CONSTRUCTING A 3-TREE FOR A 3-CONNECTED MATROID

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For our friend Geoff Whittle with thanks for many years of enjoyable collaboration

Abstract. In an earlier paper with Whittle, we showed that there is a tree that displays, up to a natural equivalence, all non-trivial 3-separations of a 3-connected matroid $M$. The purpose of this paper is to give a polynomial-time algorithm for constructing such a tree for $M$.

1. Introduction

Let $M$ be a matroid with ground set $E$ and rank function $r$. The connectivity function $\lambda_M$ of $M$ is defined for all subsets $X$ of $E$ by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. For a positive integer $k$, a subset $X$ or a partition $(X, E - X)$ of $E$ is $k$-separating if $\lambda_M(X) \leq k - 1$. A $k$-separating partition $(X, E - X)$ is a $k$-separation if $|X|, |E - X| \geq k$. A $k$-separating set $X$, or a $k$-separating partition $(X, E - X)$, or a $k$-separation $(X, E - X)$ is exact if $\lambda_M(X) = k - 1$.

We shall denote the set $\{1, 2, \ldots, n\}$ by $[n]$. Let $X$ be an exactly 3-separating set of a matroid $M$. If there is an ordering $(x_1, x_2, \ldots, x_n)$ of $X$ such that, for all $i$ in $[n]$, the set $\{x_1, x_2, \ldots, x_i\}$ is 3-separating, then $X$ is sequential and the ordering $(x_1, x_2, \ldots, x_n)$ is called a sequential ordering of $X$. An exactly 3-separating partition $(X, Y)$ of $M$ is sequential if either $X$ or $Y$ is a sequential 3-separating set. For a set $X$ of $M$, we say that $X$ is fully closed if it is closed in both $M$ and $M^*$, that is, $cl(X) = X$ and $cl^*(X) = X$. The full closure of $X$, denoted $fcl(X)$, is the intersection of all fully closed sets that contain $X$. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exactly 3-separating partitions $(A_1, B_1)$ and $(A_2, B_2)$ of $M$ are equivalent, written $(A_1, B_1) \cong (A_2, B_2)$, if $fcl(A_1) = fcl(A_2)$ and $fcl(B_1) = fcl(B_2)$.

The main theorem of [6], Theorem 9.1, shows that every 3-connected matroid $M$ with at least nine elements has a tree decomposition that displays, up to equivalence, all non-sequential 3-separations. While the proof of that theorem does yield an algorithm for finding such a tree decomposition, that algorithm does not appear to be polynomial in $|E(M)|$. In this paper, we will describe such a polynomial algorithm. The proof that this algorithm works gives an alternative proof of [6, Theorem 9.1]. This paper will make repeated reference to the results of [6].

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2. Main Result

In this section, we state the main theorem of the paper together with the main result of [6]. The section begins by introducing the concepts and terminology needed to make these statements meaningful. Our terminology will follow Oxley [5]. We write \( x \in \text{cl}^*(Y) \) to mean that \( x \in \text{cl}(Y) \) or \( x \in \text{cl}^*(Y) \).

Let \((P_1, P_2, \ldots, P_n)\) be a flower \( \Phi \) in a 3-connected matroid \( M \), that is, \((P_1, P_2, \ldots, P_n)\) is an ordered partition of \( E(M) \) such that \( \lambda_M(P_i) = 2 = \lambda_M(P_i \cup P_{i+1}) \) for all \( i \) in \([n]\), where all subscripts are interpreted modulo \( n \). The sets \( P_1, P_2, \ldots, P_n \) are the petals of \( \Phi \). Each must have at least two elements. It is shown in [6, Theorem 4.1] that every flower in a 3-connected matroid is either an anemone or a daisy. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering \((P_1, P_2, \ldots, P_n)\). A 3-separation \((X, Y)\) is displayed by a flower if \( X \) is a union of petals of the flower.

Let \( \Phi_1 \) and \( \Phi_2 \) be flowers in a matroid \( M \). A natural quasi ordering on the set of flowers of \( M \) is obtained by setting \( \Phi_1 \preceq \Phi_2 \) if every non-sequential 3-separation displayed by \( \Phi_2 \) is equivalent to one displayed by \( \Phi_1 \). If \( \Phi_1 \preceq \Phi_2 \) and \( \Phi_2 \preceq \Phi_1 \), then \( \Phi_1 \) and \( \Phi_2 \) are equivalent flowers. Such flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations of \( M \). Let \( \Phi \) be a flower of \( M \). The order of \( \Phi \) is the minimum number of petals in a flower equivalent to \( \Phi \). An element \( e \) of \( M \) is loose in \( \Phi \) if \( e \in \text{cl}(P_i) - P_i \) for some petal \( P_i \) of \( \Phi \); otherwise \( e \) is tight. A petal \( P_i \) is loose if all its elements are loose; and \( P_i \) is tight otherwise. A flower of order at least 3 is tight if all of its petals are tight. A flower of order 2 or 1 is tight if it has two petals or one petal, respectively. A flower \( \Phi \) is maximal if \( \Phi \) is equivalent to \( \Phi' \) for every flower \( \Phi' \) such that \( \Phi \preceq \Phi' \).

The classes of anemones and daisies can be further refined using a useful companion function to the connectivity function. The local connectivity, \( \cap(X, Y) \), is defined for all sets \( X \) and \( Y \) in a matroid \( M \) by

\[
\cap(X, Y) = r(X) + r(Y) - r(X \cup Y).
\]

Let \((P_1, P_2, \ldots, P_n)\) be a flower \( \Phi \) with \( n \geq 3 \). If \( \Phi \) is an anemone, then \( \cap(P_i, P_j) \) takes a fixed value \( k \) in \( \{0, 1, 2\} \) for all distinct \( i, j \) in \([n]\). We call \( \Phi \) a paddle if \( k = 2 \), a copaddle if \( k = 0 \), and a spike-like flower if \( k = 1 \) and \( n \geq 4 \). Similarly, if \( \Phi \) is a daisy, then \( \cap(P_i, P_j) = 1 \) for all consecutive \( i \) and \( j \). We say \( \Phi \) is swirl-like if \( n \geq 4 \) and \( \cap(P_i, P_j) = 0 \) for all non-consecutive \( i \) and \( j \); and \( \Phi \) is Vámos-like if \( n = 4 \) and \( \{\cap(P_1, P_3), \cap(P_2, P_4)\} = \{0, 1\} \).

If \((P_1, P_2, P_3)\) is a flower \( \Phi \) and \( \cap(P_i, P_j) = 1 \) for all distinct \( i \) and \( j \), we call \( \Phi \) ambiguous if it has no loose elements, spike-like if there is an element in \( \text{cl}(P_1) \cap \text{cl}(P_2) \cap \text{cl}(P_3) \) or \( \text{cl}^*(P_1) \cap \text{cl}^*(P_2) \cap \text{cl}^*(P_3) \), and swirl-like otherwise. Every flower with at least three petals is of one of these six types: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous [6].
To visualize a flower geometrically, it is helpful to think of a collection of lines in projective space along which the petals of the flower are attached. For example, we can obtain a paddle by gluing the petals along a single common line. Fig. 1 represents a 5-petal paddle in which each petal is a plane with enough structure to make the matroid 3-connected. This matroid has rank 7. Furthermore, Fig. 2 represents a 4-petal swirl-like flower. Again each petal is a plane. In that figure, the lines of attachment are the lines spanned by \{b_1, b_2\}, \{b_2, b_3\}, \{b_3, b_4\}, and \{b_4, b_1\}, where \{b_1, b_2, b_3, b_4\} is an independent set and each of the elements in this set may or may not be in the matroid. The rank of this matroid is 8.

Flowers provide a way of representing 3-separations in a 3-connected matroid \(M\). It was shown in [6] that, by using a certain type of tree, one can simultaneously display a representative of each equivalence class of non-sequential 3-separations of \(M\). We now describe the type of tree that is used. Let \(\pi\) be a partition of a finite set \(E\). Let \(T\) be a tree such that every member of \(\pi\) labels a vertex of \(T\); some vertices may be unlabelled but no vertex is multiply labelled. We say that \(T\) is a \(\pi\)-labelled tree; labelled vertices are called bag vertices and members of \(\pi\) are called bags. If \(B\) is a bag vertex of \(T\), then \(\pi(B)\) denotes the subset of \(E\) that labels it. If the degree of \(B\) is at most one, then \(B\) is a terminal bag vertex; otherwise \(B\) is non-terminal.
Let $G$ be a subgraph of $T$ with components $G_1, G_2, \ldots, G_m$. Let $X_i$ be the union of those bags that label vertices of $G_i$. Then the subsets of $E$ displayed by $G$ are $X_1, X_2, \ldots, X_m$. In particular, if $V(G) = V(T)$, then $\{X_1, X_2, \ldots, X_m\}$ is the partition of $E$ displayed by $G$. Let $e$ be an edge of $T$. The partition of $E$ displayed by $e$ is the partition displayed by $T \setminus e$. If $e = v_1v_2$ for vertices $v_1$ and $v_2$, then $(Y_1, Y_2)$ is the (ordered) partition of $E(M)$ displayed by $v_1v_2$ if $Y_1$ is the union of the bags in the component of $T \setminus v_1v_2$ containing $v_1$. Let $v$ be a vertex of $T$ that is not a bag vertex. The partition of $E$ displayed by $v$ is the partition displayed by $T - v$. The edges incident with $v$ correspond to the components of $T - v$, and hence to the members of the partition displayed by $v$. In what follows, if a cyclic ordering $(e_1, e_2, \ldots, e_n)$ is imposed on the edges incident with $v$, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by $v$.

Let $M$ be a 3-connected matroid with ground set $E$. Let $T$ be a $\pi$-labelled tree for $M$, where $\pi$ is a partition of $E$ such that:

(I) For each edge $e$ of $T$, the partition $(X, Y)$ of $E$ displayed by $e$ is 3-separating, and, if $e$ is incident with two bag vertices, then $(X, Y)$ is a non-sequential 3-separation.

(II) Every non-bag vertex $v$ is labelled either $D$ or $A$; if $v$ is labelled $D$, then there is a cyclic ordering on the edges incident with $v$.

(III) If a vertex $v$ is labelled $A$, then the partition of $E$ displayed by $v$ is an anemone of order at least 3.

(IV) If a vertex $v$ is labelled $D$, then the partition of $E$ displayed by $v$, with the cyclic order induced by the cyclic ordering on the edges incident with $v$, is a daisy of order at least 3.

By conditions (III) and (IV), a vertex $v$ labelled $D$ or $A$ corresponds to a flower of $M$. The 3-separations displayed by this flower are the 3-separations displayed by $v$. A vertex of $T$ is referred to as a daisy vertex or an anemone vertex if it is labelled $D$ or $A$, respectively. A vertex labelled either $D$ or $A$ is a flower vertex. A 3-separation is displayed by $T$ if it is displayed by some edge or some flower vertex of $T$. A 3-separation $(R, G)$ of $M$ conforms with $T$ if either $(R, G)$ is equivalent to a 3-separation that is displayed by a flower vertex or an edge of $T$, or $(R, G)$ is equivalent to a 3-separation $(R', G')$ with the property that either $R'$ or $G'$ is contained in a bag of $T$.

A $\pi$-labelled tree $T$ for $M$ satisfying (I)–(IV) is a conforming tree for $M$ if every non-sequential 3-separation of $M$ conforms with $T$. A conforming tree $T$ is a partial 3-tree if, for every flower vertex $v$ of $T$, the partition of $E$ displayed by $v$ is a tight maximal flower of $M$.

We now define a quasi order on the set of partial 3-trees for $M$ clarifying the corresponding definition in [6, 7]. Let $T_1$ and $T_2$ be partial 3-trees for $M$. Define $T_1 \preceq T_2$ if every non-sequential 3-separation displayed by $T_1$ is equivalent to one displayed by $T_2$. If $T_1 \preceq T_2$ and $T_2 \preceq T_1$, then $T_1$ and $T_2$ are equivalent partial 3-trees. A partial 3-tree is maximal if it is maximal with respect to this quasi order. We shall call a maximal partial 3-tree a 3-tree. Note that this terminology differs
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Figure 3. The 3-tree $T$.

from that used in [7] where we use the term ‘3-tree’ for a particular type of maximal
3-tree defined in that paper.

As an example, for $n \geq 3$ and $k \geq 2$, the free $(n, k)$-swirl is the matroid that
is obtained by beginning with a basis $\{1, 2, \ldots, n\}$, adding $k$ points freely on each
of the $n$ lines spanned by $\{1, 2\}$, $\{2, 3\}$, $\ldots$, $\{n, 1\}$, and then deleting $\{1, 2, \ldots, n\}$. The usual free $n$-swirl coincides with the free $(n, 2)$-swirl. We observe that, when
$n + k \geq 5$, the free $(n, k)$-swirl can be viewed as a swirl-like flower whose $n$ petals
consist of the sets of $k$ points that were freely placed on the $n$ lines above. The spine of a paddle $(P_1, P_2, \ldots, P_n)$ is the set $\text{cl}(P_1) \cap \text{cl}(P_2) \cap \cdots \cap \text{cl}(P_n)$, which
coincides with each of the sets $\text{cl}(P_i) \cap \text{cl}(P_j)$ with $1 \leq i < j \leq n$.

Now, beginning with a free $(5, 4)$-swirl $S = (V_1, V_2, V_3, V_4, L)$, where each of $V_1,$ $V_2$, $V_3$, $V_4$, and $L$ is a line of $S$, use $L$ as the spine of a paddle to which we attach three $(4, 4)$-swirls $(X_1, X_2, X_3, L)$, $(Y_1, Y_2, Y_3, L)$, and $(Z_1, Z_2, Z_3, L)$. A possible 3-tree $T$ for this matroid $M$ is shown in Fig. 3, where large open circles represent bag vertices. At the end of Section 5, we will use this example, which is taken from [7], to illustrate our polynomial-time algorithm for finding a 3-tree. The 3-tree for $M$

is not unique. Indeed, we can move the bag vertex labelled by $L$ so that it occurs
on one of the other edges incident with the anemone vertex of $T$ to obtain another
3-tree for $M$.

The following theorem is the main result of [6, Theorem 9.1].

**Theorem 2.1.** Let $M$ be a 3-connected matroid with $|E(M)| \geq 9$. Then $M$ has
a 3-tree $T$. Moreover, every non-sequential 3-separation of $M$ is equivalent to a
3-separation displayed by $T$.

Throughout, we shall assume that each matroid $M$ that we deal with is specified
by a rank oracle, that is, a subroutine that, in unit time, gives the rank of any
specified subset $X$ of $E(M)$. The following is the main result of this paper.
Theorem 2.2. Let $M$ be a 3-connected matroid specified by a rank oracle and suppose that $|E(M)| \geq 9$. Then there is a polynomial-time algorithm for finding a 3-tree for $M$.

The next section contains a number of preliminaries that we use to prove the last theorem. In Section 4, we use a result of Cunningham and Edmonds to show that, for a 3-connected matroid $M$ with $n$ elements, there is a polynomial $p(n)$ such that by making at most $p(n)$ calls to a rank oracle, we can either find a non-sequential 3-separation in $M$ or show that no such 3-separation exists. Section 5 presents our algorithm for finding a 3-tree. In Section 6, we prove the correctness of the algorithm and thereby prove Theorem 2.2. Finally, Section 7 discusses why the proof of Theorem 2.1 in [6] does not appear to yield the desired polynomial-time algorithm for finding a 3-tree.

3. Preliminaries

In this section, we prove a number of lemmas needed to establish the main result. The first lemma is routine and often freely used.

Lemma 3.1. Let $(X, Y)$ be an exactly 3-separating partition of a matroid $M$.

(i) For $e \in E(M)$, the partition $(X \cup e, Y - e)$ is 3-separating if and only if $e \in \cl^*(X)$.

(ii) For $e \in Y$, the partition $(X \cup e, Y - e)$ is exactly 3-separating if and only if $e$ is in exactly one of $\cl(X) \cap \cl(Y - e)$ and $\cl^*(X) \cap \cl^*(Y - e)$.

(iii) The elements of $\fcl(X) - X$ can be ordered $(x_1, x_2, \ldots, x_n)$ so that $X \cup \{x_1, x_2, \ldots, x_i\}$ is 3-separating for all $i$ in $[n]$.

The connectivity function $\lambda_M$ of a matroid $M$ has many attractive properties. Clearly $\lambda_M(X) = \lambda_M(E - X)$. Moreover, one easily checks that $\lambda_M(X) = r(X) + r^*(X) - |X|$ for all subsets $X$ of $E(M)$. Hence $\lambda_M(X) = \lambda_{M^*}(X)$. We often abbreviate $\lambda_M$ as $\lambda$. This function is submodular, that is, $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E(M)$. The next lemma is a consequence of this. We make frequent use of it here and write by uncrossing to mean “by an application of Lemma 3.2.”

Lemma 3.2. Let $M$ be a 3-connected matroid, and let $X$ and $Y$ be 3-separating subsets of $E(M)$.

(i) If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.

(ii) If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.

The next two lemmas were established in [8, Lemma 2.7] and [6, Lemma 5.9].

Lemma 3.3. Let $(X, Y)$ be a 3-separation in a 3-connected matroid $M$ and let $Y'$ be a non-sequential 3-separating set in $M$. If $Y' \subseteq Y$, then $Y$ is non-sequential.

Lemma 3.4. Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight flower of order at least 3.
(i) If $1 \leq j \leq n - 2$, then
\[
\text{fcl}(P_1 \cup P_2 \cup \cdots \cup P_j) - (P_1 \cup P_2 \cup \cdots \cup P_j) \subseteq (\text{fcl}(P_1) - P_1) \cup (\text{fcl}(P_j) - P_j)
\]
and every element of $(\text{fcl}(P_1) - P_1) \cup (\text{fcl}(P_j) - P_j)$ is loose.

(ii) If $2 \leq j \leq n - 1$, then $P_1 \cup P_2 \cup \cdots \cup P_j$ is a non-sequential 3-separating set.
If, in addition, $j \leq n - 2$, then $(P_1 \cup P_2 \cup \cdots \cup P_j, P_{j+1} \cup P_{j+2} \cup \cdots \cup P_n)$ is a non-sequential 3-separation.

The next result is a consequence of the last lemma.

**Corollary 3.5.** Let $\Phi$ be a tight flower in a 3-connected matroid and $(U, V)$ be a non-sequential 3-separation such that $U$ is a union of petals of $\Phi$. Then no petal of $\Phi$ is in the full closure of both $U$ and $V$.

**Proof.** Let $P$ be a petal of $\Phi$ such that $P \subseteq U$ and $P \subseteq \text{fcl}(V)$. Then $P$ is a proper subset of $U$ as $(U, V)$ is non-sequential. Hence $\Phi$ has at least three petals. Therefore, by [6, Corollary 5.10], $\Phi$ has order at least three. Thus, by Lemma 3.4(i), $P$ is loose; a contradiction. \qed

The next lemma was proved in [8, Lemma 3.1].

**Lemma 3.6.** Let $(P_1, P_2, \ldots, P_k)$ be a flower in a 3-connected matroid. If $P_2$ is loose and $P_1$ is tight, then $P_2 \subseteq \text{fcl}(P_1)$.

An ordered partition $(Z_1, Z_2, \ldots, Z_k)$ of the elements of a 3-connected matroid is a 3-sequence if, for all $i$ in $[k - 1]$, the set $\cup_{j=1}^{i} Z_j$ is 3-separating. When a set $Z_i$ consists of a single element $z_i$, we shall write $z_i$ rather than $\{z_i\}$ in the 3-sequence.

**Lemma 3.7.** Let $U$ and $Y$ be disjoint subsets of the ground set $E$ of a 3-connected matroid $M$. Suppose that $U$ and $U \cup Y$ are 3-separating and $Y \subseteq \text{fcl}(U)$. If $\text{fcl}(U) \neq E$, then there is an ordering $(y_1, y_2, \ldots, y_k)$ of the elements of $Y$ such that $(U, y_1, y_2, \ldots, y_k, E - (U \cup Y))$ is a 3-sequence.

**Proof.** Let $(u_1, u_2, \ldots, u_l)$ be an ordering of $\text{fcl}(U) - U$ such that $U \cup \{y_1, u_2, \ldots, u_l\}$ is 3-separating for all $i$ in $[l]$. Let $(y_1', y_2', \ldots, y_k')$ be the ordering of the elements of $Y$ induced by this ordering of $\text{fcl}(U) - U$. As $\text{fcl}(U) \neq E$, we have $|E - \text{fcl}(U)| \geq 4$ so, by uncrossing, $U \cup \{y_1', y_2', \ldots, y_j'\}$ is 3-separating for all $j$ in $[k]$. In particular, $(U, y_1', y_2', \ldots, y_k', E - (U \cup Y))$ is a 3-sequence in $M$. \qed

In [6], our approach to finding a 3-tree for a 3-connected matroid $M$ relied on first constructing a maximal flower in $M$. As we shall see in Section 7, it is not clear how this approach can be used to produce a 3-tree for $M$ in polynomial time. The basis of the algorithm that we shall introduce here will be to first find, if possible, a non-sequential 3-separation $(X, Y)$ in $M$. Next we determine whether $X$ has a partition $(X', X'')$ so that $(X', X'' \cup Y)$ is a non-sequential 3-separation that is not equivalent to $(X, Y)$. To facilitate our discussion of this process, we next introduce the notion of a 3-path. After formally defining this concept, we devote the rest of this section to proving various properties of 3-paths that we shall need.
Let $M$ be a 3-connected matroid with ground set $E$. A 3-path in $M$ is an ordered partition $(X_1, X_2, \ldots, X_m)$ of $E$ into non-empty sets, called parts, such that

(i) $(\cup_{j=1}^i X_j, \cup_{j=i+1}^m X_j)$ is a non-sequential 3-separation of $M$ for all $i$ in $[m-1]$; and
(ii) for all $i$ in $\{2, 3, \ldots, m - 1\}$, the set $X_i$ is not in the full closure of either $\cup_{j=1}^{i-1} X_j$ or of $\cup_{j=i+1}^m X_j$.

Condition (ii) is equivalent to the assertion that the non-sequential 3-separations $(\cup_{j=1}^i X_j, \cup_{j=i+1}^m X_j)$ and $(\cup_{j=i+1}^{i+1} X_j, \cup_{j=i+2}^m X_j)$ are inequivalent for all $i$ in $[m-2]$. For a subset $X_0$ of $E$, an $X_0$-rooted 3-path is a 3-path of the form $(X_0 \cup X_1, X_2, \ldots, X_m)$ where $X_0 \cap X_1 = \emptyset$. Thus a 3-path is just a $\emptyset$-rooted 3-path. An $X_0$-rooted 3-path is maximal if

(i) none of the sets $X_i$ with $i \geq 2$ can be partitioned into sets $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$ for some $k \geq 2$ such that $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_{i,1}, X_{i,2}, \ldots, X_{i,k}, X_{i+1}, \ldots, X_m)$ is a 3-path; and
(ii) $X_1$ cannot be partitioned into sets $X_{1,1}, X_{1,2}, \ldots, X_{1,k}$ for some $k \geq 2$ such that $(X_0 \cup X_{1,1}, X_{1,2}, \ldots, X_{1,k}, X_2, \ldots, X_m)$ is a 3-path.

Observe that, in (ii), the set $X_{1,1}$ may be empty when $X_0$ is non-empty although all of $X_{1,2}, X_{1,3}, \ldots, X_{1,k}$ must be non-empty.

An $X_0$-rooted 3-path is left-justified if, for all $i$ in $\{2, 3, \ldots, m\}$, no element of $X_i$ is in the full closure of $\cup_{j=0}^{i-1} X_j$. In a 3-path $(X_1, X_2, \ldots, X_m)$, for each $i$ in $[m]$, we denote the sets $\cup_{j=1}^{i-1} X_j$ and $\cup_{j=i+1}^m X_j$ by $X_i^-$ and $X_i^+$, respectively. In particular, $X_1^- = \emptyset = X_m^+$. Observe that, in a 3-path $(X_1, X_2, \ldots, X_m)$, each of $X_1$ and $X_m$ has at least four elements as neither set is sequential, and each of $X_2, X_3, \ldots, X_{m-1}$ has at least two elements by (ii).

In what follows, we shall frequently be referring to a 3-separation $(R, G)$ of a 3-connected matroid $M$. In general, we shall view $(R, G)$ as a colouring of the elements of $E(M)$, the elements in $R$ and $G$ being red and green, respectively. A non-empty subset $X$ of $E$ is bichromatic if it meets both $R$ and $G$; otherwise it is monochromatic. We shall view the empty set as being monochromatic. In the lemmas that follow, we shall make repeated use of the fact [6, Lemma 3.3] that if $(R, G)$ is non-sequential and $(R', G')$ is a partition of $E(M)$ such that $\text{fcl}(R') = \text{fcl}(R)$ or $\text{fcl}(G') = \text{fcl}(G)$, then $(R', G')$ is a non-sequential 3-separation of $M$.

**Lemma 3.8.** Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be a left-justified maximal $X_0$-rooted 3-path in a 3-connected matroid $M$. Let $(R, G)$ be a non-sequential 3-separation in $M$. If, for some $i$ in $\{2, 3, \ldots, m - 1\}$, both $X_i^-$ and $X_i^+$ contain at least two red and at least two green elements, then $X_i$ is monochromatic.

**Proof.** Assume that $X_i$ is bichromatic. Now $|X_j^+ \cap G| \geq 2$. Thus, by uncrossing, as $R$ and $X_i^+ \cup X_i$ are both 3-separating, so is their intersection, $(X_i^- \cup X_i) \cap R$. Again, by uncrossing, the union of the last set with $X_i^-$, which equals $X_i^- \cup (X_i \cap R)$, is 3-separating. By maximality, $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap R, X_i \cap G, X_{i+1}, \ldots, X_m)$
Proof. Suppose first that \( |X_i^+ \cap R| = 1 \). As \((E - X_i^+ , X_i^+)\) and \((R,G)\) are non-sequential, \(|X_i^+| \geq 4\) and \(|R \cap (E - X_i^+)| \geq 3\). Thus, by uncrossing, \(G \cap X_i^+\) is 3-separating. Since \(X_i^+\) is also 3-separating, the one red element in \(X_i^+\) can be recoloured green producing a 3-separation equivalent to \((R,G)\) with fewer bichromatic parts; a contradiction. Hence \(|X_i^+ \cap R| \geq 2\). A symmetric argument establishes that if \(|X_i^- \cap R| \geq 1\), then \(|X_i^- \cap R| \geq 2\). We note here that if \(|X_i^- \cap R| = 1\) and the unique element of this set is in \(X_0\), then \(|X_0| = 1\) as \(X_0\) is monochromatic. Thus \(X_0\) stays monochromatic when the element of \(X_i^- \cap R\) is recoloured and, as \(X_0 \subseteq X_1\), we produce a 3-separation equivalent to \((R,G)\) with fewer bichromatic parts.

Lemma 3.9. Let \((X_1, X_2, \ldots, X_m)\) be a 3-path in a 3-connected matroid \(M\). Let \(X_0\) be a subset of \(X_1\), and \((R, G)\) be a non-sequential 3-separation in \(M\) for which \(X_0\) is monochromatic and no equivalent 3-separation in which \(X_0\) is monochromatic has fewer bichromatic parts. Suppose that, for some \(i\) in \([m]\), the set \(X_i\) is bichromatic. If, for some \(Z\) in \(\{X_i^- , X_i^+\}\), there is at least one red element in \(Z\), then there are at least two red elements in \(Z\).

Proof. Assume that \(X_i\) is 3-separating and that \(X_i^- \cup X_i^+\) is not monochromatic. By Lemmas 3.8 and 3.9, \(X_i^-\) or \(X_i^+\) is monochromatic and is green, say. Then, by Lemma 3.9, \(X_i^+\) or \(X_i^-\), respectively, contains at least two red elements. If \(X_i\) contains a single red element \(x\), then \(x\) is the unique red element of some \(Y\) in \(\{X_i^- \cup X_i, X_i^+ \cup X_i\}\). By uncrossing \(Y\) and \(G\), we see that \(x\) can be recoloured green to produce a 3-separation equivalent to \((R,G)\) with fewer bichromatic parts. If \(X_i\) contains a single green element, \(g\), but more than one red element, then, by uncrossing, \(X_i - g\) is 3-separating, so \(g \in cl(X_i - g)\) and we can recolour \(g\) red to reduce the number of bichromatic parts. We conclude that both \(X_i \cap R\) and \(X_i \cap G\) contain at least two elements. Now either \(X_i^-\) or \(X_i^+\) is green. In the first case, by uncrossing, \(X_i^- \cup (X_i \cap G)\) is 3-separating. As \((X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap G, X_i \cap R, X_{i+1}, \ldots, X_m)\) is not a 3-path, but the original 3-path is left-justified, it follows that \(X_i \cap R \subseteq fcl(X_i^+)\). It is straightforward to check that if \(X_i^+\) is green, then \(X_i \cap G \subseteq fcl(X_i^+)\). Thus some \(Z\) in \(\{X_i \cap R, X_i \cap G\}\) is a subset of \(fcl(X_i^- \cup X_i^+)\).

Suppose that \(X_i \not\subseteq fcl(X_i^- \cup X_i^+)\). Then, by Lemma 3.7, there is an ordering \((z_1, z_2, \ldots, z_k)\) of the elements of \(Z\) such that \((X_i^- \cup X_i^+, z_1, z_2, \ldots, z_k, X_i - Z)\) is a 3-sequence. Therefore \(Z \subseteq fcl(X_i - Z)\), so we can change the colour of all the
elements of \( Z \) to give a 3-separation that is equivalent to \((R, G)\) but has fewer bichromatic parts; a contradiction. We may now assume that \( X_i \subseteq \text{fcl}(X_i^- \cup X_i^+) \).

Then there is an ordering \((z_1, z_2, \ldots, z_k)\) of the elements of \( X_i \) such that \((X_i^- \cup X_i^+, z_1, z_2, \ldots, z_k)\) is a 3-sequence. We can reorder the last three elements of this 3-sequence if necessary to obtain a 3-sequence whose last two elements are the same colour. Then we can recolour all of the elements of \( X_i \) this colour to get a 3-separation that is equivalent to \((R, G)\) but has fewer bichromatic parts, again getting a contradiction.

\[ \square \]

**Lemma 3.11.** Let \((X_0 \cup X_1, X_2, \ldots, X_m)\) be a left-justified maximal \(X_0\)-rooted 3-path in a 3-connected matroid \(M\). Let \((R, G)\) be a non-sequential 3-separation in \(M\) for which \(X_0\) is monochromatic and no equivalent 3-separation in which \(X_0\) is monochromatic has fewer bichromatic parts. If, for some \(i \in \{2, 3, \ldots, m-1\}\), the set \(X_i^-\) is monochromatic but \(X_i\) is bichromatic, then \(X_i^- \cup X_i^+\) is monochromatic.

**Proof.** Assume that \(X_i^-\) is green but \(X_i^- \cup X_i^+\) is bichromatic. Then, by Lemma 3.9, \(X_i^+\) contains at least two red elements. Thus, by uncrossing, \(X_i^- \cup (X_i \cap G)\) is 3-separating. As the 3-path \((X_0 \cup X_1, X_2, \ldots, X_m)\) is maximal and left-justified, it follows that \(X_i \cap R \subseteq \text{fcl}(X_i^- \cup (X_i \cap G))\), so \(X_i \cap R \subseteq \text{fcl}(G)\). Hence we can recolour all the elements in \(X_i \cap R\) green thereby reducing the number of bichromatic parts; a contradiction.

\[ \square \]

**Lemma 3.12.** Let \((Z_0, Z_1, Z_2, \ldots, Z_m)\) be a 3-path in a 3-connected matroid \(M\) where \(m \geq 2\). Let \((R, G)\) be a non-sequential 3-separation of \(M\) such that

- (i) each of \(Z_1, Z_2, \ldots, Z_m\) is monochromatic;
- (ii) \(Z_{m-1} \cup Z_m\) is bichromatic;
- (iii) either
  - (a) \(Z_0\) is monochromatic but \(Z_0 \cup Z_1\) is not; or
  - (b) \(Z_0\) is bichromatic and \(\min\{|Z_0 \cap R|, |Z_0 \cap G|\} \geq 2\).

Then \(M\) has a flower \((Z_0, Z_1, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_{m}, Z_{j,1}, Z_{j,2}, \ldots, Z_{j,t})\) where each of \(Z_{i,1} \cup Z_{i,2} \cup \cdots \cup Z_{i,s}\) and \(Z_{j,1} \cup Z_{j,2} \cup \cdots \cup Z_{j,t}\) is monochromatic; each of \((Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s})\) and \((Z_{j,1}, Z_{j,2}, \ldots, Z_{j,t})\) is a subsequence of \((Z_1, Z_2, \ldots, Z_{m-1})\); and \(\{Z_1, Z_2, \ldots, Z_{m-1}\} = \{Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}\} \cup \{Z_{j,1}, Z_{j,2}, \ldots, Z_{j,t}\}\). Moreover, when \(Z_0\) is bichromatic, this flower can be refined so that \((Z'_0, Z''_0, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_{m}, Z_{j,1}, Z_{j,2}, \ldots, Z_{j,t})\) is a flower where \(\{Z'_0, Z''_0\} = \{Z_0 \cap R, Z_0 \cap G\}\) and \(Z'_0 \cup Z_{i,1}\) and \(Z'_0 \cup Z_{j,1}\) are monochromatic.

**Proof.** Without loss of generality, we may assume that \(Z_m \subseteq G\) and \(Z_{m-1} \subseteq R\). By assumption, \(Z_0 \cup Z_1\) is bichromatic containing at least two red elements and at least two green elements. Let the subsequence of \((Z_2, Z_3, \ldots, Z_m)\) consisting of red sets be \((Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k})\). Then \(p_k = m - 1\). By repeated applications of uncrossing, we get that \(Z_{p_1} \cup Z_{p_{a+1}} \cup \cdots \cup Z_{p_k}\) is 3-separating for all \(a \in [k]\). As \(Z_0 \cup Z_1 \cup \cdots \cup Z_k\) is 3-separating for all \(b \in [m-1]\), we deduce, by uncrossing, that each of \(Z_{p_1}, Z_{p_2}, \ldots, Z_{p_{a+1}}, Z_{p_1} \cup Z_{p_{a+2}}, Z_{p_2} \cup Z_{p_3}, \ldots, Z_{p_{a+1}} \cup Z_{p_k}\) is 3-separating. Moreover, \(Z_{p_k} \cup Z_m = Z_{m-1} \cup Z_m\), so it is 3-separating.
Now let the subsequence of \((Z_2, Z_3, \ldots, Z_m)\) consisting of green sets be \((Z_{q_1}, Z_{q_2}, \ldots, Z_{q_l})\). Then \(q_l = m\), so \(Z_q\) is 3-separating and, by uncrossing again, we deduce that each of \(Z_{q_1}, Z_{q_2}, \ldots, Z_{q_{l-1}}, Z_{q_1} \cup Z_{q_2}, Z_{q_2} \cup Z_{q_3}, \ldots, Z_{q_{l-1}} \cup Z_{q_l}\) is 3-separating.

As each of \(Z_{p_1} \cup Z_{p_2}, Z_{p_2} \cup Z_{p_3}, \ldots, Z_{p_k} \cup Z_{p_{k+1}}, Z_{q_1} \cup Z_{q_{l-1}}, Z_{q_{l-1}} \cup Z_{q_l}\) is 3-separating, the union of all but the last of these sets is 3-separating and hence so is its complement, \(Z_0 \cup Z_1 \cup \cdots \cup Z_q\). Similarly, \(Z_0 \cup Z_1 \cup Z_{p_1}\) is 3-separating. We deduce that \((Z_0, Z_1, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k}, Z_{l-1}, \ldots, Z_{k_1})\) is a flower. If \(Z_1\) is red, then, by uncrossing, \(Z_1 \cup Z_{p_1} \cup \cdots \cup Z_{p_k}\) is 3-separating, as are \(Z_0 \cup Z_1\) and \(Z_0 \cup Z_1 \cup Z_{p_1}\), so \(Z_1\) and \(Z_1 \cup Z_{p_1}\) are 3-separating. Also, \(E - (Z_1 \cup Z_{p_1} \cup \cdots \cup Z_{p_k})\) is 3-separating and, by uncrossing, so too is \(Z_0 \cup Z_{q_1}\). Hence \((Z_0, Z_1, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_k}, Z_{l-1}, \ldots, Z_{k_1})\) is a flower.

We conclude, using the notation in statement of the lemma, that \((Z_0, Z_{i_1}, Z_{i_2}, \ldots, Z_{i_t}, Z_{j_t}, Z_{j_{t-1}}, \ldots, Z_{j_1})\) is a flower.

Finally, assume that \(Z_0\) is bichromatic. Then, by uncrossing, \(Z_0 \cap R\) and \(Z_0 \cap G\) are both 3-separating and the argument at the end of the last paragraph implies that \((Z_0 \cap G, Z_0 \cap R, Z_{i_1}, Z_{i_2}, \ldots, Z_{i_t}, Z_{j_t}, Z_{j_{t-1}}, \ldots, Z_{j_1})\) is a flower. □

In our algorithm, we shall construct maximal flowers from 3-paths. The next lemma is designed to cope with the fact that, whereas each 3-separation displayed by a 3-path is non-sequential, a maximal flower may have sequential petals.

**Lemma 3.13.** Let \(M\) be a 3-connected matroid with at least nine elements and \(X\) be a non-sequential 3-separating set in \(M\). Let \((R, G)\) be a non-sequential 3-separation such that both \(R \cap X\) and \(G \cap X\) are sequential 3-separating sets. Let \((U, V)\) be a non-sequential 3-separation with \(\min\{|U - X|, |V - X|\} \geq 2\) such that \(U \cap X \not\subseteq \text{fcl}(U - X)\) and \(V \cap X \not\subseteq \text{fcl}(V - X)\). Then some of the elements of \(X\) can be recoloured to give a 3-separation \((R', G')\) equivalent to \((R, G)\) such that both \(U \cap X\) and \(V \cap X\) are monochromatic.

**Proof.** Since \(X\) is non-sequential, \(|X| \geq 4\). As \(U \cap X \not\subseteq \text{fcl}(U - X)\) and \(V \cap X \not\subseteq \text{fcl}(V - X)\), it follows that neither \(U \cap X\) nor \(V \cap X\) is empty. If \(|U \cap X| = 1\), then \(|V \cap X| \geq 2\) so, by uncrossing, \(U - X\) is 3-separating and then \(U \cap X \subseteq \text{fcl}(U - X)\); a contradiction. Hence \(|U \cap X| \geq 2\) and, by symmetry, \(|V \cap X| \geq 2\).

Let \((r_1, r_2, \ldots, r_k)\) and \((g_1, g_2, \ldots, g_l)\) be sequential orderings of \(R \cap X\) and \(G \cap X\), respectively. Observe that the lemma trivially holds if either \(k = 0\) or \(l = 0\). Thus we may assume that \(k, l \geq 1\). If \(k = 1\), then, as \(G \cap X\) is 3-separating, it follows by Lemma 3.1 that \(r_1 \in \text{cl}(X)(G \cap X)\). Thus the lemma holds by recolouring \(r_1\) green. Similarly, the lemma holds if \(l = 1\), so we may assume that \(k, l \geq 2\).

Suppose that \(|R \cap X| \geq 3\). Then we may assume that \(|\{r_1, r_2, r_3\} \cap U| \geq 2\). As \(|U - X| \geq 2\), it follows by uncrossing that \(X \cap V\) is 3-separating. Since \(|R \cap (E - (X \cap V))| \geq 2\), it follows by another application of uncrossing that \(G \cap (X \cap V)\) is 3-separating. As \(|\{r_1, r_2, r_3\} \cap U| \geq 2\), it follows that \(R \cap (X \cap V) \subseteq \text{fcl}(U)\) and so, by Lemma 3.7, there is an ordering \((r'_1, r'_2, \ldots, r'_{k'}\) of the elements in \(R \cap (X \cap V)\)
such that

\[(U \cup (V - X), r'_1, r'_2, \ldots, r'_{\ell'}, G \cap (X \cap V))\]

is a 3-sequence in \(M\). Hence \(R \cap (X \cap V) \subseteq \text{fcl}(G \cap (X \cap V))\) so we can recolour the elements of \(R \cap (X \cap V)\) green to obtain an equivalent 3-separation \((R', G')\) in which \(X \cap V\) is green, that is, \(X \cap V \subseteq G'\). If \(X \cap U \subseteq R'\), then the required result holds, so we may assume that \(X \cap U \cap G' = \emptyset\). Thus \(|X \cap G'| \geq 3\). Since \(G \cap X\) is sequential, \(G' \cap X\) is sequential. Take a sequential ordering \((g_1', g_2', \ldots, g_{\ell}'\)) of \(G' \cap X\). If at least two of \(g_1', g_2', \) and \(g_3'\) are in \(U\), then there is a 3-sequence in \(M\) of the form \((U, e_1, e_2, \ldots, e_t, V - X)\) where \(\{e_1, e_2, \ldots, e_t\} = V \cap X\). Hence \(V \cap X \subseteq \text{fcl}(V - X)\); a contradiction. We deduce that at least two of \(g_1', g_2',\) and \(g_3'\) are in \(V\). Then there is a sequential ordering of \(G' \cap X\) that first uses all of the elements of \(V \cap X\).

Let this ordering be \((v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_b)\) where \(\{v_1, v_2, \ldots, v_a\} \subseteq V\) and \(\{u_1, u_2, \ldots, u_b\} \subseteq U\). Then, by uncrossing, \((X \cap U \cap R') \cup \{u_b, u_{b-1}, \ldots, u_1\}\) is 3-separating for all \(i\) in \([b]\). Thus we can recolour the elements of \(\{u_b, u_{b-1}, \ldots, u_1\}\) red to get that \(U \cap X\) is red and \(V \cap X\) is green as required.

We may now assume that \(|R \cap X| = 2\) and, by symmetry, that \(|G \cap X| = 2\). The required result follows unless \(U \cap X = \{r_1, g_1\}\) and \(V \cap X = \{r_2, g_2\}\) where \(\{r_1, r_2\} = R \cap X\) and \(\{g_1, g_2\} = G \cap X\).

Since \(|R|, |G| \geq 4\) and \(|E(M)| \geq 9\), we may assume that \(|R \cap X| \geq 3\). Then, without loss of generality, we may suppose that \(U - X\) contains at least two red elements. Assume that \(V - X\) contains at least one green element. Then, by uncrossing, \(U \cap R\) is 3-separating and so \((U - X) \cup r_1\) is 3-separating. As \((U - X) \cup r_1 \cup g_1\) is 3-separating, it follows that \(U \cap X \subseteq \text{fcl}(U - X)\); a contradiction. We deduce that \((V - X) \cap G = \emptyset\), so \(|(V - X) \cap R| \geq 2\). Then, by arguing as above, we get that \((U - X) \cap G = \emptyset\). Hence \(E(M) - X \subseteq R\), so \(|G| = 2\); a contradiction. \(\square\)

**Lemma 3.14.** Let \((X_0 \cup X_1, X_2, \ldots, X_m)\) be a left-justified maximal \(X_0\)-rooted 3-path in a 3-connected matroid \(M\). Let \((R, G)\) be a non-sequential 3-separation in \(M\) for which \(X_0\) is monochromatic and no equivalent 3-separation in which \(X_0\) is monochromatic has fewer bichromatic parts. Suppose that \(m \geq 2\) and that \(X_m\) and \(X_m^-\) are bichromatic. Then both \(R \cap X_m\) and \(G \cap X_m\) are sequential 3-separating sets.

**Proof.** By Lemma 3.9, \(|R \cap X_m|, |G \cap X_m| \geq 2\). Therefore, as \(R\) and \(X_m\) are 3-separating and \(|E(M) - (R \cup X_m)| \geq 2\), we have \(R \cap X_m\) is 3-separating. Similarly, \(G \cap X_m\) is 3-separating. If \((E(M) - (R \cap X_m), R \cap X_m)\) is non-sequential, then, as \((X_0 \cup X_1, X_2, \ldots, X_m)\) is left-justified and maximal, \(\text{fcl}(R \cap X_m) = \text{fcl}(X_m)\). In particular, by Lemma 3.7, we can recolour all the elements in \(G \cap X_m\) red to give a 3-separation equivalent to \((R, G)\) with fewer bichromatic parts; a contradiction. Thus \((E(M) - (R \cap X_m), R \cap X_m)\) is sequential, in particular, by Lemma 3.3, \(R \cap X_m\) is sequential. Similarly, \(G \cap X_m\) is sequential. \(\square\)

**Lemma 3.15.** Let \((X_0 \cup X_1, X_2, \ldots, X_m)\) be a left-justified maximal \(X_0\)-rooted 3-path in a 3-connected matroid \(M\). Let \((R, G)\) be a non-sequential 3-separation in \(M\) for which \(X_0\) is monochromatic and no equivalent 3-separation in which \(X_0\) is monochromatic has fewer bichromatic parts. Suppose that \(\{2, 3, \ldots, m - 1\}\) contains an element \(j\) such that \(X_j^+\) and \(X_j^-\) are bichromatic, but \(X_j^+\) is red. Then \(R \cap X_j \subseteq\)
fcl(\(X_j^+\)). Furthermore, there is a 3-separation \((R', G')\) equivalent to \((R, G)\) such that \(R' \cap X_j = X_j \cap fcl(X_j^+)\) while \(R' \cap X_i = R \cap X_i\) and \(G' \cap X_i = G \cap X_i\) for all \(i \neq j\).

**Proof.** By Lemma 3.9, \(|G \cap X_j^-| \geq 2\) as \(G \cap X_j^-\) is non-empty. Therefore, as \(R\) and \(X_j \cup X_j^+\) are both 3-separating and avoid \(G \cap X_j^-\), it follows by uncrossing that \((X_j^- \cup (G \cap X_j), R \cap (X_j \cup X_j^+))\) is a 3-separation. By Lemma 3.3, this 3-separation is non-sequential. But \((X_0 \cup X_1, X_2, \ldots, X_m)\) is maximal and left-justified, so \((X_j^- \cup (G \cap X_j), R \cap (X_j \cup X_j^+))\) is equivalent to \((X_j^- \cup X_j, X_j^+)\). Therefore \(R \cap X_j \subseteq \text{fcl}(X_j^+)\). Furthermore, by Lemma 3.7, recolouring all the elements in \((G \cap X_j) \cap \text{fcl}(X_j^+)\) red, we have a 3-separation \((R', G')\) equivalent to \((R, G)\) with the desired properties. \(\square\)

## 4. FINDING A NON-SEQUENTIAL 3-Separation

Finding a 3-tree for a 3-connected matroid \(M\) depends crucially on being able to find a non-sequential 3-separation for \(M\) or showing that \(M\) has no such 3-separation. We rely heavily on a polynomial-time algorithm of Cunningham and Edmonds (in Cunningham 1973) that, for any fixed positive integer \(k\), will either find a \(k\)-separation in a matroid or will show that no such \(k\)-separation exists. Underlying this algorithm is the following result of Edmonds [4], which specifies the size of a largest common independent set of two matroids that share a common ground set.

**Theorem 4.1.** Let \(M_1\) and \(M_2\) be matroids with rank functions \(r_1\) and \(r_2\) and a common ground set \(E\). Then

\[
\max\{|I| : I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)\} = \min\{r_1(T) + r_2(E - T) : T \subseteq E\}.
\]

The next result (see, for example, [5, Proposition 13.4.7]) provides the link between the existence of a certain \(k\)-separation and a common independent set of two matroids.

**Proposition 4.2.** Let \(M\) be a matroid and \(k\) be a positive integer. If \(X_1\) and \(X_2\) are disjoint subsets of \(E(M)\) each having at least \(k\) elements, then \(M\) has a \(k\)-separation \((Y_1, Y_2)\) with \(X_1 \subseteq Y_1\) and \(X_2 \subseteq Y_2\) if and only if \(M/X_1/X_2\) and \(M/X_2/X_1\) do not have a common \(t\)-element independent set where \(t = r(M) + k - r(X_1) - r(X_2)\).

The matroid intersection algorithm finds, in polynomial time, not only a maximum-sized common independent set \(I\) of two matroids \(M_1\) and \(M_2\) on the same set \(E\), but also a subset \(X\) of \(E\) that minimizes \(r_1(X) + r_2(E - X)\), where \(r_i\) is the rank function of \(M_i\). By Theorem 4.1, each of \(I\) and \(X\) verifies that the other has the specified property. By applying this algorithm to all pairs \(M/X_1/X_2\) and \(M/X_2/X_1\) for which \(X_1\) and \(X_2\) are disjoint 3-element subsets of \(E(M)\), we get a polynomial-time algorithm for either finding a 3-separation in \(M\) or showing that no 3-separation exists. The difficulty with this process is that it may produce a sequential 3-separation and we want a non-sequential 3-separation. We show below...
how a minor modification of the algorithm will find a non-sequential 3-separation if one exists. First, we note that the basic idea in the matroid intersection algorithm is similar to that used in the algorithm for finding a maximum-sized matching in a bipartite graph: construction of an augmenting path. For a detailed description of the matroid intersection algorithm, the reader is referred to Cook, Cunningham, Pulleyblank, and Schrijver [1].

In order to find a non-sequential 3-separation in a matroid $M$ if one exists, we begin by finding the set $F$ of all maximal sequential 3-separating sets. To do this, we begin by finding all triangles and triads of $M$ by determining which 3-element subsets $X$ of $E(M)$ have $r(X)$ or $r^*(X)$ equal to 2, where $r^*(X) = r(E - X) - r(M) + 3$. We then find the full closure of each triangle and each triad by taking the closure of each such set, the coclosure of the result, the closure of the result, and so on until two consecutive terms are equal. For a given triangle or triad $X$ in an $n$-element matroid, we can find $\text{fcl}(X)$ by using $O(n^2)$ calls to the rank oracle. Observe that $F$ consists of the maximal members of $\{\text{fcl}(X) : X$ is a triangle or triad$\}$ and that the latter set has $O(n^3)$ members.

The next result is a straightforward consequence of Lemma 3.3 and we omit the proof. We use this corollary in the proof of the subsequent lemma.

**Corollary 4.3.** In a 3-connected matroid $M$, a 3-separating set $X$ is non-sequential if and only if no member of $F$ contains $X$.

The next lemma is key to finding a non-sequential 3-separation of a 3-connected matroid.

**Lemma 4.4.** Let $(U, V)$ be a 3-separation in a 3-connected matroid $M$ and suppose $k \in \{3, 4\}$. Then $(U, V)$ is non-sequential if and only if there are $k$-element subsets $U'$ and $V'$ of $U$ and $V$, respectively, such that no member of $F$ contains $U'$ or $V'$.

**Proof.** Suppose $(U, V)$ is non-sequential. Then $(U - \text{fcl}(V), \text{fcl}(V))$ is also non-sequential. Clearly $|U - \text{fcl}(V)| \geq 4$. Let $U_1$ be a $k$-element subset of $U - \text{fcl}(V)$. We take $U' = U_1$ unless $U_1$ is contained in some member $F$ of $F$. Consider the exceptional case. We have $F = \text{fcl}(T)$ for some triangle or triad $T$. Clearly $|T \cap \text{fcl}(V)| \leq 1$. Take $\{a, b\} \subseteq T - \text{fcl}(V)$. Clearly $\text{fcl}(\{a, b\}) = \text{fcl}(T) = F$. If $F$ contains $U - \text{fcl}(V)$, then, by Lemma 3.3, $U - \text{fcl}(V)$ is sequential; a contradiction. Thus $U - \text{fcl}(V) - F$ is non-empty. Suppose this set contains a single element $c$. Then $F$ and $U - \text{fcl}(V)$ are 3-separating. By uncrossing, so is their intersection, $U - \text{fcl}(V) - c$. As $U - \text{fcl}(V) - c$ and $U - \text{fcl}(V)$ are 3-separating, $c \in \text{cl}^{(k)}(U - \text{fcl}(V) - c)$, so $c \in F$; a contradiction. We deduce that $U - \text{fcl}(V) - F$ contains at least two distinct elements, $c$ and $d$. If $k = 3$, let $U' = \{a, b, c\}$; if $k = 4$, let $U' = \{a, b, c, d\}$. If $U'$ is contained in a member $F'$ of $F$, then $F'$ contains $T$ and hence contains $F$. Thus $F' = F$, but $c \in F' - F$; a contradiction. Hence no member of $F$ contains $U'$. We now know how to construct $U'$. We construct $V'$ symmetrically from $V - \text{fcl}(U)$.

The converse is an immediate consequence of the last corollary. □
Now to obtain a non-sequential 3-separation of \( M \), we apply the procedure described above for finding a 3-separation with the modification that the disjoint sets \( X_1 \) and \( X_2 \) are chosen to be 3-element sets that are not contained in any member of \( \mathcal{F} \). By the last lemma, if \((Y_1, Y_2)\) is a 3-separation with \( X_1 \subseteq Y_1 \) and \( X_2 \subseteq Y_2 \), then \((Y_1, Y_2)\) is non-sequential. Moreover, if, after searching through all such pairs \( \{X_1, X_2\} \) of sets, we find no 3-separation \((Y_1, Y_2)\) with \( X_1 \subseteq Y_1 \) and \( X_2 \subseteq Y_2 \), then \( M \) has no non-sequential 3-separations.

5. The Algorithm

In this section, we present the algorithm 3-Tree for constructing a 3-tree of a 3-connected matroid. To do this, we shall need some additional terminology. We shall also provide an informal description of the algorithm and an example to illustrate it.

Let \( M \) be a 3-connected matroid. Let \((P_1, P_2, \ldots, P_k)\) be a tight flower \( \Phi \) in \( M \), where \( k \geq 3 \). Consider how \( \Phi \) might arise in a 3-path where the petals of \( \Phi \) are the parts of the 3-path. Let \( P_1 \) and \( P_j \) be the first and last petals of \( \Phi \) occurring in the 3-path. Then the definition of a 3-path requires that both \( P_1 \) and \( P_j \) are non-sequential. Clearly \( j \in \{2, 3, \ldots, k\} \). Now \((P_1, Q_1, Q_2, \ldots, Q_{k-2}, P_j)\)

is a 3-path provided that \( \{Q_1', Q_2', \ldots, Q_{k-2}'\} = \{P_2, P_3, \ldots, P_k\} \) - \( \{P_j\} \), and both \((P_2, P_3, \ldots, P_{j-1})\) and \((P_k, P_{k+1}, \ldots, P_{j-1})\) are subsequences of \((Q_1', Q_2', \ldots, Q_{k-2}')\). If, for example, each petal of \( \Phi \) is sequential, then there is no 3-path whose parts coincide with the petals of \( \Phi \). But \((P_1 \cup P_2, P_3, \ldots, P_{k-2}, P_{k-1} \cup P_k)\) is one of many 3-paths arising from \( \Phi \). We now generalize the notion of a 3-path to indicate the presence of flowers including those with sequential petals.

Let \( \tau \) be a 3-path \((P_{1,1}, P_{1,2}, \ldots, P_{1,s}, Q_1', Q_2', \ldots, Q_{k-2}', P_{3,1}, P_{3,2}, \ldots, P_{3,t})\) in \( M \) such that there is a flower \( \Phi = (P_1, P_2, \ldots, P_k) \) with \( P_1 = P_{1,1} \cup P_{1,2} \cup \cdots \cup P_{1,s} \) and \( P_j = P_{3,1} \cup P_{3,2} \cup \cdots \cup P_{3,t} \) where \( \{Q_1', Q_2', \ldots, Q_{k-2}'\} = \{P_2, P_3, \ldots, P_k\} \) - \( \{P_j\} \). We call \( P_1 \) and \( P_j \) the entry and exit petals, respectively, of \((P_1, P_2, \ldots, P_k)\). When \( j \neq k \), we denote this flower \( \Phi \) in \( \tau \) by replacing the subsequence \( Q_1', Q_2', \ldots, Q_{k-2}' \) by \([P_{2,1}, P_{2,3}, \ldots, P_{j-1}]\) and \((P_{2}, P_{3}, \ldots, P_{j-1})\); and we call \( P_2, P_3, \ldots, P_{j-1} \) and \( P_{k-1}, \ldots, P_{j+1} \) the clockwise and anticlockwise petals, respectively, of \( \Phi \). If \( j = k \), then we replace \( Q_1', Q_2', \ldots, Q_{k-2}' \) by \([P_{2,1}, P_{2,3}, \ldots, P_{k-1}]\). In this case, we call \( P_2, P_3, \ldots, P_{k-1} \) the clockwise petals of \( \Phi \) and say that \( \Phi \) has no anticlockwise petals. Such modified 3-paths are examples of generalized 3-paths. There are three further elementary modifications of a 3-path which we shall want our notion of a generalized 3-path to encompass. Each of these occurs at the end of a 3-path and will be called an end move. Suppose \((Z_1, Z_2, \ldots, Z_m)\) is a 3-path in \( M \) and that there is a partition \((Z_m', Z_m'')\) of \( Z_m \) such that \((Z_1 \cup Z_2 \cup \cdots \cup Z_{m-2}, Z_{m-1}, Z_m', Z_m'')\) is a tight flower \( \Psi \). Then, in \((Z_1, Z_2, \ldots, Z_m)\), we replace \( Z_m, Z_m' \) by \([Z_{m-1}', Z_m'']\).
a tight flower. Then \((Z'_1, ([Z'_2, Z'_3], Z'_4))\) is a generalized 3-path. In the first and second type of end move, we refer to \(Z_m\) and \(Z_1\), respectively, as the split part, while in the third type of end move, we refer to \(Z_1\) and \(Z_2\) as the split parts.

The moves described in the last paragraph indicate how we modify a 3-path \(\tau\) when we detect a single flower arising from it. The algorithm describes a systematic way in which we repeat the above steps for every flower occurring in \(\tau\) each time modifying the current generalized 3-path to produce a new structure which we will also view as a generalized 3-path. The flowers that arise here are dealt with in order, starting from the far end of a 3-path. As we shall prove, the procedure we follow ensures that each flower we construct is tight and maximal.

Let \(\tau\) be a generalized 3-path in a 3-connected matroid \(M\) with ground set \(E\). Within \(\tau\), certain subsets of \(E\) are enclosed between the same pair of square brackets. Let \(\tau'\) be the ordered sequence obtained from \(\tau\) by, for each pair of corresponding square brackets, replacing these brackets and all the sets between them by the union of all the enclosed sets. Say \(\tau' = (Y_1, Y_2, \ldots, Y_p)\). Note that \(\tau'\) is a 3-path unless \(Y_1\) or \(Y_p\) is sequential as may occur if we apply an end move. Let \(P\) denote the \(\pi\)-labelled tree consisting of a path of \(p\) bag vertices labelled, in order, \(Y_1, Y_2, \ldots, Y_p\). Now modify \(P\) as follows. For each \(Y_j\) that is the union of \(s\) clockwise petals and \(t\) anticlockwise petals of a flower, replace the bag vertex labelled \(Y_j\) with a flower vertex \(v\) and adjoin \(s + t\) new bag vertices to \(v\) each via a new edge so that the cyclic ordering induced by the cyclic ordering on the edges incident with \(v\) preserves the ordering of the flower \(\Phi_j\) to which \(Y_j\) corresponds. Label the vertex \(v\) by \(D\) or \(A\) depending on whether \(\Phi_j\) is a daisy or an anemone respectively. We refer to the resulting modification of \(P\) as a path realization of \(\tau\).

To deal with generalized 3-paths, it will be useful to have some more terminology. Let \(Z\) be a term in a generalized 3-path \(\tau\) and assume that \(Z\) is not enclosed between two square brackets. We can then write \(\tau\) as \((\tau(Z^-), Z, \tau(Z^+))\) so \(\tau(Z^-)\) and \(\tau(Z^+)\) denote, respectively, the portions of \(\tau\) that occur before and after \(Z\). In this case, as in a 3-path, we shall denote by \(Z^-\) and \(Z^+\) the union of all of the sets in \(\tau\) that occur, respectively, before and after \(Z\).

We now give an informal description of our algorithm. An example to illustrate it is given at the end of the section. From the last section, we can test whether or not a given matroid \(M\) is 3-connected by making polynomially many calls to a rank oracle. We may now assume that \(M\) is 3-connected having ground set \(E\). Starting with a single unmarked bag vertex labelled \(E\), the algorithm 3-TREE recursively builds a \(\pi\)-labelled tree by selecting an unmarked bag vertex \(B\) and deciding if there is a non-sequential 3-separation \((Y, Z)\) such that either \(Y \subseteq \pi(B)\) or \(Z \subseteq \pi(B)\). If there is no such 3-separation, the vertex is marked. If there is such a 3-separation, 3-TREE calls the first of its two subroutines, FORWARDSweep, which constructs a left-justified maximal \((E - \pi(B))\)-rooted 3-path. Once such a 3-path, say \(\tau\), is constructed, FORWARDSweep ends and 3-TREE calls its second subroutine, BACKWARDSweep. This subroutine starts at the non-root end of \(\tau\) and recursively works its way towards the root end uncovering flower structure. Eventually, BACKWARDSweep outputs a generalized 3-path \(\tau'\). Lastly, 3-TREE takes a path realization of \(\tau'\) and adjoins it to the bag vertex \(B\). The algorithm now repeats this process by
selecting another unmarked bag vertex. When all bag vertices are marked, 3-
Tree outputs a $\pi$-labelled tree. We end with two remarks. Firstly, some flower
subtleties need to be dealt with at the non-root end of $\tau$ and also, in the first call to
BackwardSweep, at the root end of $\tau$. These subtleties correspond to applying
end moves. Secondly, the fact that ForwardSweep constructs a left-justified
maximal 3-path is established in Lemma 6.1.

**Algorithm:** 3-Tree($M$)

**Input:** A 3-connected matroid $M$ with ground set $E$ and $|E| \geq 9$.

**Output:** A 3-tree for $M$.

1. Construct the collection $\mathcal{F}$ of maximal sequential 3-separating sets of $M$.

2. Let $T_0$ denote the $\pi$-labelled tree consisting of a single (unmarked) bag vertex
   labelled $E$.

3. Search through pairs $\{(y_1, y_2, y_3), \{z_1, z_2, z_3\}\}$ of disjoint subsets of $E$ neither of
   which is contained in a member of $\mathcal{F}$ and find a 3-separation $(Y, Z)$ of $M$ such
   that $Y$ and $Z$ contain $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$, respectively.

   (i) If there is no such 3-separation, mark $E$, and output $T_0$.

   (ii) Otherwise, do the following:

      (a) Set $X_0 = \emptyset$, set $X_1 = \text{fcl}(Y)$, and set $X_2 = Z - \text{fcl}(Y)$. Call
          ForwardSweep($M$, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).

      (b) Call BackwardSweep($M$, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$), where $(X_0 \cup
          Z_1, Z_2, \ldots, Z_m)$ is the 3-path of $M$ outputted by ForwardSweep($M$,
          $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).

      (c) Set $i = 1$ and set $T_1$ to be the path realization of the generalized
          3-path outputted by BackwardSweep($M$, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$)
          with each bag vertex unmarked.

4. If there is no unmarked bag vertex, output $T_i$. Otherwise, choose an unmarked bag
   vertex $B$ of $T_i$.

5. If $B$ is a non-terminal bag vertex, find a 3-separation $(Y, Z)$ such that $Y$
   contains $\text{fcl}(E - \pi(B))$, and $Z$ contains a subset $\{z_1, z_2, z_3\}$ of
   $\pi(B) - \text{fcl}(E - \pi(B))$ with no member of $\mathcal{F}$ containing $\{z_1, z_2, z_3\}$. If $B$
   is a terminal bag vertex, find a 3-separation $(Y, Z)$ such that $Y$
   contains $\text{fcl}(E - \pi(B))$ and an element $y$ of $\pi(B)$
   with $y \notin \text{fcl}(E - \pi(B))$, and $Z$ contains a subset $\{z_1, z_2, z_3\}$ of
   $\pi(B) - \text{fcl}(E - \pi(B)) - \{y\}$ with no member of $\mathcal{F}$ containing $\{z_1, z_2, z_3\}$. Now do the following:

   (i) If there is no such 3-separation, mark $B$ and return to Step 4.

   (ii) Otherwise, do the following:

      (a) Set $X_0 = E - \pi(B)$, set $X_1 = \pi(B) \cap \text{fcl}(Y)$, and set $X_2 = \pi(B) -
          \text{fcl}(Y)$. Call ForwardSweep($M$, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).

      (b) Call BackwardSweep($M$, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$), where $(X_0 \cup
          Z_1, Z_2, \ldots, Z_m)$ is the 3-path of $M$ outputted by ForwardSweep($M$,
          $(X_0 \cup X_1, X_2)$, $\mathcal{F}$).
If $s = m$, find a 3-separation $(Y, Z)$ such that $Y$ contains $X_0 \cup X_0' \cup \cdots \cup X_{s-1}'$ and an element $y$ of $X_s' = \text{fcl}(X_0 \cup X_0' \cup \cdots \cup X_{s-1}')$, and $Z$ contains a subset $\{z_1, z_2, z_3\}$ of $X_s' - \text{fcl}(X_0 \cup X_0' \cup \cdots \cup X_{s-1}') - \{y\}$ such that no member of $\mathcal{F}$ contains $\{z_1, z_2, z_3\}$.

(d) Return to Step 4.

**Algorithm:** FORWARDSWEEP$(M, (X_0 \cup X_1, X_2), \mathcal{F})$

**Input:** A 3-connected matroid $M$ with ground set $E$ and $|E| \geq 9$, a 3-path $(X_0 \cup X_1, X_2)$ of $M$, and the collection $\mathcal{F}$ of maximal sequential 3-separating sets of $M$.

**Output:** A 3-path $(X_0 \cup X_1', X_2', \ldots, X_m')$ of $M$ that is a refinement of $(X_0 \cup X_1, X_2)$.

1. Let $\tau_0 = (X_0 \cup X_1, X_2)$, set $(i, s, m) = (1, 1, 2)$, and set $(X'_1, X'_2) = (X_1, X_2)$.

2. If $s \neq m$, do the following:
   (i) If $X_0 = \emptyset$ and $s = 1$, find a 3-separation $(Y, Z)$ such that $Y$ contains a subset $\{y_1, y_2, y_3\}$ of $X_1'$ with no member of $\mathcal{F}$ containing $\{y_1, y_2, y_3\}$, and $Z$ contains $X'_2 \cup \cdots \cup X'_m$ and an element $z$ of $X'_1$ with $z \notin \text{fcl}(X'_2 \cup \cdots \cup X'_m) \cup \{y_1, y_2, y_3\}$.
      (a) If there is no such 3-separation, go to Step 4.
      (b) Otherwise, increase $m$ by 1 and, for each $t > 1$, set $X'_t$ to be $X'_{t+1}$. Furthermore, set $X'_1$ to be $X'_1 \cap (E - \text{fcl}(Y))$ and then set $X'_1$ to be $X'_1 \cap \text{fcl}(Y)$. Go to Step 5.
   (ii) If $X_0 \neq \emptyset$ and $s = 1$, find a 3-separation $(Y, Z)$ such that $Y$ contains $\text{fcl}(X_0)$, and $Z$ contains $X'_2 \cup \cdots \cup X'_m$ and an element $z$ of $X'_1$ with $z \notin \text{fcl}(X'_2 \cup \cdots \cup X'_m)$.
      (a) If there is no such 3-separation, go to Step 4.
      (b) Otherwise, increase $m$ by 1 and, for each $t > 1$, set $X'_t$ to be $X'_{t+1}$. Furthermore, set $X'_2$ to be $X'_2 \cap (E - \text{fcl}(Y))$ and then set $X'_1$ to be $X'_1 \cap \text{fcl}(Y)$. Go to Step 5.
   (iii) Otherwise, find a 3-separation $(Y, Z)$ such that $Y$ contains $X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}$ and an element $y$ of $X'_s = \text{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1})$, and $Z$ contains $X'_{s+1} \cup \cdots \cup X'_m$ and an element $z$ of $X'_s$ with $z \notin \text{fcl}(X'_{s+1} \cup \cdots \cup X'_m) \cup \{y\}$.
      (a) If there is no such 3-separation, go to Step 4.
      (b) Otherwise, increase $m$ by 1 and, for each $t > s$, set $X'_t$ to be $X'_{t+1}$. Furthermore, set $X'_{s+1}$ to be $X'_s \cap (E - \text{fcl}(Y))$ and then set $X'_s$ to be $X'_s \cap \text{fcl}(Y)$. Go to Step 5.

3. If $s = m$, find a 3-separation $(Y, Z)$ such that $Y$ contains $X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}$ and an element $y$ of $X'_s = \text{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1})$, and $Z$ contains a subset $\{z_1, z_2, z_3\}$ of $X'_s - \text{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}) - \{y\}$ such that no member of $\mathcal{F}$ contains $\{z_1, z_2, z_3\}$.
(i) If there is no such 3-separation, then output $\tau_i$.

(ii) Otherwise, increase $m$ by 1. Furthermore, set $X_{s+1}'$ to be $X_s' \cap (E - \text{fcl}(Y))$ and then set $X_s'$ to be $X_s' \cap \text{fcl}(Y)$. Go to Step 5.

4. Increase $s$ by 1. Return to Step 2.

5. Increase $i$ by 1 and set $\tau_i$ to be $(X_0 \cup X_1', X_2', \ldots, X_m')$. Return to Step 2.

Algorithm: BackwardSweep($M, (X_0 \cup Z_1, Z_2, \ldots, Z_m)$)

Input: A matroid $M$ with ground set $E$ and $|E| \geq 9$, and a 3-path $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ of $M$, where $m \geq 2$.

Output: A generalized 3-path of $M$.

1. Let $\tau_m = (X_0 \cup Z_1, Z_2, \ldots, Z_m)$.

2. If $m = 2$ and $X_0$ is empty, find a 3-separation $(U, V)$ for which $U$ and $V$ contain subsets $U'$ and $V'$ such that no member of $\mathcal{F}$ contains $U'$ or $V'$ and $|U' \cap Z_1| = |U' \cap Z_2| = |V' \cap Z_1| = |V' \cap Z_2| = 2$.

   (i) If there is no such 3-separation, output $\tau_m$.

   (ii) Otherwise, output

   $$(V \cap Z_1, [(U \cap Z_1, U \cap Z_2)], V \cap Z_2).$$

3. If $m = 2$ and $X_0$ is non-empty, output $\tau_m$.

4. If $m \geq 3$, set $i = m - 1$.

5. If $Z_{m-1}$ is 3-separating, find a 3-separation $(U, V)$ such that $U$ contains $Z_{m-1}$ and $|U \cap Z_m| \geq 2$, and $V$ contains $Z_{m-1}$ and $|V \cap Z_m| \geq 2$.

   (i) If there is such a 3-separation, set

   $$\tau_{m-1} = (\tau_m(Z_{m-1}^-), [(Z_{m-1}, Z_m \cap U)], Z_m \cap V)$$

   and go to Step 7.

   (ii) Otherwise, set

   $$\tau_{m-1} = (\tau_m(Z_{m-1}^-), [(Z_{m-1})], Z_m)$$

   and go to Step 7.

6. If $Z_{m-1}$ is not 3-separating, do the following:

   (i) If $Z_{m-1} = \text{fcl}(Z_m)$ is 3-separating, set

   $$\tau_{m-1} = (\tau_m(Z_{m-1}^-), [(Z_{m-1} - \text{fcl}(Z_m))], Z_{m-1} \cap \text{fcl}(Z_m), Z_m)$$

   and go to Step 7.

   (ii) Otherwise, set $\tau_{m-1}$ to be $\tau_m$ and go to Step 7.

7. (i) If $i \neq 2$, decrease $i$ by 1 and go to Step 8.

   (ii) Otherwise, go to Step 10.

8. If $Z_i$ is 3-separating, do the following:

   (i) If $\tau_{i+1} = (X_0 \cup Z_1, Z_2, \ldots, Z_i, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \ldots)$, where $p \geq 1$, do the following:
(a) If $Z_i \cup P_1$ is 3-separating, set

\[ \tau_i = (\tau_{i+1}(Z_i^-), [(Z_i, P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \tau_{i+1}([[P_1, \ldots, P_p), (Q_1, \ldots, Q_q)]^+)) \]

and return to Step 7.

(b) If $Z_i \cup P_1$ is not 3-separating but $q \geq 1$ and $Z_i \cup Q_1$ is 3-separating, set

\[ \tau_i = (\tau_{i+1}(Z_i^-), [(P_1, \ldots, P_p), (Z_i, Q_1, \ldots, Q_q)], \tau_{i+1}([[P_1, \ldots, P_p), (Q_1, \ldots, Q_q)]^+)) \]

and return to Step 7.

(c) If $Z_i \cup P_1$ is not 3-separating but $q = 0$ and the union of $Z_i$ with $\tau_{i+1}([[P_1, \ldots, P_p])^+$ is 3-separating, set

\[ \tau_i = (\tau_{i+1}(Z_i^-), [(P_1, \ldots, P_p), (Z_i)], \tau_{i+1}([[P_1, \ldots, P_p])^+)) \]

and return to Step 7.

(d) Otherwise, set

\[ \tau_i = (\tau_{i+1}(Z_i^-), [(Z_i)], [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \tau_{i+1}([[P_1, \ldots, P_p), (Q_1, \ldots, Q_q)]^+)) \]

and return to Step 7.

(ii) Otherwise, set

\[ \tau_i = (\tau_{i+1}(Z_i^-), [(Z_i)], \tau_{i+1}(Z_i^+)) \]

and return to Step 7.

9. If $Z_i$ is not 3-separating, do the following:

(i) If $Z_i - \text{fcl}(Z_i^+)$ is 3-separating, set

\[ \tau_i = (\tau_{i+1}(Z_i^-), [(Z_i - \text{fcl}(Z_i^+))], Z_i \cap \text{fcl}(Z_i^+), \tau_{i+1}(Z_i^+)) \]

and return to Step 7.

(ii) Otherwise, set $\tau_i$ to be $\tau_{i+1}$ and return to Step 7.

10. (i) If $X_0$ is empty and $\tau_2 = (Z_i, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \ldots)$, find a 3-separation $(U, V)$ for which $U$ contains $P_1$ and an element $u$ of $Z_i$ such that $u \notin \text{fcl}(P_1)$, and $V$ contains $E - (Z_i \cup P_1)$ and an element $v$ of $Z_i - u$ such that $v \notin \text{fcl}(E - (Z_i \cup P_1))$, and do the following:

(a) If there is such a 3-separation, set $\tau_1$ to be

\[ (Z_i \cap V, [(Z_i \cap U, P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], \tau_2([[P_1, \ldots, P_p), (Q_1, \ldots, Q_q)]^+)) \]

and output $\tau_1$.

(b) Otherwise, output $\tau_2$.

(ii) Otherwise, output $\tau_2$.

As an example to illustrate the key ideas in 3-Tree, consider the matroid $M$, and the 3-tree for $M$ shown in Fig. 3. Let $(X, Y, Z) = (X_1 \cup X_2 \cup X_3, Y_1 \cup Y_2 \cup Y_3, Z_1 \cup Z_2 \cup Z_3)$. Suppose that 3-Tree is applied to $M$. If $(V_2 \cup V_3 \cup V_4, V_1 \cup L \cup X \cup Y \cup Z)$
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Figure 4. The path realization $T_1$.

is the 3-separation found in Step 3 in 3-TREE, then a possible 3-path outputted by the first call to ForwardSweep is

$$(V_2 \cup V_3, V_4, V_1 \cup L, X, Z, Y_1, Y_2 \cup Y_3).$$

Observe that the 3-path is left-justified and maximal. With this 3-path, a possible generalized 3-path outputted by the immediate subsequent call to BackwardSweep is

$$(V_3, [(V_2, V_3), (V_4)], L, [(X, Z)], [(Y_1, Y_2)], Y_3).$$

Comparing the 3-path and the generalized 3-path, both $V_2 \cup V_3$ and $Y_2 \cup Y_3$ are split parts. The splitting of $Y_2 \cup Y_3$ and $V_2 \cup V_3$ is the result of end moves performed in Steps 5 and 10 in BackwardSweep, respectively. The path realization $T_1$ of this generalized 3-path, produced in Step 3(ii)c in 3-TREE, is shown in Fig. 4, where we note that $X$ and $Z$ are petals of an anemone. The algorithm now starts to repeatedly apply Steps 4 and 5 in 3-TREE.

Since all bag vertices in $T_1$ are unmarked, Step 5 in 3-TREE selects a bag vertex and, depending upon whether it is a non-terminal or terminal bag, attempts to find a particular type of 3-separation. If there is no such 3-separation, such as when one of the bag vertices labelled $V_1$, $V_2$, $V_3$, $V_4$, $L$, $Y_1$, $Y_2$, or $Y_3$ is selected, the bag vertex is marked at Step 5i in 3-TREE. On the other hand, if there is such a 3-separation, such as when one of the bag vertices labelled $X$ or $Z$ is selected, then Step 5ii is invoked and 3-TREE calls ForwardSweep, BackwardSweep, and then updates the current $\pi$-labelled tree. For example, assume the bag vertex labelled $X$ is selected before the bag vertex labelled $Z$. When this happens, Step 5 in 3-TREE finds an appropriate 3-separation and then calls ForwardSweep using this 3-separation. The subroutine BackwardSweep is subsequently called and a possible generalized 3-path outputted by this call is

$$(E - X, [(X_1, X_2)], X_3).$$

A path realization of this generalized 3-path is then merged with the current $\pi$-labelled tree, in this case $T_1$, in Step 5(ii)c in 3-TREE to produce the $\pi$-labelled tree $T_2$ shown in Fig. 5. This process continues until all bag vertices are marked. The 3-tree finally outputted by this application of 3-TREE is shown in Fig. 3.
6. Correctness of the Algorithm and the Proof of Theorem 2.2

Let \( M \) be a 3-connected matroid with ground set \( E \), where \(|E| \geq 9\), and let \( T \) be the \( \pi \)-labelled tree outputted by 3-Tree when applied to \( M \). In this section, we prove that \( T \) is a 3-tree for \( M \) and that this application takes time polynomial in \(|E|\).

We begin with several lemmas, the first of which specifies the type of ordered partition outputted by ForwardSweep.

**Lemma 6.1.** Let \((X_0 \cup X_1, X_2)\) be a 3-path in \( M \) with \( X_0 \cup X_1 \) fully closed and let \( \mathcal{F} \) be the set of maximal sequential 3-separating sets of \( M \). Let \((X_0 \cup X'_1, X'_2, \ldots, X'_m)\) be the output of ForwardSweep when applied to \((M, (X_0 \cup X_1, X_2), \mathcal{F})\). Then \((X_0 \cup X'_1, X'_2, \ldots, X'_m)\) is a left-justified maximal \( X_0 \)-rooted 3-path of \( M \).

**Proof.** By construction, \((X_0 \cup X'_1, X'_2, \ldots, X'_m)\) is a left-justified \( X_0 \)-rooted 3-path. Thus if the lemma fails, then there is a partition \((Y_j, Z_j)\) of \( X'_j \) for some \( j \) in \([m]\) such that \((X_0 \cup X'_1 \cup \cdots \cup X'_{j-1} \cup Y_j, Z_j \cup X'_{j+1} \cup \cdots \cup X'_m)\) is a non-sequential 3-separation of \( M \). We need to show that this 3-separation is equivalent to \((X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)\) or \((X_0 \cup X'_1 \cup \cdots \cup X'_{j+1}, X'_j \cup \cdots \cup X'_m)\).

If \( j = m \), then the result follows immediately from Step 3 of ForwardSweep. Now assume that \( j < m \).

Suppose \( X_0 = \emptyset \) and \( j = 1 \). Then, because \((X_0 \cup Y_1, Z_1 \cup X'_2 \cup \cdots \cup X'_m)\) is a non-sequential 3-separation of \( M \), there is a 3-element subset \\{\( y_1, y_2, y_3 \)\} of \( Y_1 \) that is not contained in any member of \( \mathcal{F} \), and \( Z_1 \cup X'_2 \cup \cdots \cup X'_m \) clearly contains \( X'_2 \cup \cdots \cup X'_m \). Step 2i of ForwardSweep implies that every element of \( Z_1 \) is in \( \text{fcl}(X'_2 \cup \cdots \cup X'_m) \) otherwise Step 2(i) would further refine the 3-path; a contradiction. Hence every element of \( Z_1 \) is in \( \text{fcl}(Y_1) \) and \((X_0 \cup Y_1, Z_1 \cup X'_2 \cup \cdots \cup X'_m)\) is equivalent to \((X_0 \cup X'_1, X'_2 \cup \cdots \cup X'_m)\), as required.
We may now assume that either $X_0 \neq \emptyset$ or $j \geq 2$. Then, to prevent Steps 2(ii)b and 2(iii)b of ForwardSweep from further refining the 3-path, either every element of $Y_j$ is in $\mathrm{fcl}(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1})$ or every element of $Z_j$ is in $\mathrm{fcl}(X'_{j+1} \cup \cdots \cup X'_m)$. Hence $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}) \cup \bar{Y}_j, Z_j \cup X'_{j+1} \cup \cdots \cup X'_m)$ is equivalent to $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)$ or $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)$, as required.

In the rest of this section, we freely use Lemma 6.1.

**Lemma 6.2.** Let $i \geq 0$, and let $T_i$ and $T_{i+1}$ be $\pi$-labelled trees constructed by 3-Tree in Steps 3(ii)c and 5(ii)c. Suppose that $T_i$ is a conforming tree for $M$, and $T_{i+1}$ satisfies (I)-(IV) but is not a conforming tree for $M$. Let $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ be the 3-path outputted when ForwardSweep is applied in Step 3(ii)(a) or Step 5(ii)(a) of 3-Tree depending on whether $i = 0$ or $i$ is positive. Let $(R, G)$ be a non-sequential 3-separation in $M$ that does not conform with $T_{i+1}$ for which $X_0$ is monochromatic and no equivalent 3-separation in which $X_0$ is monochromatic has fewer bichromatic parts in $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$. Then $X_0 \cup X'_1$ is monochromatic unless $i = 0$. In the exceptional case, either $X'_1$ is monochromatic, or both $R \cap X'_1$ and $G \cap X'_1$ are sequential 3-separating sets with $|R \cap X'_1|, |G \cap X'_1| \geq 2$.

**Proof.** Assume that $X_0 \cup X'_1$ is bichromatic. First suppose that $i \geq 1$. Then $X_0$ is non-empty. Then, as $X_0$ is monochromatic, we may assume that $X_0 \subseteq G$. Furthermore, as $(R, G)$ does not conform with $T_{i+1}$, we have $|R \cap (X'_2 \cup \cdots \cup X'_m)| \geq 1$. Since $X_0 \cup X'_1$ is bichromatic, it follows by Lemma 3.9 that $|R \cap (X'_2 \cup \cdots \cup X'_m)| \geq 2$.

Since $G$ and $X_0 \cup X'_1$ are both 3-separating and $|R \cap (X'_2 \cup \cdots \cup X'_m)| \geq 2$, it follows by uncrossing that $G \cap (X_0 \cup X'_1)$, which equals $X_0 \cup (G \cap X'_1)$, is 3-separating. Therefore $(X_0 \cup (G \cap X'_1), (R \cap X'_1) \cup X'_2 \cup \cdots \cup X'_m)$ is a 3-separation in $M$. If this 3-separation is non-sequential, then, by Lemma 6.1, it is equivalent to $(X_0 \cup X'_1, X'_2 \cup \cdots \cup X'_m)$ and so $R \cap X'_1 \subseteq \mathrm{fcl}(G)$. In this case, we recolour all the elements in $R \cap X'_1$ green thereby reducing the number of bichromatic parts; a contradiction. Therefore either $X_0 \cup (G \cap X'_1)$ or $(R \cap X'_1) \cup X'_2 \cup \cdots \cup X'_m$ is sequential. By Lemma 3.3, the last set is not sequential as $X'_2 \cup X'_3 \cup \cdots \cup X'_m$ is non-sequential. Thus $X_0 \cup (G \cap X'_1)$ is sequential. But, as $i \geq 1$, the set $X_0$ contains at least one non-sequential 3-separation, contradicting Lemma 3.3.

Now suppose that $i = 0$. Then $X_0$ is empty. If $R \cap X'_1 = \{z\}$, then $|R \cap (E - X'_1)| \geq 2$ and so, as $G$ and $X'_1$ are both 3-separating, by uncrossing, $G \cap X'_1$ is 3-separating. Therefore, as $X'_1$ is 3-separating, it follows by Lemma 3.1 that $z \in \mathrm{cl}^{(\infty)}(G \cap X'_1)$. Thus we can recolour $z$ green thereby reducing the number of bichromatic parts; a contradiction. Hence $|R \cap X'_1| \geq 2$ and, by symmetry, $|G \cap X'_1| \geq 2$. If $R \cap (E - X'_1)$ is empty, then, as $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ is a maximal $X_0$-rooted 3-path, $(R, G)$ is equivalent to $(X'_1, E - X'_1)$. Hence $G \cap X'_1 \subseteq \mathrm{fcl}(R)$ and so we can recolour the elements in $G \cap X'_1$ red, reducing the number of bichromatic parts; a contradiction. Thus $|R \cap (E - X'_1)| \geq 1$ and so, by Lemma 3.9, $|R \cap (E - X'_1)| \geq 2$. Similarly, $|G \cap (E - X'_1)| \geq 2$. It now follows by uncrossing that both $G \cap X'_1$ and $R \cap X'_2$ are 3-separating.
Consider the 3-separation \((G \cap X'_1, E - (G \cap X'_1))\). If this 3-separation is non-sequential, then, by Lemma 6.1, it is equivalent to \((X'_1, E - X'_1)\) and so \(R \cap X'_1 \subseteq \text{fcl}(G \cap X'_1) \subseteq \text{fcl}(G)\). Thus we can recolour all the elements in \(R \cap X'_1\) green thereby reducing the number of bichromatic parts; a contradiction. Hence either \(G \cap X'_1\) or \(E - (G \cap X'_1)\) is sequential. As \(E - (G \cap X'_1)\) contains the non-sequential set \(X'_2 \cup X'_3 \cup \ldots \cup X'_m\), it follows by Lemma 3.3 that \(G \cap X'_1\) is sequential. By symmetry, \(R \cap X'_1\) is sequential, and the lemma follows.

\[\square\]

**Lemma 6.3.** The \(\pi\)-labelled tree \(T\) outputted by \(3\)-Tree is a conforming tree for \(M\). Furthermore, if \(v\) is a flower vertex of \(T\), then the flower corresponding to \(v\) is tight.

**Proof.** Let \(E\) denote the ground set of \(M\). We prove the lemma by showing that each of the \(\pi\)-labelled trees \(T_p\) constructed in Steps 3(ii)c and 5(ii)c in 3-Tree is a conforming tree for \(M\) in which the flower corresponding to each flower vertex is tight. Since \(T_0\) consists of a single bag vertex labelled \(E\), the result trivially holds if \(p = 0\). Now suppose that \(T_p\) is a conforming tree for \(M\) with the property that if \(v\) is a flower vertex of \(T_p\), then the flower corresponding to \(v\) is tight. We will show that \(T_{p+1}\) is a conforming tree for \(M\) with this additional property on its flower vertices.

It follows by induction, Lemma 6.1, and the construction in BackwardSweep that \(T_{p+1}\) satisfies (I) in the definition of a conforming tree. Furthermore, \(T_{p+1}\) trivially satisfies (II) in this definition. To see that (III) and (IV) holds for \(T_{p+1}\), let \(\Phi = (Q_1, Q_2, \ldots, Q_k)\) be a flower in \(M\) corresponding to a flower vertex \(v\) in the path realization of the generalized 3-path outputted by BackwardSweep in the construction of \(T_{p+1}\) from \(T_p\). By induction, to show that (III) and (IV) holds for \(T_{p+1}\), it suffices to show that \(v\) satisfies either (III) or (IV) depending upon whether it is labelled \(A\) or \(D\), respectively. Without loss of generality, we may assume that, relative to this generalized 3-path, \(Q_1\) is the entry petal. By construction, each petal of \(\Phi\) is 3-separating and, apart from at most one of \(Q_1 \cup Q_2\) and \(Q_1 \cup Q_k\), each pair of consecutive petals is 3-separating. Thus, by symmetry, it suffices to check that \(Q_1 \cup Q_2\) is 3-separating. This check is done by induction by showing, for all \(i\) in \(\{3, 4, \ldots, k\}\), that \(Q_3 \cup Q_4 \cup \ldots \cup Q_i\) is 3-separating. In particular, this will show that \(Q_3 \cup Q_4 \cup \ldots \cup Q_k\) is 3-separating, and so \(Q_1 \cup Q_2\) is 3-separating. Clearly, \(Q_3\) and \(Q_3 \cup Q_4\) are 3-separating. Now let \(i \geq 5\) and assume that the check holds for \(i - 1\). Then \(Q_3 \cup Q_4 \cup \ldots \cup Q_{i-1}\) and \(Q_{i-1} \cup Q_i\) are 3-separating. Therefore, as their intersection contains at least two elements, it follows by uncrossing that their union \(Q_3 \cup Q_4 \cup \ldots \cup Q_i\) is 3-separating, and we get the desired result.

To complete the proof that \(T_{p+1}\) is a conforming tree for \(M\), suppose there is a non-sequential 3-separation \((R', G')\) that does not conform with \(T_{p+1}\). Because this 3-separation does conform with \(T_p\), it is equivalent to a 3-separation \((R, G)\) such that \(R\) or \(G\) is contained in a bag of \(T_p\). Only one bag of \(T_p\) is affected in the construction of \(T_{p+1}\), so we may assume that \(R\) or \(G\) is contained in this bag \(B\). As \(X_0 = E - \pi(B)\), which may be empty, we deduce that, with respect to \((R, G)\), the set \(X_0\) is monochromatic. Thus \((R, G)\) is a non-sequential 3-separation that does not conform with \(T_{p+1}\) and has \(X_0\) monochromatic. From among the collection of
choices for \((R, G)\) satisfying these conditions, choose one such that no equivalent 3-
separation in which \(X_0\) is monochromatic has fewer bichromatic parts with respect to the left-justified maximal \(X_0\)-rooted 3-path \((X_0 \cup Z_1, Z_2, \ldots, Z_m)\) outputted by \textsc{ForwardSweep} during the construction of \(T_{p+1}\) from \(T_p\). By Lemma 6.2, we may further assume that if \(p \geq 1\), then \(X_0 \cup Z_1\) is monochromatic and, if \(p = 0\), in which case \(X_0\) is empty, either \(Z_1\) is monochromatic, or \(|R \cap Z_1|, |G \cap Z_1| \geq 2\) and each of \(R \cap Z_1\) and \(G \cap Z_1\) is a sequential 3-separating set.

First suppose that \(X_0 \cup Z_1\) is monochromatic. Without loss of generality, we may assume that \(X_0 \cup Z_1 \subseteq G\). Let \(b\) be the number of bichromatic parts amongst \(Z_2, \ldots, Z_m\). Assume \(b \geq 2\) and let \(Z_i\) be the bichromatic part with smallest subscript. If \(Z_i^- \cap R\) is non-empty, then, by Lemmas 3.8 and 3.9, \(Z_i\) is monochromatic; a contradiction. Therefore \(Z_i^- \subseteq G\). But then, by Lemma 3.11, \(Z_i^+\) is monochromatic; a contradiction as there is a bichromatic part \(Z_j\) with \(j > i\). Thus \(b \in \{0, 1\}\).

Assume \(b = 1\) and \(Z_i\) is bichromatic. We first consider \(i \neq m\). If \(Z_i^+\) is not monochromatic, then, by Lemma 3.11, \(Z_i^-\) is not monochromatic. Therefore, by Lemma 3.9, \(|R \cap Z_i^-|, |G \cap Z_i^-|, |R \cap Z_i^+|, |G \cap Z_i^+| \geq 2\). But then, by Lemma 3.8, \(Z_i\) is monochromatic; a contradiction. Thus we may assume that \(Z_i^+\) is monochromatic.

We next eliminate a special case. Say \(Z_i^-, Z_i^+ \subseteq G\). Then \(R \subseteq Z_i\). The only steps in \textsc{BackwardSweep} that do not leave \(Z_i\) intact are Steps 6i (if \(i = m - 1\) and 9i. As \((R, G)\) does not conform with \(T_{p+1}\), we may assume that one of these is invoked. Then both \(R \cap (Z_i^- \cap \text{fcl}(Z_i^+))\) and \(R \cap (Z_i \cap \text{fcl}(Z_i^+))\) are non-empty. But, as \(R \cap (Z_i \cap \text{fcl}(Z_i^+)) \subseteq \text{fcl}(Z_i^+),\) it follows that \(R \cap (Z_i \cap \text{fcl}(Z_i^+)) \subseteq \text{fcl}(G)\). Therefore we can recolour all the elements in \(R \cap (Z_i \cap \text{fcl}(Z_i^+))\) green thereby obtaining an equivalent 3-separation in which all the red elements are all in \(Z_i - \text{fcl}(Z_i^+)\), a single bag of \(T_{p+1}\). It now follows that if \(Z_i^+ \subseteq G\), then \(Z_i^- \cap R\) is non-empty.

Consider the case when \(Z_i^+ \subseteq R\). If \(Z_i^- \subseteq G\), then, by Lemma 3.11, \(Z_i^- \subseteq G\); a contradiction. Therefore \(Z_i^- \cap R \neq \emptyset\) and so, by Lemma 3.9, \(|Z_i^- \cap R| \geq 2\). Now, by Lemma 3.15, \(R \cap Z_i \subseteq \text{fcl}(Z_i^+)\). Furthermore, by recolouring if necessary, we may assume that \(R \cap Z_i = Z_i \cap \text{fcl}(Z_i^+)\). Since \(Z_i \cap Z_i^+\) and \(G\) are both 3-separating, and since \(|Z_i^- \cap R| \geq 2\), it follows by uncrossing that \(G \cap Z_i\) is 3-separating. Furthermore, by Lemma 3.10, \(Z_i\) is not 3-separating. Therefore the generalized 3-path \(\tau_i\) at the end of the iteration of \textsc{BackwardSweep} in which \(Z_i\) is considered is

\[
\tau_i = (X_0 \cup Z_1, Z_2, \ldots, Z_{i-1}, ([Z_i - \text{fcl}(Z_i^+)],[Z_i \cap \text{fcl}(Z_i^+), \tau_{i+1}(Z_i^+)])\).
\]

Now \(Z_i - \text{fcl}(Z_i^+) \subseteq G\) and \((Z_i \cap \text{fcl}(Z_i^+)) \cup Z_i^+ \subseteq R\). Let \(h\) be the smallest index for which \(Z_h^- \subseteq G\), but \(Z_h \subseteq R\). Since \(X_0 \cup Z_1 \subseteq G\) and \(|R \cap Z_i^-| \geq 2\), we have \(2 \leq h \leq i - 1\). By applying Lemma 3.12 to the 3-path \((Z_h^-, Z_h, Z_{h+1}, \ldots, Z_{i-1}, Z_i - \text{fcl}(Z_i^+), (Z_i \cap \text{fcl}(Z_i^+) \cup Z_i^+))\), we deduce that \(M\) has a flower in which the parts of the 3-path are petals of a flower. It now follows by Lemma 3.12 and the construction in \textsc{BackwardSweep} that \(T_{p+1}\) displays \((R, G)\), and so \((R, G)\) conforms with \(T_{p+1}\). This contradiction implies that we may assume \(Z_i^+ \subseteq G\).

The case when \(Z_i^+ \subseteq G\) is handled similarly to that when \(Z_i^+ \subseteq R\). Note that \(Z_i^- \cap R\) is non-empty as a result of the consideration of the above special case.
Now suppose that \( i = m \). If \( Z_m^- \) is monochromatic, that is, \( Z_m^- \subseteq G \), then either \( (X_0 \cup Z_1, Z_2, \ldots, Z_m) \) is not left-justified or it is not maximal; a contradiction. Therefore \( Z_m^- \) is not monochromatic, and so \( m \geq 3 \). Furthermore, as \( |G \cap Z_m^-| \geq 2 \) and both \( Z_m \) and \( R \) are 3-separating, uncrossing implies that \( R \cap Z_m \) is 3-separating. Therefore if \( |G \cap Z_m| = 1 \), then \( Z_m \subseteq \text{fcl}(R \cap Z_m) \) by Lemma 3.1, and so we can recolour the element of \( G \cap Z_m \) red to obtain a 3-separation equivalent to \((R, G)\) with fewer bichromatic parts; a contradiction. Thus \( |G \cap Z_m| \geq 2 \). A similar argument shows that \( |R \cap Z_m| \geq 2 \).

We show next that \( Z_{m-1}^- \) is 3-separating. Say \( Z_{m-1} \subseteq R \). Then, as \( R \) and \( Z_m \cup Z_{m-1} \) are both 3-separating and \( |G \cap Z_{m-1}| \geq 2 \), it follows that \( R \cap (Z_m \cup Z_{m-1}) \) is 3-separating. Therefore, as \( Z_m^- \) is 3-separating and \( |G \cap Z_m^-| \geq 2 \), it follows by uncrossing again that \( Z_{m-1}^- \) is 3-separating. Using the fact that \( Z_m^- \) is not monochromatic, the same argument shows that if \( Z_{m-1}^- \subseteq G \), then \( Z_{m-1}^- \) is 3-separating. Thus Step 5 in BACKWARDSWEEP is invoked. Furthermore, as \( Z_{m-1}^- \cup (R \cap Z_m^-) \) is a non-sequential 3-separating set if \( Z_{m-1}^- \subseteq R \) and, similarly, \( Z_{m-1}^- \cup (G \cap Z_m^-) \) is a non-sequential 3-separating set if \( Z_{m-1}^- \subseteq G \), it follows that Step 5 finds a 3-separation \((U, V)\) as described in that step. By Lemma 3.14, \( R \cap Z_m \) and \( G \cap Z_m \) are sequential 3-separating sets. Hence, by Lemma 3.13, we may assume, by recolouring if necessary, that both \( U \cap Z_m \) and \( V \cap Z_m \) are monochromatic. Let \( h \) denote the smallest index for which \( Z_h^- \subseteq G \), but \( Z_h \subseteq R \). Then, by Lemma 3.12, \( M \) has a flower with petals \( Z_h^- \cup Z_{h+1} \cup \ldots \cup Z_m \cup Z_{m-1} \cup U \cap Z_m \cup V \cap Z_m \). Thus, by Lemma 3.12 and the construction in BACKWARDSWEEP, \( T_{p+1} \) displays \((R, G)\), and so \((R, G)\) conforms with \( T_{p+1} \); a contradiction.

Now assume \( b = 0 \). Let \( h \) denote the smallest index for which \( Z_h^- \subseteq G \), but \( Z_h \subseteq R \). Say \( Z_h \cup Z_h^+ \) is not monochromatic. Let \( h' \) denote the largest index for which \( Z_h \cup Z_h^+ \) is not monochromatic, but \( Z_h^- \subseteq G \). Note that \( h' \geq h \). Then it follows by Lemma 3.12 that each of the sets \( Z_h \cup Z_{h+1} \cup \ldots \cup Z_{h'} \) is 3-separating and so, by the construction in BACKWARDSWEEP and Lemma 3.12, \( T_{p+1} \) displays \((R, G)\) as the petals of a flower; a contradiction. Now say \( Z_h \cup Z_h^+ \) is monochromatic. It follows from the construction in BACKWARDSWEEP that the only way in which \((R, G)\) does not conform with \( T_{p+1} \) is when \( h \geq 3 \) and Step 9 of BACKWARDSWEEP is invoked when \( Z_{h-1}^- \) is considered. But then we can recolour all the elements in \( Z_{h-1}^- \cap \text{fcl}(Z_h \cup Z_h^+) \) red giving a 3-separation equivalent to \((R, G)\), thereby resulting in \( T_{p+1} \) displaying \((R, G)\); a contradiction. This completes the analysis for when \( X_0 \cup Z_1 \) is monochromatic.

Suppose that \( p = 0 \) and \( Z_1 \) is bichromatic. Recall that \( X_0 \) is empty and that \( |R \cap Z_1|, |G \cap Z_1| \geq 2 \) and each of \( R \cap Z_1 \) and \( G \cap Z_1 \) is a sequential 3-separating set. Let \( b \) denote the number of bichromatic parts amongst \( Z_1, \ldots, Z_m \). By Lemmas 3.8 and 3.9, \( b \in \{1, 2\} \). First assume that \( b = 2 \), and let \( Z_i \) denote the bichromatic part with \( i > 1 \). Say \( i \neq m \). By Lemmas 3.8 and 3.9, \( Z_i^+ \) is monochromatic. Without loss of generality, we may assume that \( Z_i^- \subseteq R \). By Lemma 3.10, \( Z_i \) is not 3-separating. Furthermore, by Lemma 3.15, \( R \cap Z_i \subseteq \text{fcl}(Z_i^+) \). By recolouring if necessary and moving to a 3-separation equivalent to \((R, G)\), we may assume that \( R \cap Z_i = Z_i \cap \text{fcl}(Z_i^+) \). Since \( |R \cap Z_i^-| \geq 2 \) and since both \( G \) and \( Z_i \cup Z_i^+ \) are 3-separating, it follows by uncrossing that \( G \cap Z_i \), which equals \( Z_i - \text{fcl}(Z_i^+) \),
is 3-separating. Thus, by the construction in \textsc{BackwardSweep}, the generalized 3-path \( \tau_i \) at the end of the iteration in which \( Z_i \) is considered is 
\[
\tau_i = (Z_1, Z_2, \ldots, Z_{i-1}, [Z_i - \text{fcl}(Z_i^+)], Z_i \cap \text{fcl}(Z_i^+), \tau_{i+1}(Z_i^+)).
\]
Now \( Z_i - \text{fcl}(Z_i^+) \subseteq G \) and \( (Z_i \cap \text{fcl}(Z_i^+)) \cup Z_i^+ \subseteq R \) and so, by Lemma 3.12, \( M \) has a flower with petals \( Z_1, Z_2, \ldots, Z_{i-1}, Z_i - \text{fcl}(Z_i^+), (Z_i \cap \text{fcl}(Z_i^+)) \cup Z_i^+ \). By the construction in \textsc{BackwardSweep} and Lemma 3.12, \( \tau_2 \) is eventually constructed and is of the form
\[
\tau_2 = (Z_1, ([P_1, \ldots, P_p], (Q_1, \ldots, Q_q)], (Z_i \cap \text{fcl}(Z_i^+)) \cup Z_i^+),
\]
where \( \{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} = \{Z_2, \ldots, Z_{i-1}, Z_i - \text{fcl}(Z_i^+)\} \). Thus Step 10i is invoked. As the second petal on the last list is monochromatic, it follows by uncrossing that Step 10i finds a 3-separation \((U, V)\) as described in that step. By Lemma 3.13, we may assume that both \( U \cap Z_1 \) and \( V \cap Z_1 \) are monochromatic. Thus, by Lemma 3.12 again, it follows that \( M \) has a flower with petals
\[
V \cap Z_1, U \cap Z_1, Z_2, \ldots, Z_{i-1}, Z_i - \text{fcl}(Z_i^+), (Z_i \cap \text{fcl}(Z_i^+)) \cup Z_i^+.
\]
Therefore, by Lemma 3.12 and construction, \((R, G)\) is displayed by \( T_{p+1} \); a contradiction.

Now say \( i = m \). Since \(|G \cap Z_1| \geq 2\) and both \( R \) and \( Z_m \) are 3-separating, it follows by uncrossing that \( R \cap Z_m \) is 3-separating. Therefore if \(|G \cap Z_m| = 1\), then \( Z_m \subseteq \text{fcl}(R \cap Z_m) \) by Lemma 3.1. Thus we can recolour the single green element in \( Z_m \) red thereby obtaining an equivalent 3-separation with fewer bichromatic parts; a contradiction. Hence \(|G \cap Z_m| \geq 2\) and, by symmetry, \(|R \cap Z_m| \geq 2\).

There are two cases depending upon whether \( m = 2 \) or \( m \geq 3 \). If \( m \geq 3 \), then, without loss of generality, we may assume that \( Z_{m-1} \subseteq R \). Since \(|G \cap Z_1| \geq 2\) and since \( R \) and \( Z_{m-1} \cup Z_m \) are both 3-separating, it follows by uncrossing that \( R \cap (Z_{m-1} \cup Z_m) \) is 3-separating. Therefore, as \( Z_m \) is 3-separating and \(|G \cap Z_m| \geq 2\), an application of uncrossing implies that \( Z_{m-1} \) is 3-separating. Thus Step 5 of \textsc{BackwardSweep} is invoked. Since \( Z_{m-1} \cup (R \cap Z_m) \) is a 3-separating set, it follows that Step 5 of \textsc{BackwardSweep} finds a 3-separation \((U, V)\) as described in that step. By Lemma 3.14, \( R \cap Z_m \) and \( G \cap Z_m \) are sequential 3-separating sets. Therefore, by Lemma 3.13, we can recolour some elements of \( Z_m \) if necessary to get an equivalent 3-separation in which both \( U \cap Z_m \) and \( V \cap Z_m \) are monochromatic. Lemma 3.12 now implies that \( M \) has a flower with petals \( Z_1, Z_2, \ldots, Z_{m-1}, U \cap Z_m, V \cap Z_m \). By the construction in \textsc{BackwardSweep} and Lemma 3.12, \( \tau_2 \) is eventually constructed and is of the form
\[
\tau_2 = (Z_1, ([P_1, \ldots, P_p], (Q_1, \ldots, Q_q)], W \cap Z_m),
\]
where \( \{P_1, \ldots, P_p, Q_1, \ldots, Q_q, W \cap Z_m\} = \{Z_2, \ldots, Z_{m-1}, U \cap Z_m, V \cap Z_m\} \) and \( W \in \{U, V\} \). Thus Step 10i is invoked. As \( Z_2 \) is monochromatic, it follows by uncrossing that Step 10i finds a 3-separation \((U', V')\) as described in that step. By Lemma 3.13, we may assume that \( U' \cap Z_1 \) and \( V' \cap Z_1 \) are monochromatic. Therefore, by Lemma 3.12, it follows that \( M \) has a flower with petals
\[
V' \cap Z_1, U' \cap Z_1, Z_2, \ldots, Z_{m-1}, U \cap Z_m, V \cap Z_m.
\]
Thus, by Lemma 3.12 and construction, \((R, G)\) is displayed by \(T_{p+1}\); a contradiction. A similar analysis holds when \(m = 2\), where we invoke Step 2 instead of Steps 5 and 10 in BackwardSweep.

Now assume that \(b = 1\). Then \(Z_1\) is the only bichromatic part. Since \(R \cap Z_1\) and \(G \cap Z_1\) are sequential 3-separating sets, \(Z_1^+\) is not monochromatic. So \(m \geq 3\). Let \(h\) denote the largest index for which \(Z_h \cup Z_h^+\) is not monochromatic, but \(Z_h^+\) is monochromatic. By Lemma 3.12, \(M\) has a flower with petals \(Z_1, Z_2, \ldots, Z_h, Z_h^+\). Therefore, by the construction and Lemma 3.12, \(\tau_2\) is eventually constructed and is of the form

\[
\tau_2 = (Z_1, [(P_1, \ldots, P_p), (Q_1, \ldots, Q_q)], Z_h^+)
\]

where \(\{P_1, \ldots, P_p, Q_1, \ldots, Q_q\} = \{Z_2, \ldots, Z_h\}\). Thus Step 10i is invoked. Since \(Z_2\) is monochromatic, it follows by uncrossing that Step 10i finds a 3-separation \((U, V)\) as described in that step. By Lemma 3.13, we may assume that \(U \cap Z_1\) and \(V \cap Z_1\) are monochromatic. Thus, by Lemma 3.12, \(M\) has a flower with petals \(V \cap Z_1, U \cap Z_1, Z_2, \ldots, Z_h, Z_h^+\). Thus, by Lemma 3.12 and construction, \(T_{p+1}\) displays \((R, G)\); a contradiction. We conclude that \(T_{p+1}\) is a conforming tree for \(M\).

We next show that if \(v\) is a flower vertex of \(T_{p+1}\), then the flower corresponding to \(v\) is tight. By induction, \(T_p\) has this property on its flower vertices. Therefore, by construction, it suffices to only consider the flower vertices on the path realization, \(P_{p+1}\) say, of the generalized 3-path outputted by BackwardSweep in Step 5 of 3-Tree in the construction of \(T_{p+1}\) from \(T_p\). Let \((X_0 \cup X_1, X_2, \ldots, X_m)\) be the left-justified maximal \(X_0\)-rooted 3-path outputted by ForwardSweep in the construction of \(T_{p+1}\) from \(T_p\) of 3-Tree. Let \(v\) be a flower vertex of \(P_{p+1}\) and suppose that \(\Phi\), the flower corresponding to \(v\), is not tight. By definition, we may assume that \(v\) has degree at least three. For clarity, we will assume that Step 9i in BackwardSweep is not invoked in the construction of \(\Phi\). The straightforward extension of the proof below to include the case when this step is invoked is omitted.

It follows from the description of BackwardSweep that if no end moves are performed, then, for some \(i\) and \(j\) with \(1 \leq i \leq j \leq m\), the sets \(X_i^-\) and \(X_j^+\) are the entry and exit petals of \(\Phi\), respectively, and \(\{X_i, X_{i+1}, \ldots, X_j\}\) is the union of the sets of clockwise and anticlockwise petals of \(\Phi\). If end moves are performed, then either \(X_i^- \cup X_i = X_0 \cup X_1\), or \(X_j \cup X_j^+ = X_m\). Ignoring the possibility of end moves for now, if \(X_i^-\) is loose, then \(X_i^- \subseteq \text{fcl}(X_i \cup X_i^+)\), and so \((X_i^-, X_i \cup X_i^+)\) is sequential; a contradiction. Similarly, we get a contradiction if \(X_j^+\) is loose. Now assume that, for some \(i \leq s \leq j\), the petal \(X_s\) is loose. Without loss of generality, we may assume that \(X_{s-1}\) is tight where, if \(s = i\), we take \(X_{s-1}\) to be \(X_i^-\). By Lemma 3.6, \(X_s \subseteq \text{fcl}(X_{s-1})\) and so \(X_s \subseteq \text{fcl}(X_s^-)\). But then \((X_s^-, X_s \cup X_s^+)\) is equivalent to \((X_s^- \cup X_s, X_s^+)\), contradicting that \((X_0 \cup X_1, X_2, \ldots, X_m)\) is a 3-path.

Now consider the possibility of end moves. If \(X_i^- \cup X_i = X_0 \cup X_1\), then Step 10i of BackwardSweep is invoked, in which case, \(X_i^-\) and \(X_i\) are both sequential. Say \(X_i^-\) is loose. By concatenating the petals \(X_{i+1}, \ldots, X_m\) into a single petal, \(X_{i+1} \cup \cdots \cup X_m\) is a tight petal in the resulting flower, while \(X_i^-\) remains loose. Thus, by Lemma 3.6, \(X_i^- \subseteq \text{fcl}(X_{i+1} \cup \cdots \cup X_m)\). Therefore, by Lemma 3.7, there is an
ordering $x_1, x_2, \ldots, x_l$ of the elements of $X_-$ such that $(X_1, x_l, x_{l-1}, \ldots, x_1, X_{i+1} \cup \cdots \cup X_m)$ is a 3-sequence in $M$. But $X_i$ is sequential and it follows that $X^- \cup X_i = X_0 \cup X_1$ is sequential; a contradiction. Hence $X_-$ is tight and, similarly, $X_i$ is tight. The case $X_j \cup X_+ = X_m$ is handled analogously. We conclude that if $v$ is a flower vertex of $T_{p+1}$, then the flower corresponding to $v$ is tight. This completes the proof of the lemma. □

It follows by Lemma 6.3 that $T$ is a conforming tree for $M$. The following is a straightforward consequence of the way in which flowers are constructed in the algorithm.

**Lemma 6.4.** The conforming tree $T$ for $M$ outputted by 3-Tree has the property that every flower corresponding to a flower vertex in $T$ displays at least two inequivalent non-sequential 3-separations.

**Proof.** First note that, by construction, all flower vertices in $T$ have degree at least three. Now, except when we invoke an end move, every flower that is constructed in the algorithm has an entry petal and an exit petal and these correspond to inequivalent non-sequential 3-separations. When an end move is invoked, we already have one non-sequential 3-separation and it is easily checked that there is a second inequivalent one $(U, V)$ with the split part, or parts in the case $m = 2$, having non-empty intersection with $U$ and $V$. □

**Lemma 6.5.** The conforming tree $T$ for $M$ outputted by 3-Tree has the property that every flower corresponding to a flower vertex in $T$ is a tight maximal flower.

**Proof.** Let $\Phi$ be a flower corresponding to a flower vertex in $T$. By Lemma 6.3, $\Phi$ is tight. Assume that $\Phi$ is not maximal. Then there is a tight maximal flower $\Phi_m$ that displays, up to equivalence, all non-sequential 3-separations that are displayed by $\Phi$ as well as at least one non-sequential 3-separation $(R, G)$ that, up to equivalence, is not displayed by $\Phi$. In particular:

**6.5.1.** For every union $U$ of petals of $\Phi$ such that $(U, E - U)$ is a non-sequential 3-separation in $M$, there is a union $U'$ of petals of $\Phi_m$ such that $(U, E - U) \equiv (U', E - U')$.

We may assume that $\Phi_m = (Q_1, Q_2, \ldots, Q_n)$ and that $R = Q_1 \cup Q_2 \cup \cdots \cup Q_k$ for some $k \leq n - 1$. As $(R, G)$ is not displayed by $\Phi$, an equivalent 3-separation $(R', G')$ must conform with $T$. Hence we may assume that $R'$ is properly contained in some petal $P$ of $\Phi$. Then, by Lemma 3.3, $P$ is non-sequential. If $E - P$ is sequential, then it follows by Lemma 3.3 that $\Phi$ displays no non-sequential 3-separations; a contradiction. Hence $(P, E - P)$ is non-sequential and $\Phi_m$ displays an equivalent 3-separation $(P', E - P')$. Thus $(P', E - P') = (\cup_{i \in I} Q_i, \cup_{j \in [n] - I} Q_j)$ for some subset $I$ of $[n]$. Suppose $|[n] - I| = 1$. By Lemma 6.4, $\Phi$ displays a non-sequential 3-separation $(O, E - O)$ that is not equivalent to $(P, E - P)$. As $P$ is a petal of $\Phi$, we must have that fcl$(P)$ is a proper subset of fcl$(O)$ or fcl$(E - O)$. Some 3-separation $(O', E - O')$ equivalent to $(O, E - O)$ is displayed by $\Phi_m$. Since $\Phi_m$ has only one petal outside of $P'$, (6.5.1) implies that $O'$ or $E - O'$ is contained in $P'$. Hence
Since \( \text{fcl}(R) = \text{fcl}(Q_1 \cup Q_2 \cup \cdots \cup Q_k) = \text{fcl}(P') \subseteq \text{fcl}(P') = \text{fcl}(\cup_{i \in I} Q_i) \) and \( \Phi_m \) is a tight flower, it follows that \( [k] \subseteq I \). Moreover, \( I \) must contain at least one element not in \( [k] \) since no 3-separation equivalent to \((R,G)\) is displayed by \( \Phi \). Thus we may assume that \( I = \{ n-s+1, \ldots, n, 1, 2, \ldots, k, k+1, \ldots, k+t \} \) where \( s > 0 \) and \( k+t \leq n-s-2 \). Now let \( Q = Q_1 \cup Q_2 \cup \cdots \cup Q_{k+t+1} \). This is a union of consecutive petals of \( \Phi_m \) that contains at least two petals and avoids at least two petals. Hence, by Lemma 3.4(ii) \((Q,E-Q)\) is a non-sequential 3-separation in \( M \). Thus \((Q,E-Q)\) is equivalent to a 3-separation \((Q',E-Q')\) that conforms with \( T \). Hence either

(i) \((Q',E-Q')\) is displayed by \( \Phi \); or
(ii) \(Q'\) or \(E-Q'\) is contained in a petal of \( \Phi \).

Let \( \Phi = (P_1, P_2, \ldots, P_m) \). Recall that \( \text{fcl}(P) = \text{fcl}(P') = \text{fcl}(\cup_{i \in I} Q_i) \) where \( I = \{ n-s+1, \ldots, n, 1, 2, \ldots, k \} \). Suppose first that (i) holds. Then we may assume that \( Q' = \cup_{i \in K} P_i \) for some proper subset \( K \) of \( [m] \). Now \( \text{fcl}(E-Q') = \text{fcl}(Q_{k+t+2} \cup \cdots \cup Q_n) \) so \( \text{fcl}(E-Q') \) does not contain \( Q_1 \); otherwise, by Lemma 3.4(i), \( Q_1 \) is loose. But \( Q_1 \subseteq \text{fcl}(P) \) for \( P \in \{ P_i : i \in K \} \). Then \( Q_n \subseteq \text{fcl}(P) \subseteq \text{fcl}(\cup_{i \in K} P_i) = \text{fcl}(Q') = \text{fcl}(Q_1 \cup Q_2 \cup \cdots \cup Q_{k+t+1}) \). It follows by Lemma 3.4(i) that \( Q_n \) is loose; a contradiction. We deduce that (ii) does not hold so (ii) holds.

Assume that \( Q' \subseteq P_1 \). Then \( Q_{k+t+1} \subseteq \text{fcl}(Q') \subseteq \text{fcl}(P_1) \). But \( Q_{k+t+1} \nsubseteq \text{fcl}(P) \), otherwise, by Lemma 3.4(ii), \( Q_{k+t+1} \) is loose. So \( P \neq P_1 \). Now, as \( Q' \subseteq P_1 \) and \( R' \subseteq P \subseteq E - P_1 \), it follows by Lemma 3.3 that \((P_1,E-P_1)\) is non-sequential. Thus, by (6.5.1), there is a union \( \cup_{j \in J} Q_j \) of petals of \( \Phi_m \) such that \((P_1,E-P_1) \cong (\cup_{j \in J} Q_j, \cup_{j \in [n]-J} Q_j) \). Now \( Q_1 \subseteq \text{fcl}(Q') \subseteq \text{fcl}(P_1) = \text{fcl}(\cup_{j \in J} Q_j) \) and \( Q_1 \subseteq \text{fcl}(P) \subseteq (E-P_1) \subseteq \text{fcl}(\cup_{j \in [n]-J} Q_j) \). Thus we have a contradiction to Corollary 3.5.

We may now assume that \( E-Q' \subseteq P_1 \). Suppose first that \( P \neq P_1 \). Then \( P \subseteq Q' \), so \( Q_n \subseteq \text{fcl}(P) \subseteq \text{fcl}(Q_1 \cup Q_2 \cup \cdots \cup Q_{k+t+1}) \). Hence, by Lemma 3.4(i), \( Q_n \) is loose; a contradiction. We deduce that \( P = P_1 \). Recall that \( k+t \leq n-s-2 \). Thus we have \( Q_{k+t+2} \subseteq \text{fcl}(E-Q') \subseteq \text{fcl}(P) = \text{fcl}(Q_{n-s-1} \cup \cdots \cup Q_n \cup Q_1 \cup \cdots Q_{k+1}) \), so, by Lemma 3.4(i) again, \( Q_{k+t+2} \) is loose; a contradiction.

**Proof of Theorem 2.2.** To prove the theorem, we show that 3-TREE is a polynomial-time algorithm for finding a 3-tree for \( M \). Let \( T \) be the tree outputted by an application of 3-TREE to \( M \). Then every vertex of \( T \) is marked. Moreover, by Lemmas 6.3 and 6.5, \( T \) is a partial 3-tree for \( M \). Now \( T \) is a 3-tree for \( M \) unless there is a non-sequential 3-separation of \( M \) with the property that no equivalent 3-separation is displayed by \( T \). So assume there is such a 3-separation \((R,G)\). Since \( T \) is conforming, we may assume, by taking an equivalent 3-separation if necessary, that \( G \) is contained in a bag \( B \) of \( T \). If \( T \) consists of the single bag vertex \( B \), then Step 3 of 3-TREE would have found a non-sequential 3-separation \((Y,Z)\) of \( M \); a contradiction. So assume that \( T \) consists of at least two vertices. Then Step 5
of 3-Tree would have found a non-sequential 3-separation \((Y, Z)\) of \(M\) with the property that \(Z \subseteq \pi(B)\), contradicting the fact that \(B\) is marked. Hence \(T\) is a 3-tree for \(M\).

We next show that 3-Tree runs in polynomial time in the size \(n\) of \(|E(M)|\). We showed in Section 4 that the collection \(F\) of maximal sequential 3-separating sets of \(M\) can be constructed in polynomial time in \(n\) and that, for fixed disjoint subsets \(Y_1\) and \(Z_1\) of \(E(M)\), we can find a 3-separation \((Y, Z)\) with \(Y_1 \subseteq Y\) and \(Z_1 \subseteq Z\), if one exists, in polynomial time in \(n\). Extending this, we see that whenever 3-Tree is called upon to find a particular type of 3-separation, it either finds such a 3-separation or correctly determines that there is no such 3-separation in time polynomial in \(n\). Therefore, as every 3-path of \(M\) has length \(O(n)\), it follows by Lemma 6.1 that each complete call from 3-Tree to ForwardSweep takes time polynomial in \(n\). Now consider a call from 3-Tree to the subroutine BackwardSweep. Starting with \(Z_m\), each iteration of BackwardSweep considers either a subset \(Z_i\) of \(E(M)\) where \(i \in \{2, 3, \ldots, m\}\), or the subset \(X_0 \cup Z_1\) of \(E(M)\). In the cases of \(Z_m\) and \(X_0 \cup Z_1\), BackwardSweep determines if there is a 3-separation \((U, V)\) with each of \(U\) and \(V\) containing certain subsets of \(E(M)\). As above, it follows that the time taken for BackwardSweep to consider each of \(Z_m\) and \(X_0 \cup Z_1\) is polynomial in \(n\). For each of the subsets \(Z_2, Z_3, \ldots, Z_m\), it is clear that their consideration is also polynomial time in \(n\). Note that finding the full closure of a subset \(X\) of \(E(M)\) as in Step 9 of BackwardSweep takes time \(O(n^2)\). Since \(m \leq n\), it follows that each complete call from 3-Tree to BackwardSweep takes time polynomial in \(n\). At the completion of each call to BackwardSweep, the algorithm 3-Tree extends the current \(\pi\)-labelled tree to a new \(\pi\)-labelled tree in polynomial time in \(n\). This extension is non-trivial in that at least one new edge is created. Since the terminal bags of each such constructed \(\pi\)-labelled tree contain at least two elements of \(E(M)\) and there is no empty bag vertex of degree two, the number of edges of each constructed \(\pi\)-labelled tree is linear in \(n\), and so the total number of calls to ForwardSweep and BackwardSweep from 3-Tree is \(O(n)\).

As marked bags are never reconsidered, we deduce that 3-Tree terminates in time polynomial in \(n\). This completes the proof of the theorem.

\[\square\]

7. An Alternative Approach

The algorithm implicit in [6] for finding a 3-tree for a 3-connected matroid \(M\) with at least nine elements begins by constructing a tight maximal flower \(\Phi\) for \(M\) and uses the fact that \(\Phi\) is a partial 3-tree. This partial 3-tree is then modified to display more and more of the non-sequential 3-separations of \(M\) until eventually a 3-tree is obtained. However, it is not clear how to construct a tight maximal flower in polynomial time. We can certainly find a non-sequential 3-separation \((X, Y)\) quickly if one exists. The problem arises with testing in polynomial time whether \((X, Y)\) is a tight maximal flower or whether it can be refined. Curiously, once we have a tight flower with at least three petals, we can modify the techniques used above to quickly test whether it can be refined and, if so, to find such a refinement. Furthermore, when \((X, Y)\) can be refined to a paddle with at least three petals, we can detect that by finding a 1-separation in one of \(si(M/X)\) and \(si(M/Y)\) and this can be done quickly by using Proposition 4.2. By duality, we can deal with the case
when \((X,Y)\) can be refined to a copaddle with at least three petals. What seems difficult to detect in polynomial time is whether \((X,Y)\) can be refined to a flower with at least three petals in which the local connectivity between the petals is one. Even if this approach could be made to work, it seems more complicated than the approach we have adopted here although both approaches rely on the same basic technique for finding 3-separations.

Lastly, Step 3 of 3-TREE locates a non-sequential 3-separation of a 3-connected matroid \(M\) and uses this to begin the construction of a 3-tree for \(M\). If we already know some 3-separation for \(M\), we can use it as \((Y,Z)\) in this step of the algorithm and proceed with the rest of the algorithm as stated.

**References**


