begin by finding the set $\mathcal{F}$ of all maximal sequential 3-separating sets. To do this, we begin by finding all triangles and triads of M by determining which 3-element subsets X of $E(M)$ have $r(X)$ or $r^*(X)$ equal to 2, where $r^*(X) = r(E-X) - r(M) + 3$. We then find the full closure of each triangle and each triad by taking the closure of each such set, the coclosure of the result, the closure of the result, and so on until two consecutive terms are equal. For a given triangle or triad X in an n-element matroid, we can find $\mathrm{fcl}(X)$ by using $O(n^2)$ calls to the rank oracle. Observe that $\mathcal{F}$ consists of the maximal members of $\{\mathrm{fcl}(X) : X \text{ is a triangle or triad}\}$ and that the latter set has $O(n^3)$ members.

The next result is a straightforward consequence of Lemma 3.3 and we omit the proof. We use this corollary in the proof of the subsequent lemma.

Corollary 4.3. *In a 3-connected matroid M, a 3-separating set X is non-sequential if and only if no member of $\mathcal{F}$ contains X.*

The next lemma is key to finding a non-sequential 3-separation of a 3-connected matroid.

Lemma 4.4. *Let (U,V) be a 3-separation in a 3-connected matroid M and suppose $k \in \{3,4\}$. Then (U,V) is non-sequential if and only if there are k-element subsets U' and V' of U and V, respectively, such that no member of $\mathcal{F}$ contains U' or V'.*

Proof. Suppose (U,V) is non-sequential. Then $(U-\mathrm{fcl}(V), \mathrm{fcl}(V))$ is also non-sequential. Clearly $|U-\mathrm{fcl}(V)| \geq 4$. Let U_1 be a k-element subset of $U-\mathrm{fcl}(V)$. We take $U' = U_1$ unless U_1 is contained in some member F of $\mathcal{F}$. Consider the exceptional case. We have $F = \mathrm{fcl}(T)$ for some triangle or triad T. Clearly $|T \cap \mathrm{fcl}(V)| \leq 1$. Take $\{a,b\} \subseteq T-\mathrm{fcl}(V)$. Clearly $\mathrm{fcl}(\{a,b\}) = \mathrm{fcl}(T) = F$. If F contains $U-\mathrm{fcl}(V)$, then, by Lemma 3.3, $U-\mathrm{fcl}(V)$ is sequential; a contradiction. Thus $U-\mathrm{fcl}(V)-F$ is non-empty. Suppose this set contains a single element c. Then F and $U-\mathrm{fcl}(V)$ are 3-separating. By uncrossing, so is their intersection, $U-\mathrm{fcl}(V)-c$. As $U-\mathrm{fcl}(V)-c$ and $U-\mathrm{fcl}(V)$ are 3-separating, $c \in \mathrm{cl}^{(*)}(U-\mathrm{fcl}(V)-c)$, so $c \in F$; a contradiction. We deduce that $U-\mathrm{fcl}(V)-F$ contains at least two distinct elements, c and d. If $k=3$, let $U' = \{a,b,c\}$; if $k=4$, let $U' = \{a,b,c,d\}$. If U' is contained in a member F' of $\mathcal{F}$, then F' contains T and hence contains F. Thus $F' = F$, but $c \in F'-F$; a contradiction. Hence no member of $\mathcal{F}$ contains U'. We now know how to construct U'. We construct V' symmetrically from $V-\mathrm{fcl}(U)$.

The converse is an immediate consequence of the last corollary. □

Now to obtain a non-sequential 3-separation of M, we apply the procedure described above for finding a 3-separation with the modification that the disjoint sets X_1 and X_2 are chosen to be 3-element sets that are not contained in any member of $\mathcal{F}$. By the last lemma, if (Y_1, Y_2) is a 3-separation with $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then (Y_1, Y_2) is non-sequential. Moreover, if, after searching through all such pairs $\{X_1, X_2\}$ of sets, we find no 3-separation (Y_1, Y_2) with $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then M has no non-sequential 3-separations.

5. The Algorithm

In this section, we present the algorithm 3-Tree for constructing a 3-tree of a 3-connected matroid. To do this, we shall need some additional terminology. We shall also provide an informal description of the algorithm and an example to illustrate it.

Let M be a 3-connected matroid. Let $(P_1, P_2, \ldots, P_k)$ be a tight flower Φ in M, where $k \geq 3$. Consider how Φ might arise in a 3-path where the petals of Φ are the parts of the 3-path. Let P_1 and P_j be the first and last petals of Φ occurring in the 3-path. Then the definition of a 3-path requires that both P_1 and P_j are non-sequential. Clearly $j \in \{2, 3, \ldots, k\}$. Now $(P_1, Q'_1, Q'_2, \ldots, Q'_{k-2}, P_j)$ is a 3-path provided that $\{Q'_1, Q'_2, \ldots, Q'_{k-2}\} = \{P_2, P_3, \ldots, P_k\} - \{P_j\}$, and both $(P_2, P_3, \ldots, P_{j-1})$ and $(P_k, P_{k-1}, \ldots, P_{j+1})$ are subsequences of $(Q'_1, Q'_2, \ldots, Q'_{k-2})$. If, for example, each petal of Φ is sequential, then there is no 3-path whose parts coincide with the petals of Φ. But $(P_1 \cup P_2, P_3, P_4, \ldots, P_{k-2}, P_{k-1} \cup P_k)$ is one of many 3-paths arising from Φ. We now generalize the notion of a 3-path to indicate the presence of flowers including those with sequential petals.

Let τ be a 3-path $(P_{1,1}, P_{1,2}, \ldots, P_{1,s}, Q'_1, Q'_2, \ldots, Q'_{k-2}, P_{j,1}, P_{j,2}, \ldots, P_{j,t})$ in M such that there is a flower $\Phi = (P_1, P_2, \ldots, P_k)$ with $P_1 = P_{1,1} \cup P_{1,2} \cup \cdots \cup P_{1,s}$ and $P_j = P_{j,1} \cup P_{j,2} \cup \cdots \cup P_{j,t}$ where $\{Q'_1, Q'_2, \ldots, Q'_{k-2}\} = \{P_2, P_3, \ldots, P_k\} - \{P_j\}$. We call P_1 and P_j the *entry* and *exit petals*, respectively, of $(P_1, P_2, \ldots, P_k)$. When $j \neq k$, we denote this flower Φ in τ by replacing the subsequence $Q'_1, Q'_2, \ldots, Q'_{k-2}$ by $[(P_2, P_3, \ldots, P_{j-1}), (P_k, P_{k-1}, \ldots, P_{j+1})]$; and we call $P_2, P_3, \ldots, P_{j-1}$ and $P_k, P_{k-1}, \ldots, P_{j+1}$ the *clockwise* and *anticlockwise petals*, respectively, of Φ. If $j = k$, then we replace $Q'_1, Q'_2, \ldots, Q'_{k-2}$ by $[(P_2, P_3, \ldots, P_{k-1})]$. In this case, we call $P_2, P_3, \ldots, P_{k-1}$ the *clockwise petals* of Φ and say that Φ has no *anticlockwise petals*. Such modified 3-paths are examples of *generalized 3-paths*. There are three further elementary modifications of a 3-path which we shall want our notion of a generalized 3-path to encompass. Each of these occurs at the end of a 3-path and will be called an *end move*. Suppose $(Z_1, Z_2, \ldots, Z_m)$ is a 3-path in M and that there is a partition (Z'_m, Z''_m) of Z_m such that $(Z_1 \cup Z_2 \cup \cdots \cup Z_{m-2}, Z_{m-1}, Z'_m, Z''_m)$ is a tight flower Ψ. Then, in $(Z_1, Z_2, \ldots, Z_m)$, we replace Z_{m-1}, Z_m by $[(Z_{m-1}, Z'_m)], Z''_m$ and call $Z_1 \cup Z_2 \cup \cdots \cup Z_{m-2}$ and Z''_m the entry and exit petals of Ψ, and Z_{m-1}, Z'_m the clockwise petals of Ψ. We will also view $(Z_1, Z_2, \ldots, Z_{m-2}, [(Z_{m-1}, Z'_m)], Z''_m)$ as a generalized 3-path. Symmetrically, if there is a partition (Z'_1, Z''_1) of Z_1 such that $(Z'_1, Z''_1, Z_2, Z_3 \cup \cdots \cup Z_m)$ is a tight flower, we view $(Z'_1, [(Z''_1, Z_2)], Z_3, \ldots, Z_m)$ as a generalized 3-path. A combination of the last two end moves arises when $m = 2$ if Z_1 and Z_2 have partitions (Z'_1, Z''_1) and (Z'_2, Z''_2) such that $(Z'_1, Z''_1, Z'_2, Z''_2)$ is a tight flower. Then $(Z_1', [(Z_1'', Z_2')], Z_2'')$ is a generalized 3-path. In the first and second type of end move, we refer to Z_m and Z_1, respectively, as the *split part*, while in the third type of end move, we refer to Z_1 and Z_2 as the *split parts*.

The moves described in the last paragraph indicate how we modify a 3-path τ when we detect a single flower arising from it. The algorithm describes a systematic way in which we repeat the above steps for every flower occurring in τ each time modifying the current generalized 3-path to produce a new structure which we will also view as a generalized 3-path. The flowers that arise here are dealt with in order, starting from the far end of a 3-path. As we shall prove, the procedure we follow ensures that each flower we construct is tight and maximal.

Let τ be a generalized 3-path in a 3-connected matroid M with ground set E. Within τ, certain subsets of E are enclosed between the same pair of square brackets. Let τ' be the ordered sequence obtained from τ by, for each pair of corresponding square brackets, replacing these brackets and all the sets between them by the union of all the enclosed sets. Say $\tau' = (Y_1, Y_2, \ldots, Y_p)$. Note that τ' is a 3-path unless Y_1 or Y_p is sequential as may occur if we apply an end move. Let P denote the π-labelled tree consisting of a path of p bag vertices labelled, in order, $Y_1, Y_2, \ldots, Y_p$. Now modify P as follows. For each Y_j that is the union of s clockwise petals and t anticlockwise petals of a flower, replace the bag vertex labelled Y_j with a flower vertex v and adjoin $s+t$ new bag vertices to v each via a new edge so that the cyclic ordering induced by the cyclic ordering on the edges incident with v preserves the ordering of the flower Φ_j to which Y_j corresponds. Label the vertex v by D or A depending on whether Φ_j is a daisy or an anemone respectively. We refer to the resulting modification of P as a *path realization* of τ.

To deal with generalized 3-paths, it will be useful to have some more terminology. Let Z be a term in a generalized 3-path τ and assume that Z is not enclosed between two square brackets. We can then write τ as $(\tau(Z^-), Z, \tau(Z^+))$ so $\tau(Z^-)$ and $\tau(Z^+)$ denote, respectively, the portions of τ that occur before and after Z. In this case, as in a 3-path, we shall denote by Z^- and Z^+ the union of all of the sets in τ that occur, respectively, before and after Z.

We now give an informal description of our algorithm. An example to illustrate it is given at the end of the section. From the last section, we can test whether or not a given matroid M is 3-connected by making polynomially many calls to a rank oracle. We may now assume that M is 3-connected having ground set E. Starting with a single unmarked bag vertex labelled E, the algorithm 3-Tree recursively builds a π-labelled tree by selecting an unmarked bag vertex B and deciding if there is a non-sequential 3-separation (Y, Z) such that either $Y \subseteq \pi(B)$ or $Z \subseteq \pi(B)$. If there is no such 3-separation, the vertex is marked. If there is such a 3-separation, 3-Tree calls the first of its two subroutines, ForwardSweep, which constructs a left-justified maximal $(E - \pi(B))$-rooted 3-path. Once such a 3-path, say τ, is constructed, ForwardSweep ends and 3-Tree calls its second subroutine, BackwardSweep. This subroutine starts at the non-root end of τ and recursively works its way towards the root end uncovering flower structure. Eventually, BackwardSweep outputs a generalized 3-path τ'. Lastly, 3-Tree takes a path realization of τ' and adjoins it to the bag vertex B. The algorithm now repeats this process by selecting another unmarked bag vertex. When all bag vertices are marked, 3-Tree outputs a π-labelled tree. We end with two remarks. Firstly, some flower subtleties need to be dealt with at the non-root end of τ and also, in the first call to BackwardSweep, at the root end of τ. These subtleties correspond to applying end moves. Secondly, the fact that ForwardSweep constructs a left-justified maximal 3-path is established in Lemma 6.1.

Algorithm: 3-Tree(M)
Input: A 3-connected matroid M with ground set E and $|E| \geq 9$.
Output: A 3-tree for M.

1. Construct the collection $\mathcal{F}$ of maximal sequential 3-separating sets of M.

2. Let T_0 denote the π-labelled tree consisting of a single (unmarked) bag vertex labelled E.

3. Search through pairs $(\{y_1, y_2, y_3\}, \{z_1, z_2, z_3\})$ of disjoint subsets of E neither of which is contained in a member of $\mathcal{F}$ and find a 3-separation (Y, Z) of M such that Y and Z contain $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$, respectively.

- **(i)** If there is no such 3-separation, mark E, and output T_0.
- **(ii)** Otherwise, do the following:
  - **(a)** Set $X_0 = \emptyset$, set $X_1 = \mathrm{fcl}(Y)$, and set $X_2 = Z - \mathrm{fcl}(Y)$. Call ForwardSweep($M$, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).
  - **(b)** Call BackwardSweep(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$), where $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ is the 3-path of M outputted by ForwardSweep(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).
  - **(c)** Set $i = 1$ and set T_1 to be the path realization of the generalized 3-path outputted by BackwardSweep(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$) with each bag vertex unmarked.

4. If there is no unmarked bag vertex, output T_i. Otherwise, choose an unmarked bag vertex B of T_i.

5. If B is a non-terminal bag vertex, find a 3-separation (Y, Z) such that Y contains $\mathrm{fcl}(E - \pi(B))$, and Z contains a subset $\{z_1, z_2, z_3\}$ of $\pi(B) - \mathrm{fcl}(E - \pi(B))$ with no member of $\mathcal{F}$ containing $\{z_1, z_2, z_3\}$. If B is a terminal bag vertex, find a 3-separation (Y, Z) such that Y contains $\mathrm{fcl}(E - \pi(B))$ and an element y of $\pi(B)$ with $y \notin \mathrm{fcl}(E - \pi(B))$, and Z contains a subset $\{z_1, z_2, z_3\}$ of $\pi(B) - \mathrm{fcl}(E - \pi(B)) - \{y\}$ with no member of $\mathcal{F}$ containing $\{z_1, z_2, z_3\}$. Now do the following:

- **(i)** If there is no such 3-separation, mark B and return to Step 4.
- **(ii)** Otherwise, do the following:
  - **(a)** Set $X_0 = E - \pi(B)$, set $X_1 = \pi(B) \cap \mathrm{fcl}(Y)$, and set $X_2 = \pi(B) - \mathrm{fcl}(Y)$. Call ForwardSweep($M$, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).
  - **(b)** Call BackwardSweep(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$), where $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ is the 3-path of M outputted by ForwardSweep(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).

(c) Increase i by 1 and set T_i to be the π-labelled tree obtained from T_{i-1} and a path realization of the generalized 3-path outputted by BackwardSweep(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$) by identifying the vertex B of T_{i-1} with the vertex of the path realization labelled $X_0 \cup Z_1$, where the resulting composite vertex is labelled Z_1. If $Z_1 = \emptyset$ and the composite vertex has degree two, then suppress the composite vertex. Each bag vertex originating from the path realization, including the identified vertex, is unmarked.

(d) Return to Step 4.

Algorithm: ForwardSweep(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$)
Input: A 3-connected matroid M with ground set E and $|E| \geq 9$, a 3-path $(X_0 \cup X_1, X_2)$ of M, and the collection $\mathcal{F}$ of maximal sequential 3-separating sets of M.
Output: A 3-path $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ of M that is a refinement of $(X_0 \cup X_1, X_2)$.

1. Let $\tau_0 = (X_0 \cup X_1, X_2)$, set $(i, s, m) = (1, 1, 2)$, and set $(X'_1, X'_2) = (X_1, X_2)$.

2. If $s \neq m$, do the following:

(i) If $X_0 = \emptyset$ and $s = 1$, find a 3-separation (Y, Z) such that Y contains a subset $\{y_1, y_2, y_3\}$ of X'_1 with no member of $\mathcal{F}$ containing $\{y_1, y_2, y_3\}$, and Z contains $X'_2 \cup \cdots \cup X'_m$ and an element z of X'_1 with $z \notin \mathrm{fcl}(X'_2 \cup \cdots \cup X'_m) \cup \{y_1, y_2, y_3\}$.

(a) If there is no such 3-separation, go to Step 4.

(b) Otherwise, increase m by 1 and, for each $t > 1$, set X'_t to be X'_{t+1}. Furthermore, set X'_2 to be $X'_1 \cap (E - \mathrm{fcl}(Y))$ and then set X'_1 to be $X'_1 \cap \mathrm{fcl}(Y)$. Go to Step 5.

(ii) If $X_0 \neq \emptyset$ and $s = 1$, find a 3-separation (Y, Z) such that Y contains $\mathrm{fcl}(X_0)$, and Z contains $X'_2 \cup \cdots \cup X'_m$ and an element z of X'_1 with $z \notin \mathrm{fcl}(X'_2 \cup \cdots \cup X'_m)$.

(a) If there is no such 3-separation, go to Step 4.

(b) Otherwise, increase m by 1 and, for each $t > 1$, set X'_t to be X'_{t+1}. Furthermore, set X'_2 to be $X'_1 \cap (E - \mathrm{fcl}(Y))$ and then set X'_1 to be $X'_1 \cap \mathrm{fcl}(Y)$. Go to Step 5.

(iii) Otherwise, find a 3-separation (Y, Z) such that Y contains $X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}$ and an element y of $X'_s - \mathrm{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1})$, and Z contains $X'_{s+1} \cup \cdots \cup X'_m$ and an element z of X'_s with $z \notin \mathrm{fcl}(X'_{s+1} \cup \cdots \cup X'_m) \cup \{y\}$.

(a) If there is no such 3-separation, go to Step 4.

(b) Otherwise, increase m by 1 and, for each $t > s$, set X'_t to be X'_{t+1}. Furthermore, set X'_{s+1} to be $X'_s \cap (E - \mathrm{fcl}(Y))$ and then set X'_s to be $X'_s \cap \mathrm{fcl}(Y)$. Go to Step 5.

3. If $s = m$, find a 3-separation (Y, Z) such that Y contains $X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}$ and an element y of $X'_s - \mathrm{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1})$, and Z contains a subset $\{z_1, z_2, z_3\}$ of $X'_s - \mathrm{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}) - \{y\}$ such that no member of $\mathcal{F}$ contains $\{z_1, z_2, z_3\}$.

(i) If there is no such 3-separation, then output τ_i.

(ii) Otherwise, increase m by 1. Furthermore, set X'_{s+1} to be $X'_s \cap (E - \mathrm{fcl}(Y))$ and then set X'_s to be $X'_s \cap \mathrm{fcl}(Y)$. Go to Step 5.

4. Increase s by 1. Return to Step 2.

5. Increase i by 1 and set τ_i to be $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$. Return to Step 2.

Algorithm: BackwardSweep(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$)
Input: A matroid M with ground set E and $|E| \geq 9$, and a 3-path $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ of M, where $m \geq 2$.
Output: A generalized 3-path of M.

1. Let $\tau_m = (X_0 \cup Z_1, Z_2, \ldots, Z_m)$.
2. If $m = 2$ and X_0 is empty, find a 3-separation (U, V) for which U and V contain subsets U' and V' such that no member of $\mathcal{F}$ contains U' or V' and $|U' \cap Z_1| = |U' \cap Z_2| = |V' \cap Z_1| = |V' \cap Z_2| = 2$.

(i) If there is no such 3-separation, output τ_m.

(ii) Otherwise, output

$$(V \cap Z_1, [(U \cap Z_1, U \cap Z_2)], V \cap Z_2).$$

3. If $m = 2$ and X_0 is non-empty, output τ_m.

4. If $m \geq 3$, set $i = m - 1$.

5. If Z_{m-1} is 3-separating, find a 3-separation (U, V) such that U contains Z_{m-1} and $|U \cap Z_m| \geq 2$, and V contains Z^-_{m-1} and $|V \cap Z_m| \geq 2$.

(i) If there is such a 3-separation, set

$$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, Z_m \cap U)], Z_m \cap V\big)$$

and go to Step 7.

(ii) Otherwise, set

$$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1})], Z_m\big)$$

and go to Step 7.

6. If Z_{m-1} is not 3-separating, do the following:

(i) If $Z_{m-1} - \mathrm{fcl}(Z_m)$ is 3-separating, set

$$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1} - \mathrm{fcl}(Z_m))], Z_{m-1} \cap \mathrm{fcl}(Z_m), Z_m\big)$$

and go to Step 7.

(ii) Otherwise, set τ_{m-1} to be τ_m and go to Step 7.

7. **(i)** If $i \neq 2$, decrease i by 1 and go to Step 8.

(ii) Otherwise, go to Step 10.

8. If Z_i is 3-separating, do the following:

(i) If $\tau_{i+1} = (X_0 \cup Z_1, Z_2, \ldots, Z_i, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \ldots)$, where $p \geq 1$, do the following:

(a) If $Z_i \cup P_1$ is 3-separating, set

$$\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_{i+1}([(P_1, \dots, P_p), (Q_1, \dots, Q_q)]^+)\big)$$

and return to Step 7.

(b) If $Z_i \cup P_1$ is not 3-separating but $q \geq 1$ and $Z_i \cup Q_1$ is 3-separating, set

$$\tau_i = \big(\tau_{i+1}(Z_i^-), [(P_1, \dots, P_p), (Z_i, Q_1, \dots, Q_q)], \tau_{i+1}([(P_1, \dots, P_p), (Q_1, \dots, Q_q)]^+)\big)$$

and return to Step 7.

(c) If $Z_i \cup P_1$ is not 3-separating but $q = 0$ and the union of Z_i with $\tau_{i+1}([(P_1, \dots, P_p)]^+)$ is 3-separating, set

$$\tau_i = \big(\tau_{i+1}(Z_i^-), [(P_1, \dots, P_p), (Z_i)], \tau_{i+1}([(P_1, \dots, P_p)]^+)\big)$$

and return to Step 7.

(d) Otherwise, set

$$\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i)], [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_{i+1}([(P_1, \dots, P_p), (Q_1, \dots, Q_q)]^+)\big)$$

and return to Step 7.

(ii) Otherwise, set

$$\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i)], \tau_{i+1}(Z_i^+)\big)$$

and return to Step 7.

9. If Z_i is not 3-separating, do the following:

(i) If $Z_i - \mathrm{fcl}(Z_i^+)$ is 3-separating, set

$$\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i - \mathrm{fcl}(Z_i^+))], Z_i \cap \mathrm{fcl}(Z_i^+), \tau_{i+1}(Z_i^+)\big)$$

and return to Step 7.

(ii) Otherwise, set τ_i to be τ_{i+1} and return to Step 7.

10. **(i)** If X_0 is empty and $\tau_2 = (Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \dots)$, find a 3-separation (U, V) for which U contains P_1 and an element u of Z_1 such that $u \notin \mathrm{fcl}(P_1)$, and V contains $E - (Z_1 \cup P_1)$ and an element v of $Z_1 - u$ such that $v \notin \mathrm{fcl}(E - (Z_1 \cup P_1))$, and do the following:

(a) If there is such a 3-separation, set τ_1 to be

$$\big(Z_1 \cap V, [(Z_1 \cap U, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2([(P_1, \dots, P_p), (Q_1, \dots, Q_q)]^+)\big)$$

and output τ_1.

(b) Otherwise, output τ_2.

(ii) Otherwise, output τ_2.

□

As an example to illustrate the key ideas in 3-Tree, consider the matroid M, and the 3-tree for M shown in Fig. 3. Let $(X, Y, Z) = (X_1 \cup X_2 \cup X_3, Y_1 \cup Y_2 \cup Y_3, Z_1 \cup Z_2 \cup Z_3)$. Suppose that 3-Tree is applied to M. If $(V_2 \cup V_3 \cup V_4, V_1 \cup L \cup X \cup Y \cup Z)$

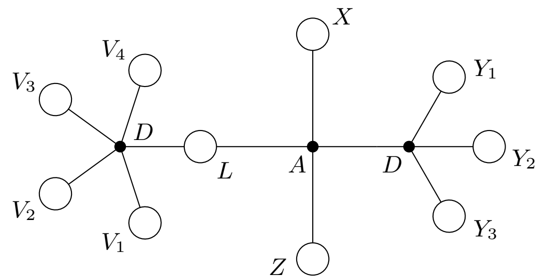

FIGURE 4. The path realization T_1.

is the 3-separation found in Step 3 in 3-TREE, then a possible 3-path outputted by the first call to FORWARDSWEEP is

$$(V_2 \cup V_3, V_4, V_1 \cup L, X, Z, Y_1, Y_2 \cup Y_3).$$

Observe that the 3-path is left-justified and maximal. With this 3-path, a possible generalized 3-path outputted by the immediate subsequent call to BACKWARDSWEEP is

$$(V_3, [(V_2, V_1), (V_4)], L, [(X, Z)], [(Y_1, Y_2)], Y_3).$$

Comparing the 3-path and the generalized 3-path, both $V_2 \cup V_3$ and $Y_2 \cup Y_3$ are split parts. The splitting of $Y_2 \cup Y_3$ and $V_2 \cup V_3$ is the result of end moves performed in Steps 5 and 10 in BACKWARDSWEEP, respectively. The path realization T_1 of this generalized 3-path, produced in Step 3(ii)c in 3-TREE, is shown in Fig. 4, where we note that X and Z are petals of an anemone. The algorithm now starts to repeatedly apply Steps 4 and 5 in 3-TREE.

Since all bag vertices in T_1 are unmarked, Step 5 in 3-TREE selects a bag vertex and, depending upon whether it is a non-terminal or terminal bag, attempts to find a particular type of 3-separation. If there is no such 3-separation, such as when one of the bag vertices labelled V_1, V_2, V_3, V_4, L, Y_1, Y_2, or Y_3 is selected, the bag vertex is marked at Step 5i in 3-TREE. On the other hand, if there is such a 3-separation, such as when one of the bag vertices labelled X or Z is selected, then Step 5ii is invoked and 3-TREE calls FORWARDSWEEP, BACKWARDSWEEP, and then updates the current π-labelled tree. For example, assume the bag vertex labelled X is selected before the bag vertex labelled Z. When this happens, Step 5 in 3-TREE finds an appropriate 3-separation and then calls FORWARDSWEEP using this 3-separation. The subroutine BACKWARDSWEEP is subsequently called and a possible generalized 3-path outputted by this call is

$$(E - X, [(X_1, X_2)], X_3).$$

A path realization of this generalized 3-path is then merged with the current π-labelled tree, in this case T_1, in Step 5(ii)c in 3-TREE to produce the π-labelled tree T_2 shown in Fig. 5. This process continues until all bag vertices are marked. The 3-tree finally outputted by this application of 3-TREE is shown in Fig. 3.

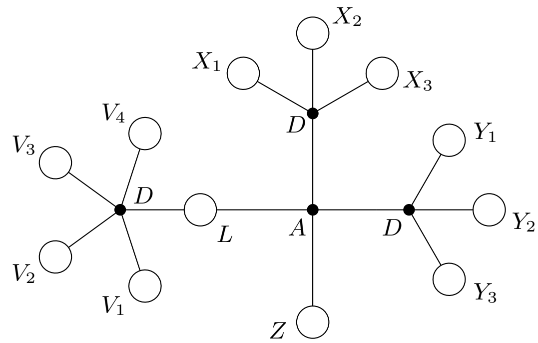

FIGURE 5. The π-labelled tree T_2.

6. Correctness of the Algorithm and the Proof of Theorem 2.2

Let M be a 3-connected matroid with ground set E, where $|E| \geq 9$, and let T be the π-labelled tree outputted by 3-TREE when applied to M. In this section, we prove that T is a 3-tree for M and that this application takes time polynomial in $|E|$.

We begin with several lemmas, the first of which specifies the type of ordered partition outputted by FORWARDSWEEP.

Lemma 6.1. *Let $(X_0 \cup X_1, X_2)$ be a 3-path in M with $X_0 \cup X_1$ fully closed and let $\mathcal{F}$ be the set of maximal sequential 3-separating sets of M. Let $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ be the output of* FORWARDSWEEP *when applied to $(M, (X_0 \cup X_1, X_2), \mathcal{F})$. Then $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ is a left-justified maximal X_0-rooted 3-path of M.*

Proof. By construction, $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ is a left-justified X_0-rooted 3-path. Thus if the lemma fails, then there is a partition (Y_j, Z_j) of X'_j for some j in $[m]$ such that $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1} \cup Y_j, Z_j \cup X'_{j+1} \cup \cdots \cup X'_m)$ is a non-sequential 3-separation of M. We need to show that this 3-separation is equivalent to $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)$ or $(X_0 \cup X'_1 \cup \cdots \cup X'_j, X'_{j+1} \cup \cdots \cup X'_m)$.

If $j = m$, then the result follows immediately from Step 3 of FORWARDSWEEP. Now assume that $j < m$.

Suppose $X_0 = \emptyset$ and $j = 1$. Then, because $(X_0 \cup Y_1, Z_1 \cup X'_2 \cup \cdots \cup X'_m)$ is a non-sequential 3-separation of M, there is a 3-element subset $\{y_1, y_2, y_3\}$ of Y_1 that is not contained in any member of $\mathcal{F}$, and $Z_1 \cup X'_2 \cup \cdots \cup X'_m$ clearly contains $X'_2 \cup \cdots \cup X'_m$. Step 2i of FORWARDSWEEP implies that every element of Z_1 is in $\mathrm{fcl}(X'_2 \cup \cdots \cup X'_m)$ otherwise Step 2(i)b will further refine the 3-path; a contradiction. Hence every element of Z_1 is in $\mathrm{fcl}(Y_1)$ and $(X_0 \cup Y_1, Z_1 \cup X'_2 \cup \cdots \cup X'_m)$ is equivalent to $(X_0 \cup X'_1, X'_2 \cup \cdots \cup X'_m)$, as required.

We may now assume that either $X_0 \neq \emptyset$ or $j \geq 2$. Then, to prevent Steps 2(ii)b and 2(iii)b of ForwardSweep from further refining the 3-path, either every element of Y_j is in $\mathrm{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1})$ or every element of Z_j is in $\mathrm{fcl}(X'_{j+1} \cup \cdots \cup X'_m)$. Hence $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1} \cup Y_j, Z_j \cup X'_{j+1} \cup \cdots \cup X'_m)$ is equivalent to $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)$ or $(X_0 \cup X'_1 \cup \cdots \cup X'_j, X'_{j+1} \cup \cdots \cup X'_m)$, as required. □

In the rest of this section, we freely use Lemma 6.1.

Lemma 6.2. *Let $i \geq 0$, and let T_i and T_{i+1} be π-labelled trees constructed by* 3-Tree *in Steps 3(ii)c and 5(ii)c. Suppose that T_i is a conforming tree for M, and T_{i+1} satisfies (I)-(IV) but is not a conforming tree for M. Let $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ be the* 3*-path outputted when* ForwardSweep *is applied in Step 3(ii)(a) or Step 5(ii)(a) of* 3-Tree *depending on whether $i = 0$ or i is positive. Let (R, G) be a non-sequential* 3*-separation in M that does not conform with T_{i+1} for which X_0 is monochromatic and no equivalent* 3*-separation in which X_0 is monochromatic has fewer bichromatic parts in $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$. Then $X_0 \cup X'_1$ is monochromatic unless $i = 0$. In the exceptional case, either X'_1 is monochromatic, or both $R \cap X'_1$ and $G \cap X'_1$ are sequential* 3*-separating sets with $|R \cap X'_1|, |G \cap X'_1| \geq 2$.*

Proof. Assume that $X_0 \cup X'_1$ is bichromatic. First suppose that $i \geq 1$. Then X_0 is non-empty. Then, as X_0 is monochromatic, we may assume that $X_0 \subseteq G$. Furthermore, as (R, G) does not conform with T_{i+1}, we have $|R \cap (X'_2 \cup \cdots \cup X'_m)| \geq 1$. Since $X_0 \cup X'_1$ is bichromatic, it follows by Lemma 3.9 that $|R \cap (X'_2 \cup \cdots \cup X'_m)| \geq 2$.

Since G and $X_0 \cup X'_1$ are both 3-separating and $|R \cap (X'_2 \cup \cdots \cup X'_m)| \geq 2$, it follows by uncrossing that $G \cap (X_0 \cup X'_1)$, which equals $X_0 \cup (G \cap X'_1)$, is 3-separating. Therefore $(X_0 \cup (G \cap X'_1), (R \cap X'_1) \cup X'_2 \cup \cdots \cup X'_m)$ is a 3-separation in M. If this 3-separation is non-sequential, then, by Lemma 6.1, it is equivalent to $(X_0 \cup X'_1, X'_2 \cup \cdots \cup X'_m)$ and so $R \cap X'_1 \subseteq \mathrm{fcl}(G)$. In this case, we recolour all the elements in $R \cap X'_1$ green thereby reducing the number of bichromatic parts; a contradiction. Therefore either $X_0 \cup (G \cap X'_1)$ or $(R \cap X'_1) \cup X'_2 \cdots \cup X'_m$ is sequential. By Lemma 3.3, the last set is not sequential as $X'_2 \cup X'_3 \cup \cdots \cup X'_m$ is non-sequential. Thus $X_0 \cup (G \cap X'_1)$ is sequential. But, as $i \geq 1$, the set X_0 contains at least one non-sequential 3-separation, contradicting Lemma 3.3.

Now suppose that $i = 0$. Then X_0 is empty. If $R \cap X'_1 = \{z\}$, then $|R \cap (E - X'_1)| \geq 2$ and so, as G and X'_1 are both 3-separating, by uncrossing, $G \cap X'_1$ is 3-separating. Therefore, as X'_1 is 3-separating, it follows by Lemma 3.1 that $z \in \mathrm{cl}^{(*)}(G \cap X'_1)$. Thus we can recolour z green thereby reducing the number of bichromatic parts; a contradiction. Hence $|R \cap X'_1| \geq 2$ and, by symmetry, $|G \cap X'_1| \geq 2$. If $R \cap (E - X'_1)$ is empty, then, as $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ is a maximal X_0-rooted 3-path, (R, G) is equivalent to $(X'_1, E - X'_1)$. Hence $G \cap X'_1 \subseteq \mathrm{fcl}(R)$ and so we can recolour the elements in $G \cap X'_1$ red, reducing the number of bichromatic parts; a contradiction. Thus $|R \cap (E - X'_1)| \geq 1$ and so, by Lemma 3.9, $|R \cap (E - X'_1)| \geq 2$. Similarly, $|G \cap (E - X'_1)| \geq 2$. It now follows by uncrossing that both $G \cap X'_1$ and $R \cap X'_2$ are 3-separating.

Consider the 3-separation $(G \cap X'_1, E - (G \cap X'_1))$. If this 3-separation is non-sequential, then, by Lemma 6.1, it is equivalent to $(X'_1, E - X'_1)$ and so $R \cap X'_1 \subseteq \mathrm{fcl}(G \cap X'_1) \subseteq \mathrm{fcl}(G)$. Thus we can recolour all the elements in $R \cap X'_1$ green thereby reducing the number of bichromatic parts; a contradiction. Hence either $G \cap X'_1$ or $E - (G \cap X'_1)$ is sequential. As $E - (G \cap X'_1)$ contains the non-sequential set $X'_2 \cup X'_3 \cup \cdots \cup X'_m$, it follows by Lemma 3.3 that $G \cap X'_1$ is sequential. By symmetry, $R \cap X'_1$ is sequential, and the lemma follows. $\square$

Lemma 6.3. *The π-labelled tree T outputted by* 3-Tree *is a conforming tree for M. Furthermore, if v is a flower vertex of T, then the flower corresponding to v is tight.*

Proof. Let E denote the ground set of M. We prove the lemma by showing that each of the π-labelled trees T_p constructed in Steps 3(ii)c and 5(ii)c in 3-Tree is a conforming tree for M in which the flower corresponding to each flower vertex is tight. Since T_0 consists of a single bag vertex labelled E, the result trivially holds if $p = 0$. Now suppose that T_p is a conforming tree for M with the property that if v is a flower vertex of T_p, then the flower corresponding to v is tight. We will show that T_{p+1} is a conforming tree for M with this additional property on its flower vertices.

It follows by induction, Lemma 6.1, and the construction in BackwardSweep that T_{p+1} satisfies (I) in the definition of a conforming tree. Furthermore, T_{p+1} trivially satisfies (II) in this definition. To see that (III) and (IV) holds for T_{p+1}, let $\Phi = (Q_1, Q_2, \ldots, Q_k)$ be a flower in M corresponding to a flower vertex v in the path realization of the generalized 3-path outputted by BackwardSweep in the construction of T_{p+1} from T_p. By induction, to show that (III) and (IV) holds for T_{p+1}, it suffices to show that v satisfies either (III) or (IV) depending upon whether it is labelled A or D, respectively. Without loss of generality, we may assume that, relative to this generalized 3-path, Q_1 is the entry petal. By construction, each petal of Φ is 3-separating and, apart from at most one of $Q_1 \cup Q_2$ and $Q_1 \cup Q_k$, each pair of consecutive petals is 3-separating. Thus, by symmetry, it suffices to check that $Q_1 \cup Q_2$ is 3-separating. This check is done by induction by showing, for all i in $\{3, 4, \ldots, k\}$, that $Q_3 \cup Q_4 \cup \cdots \cup Q_i$ is 3-separating. In particular, this will show that $Q_3 \cup Q_4 \cup \cdots \cup Q_k$ is 3-separating, and so $Q_1 \cup Q_2$ is 3-separating. Clearly, Q_3 and $Q_3 \cup Q_4$ are 3-separating. Now let $i \geq 5$ and assume that the check holds for $i - 1$. Then $Q_3 \cup Q_4 \cup \cdots \cup Q_{i-1}$ and $Q_{i-1} \cup Q_i$ are 3-separating. Therefore, as their intersection contains at least two elements, it follows by uncrossing that their union $Q_3 \cup Q_4 \cup \cdots \cup Q_i$ is 3-separating, and we get the desired result.

To complete the proof that T_{p+1} is a conforming tree for M, suppose there is a non-sequential 3-separation (R', G') that does not conform with T_{p+1}. Because this 3-separation does conform with T_p, it is equivalent to a 3-separation (R, G) such that R or G is contained in a bag of T_p. Only one bag of T_p is affected in the construction of T_{p+1}, so we may assume that R or G is contained in this bag B. As $X_0 = E - \pi(B)$, which may be empty, we deduce that, with respect to (R, G), the set X_0 is monochromatic. Thus (R, G) is a non-sequential 3-separation that does not conform with T_{p+1} and has X_0 monochromatic. From among the collection of choices for (R, G) satisfying these conditions, choose one such that no equivalent 3-separation in which X_0 is monochromatic has fewer bichromatic parts with respect to the left-justified maximal X_0-rooted 3-path $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ outputted by ForwardSweep during the construction of T_{p+1} from T_p. By Lemma 6.2, we may further assume that if $p \geq 1$, then $X_0 \cup Z_1$ is monochromatic and, if $p = 0$, in which case X_0 is empty, either Z_1 is monochromatic, or $|R \cap Z_1|, |G \cap Z_1| \geq 2$ and each of $R \cap Z_1$ and $G \cap Z_1$ is a sequential 3-separating set.

First suppose that $X_0 \cup Z_1$ is monochromatic. Without loss of generality, we may assume that $X_0 \cup Z_1 \subseteq G$. Let b be the number of bichromatic parts amongst $Z_2, \ldots, Z_m$. Assume $b \geq 2$ and let Z_i be the bichromatic part with smallest subscript. If $Z_i^- \cap R$ is non-empty, then, by Lemmas 3.8 and 3.9, Z_i is monochromatic; a contradiction. Therefore $Z_i^- \subseteq G$. But then, by Lemma 3.11, Z_i^+ is monochromatic; a contradiction as there is a bichromatic part Z_j with $j > i$. Thus $b \in \{0, 1\}$.

Assume $b = 1$ and Z_i is bichromatic. We first consider $i \neq m$. If Z_i^+ is not monochromatic, then, by Lemma 3.11, Z_i^- is not monochromatic. Therefore, by Lemma 3.9, $|R \cap Z_i^-|, |G \cap Z_i^-|, |R \cap Z_i^+|, |G \cap Z_i^+| \geq 2$. But then, by Lemma 3.8, Z_i is monochromatic; a contradiction. Thus we may assume that Z_i^+ is monochromatic.

We next eliminate a special case. Say $Z_i^-, Z_i^+ \subseteq G$. Then $R \subseteq Z_i$. The only steps in BackwardSweep that do not leave Z_i intact are Steps 6i (if $i = m-1$) and 9i. As (R, G) does not conform with T_{p+1}, we may assume that one of these is invoked. Then both $R \cap (Z_i - \mathrm{fcl}(Z_i^+))$ and $R \cap (Z_i \cap \mathrm{fcl}(Z_i^+))$ are non-empty. But, as $R \cap (Z_i \cap \mathrm{fcl}(Z_i^+)) \subseteq \mathrm{fcl}(Z_i^+)$, it follows that $R \cap (Z_i \cap \mathrm{fcl}(Z_i^+)) \subseteq \mathrm{fcl}(G)$. Therefore we can recolour all the elements in $R \cap (Z_i \cap \mathrm{fcl}(Z_i^+))$ green thereby obtaining an equivalent 3-separation in which all the red elements are all in $Z_i - \mathrm{fcl}(Z_i^+)$, a single bag of T_{p+1}. It now follows that if $Z_i^+ \subseteq G$, then $Z_i^- \cap R$ is non-empty.

Consider the case when $Z_i^+ \subseteq R$. If $Z_i^- \subseteq G$, then, by Lemma 3.11, $Z_i^+ \subseteq G$; a contradiction. Therefore $Z_i^- \cap R \neq \emptyset$ and so, by Lemma 3.9, $|Z_i^- \cap R| \geq 2$. Now, by Lemma 3.15, $R \cap Z_i \subseteq \mathrm{fcl}(Z_i^+)$. Furthermore, by recolouring if necessary, we may assume that $R \cap Z_i = Z_i \cap \mathrm{fcl}(Z_i^+)$. Since $Z_i \cup Z_i^+$ and G are both 3-separating, and since $|Z_i^- \cap R| \geq 2$, it follows by uncrossing that $G \cap Z_i$ is 3-separating. Furthermore, by Lemma 3.10, Z_i is not 3-separating. Therefore the generalized 3-path τ_i at the end of the iteration of BackwardSweep in which Z_i is considered is

$$\tau_i = \big(X_0 \cup Z_1, Z_2, \ldots, Z_{i-1}, [(Z_i - \mathrm{fcl}(Z_i^+))], Z_i \cap \mathrm{fcl}(Z_i^+), \tau_{i+1}(Z_i^+)\big).$$

Now $Z_i - \mathrm{fcl}(Z_i^+) \subseteq G$ and $(Z_i \cap \mathrm{fcl}(Z_i^+)) \cup Z_i^+ \subseteq R$. Let h be the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Since $X_0 \cup Z_1 \subseteq G$ and $|R \cap Z_i^-| \geq 2$, we have $2 \leq h \leq i-1$. By applying Lemma 3.12 to the 3-path $(Z_h^-, Z_h, Z_{h+1}, \ldots, Z_{i-1}, Z_i - \mathrm{fcl}(Z_i^+), (Z_i \cap \mathrm{fcl}(Z_i^+)) \cup Z_i^+)$, we deduce that M has a flower in which the parts of the 3-path are petals of a flower. It now follows by Lemma 3.12 and the construction in BackwardSweep that T_{p+1} displays (R, G), and so (R, G) conforms with T_{p+1}. This contradiction implies that we may assume $Z_i^+ \subseteq G$.

The case when $Z_i^+ \subseteq G$ is handled similarly to that when $Z_i^+ \subseteq R$. Note that $Z_i^- \cap R$ is non-empty as a result of the consideration of the above special case.

Now suppose that $i = m$. If Z_m^- is monochromatic, that is, $Z_m^- \subseteq G$, then either $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ is not left-justified or it is not maximal; a contradiction. Therefore Z_m^- is not monochromatic, and so $m \geq 3$. Furthermore, as $|G \cap Z_m^-| \geq 2$ and both Z_m and R are 3-separating, uncrossing implies that $R \cap Z_m$ is 3-separating. Therefore if $|G \cap Z_m| = 1$, then $Z_m \subseteq \mathrm{fcl}(R \cap Z_m)$ by Lemma 3.1, and so we can recolour the element of $G \cap Z_m$ red to obtain a 3-separation equivalent to (R, G) with fewer bichromatic parts; a contradiction. Thus $|G \cap Z_m| \geq 2$. A similar argument shows that $|R \cap Z_m| \geq 2$.

We show next that Z_{m-1} is 3-separating. Say $Z_{m-1} \subseteq R$. Then, as R and $Z_{m-1} \cup Z_m$ are both 3-separating and $|G \cap Z_{m-1}^-| \geq 2$, it follows that $R \cap (Z_{m-1} \cup Z_m)$ is 3-separating. Therefore, as Z_m^- is 3-separating and $|G \cap Z_m| \geq 2$, it follows by uncrossing again that Z_{m-1} is 3-separating. Using the fact that Z_m^- is not monochromatic, the same argument shows that if $Z_{m-1} \subseteq G$, then Z_{m-1} is 3-separating. Thus Step 5 in BackwardSweep is invoked. Furthermore, as $Z_{m-1} \cup (R \cap Z_m)$ is a non-sequential 3-separating set if $Z_{m-1} \subseteq R$ and, similarly, $Z_{m-1} \cup (G \cap Z_m)$ is a non-sequential 3-separating set if $Z_{m-1} \subseteq G$, it follows that Step 5 finds a 3-separation (U, V) as described in that step. By Lemma 3.14, $R \cap Z_m$ and $G \cap Z_m$ are sequential 3-separating sets. Hence, by Lemma 3.13, we may assume, by recolouring if necessary, that both $U \cap Z_m$ and $V \cap Z_m$ are monochromatic. Let h denote the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Then, by Lemma 3.12, M has a flower with petals $Z_h^-, Z_h, Z_{h+1}, \ldots, Z_{m-1}, U \cap Z_m, V \cap Z_m$. Thus, by Lemma 3.12 and the construction in BackwardSweep, T_{p+1} displays (R, G), and so (R, G) conforms with T_{p+1}; a contradiction.

Now assume $b = 0$. Let h denote the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Say $Z_h \cup Z_h^+$ is not monochromatic. Let h' denote the largest index for which $Z_{h'} \cup Z_{h'}^+$ is not monochromatic, but $Z_{h'}^+$ is monochromatic. Note that $h' \geq h$. Then it follows by Lemma 3.12 that each of the sets $Z_h, Z_{h+1}, \ldots, Z_{h'}$ is 3-separating and so, by the construction in BackwardSweep and Lemma 3.12, T_{p+1} displays (R, G) as the petals of a flower; a contradiction. Now say $Z_h \cup Z_h^+$ is monochromatic. It follows from the construction in BackwardSweep that the only way in which (R, G) does not conform with T_{p+1} is when $h \geq 3$ and Step 9i of BackwardSweep is invoked when Z_{h-1} is considered. But then we can recolour all the elements in $Z_{h-1} \cap \mathrm{fcl}(Z_h \cup Z_h^+)$ red giving a 3-separation equivalent to (R, G), thereby resulting in T_{p+1} displaying (R, G); a contradiction. This completes the analysis for when $X_0 \cup Z_1$ is monochromatic.

Suppose that $p = 0$ and Z_1 is bichromatic. Recall that X_0 is empty and that $|R \cap Z_1|, |G \cap Z_1| \geq 2$ and each of $R \cap Z_1$ and $G \cap Z_1$ is a sequential 3-separating set. Let b denote the number of bichromatic parts amongst $Z_1, \ldots, Z_m$. By Lemmas 3.8 and 3.9, $b \in \{1, 2\}$. First assume that $b = 2$, and let Z_i denote the bichromatic part with $i > 1$. Say $i \neq m$. By Lemmas 3.8 and 3.9, Z_i^+ is monochromatic. Without loss of generality, we may assume that $Z_i^+ \subseteq R$. By Lemma 3.10, Z_i is not 3-separating. Furthermore, by Lemma 3.15, $R \cap Z_i \subseteq \mathrm{fcl}(Z_i^+)$. By recolouring if necessary and moving to a 3-separation equivalent to (R, G), we may assume that $R \cap Z_i = Z_i \cap \mathrm{fcl}(Z_i^+)$. Since $|R \cap Z_i^-| \geq 2$ and since both G and $Z_i \cup Z_i^+$ are 3-separating, it follows by uncrossing that $G \cap Z_i$, which equals $Z_i - \mathrm{fcl}(Z_i^+)$, is 3-separating. Thus, by the construction in BackwardSweep, the generalized 3-path τ_i at the end of the iteration in which Z_i is considered is

$$\tau_i = \big(Z_1, Z_2, \dots, Z_{i-1}, [(Z_i - \mathrm{fcl}(Z_i^+))], Z_i \cap \mathrm{fcl}(Z_i^+), \tau_{i+1}(Z_i^+)\big).$$

Now $Z_i - \mathrm{fcl}(Z_i^+) \subseteq G$ and $(Z_i \cap \mathrm{fcl}(Z_i^+)) \cup Z_i^+ \subseteq R$ and so, by Lemma 3.12, M has a flower with petals $Z_1, Z_2, \dots, Z_{i-1}, Z_i - \mathrm{fcl}(Z_i^+), (Z_i \cap \mathrm{fcl}(Z_i^+)) \cup Z_i^+$. By the construction in BackwardSweep and Lemma 3.12, τ_2 is eventually constructed and is of the form

$$\tau_2 = (Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], (Z_i \cap \mathrm{fcl}(Z_i^+)) \cup Z_i^+),$$

where $\{P_1, \dots, P_p, Q_1, \dots, Q_q\} = \{Z_2, \dots, Z_{i-1}, Z_i - \mathrm{fcl}(Z_i^+)\}$. Thus Step 10i is invoked. As the second petal on the last list is monochromatic, it follows by uncrossing that Step 10i finds a 3-separation (U, V) as described in that step. By Lemma 3.13, we may assume that both $U \cap Z_1$ and $V \cap Z_1$ are monochromatic. Thus, by Lemma 3.12 again, it follows that M has a flower with petals

$$V \cap Z_1, U \cap Z_1, Z_2, \dots, Z_{i-1}, Z_i - \mathrm{fcl}(Z_i^+), (Z_i \cap \mathrm{fcl}(Z_i^+)) \cup Z_i^+.$$

Therefore, by Lemma 3.12 and construction, (R, G) is displayed by T_{p+1}; a contradiction.

Now say $i = m$. Since $|G \cap Z_1| \geq 2$ and both R and Z_m are 3-separating, it follows by uncrossing that $R \cap Z_m$ is 3-separating. Therefore if $|G \cap Z_m| = 1$, then $Z_m \subseteq \mathrm{fcl}(R \cap Z_m)$ by Lemma 3.1. Thus we can recolour the single green element in Z_m red thereby obtaining an equivalent 3-separation with fewer bichromatic parts; a contradiction. Hence $|G \cap Z_m| \geq 2$ and, by symmetry, $|R \cap Z_m| \geq 2$.

There are two cases depending upon whether $m = 2$ or $m \geq 3$. If $m \geq 3$, then, without loss of generality, we may assume that $Z_{m-1} \subseteq R$. Since $|G \cap Z_1| \geq 2$ and since R and $Z_{m-1} \cup Z_m$ are both 3-separating, it follows by uncrossing that $R \cap (Z_{m-1} \cup Z_m)$ is 3-separating. Therefore, as Z_m^- is 3-separating and $|G \cap Z_m| \geq 2$, an application of uncrossing implies that Z_{m-1} is 3-separating. Thus Step 5 of BackwardSweep is invoked. Since $Z_{m-1} \cup (R \cap Z_m)$ is a 3-separating set, it follows that Step 5 of BackwardSweep finds a 3-separation (U, V) as described in that step. By Lemma 3.14, $R \cap Z_m$ and $G \cap Z_m$ are sequential 3-separating sets. Therefore, by Lemma 3.13, we can recolour some elements of Z_m if necessary to get an equivalent 3-separation in which both $U \cap Z_m$ and $V \cap Z_m$ are monochromatic. Lemma 3.12 now implies that M has a flower with petals $Z_1, Z_2, \dots, Z_{m-1}, U \cap Z_m, V \cap Z_m$. By the construction in BackwardSweep and Lemma 3.12, τ_2 is eventually constructed and is of the form

$$\tau_2 = (Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], W \cap Z_m),$$

where $\{P_1, \dots, P_p, Q_1, \dots, Q_q, W \cap Z_m\} = \{Z_2, \dots, Z_{m-1}, U \cap Z_m, V \cap Z_m\}$ and $W \in \{U, V\}$. Thus Step 10i is invoked. As Z_2 is monochromatic, it follows by uncrossing that Step 10i finds a 3-separation (U', V') as described in that step. By Lemma 3.13, we may assume that $U' \cap Z_1$ and $V' \cap Z_1$ are monochromatic. Therefore, by Lemma 3.12, it follows that M has a flower with petals

$$V' \cap Z_1, U' \cap Z_1, Z_2, \dots, Z_{m-1}, U \cap Z_m, V \cap Z_m.$$

Thus, by Lemma 3.12 and construction, (R, G) is displayed by T_{p+1}; a contradiction. A similar analysis holds when $m = 2$, where we invoke Step 2 instead of Steps 5 and 10 in BackwardSweep.

Now assume that $b = 1$. Then Z_1 is the only bichromatic part. Since $R \cap Z_1$ and $G \cap Z_1$ are sequential 3-separating sets, Z_1^+ is not monochromatic. So $m \geq 3$. Let h denote the largest index for which $Z_h \cup Z_h^+$ is not monochromatic, but Z_h^+ is monochromatic. By Lemma 3.12, M has a flower with petals $Z_1, Z_2, \ldots, Z_h, Z_h^+$. Therefore, by the construction and Lemma 3.12, τ_2 is eventually constructed and is of the form

$$\tau_2 = (Z_1, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], Z_h^+),$$

where $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} = \{Z_2, \ldots, Z_h\}$. Thus Step 10i is invoked. Since Z_2 is monochromatic, it follows by uncrossing that Step 10i finds a 3-separation (U, V) as described in that step. By Lemma 3.13, we may assume that $U \cap Z_1$ and $V \cap Z_1$ are monochromatic. Thus, by Lemma 3.12, M has a flower with petals $V \cap Z_1, U \cap Z_1, Z_2, \ldots, Z_h, Z_h^+$. Thus, by Lemma 3.12 and construction, T_{p+1} displays (R, G); a contradiction. We conclude that T_{p+1} is a conforming tree for M.

We next show that if v is a flower vertex of T_{p+1}, then the flower corresponding to v is tight. By induction, T_p has this property on its flower vertices. Therefore, by construction, it suffices to only consider the flower vertices on the path realization, P_{p+1} say, of the generalized 3-path outputted by BackwardSweep in Step 5 of 3-Tree in the construction of T_{p+1} from T_p. Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be the left-justified maximal X_0-rooted 3-path outputted by ForwardSweep in the construction of T_{p+1} from T_p of 3-Tree. Let v be a flower vertex of P_{p+1} and suppose that Φ, the flower corresponding to v, is not tight. By definition, we may assume that v has degree at least three. For clarity, we will assume that Step 9i in BackwardSweep is not invoked in the construction of Φ. The straightforward extension of the proof below to include the case when this step is invoked is omitted.

It follows from the description of BackwardSweep that if no end moves are performed, then, for some i and j with $1 \leq i \leq j \leq m$, the sets X_i^- and X_j^+ are the entry and exit petals of Φ, respectively, and $\{X_i, X_{i+1}, \ldots, X_j\}$ is the union of the sets of clockwise and anticlockwise petals of Φ. If end moves are performed, then either $X_i^- \cup X_i = X_0 \cup X_1$, or $X_j \cup X_j^+ = X_m$. Ignoring the possibility of end moves for now, if X_i^- is loose, then $X_i^- \subseteq \text{fcl}(X_i \cup X_i^+)$, and so $(X_i^-, X_i \cup X_i^+)$ is sequential; a contradiction. Similarly, we get a contradiction if X_j^+ is loose. Now assume that, for some $i \leq s \leq j$, the petal X_s is loose. Without loss of generality, we may assume that X_{s-1} is tight where, if $s = i$, we take X_{s-1} to be X_i^-. By Lemma 3.6, $X_s \subseteq \text{fcl}(X_{s-1})$ and so $X_s \subseteq \text{fcl}(X_s^-)$. But then $(X_s^-, X_s \cup X_s^+)$ is equivalent to $(X_s^- \cup X_s, X_s^+)$, contradicting that $(X_0 \cup X_1, X_2, \ldots, X_m)$ is a 3-path.

Now consider the possibility of end moves. If $X_i^- \cup X_i = X_0 \cup X_1$, then Step 10i of BackwardSweep is invoked, in which case, X_i^- and X_i are both sequential. Say X_i^- is loose. By concatenating the petals $X_{i+1}, \ldots, X_m$ into a single petal, $X_{i+1} \cup \cdots \cup X_m$ is a tight petal in the resulting flower, while X_i^- remains loose. Thus, by Lemma 3.6, $X_i^- \subseteq \text{fcl}(X_{i+1} \cup \cdots \cup X_m)$. Therefore, by Lemma 3.7, there is an ordering $x_1, x_2, \ldots, x_l$ of the elements of X_i^- such that $(X_i, x_l, x_{l-1}, \ldots, x_1, X_{i+1} \cup \cdots \cup X_m)$ is a 3-sequence in M. But X_i is sequential and it follows that $X_i^- \cup X_i = X_0 \cup X_1$ is sequential; a contradiction. Hence X_i^- is tight and, similarly, X_i is tight. The case $X_j \cup X_j^+ = X_m$ is handled analogously. We conclude that if v is a flower vertex of T_{p+1}, then the flower corresponding to v is tight. This completes the proof of the lemma. □

It follows by Lemma 6.3 that T is a conforming tree for M. The following is a straightforward consequence of the way in which flowers are constructed in the algorithm.

Lemma 6.4. *The conforming tree T for M outputted by* 3-Tree *has the property that every flower corresponding to a flower vertex in T displays at least two inequivalent non-sequential* 3*-separations.*

Proof. First note that, by construction, all flower vertices in T have degree at least three. Now, except when we invoke an end move, every flower that is constructed in the algorithm has an entry petal and an exit petal and these correspond to inequivalent non-sequential 3-separations. When an end move is invoked, we already have one non-sequential 3-separation and it is easily checked that there is a second inequivalent one (U, V) with the split part, or parts in the case $m = 2$, having non-empty intersection with U and V. □

Lemma 6.5. *The conforming tree T for M outputted by* 3-Tree *has the property that every flower corresponding to a flower vertex in T is a tight maximal flower.*

Proof. Let Φ be a flower corresponding to a flower vertex in T. By Lemma 6.3, Φ is tight. Assume that Φ is not maximal. Then there is a tight maximal flower Φ_m that displays, up to equivalence, all non-sequential 3-separations that are displayed by Φ as well as at least one non-sequential 3-separation (R, G) that, up to equivalence, is not displayed by Φ. In particular:

6.5.1. *For every union U of petals of Φ such that $(U, E - U)$ is a non-sequential* 3*-separation in M, there is a union U' of petals of Φ_m such that $(U, E - U) \cong (U', E - U')$.*

We may assume that $\Phi_m = (Q_1, Q_2, \ldots, Q_n)$ and that $R = Q_1 \cup Q_2 \cup \cdots \cup Q_k$ for some $k \leq n-1$. As (R, G) is not displayed by Φ, an equivalent 3-separation (R', G') must conform with T. Hence we may assume that R' is properly contained in some petal P of Φ. Then, by Lemma 3.3, P is non-sequential. If $E - P$ is sequential, then it follows by Lemma 3.3 that Φ displays no non-sequential 3-separations; a contradiction. Hence $(P, E - P)$ is non-sequential and Φ_m displays an equivalent 3-separation $(P', E - P')$. Thus $(P', E - P') = (\cup_{i \in I} Q_i, \cup_{j \in [n]-I} Q_j)$ for some subset I of $[n]$. Suppose $|[n] - I| = 1$. By Lemma 6.4, Φ displays a non-sequential 3-separation $(O, E - O)$ that is not equivalent to $(P, E - P)$. As P is a petal of Φ, we must have that $\mathrm{fcl}(P)$ is a proper subset of $\mathrm{fcl}(O)$ or $\mathrm{fcl}(E - O)$. Some 3-separation $(O', E - O')$ equivalent to $(O, E - O)$ is displayed by Φ_m. Since Φ_m has only one petal outside of P', (6.5.1) implies that O' or $E - O'$ is contained in P'. Hence

$\text{fcl}(P')$ contains $\text{fcl}(O')$ or $\text{fcl}(E-O')$, so $\text{fcl}(P)$ contains $\text{fcl}(O)$ or $\text{fcl}(E-O)$; a contradiction. Thus $|[n]-I| \geq 2$.

Since $\text{fcl}(R) = \text{fcl}(Q_1 \cup Q_2 \cup \cdots \cup Q_k) = \text{fcl}(R') \subseteq \text{fcl}(P') = \text{fcl}(\cup_{i\in I} Q_i)$ and Φ_m is a tight flower, it follows that $[k] \subseteq I$. Moreover, I must contain at least one element not in $[k]$ since no 3-separation equivalent to (R,G) is displayed by Φ. Thus we may assume that $I = \{n-s+1,\ldots,n,1,2,\ldots,k,k+1,\ldots,k+t\}$ where $s > 0$ and $k+t \leq n-s-2$. Now let $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_{k+t+1}$. This is a union of consecutive petals of Φ_m that contains at least two petals and avoids at least two petals. Hence, by Lemma 3.4(ii) $(Q, E-Q)$ is a non-sequential 3-separation in M. Thus $(Q, E-Q)$ is equivalent to a 3-separation $(Q', E-Q')$ that conforms with T. Hence either

(i) $(Q', E-Q')$ is displayed by Φ; or
(ii) Q' or $E-Q'$ is contained in a petal of Φ.

Let $\Phi = (P_1, P_2, \ldots, P_m)$. Recall that $\text{fcl}(P) = \text{fcl}(P') = \text{fcl}(\cup_{i\in I} Q_i)$ where $I = \{n-s+1,\ldots,n,1,2,\ldots,k+t\}$. Suppose first that (i) holds. Then we may assume that $Q' = \cup_{i\in K} P_i$ for some proper subset K of $[m]$. Now $\text{fcl}(E-Q') = \text{fcl}(Q_{k+t+2} \cup \cdots \cup Q_n)$ so $\text{fcl}(E-Q')$ does not contain Q_1; otherwise, by Lemma 3.4(i), Q_1 is loose. But $Q_1 \subseteq \text{fcl}(P)$ so $P \in \{P_i : i \in K\}$. Then $Q_n \subseteq \text{fcl}(P) \subseteq \text{fcl}(\cup_{i\in K} P_i) = \text{fcl}(Q') = \text{fcl}(Q_1 \cup Q_2 \cup \cdots \cup Q_{k+t+1})$. It follows by Lemma 3.4(i) that Q_n is loose; a contradiction. We deduce that (i) does not hold so (ii) holds.

Assume that $Q' \subseteq P_1$. Then $Q_{k+t+1} \subseteq \text{fcl}(Q') \subseteq \text{fcl}(P_1)$. But $Q_{k+t+1} \not\subseteq \text{fcl}(P)$, otherwise, by Lemma 3.4(i), Q_{k+t+1} is loose. So $P \neq P_1$. Now, as $Q' \subseteq P_1$ and $R' \subseteq P \subseteq E - P_1$, it follows by Lemma 3.3 that $(P_1, E-P_1)$ is non-sequential. Thus, by (6.5.1), there is a union $\cup_{j\in J} Q_j$ of petals of Φ_m such that $(P_1, E-P_1) \cong (\cup_{j\in J} Q_j, \cup_{j\in[n]-J} Q_j)$. Now $Q_1 \subseteq \text{fcl}(Q') \subseteq \text{fcl}(P_1) = \text{fcl}(\cup_{j\in J} Q_j)$ and $Q_1 \subseteq \text{fcl}(P) \subseteq (E-P_1) \subseteq \text{fcl}(\cup_{j\in[n]-J} Q_j)$. Thus we have a contradiction to Corollary 3.5.

We may now assume that $E-Q' \subseteq P_1$. Suppose first that $P \neq P_1$. Then $P \subseteq Q'$, so $Q_n \subseteq \text{fcl}(P) \subseteq \text{fcl}(Q_1 \cup Q_2 \cup \cdots \cup Q_{k+t+1})$. Hence, by Lemma 3.4(i), Q_n is loose; a contradiction. We deduce that $P = P_1$. Recall that $k+t \leq n-s-2$. Thus we have $Q_{k+t+2} \subseteq \text{fcl}(E-Q') \subseteq \text{fcl}(P) = \text{fcl}(Q_{n-s+1} \cup \cdots \cup Q_n \cup Q_1 \cup \cdots Q_{k+t})$, so, by Lemma 3.4(i) again, Q_{k+t+2} is loose; a contradiction. □

Proof of Theorem 2.2. To prove the theorem, we show that 3-Tree is a polynomial-time algorithm for finding a 3-tree for M. Let T be the tree outputted by an application of 3-Tree to M. Then every vertex of T is marked. Moreover, by Lemmas 6.3 and 6.5, T is a partial 3-tree for M. Now T is a 3-tree for M unless there is a non-sequential 3-separation of M with the property that no equivalent 3-separation is displayed by T. So assume there is such a 3-separation (R,G). Since T is conforming, we may assume, by taking an equivalent 3-separation if necessary, that G is contained in a bag B of T. If T consists of the single bag vertex B, then Step 3 of 3-Tree would have found a non-sequential 3-separation (Y,Z) of M; a contradiction. So assume that T consists of at least two vertices. Then Step 5 of 3-Tree would have found a non-sequential 3-separation (Y, Z) of M with the property that $Z \subseteq \pi(B)$, contradicting the fact that B is marked. Hence T is a 3-tree for M.

We next show that 3-Tree runs in polynomial time in the size n of $|E(M)|$. We showed in Section 4 that the collection $\mathcal{F}$ of maximal sequential 3-separating sets of M can be constructed in polynomial time in n and that, for fixed disjoint subsets Y_1 and Z_1 of $E(M)$, we can find a 3-separation (Y, Z) with $Y_1 \subseteq Y$ and $Z_1 \subseteq Z$, if one exists, in polynomial time in n. Extending this, we see that whenever 3-Tree is called upon to find a particular type of 3-separation, it either finds such a 3-separation or correctly determines that there is no such 3-separation in time polynomial in n. Therefore, as every 3-path of M has length $O(n)$, it follows by Lemma 6.1 that each complete call from 3-Tree to ForwardSweep takes time polynomial in n. Now consider a call from 3-Tree to the subroutine BackwardSweep. Starting with Z_m, each iteration of BackwardSweep considers either a subset Z_i of $E(M)$ where $i \in \{2, 3, \ldots, m\}$, or the subset $X_0 \cup Z_1$ of $E(M)$. In the cases of Z_m and $X_0 \cup Z_1$, BackwardSweep determines if there is a 3-separation (U, V) with each of U and V containing certain subsets of $E(M)$. As above, it follows that the time taken for BackwardSweep to consider each of Z_m and $X_0 \cup Z_1$ is polynomial in n. For each of the subsets $Z_2, Z_3, \ldots, Z_{m-1}$, it is clear that their consideration is also polynomial time in n. Note that finding the full closure of a subset X of $E(M)$ as in Step 9 of BackwardSweep takes time $O(n^2)$. Since $m \leq n$, it follows that each complete call from 3-Tree to BackwardSweep takes time polynomial in n. At the completion of each call to BackwardSweep, the algorithm 3-Tree extends the current π-labelled tree to a new π-labelled tree in polynomial time in n. This extension is non-trivial in that at least one new edge is created. Since the terminal bags of each such constructed π-labelled tree contain at least two elements of $E(M)$ and there is no empty bag vertex of degree two, the number of edges of each constructed π-labelled tree is linear in n, and so the total number of calls to ForwardSweep and BackwardSweep from 3-Tree is $O(n)$. As marked bags are never reconsidered, we deduce that 3-Tree terminates in time polynomial in n. This completes the proof of the theorem. $\square$

7. An Alternative Approach

The algorithm implicit in [6] for finding a 3-tree for a 3-connected matroid M with at least nine elements begins by constructing a tight maximal flower Φ for M and uses the fact that Φ is a partial 3-tree. This partial 3-tree is then modified to display more and more of the non-sequential 3-separations of M until eventually a 3-tree is obtained. However, it is not clear how to construct a tight maximal flower in polynomial time. We can certainly find a non-sequential 3-separation (X, Y) quickly if one exists. The problem arises with testing in polynomial time whether (X, Y) is a tight maximal flower or whether it can be refined. Curiously, once we have a tight flower with at least three petals, we can modify the techniques used above to quickly test whether it can be refined and, if so, to find such a refinement. Furthermore, when (X, Y) can be refined to a paddle with at least three petals, we can detect that by finding a 1-separation in one of $\mathrm{si}(M/X)$ and $\mathrm{si}(M/Y)$ and this can be done quickly by using Proposition 4.2. By duality, we can deal with the case when (X, Y) can be refined to a copaddle with at least three petals. What seems difficult to detect in polynomial time is whether (X, Y) can be refined to a flower with at least three petals in which the local connectivity between the petals is one. Even if this approach could be made to work, it seems more complicated than the approach we have adopted here although both approaches rely on the same basic technique for finding 3-separations.

Lastly, Step 3 of 3-Tree locates a non-sequential 3-separation of a 3-connected matroid M and uses this to begin the construction of a 3-tree for M. If we already know some 3-separation for M, we can use it as (Y, Z) in this step of the algorithm and proceed with the rest of the algorithm as stated.

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