THE STRUCTURE OF THE 4-SEPARATIONS IN 4-CONNECTED MATROIDS

JEREMY AIKIN AND JAMES OXLEY

Abstract. For a 2-connected matroid M, Cunningham and Edmonds gave a tree decomposition that displays all of its 2-separations. When M is 3-connected, two 3-separations are equivalent if one can be obtained from the other by passing through a sequence of 3-separations each of which is obtained from its predecessor by moving a single element from one side of the 3-separation to the other. Oxley, Semple, and Whittle gave a tree decomposition that displays, up to this equivalence, all non-trivial 3-separations of M. Now let M be 4-connected. In this paper, we define two 4-separations of M to be 2-equivalent if one can be obtained from the other by passing through a sequence of 4-separations each obtained from its predecessor by moving at most two elements from one side of the 4-separation to the other. The main result of the paper proves that M has a tree decomposition that displays, up to 2-equivalence, all non-trivial 4-separations of M.

1. Introduction

The matroid terminology used here will follow Oxley [3]. The purpose of this paper is to generalize the main result of [4] by giving a tree decomposition for the 4-separations in a 4-connected matroid. Let M be a matroid with ground set E. The *connectivity function* λ_M of M is defined for all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. For a positive integer k, the set X is *k-separating* if $\lambda_M(X) \leq k - 1$. When equality holds here, we say that the set X and the partition $(X, E - X)$ are *exactly k-separating.* If X is k-separating and $\min\{|X|, |E - X|\} \geq k$, then $(X, E - X)$ is a *k-separation* of M having *sides* X and $E - X$. For an integer n exceeding one, M is *n-connected* if it has no k-separation for any $k < n$. A subset Z of E is *fully closed* if $\mathrm{cl}(Z) = Z = \mathrm{cl}^*(Z)$. The *full closure* $\mathrm{fcl}(Y)$ of a set Y is the intersection of all fully closed sets containing Y. It can be obtained from Y by first taking the closure of Y, then taking the coclosure of the result and repeating this process until no further elements can be added. The *local connectivity* $\sqcap(Y, Z)$ of subsets Y and Z of $E(M)$ is $r(Y) + r(Z) - r(Y \cup Z)$. For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

Date: May 31, 2011.
1991 *Mathematics Subject Classification.* 05B35.
Key words and phrases. 4-connected matroid, 4-separation, full 2-span, 4-flower, 4-tree.

Cunningham and Edmonds [2] (see also [3, Section 8.3]) considered the structure of 2-separations in a 2-connected matroid M and showed that there is a labelled tree that displays all 2-separations of M. When Oxley, Semple, and Whittle [4] sought to describe the structure of 3-separations in a 3-connected matroid, the way in which such 3-separations can interlock led them to define the following equivalence relation. In a 3-connected matroid M, two exactly 3-separating partitions (X_1, Y_1) and (X_2, Y_2) of $E(M)$ are *equivalent* if $\text{fcl}(X_1) = \text{fcl}(X_2)$ and $\text{fcl}(Y_1) = \text{fcl}(Y_2)$. A 3-separation (X, Y) is *sequential* if Y or X is *sequential*, that is, if $\text{fcl}(X) = E(M)$ or $\text{fcl}(Y) = E(M)$.

For all 3-connected matroids having at least nine elements, a tree decomposition is given in [4] that guarantees to display, up to equivalence, all non-sequential 3-separations of the matroid. Some of the vertices of this tree decomposition are labelled by flower vertices, a flower being a structure that was introduced to deal with crossing 3-separations. This notion was generalized by Aikin and Oxley [1]. For integers k and n exceeding one, a partition $(P_1, P_2, \dots, P_n)$ of the ground set E of a matroid M is a *k-flower* with *petals* $P_1, P_2, \dots, P_n$ if each P_i is exactly k-separating and, when $n \geq 3$, each $P_i \cup P_{i+1}$ is exactly k-separating. It is not difficult to show that $\sqcap(P_i, P_{i+1}) = \sqcap(P_j, P_{j+1})$ for all i and j in $[n]$. It is convenient to view (E) as a k-flower with a single petal. We call it a *trivial k-flower*. When M is a 3-connected matroid, a 3-flower is what is defined in [4] as a flower. Aikin and Oxley [1] generalized a result of [4] by showing that every non-trivial k-flower $(P_1, P_2, \dots, P_n)$ is either a *k-anemone* or a *k-daisy*. In the first of these, $\cup_{i \in I} P_i$ is exactly k-separating for all non-empty proper subsets I of $[n]$; in the second, $\cup_{i \in I} P_i$ is exactly k-separating only if I is a non-empty proper subset of $[n]$ whose members are consecutive in the cyclic order $(1, 2, \dots, n)$. In a 4-connected matroid, if a set X is exactly 4-separating, then $\min\{|X|, |E - X|\} \geq 3$. Thus each petal of a 4-flower in a 4-connected matroid must have at least three elements.

The connectivity function of a matroid M has a number of attractive properties. In particular, since we can rewrite $\lambda_M(X)$ as $r(X) + r^*(X) - |X|$, we see that $\lambda_M = \lambda_{M^*}$. Moreover, $\lambda_M(X) = \lambda_M(E - X)$. We often abbreviate λ_M as λ. This function is submodular, that is, $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E(M)$. The next lemma is a consequence of this. We make frequent use of it here and we write *by uncrossing the sets X and Y* or just "by uncrossing" to mean "by an application of Lemma 1.1".

Lemma 1.1. *Let M be a 4-connected matroid, and let X and Y be 4-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 3$, then $X \cup Y$ is 4-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 3$, then $X \cap Y$ is 4-separating.*

A number of the results here can be obtained from corresponding results in [4] by making the appropriate modifications to the proofs. When these changes are routine, we have omitted the details assuming that the reader has access to [4]. We concentrate here on the differences that exist between the tree descriptions of 3-separations and 4-separations. One of the primary differences is that, in order to be able to give the desired tree description of 4-separations, we impose on 4-separations a new type of equivalence, which we call 2-equivalence. The need for this new concept and its formal definition are given in Section 2. Following that, Section 3 investigates the properties of equivalence of 4-flowers, while Section 4 treats maximal 4-flowers. The main result of the paper, Theorem 5.1, is proved in Section 5.

2. A New Equivalence of 4-Separations

In this section, we begin to describe the structure of 4-separations in 4-connected matroids. Our work in [1] provides a general description of the behaviour of crossing separations in arbitrary matroids. We use these results specialized to the case of 4-flowers and in the context of 4-connected matroids. We begin by providing some examples to illustrate the complexity that can arise when looking at 4-separations in 4-connected matroids. We use these examples as motivation for developing a notion of equivalence for 4-separations in 4-connected matroids that is different from the notion of equivalence for 3-separations, which was defined above.

Our primary goal is to be able to display the 4-separations of a 4-connected matroid. We say that a 4-flower Φ *displays* a 4-separating partition (X, Y) of $E(M)$ if X is a union of petals of Φ. The structure used to display both 2-separations in [6] and 3-separations in [4] was a tree structure. It is reasonable that we expect to display 4-separations in a tree structure as well and that the 4-separations will be displayed by edges and 4-flower vertices of the tree. The number of 4-separations and the complexity of their interactions means that we will be content with imposing an equivalence relation on those 4-separations and displaying at least one representative from each equivalence class, as was done in [4].

Equivalence of 3-separations in 3-connected matroids is defined in terms of the full closure operator. Writing $e \in \mathrm{cl}^{(*)}(X)$ to indicate that e is in the closure or the coclosure of X, we note that, for an exactly 3-separating partition (X, Y) of a 3-connected matroid M with z in Y and $|Y| \geq 3$, the partition $(X \cup z, Y - z)$ is exactly 3-separating if and only if $z \in \mathrm{cl}^{(*)}(X)$. Hence we can view equivalence of 3-separations in terms of moving one element at a time from one side of a 3-separation to the other.

A key step in displaying 3-separations in a tree structure in [4] was to first prove that all 3-separations in a 3-connected matroid conform with a maximal flower. This will also be a key step in our tree decomposition of 4-connected matroids. We want to define what it means for a 4-separation to conform with a 4-flower. Since the definition will rely on our notion of equivalence, we must first decide how to define equivalence of 4-separations. In the following examples, we see the difficulty that arises if we define equivalence of 4-separations in the same way as equivalence of 3-separations, that is, in terms of one-element moves. For the time being, mimicking what is done for 3-separations in 3-flowers, we will just say that a 4-separation (X, Y) *conforms* with a 4-flower Φ if there is a 4-separation (X', Y') with $(\mathrm{fcl}(X'), \mathrm{fcl}(Y')) = (\mathrm{fcl}(X), \mathrm{fcl}(Y))$ such that either (X', Y') is displayed by Φ, or one of X' and Y' is contained in a petal of Φ.

The first example illustrates the need to impose some type of equivalence on 4-separations otherwise, just as in the case of 3-separations, there is no reasonable way to display a tightly interlocked collection of 4-separations.

Example 2.1. Beginning with the matroid $U_{4,4}$ with ground set $\{a_1, a_2, a_3, a_4\}$, freely add t points $b_1, b_2, \dots, b_t$ on the line spanned by $\{a_2, a_3\}$. Then, for some $m \geq 3$, freely add m points on each of the planes spanned by $\{a_1, a_2, a_3\}$ and $\{a_2, a_3, a_4\}$. Let the resulting matroid be M. We label by P_1 and P_2 the planes spanned by $\{a_1, a_2, a_3\}$ and $\{a_2, a_3, a_4\}$, each containing m freely added points, respectively. Let $A = P_1 - P_2$, $C = P_2 - P_1$, and let B be the line spanned by $\{a_2, a_3\}$ and containing the points $b_1, b_2, \dots, b_t$. Take $Z_1, Z_2, \dots, Z_t$ to be a collection of disjoint 3-point lines where $Z_i = \{x_i, y_i, b_i\}$. We let N be the matroid obtained by attaching each Z_i to M via 2-sum. Then N has rank $4 + t$ so the truncation $T(N)$ of N has rank $3 + t$. An illustration of the matroid $T(N)$ can be found in Figure 1.

In the matroid N, every pair $\{x_i, y_i\}$ is 2-separating, and these are the only non-trivial 2-separating sets. Since every 3-separating set in N with at least 3 elements is 4-separating in $T(N)$, the matroid $T(N)$ is 4-connected. Moreover, in $T(N)$, the planes P_1 and P_2 are exactly 4-separating sets. Now, let I be a k-element subset of $[t]$. Then $r(A \cup \bigcup_{i \in I}\{x_i, y_i\}) = 3 + k$ and $r(E(T(N)) - (A \cup \bigcup_{i \in I}\{x_i, y_i\})) = 3 + t - k$ for all $k \leq t$. Therefore, $\lambda(A \cup \bigcup_{i \in I}\{x_i, y_i\}) = (3 + k) + (3 + t - k) - (3 + t) = 3$. It follows that A together with any collection of the pairs $\{x_i, y_i\}$ is exactly 4-separating. However, these 4-separations are not equivalent in the same sense as 3-separations are equivalent. Indeed, if we consider a set such as $A \cup x_i$, we see that $r(A \cup x_i) = 4$ and $r(E(T(N)) - (A \cup x_i)) = 3 + t$. Hence $\lambda(A \cup x_i) = 4$ so $A \cup x_i$ is not 4-separating. It follows that no x_i or y_i is in the closure or the coclosure of A.

In Example 2.1, the only way to move a pair $\{x_i, y_i\}$ from one side of the separation to the other while maintaining a 4-separation, is to move both elements simultaneously. This suggests the need for a new notion of equivalence that incorporates 4-separations that differ by exactly two elements. A natural way one might consider trying to display the large number of 4-separations that can arise in a 4-connected matroid, such as $T(N)$, and that differ by exactly two elements is to relax the condition that the petals of 4-flowers must be exactly 4-separating. In fact, if we allow petals to contain exactly two elements, we would be able to display all of the 4-separations that arise in the matroid $T(N)$ from Example 2.1

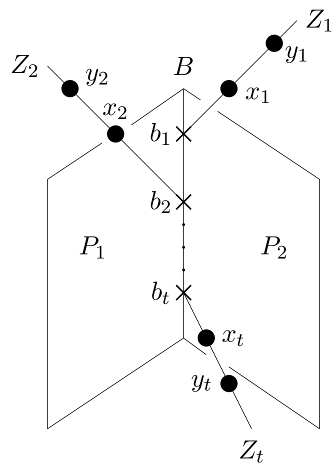

FIGURE 1. The 4-connected matroid $T(N)$.

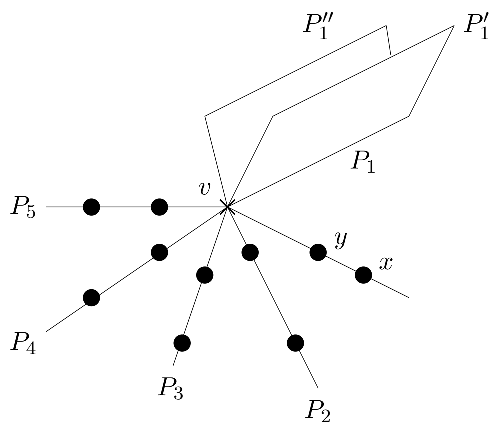

FIGURE 2. The 4-anemone Φ of rank 7.

up to ordinary one-element move equivalence, by the 4-flower $(P_1 \cup B, P_2 - B, Z_1, Z_2, \ldots, Z_t)$. As we will see in the next two examples, relaxing this condition comes at the cost of possibly not being able to display some more substantial 4-separations. Our next example consists of a 4-anemone that is constructed from a spike-like 3-flower.

Example 2.2. Let M be a rank-5 free spike with tip v, and let $P_1, P_2, \ldots, P_5$ be the legs of M such that $|P_i - v| = 2$ for all $i \neq 1$. Then M is 3-connected. Along the line P_1, we glue two planes, P_1' and P_1'', each a rank-3 uniform matroid with at least four points, none on P_1. The resulting matroid has rank 7, and the partition $(P_1' \cup P_1'', P_2, P_3, P_4, P_5)$ is a spike-like 3-flower.

Now, we add a point x freely in rank 7. On the line containing x and v, we freely place a point y. Finally, delete v and call the resulting matroid M'. It is easily verified that M' is a 4-connected matroid of rank 7. Furthermore, we assert that $\Phi = (P_1' \cup P_1'', \{x, y\}, P_2, P_3, P_4, P_5)$, pictured in Figure 2, is a 4-anemone for M' in which the local connectivity between any two distinct petals is one.

We observe that $P_1' \cup \{x, y\}$ is 4-separating, but does not conform with Φ. In order to have $P_1' \cup \{x, y\}$ conform with Φ, we would either need to move $\{x, y\}$ into the petal $P_1' \cup P_1''$, which would prevent exactly 4-separating sets such as $\{x, y\} \cup P_2$ from conforming; or we would need to make P_1' into a separate petal. However, upon calculating $\lambda(P_1' \cup \{x, y\} \cup P_2)$, we see that $P_1' \cup \{x, y\} \cup P_2$ is not 4-separating. Hence, making P_1' into a separate petal would destroy the 4-flower structure of Φ.

Because of Example 2.2, we see that, even if we were to allow 2-element petals in 4-flowers, we would still not be able to have certain 4-separations conform. In the next example, we construct a 4-daisy from a swirl-like 3-flower. This example provides insight into how one can construct arbitrarily large 4-flowers where, if we allow 2-element petals, an arbitrary number of 4-separations will not conform.

Example 2.3. We begin with a rank-8 jointed free swirl, S, having 8 segments labelled $L_1, L_2, \dots, L_8$, where $L_i = \{s_i, a_i, b_i, s_{i+1}\}$ and i is read modulo 8. The joints of S are the points $\{s_1, s_2, \dots, s_8\}$. Deleting all of the joints labelled by an even subscript, we arrive at a rank-8 semi-jointed free swirl with joints $\{s_1, s_3, s_5, s_7\}$. For each i in $\{1, 3, 5, 7\}$, we attach along L_i, two planes, $P'_{\frac{3i-1}{2}}$ and $P''_{\frac{3i-1}{2}}$, each a rank-3 uniform matroid with at least four points, none on L_i. Next we take four 3-point lines Z_1, Z_3, Z_5, Z_7 where $Z_i = \{x_i, y_i, s_i\}$ and, via 2-sum, attach each line Z_i at the point s_i. We call the resulting rank-20 matroid M.

Now, for i in $\{2, 4, 6, 8\}$, relabel L_i by $P_{\frac{3}{2}i}$ and, for i in $\{1, 3, 5, 7\}$, label the set $\{x_i, y_i\}$ by $P_{\frac{3i+1}{2}}$. The resulting partition, $(P_1' \cup P_1'', P_2, P_3, P_4' \cup P_4'', P_5, P_6, P_7' \cup P_7'', P_8, P_9, P_{10}' \cup P_{10}'', P_{11}, P_{12})$, is a swirl-like 3-flower for M. If we truncate M, we obtain the 4-connected rank-19 matroid $T(M)$ for which the swirl-like 3-flower is now a 4-daisy Φ with the same labelled partition. This is illustrated in Figure 3. The local connectivity between pairs of consecutive petals in Φ is one and, between pairs of non-consecutive petals, it is zero.

Evidently, sets of the form $P_i' \cup P_{i+1}$, for $i \equiv 1 \pmod 3$, are exactly 4-separating. Just as in Example 2.2, displaying such 4-separations or having them conform with the 4-flower comes at the cost of not being able to have other 4-separations conform, since sets of the form $P_i' \cup P_{i+1} \cup P_{i+2}$, for $i \equiv 1 \pmod 3$, are not 4-separating.

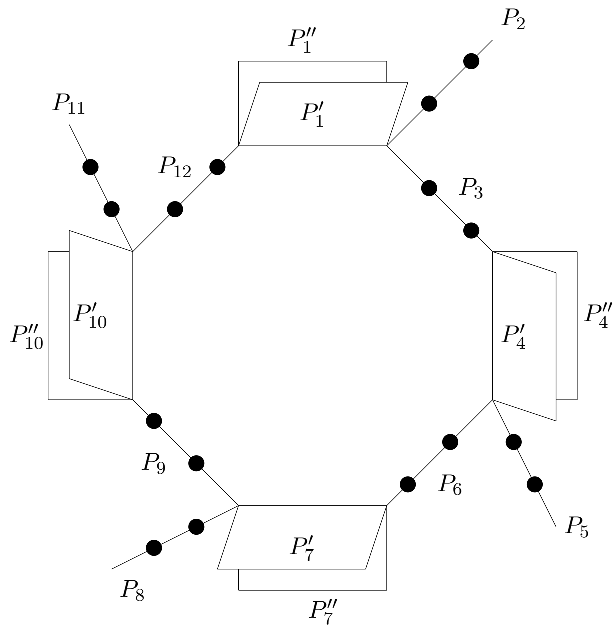

FIGURE 3. The 4-daisy Φ of rank-19.

Allowing 4-flowers to have 2-element petals clearly does not better our chances of displaying all of the 4-separations in a 4-connected matroid up to our original notion of equivalence. In fact, as the examples have shown, doing so can actually keep other 4-separations from conforming with a 4-flower. The same problem occurs if we allow single-element petals in 4-flowers. It is easily seen that, in the last example, deleting a single element from any of the petals P_i for $i \equiv 2 \pmod 3$ does not change the fundamental structure of the 4-flower in that there is still no feasible way to display all of the 4-separations. If, on the other hand, in Example 2.3, we were to regard 4-separations of the form $P_i' \cup P_i''$ and $P_i' \cup P_i'' \cup P_{i+1}$ for $i \equiv 1 \pmod 3$, as 'equivalent', then these 4-separations would conform with the 4-flower. In light of our observations, the natural course is to use an equivalence of 4-separations that incorporates moving two elements at a time across a 4-separation.

Let M be a 4-connected matroid and let X be a 4-separating subset of $E(M)$ having at least three elements. The *full* 2*-span of* X, denoted $\mathrm{fs}_2(X)$, is the set $X \cup X_1 \cup X_2 \cup \cdots \cup X_m$, where $X_1, X_2, \ldots, X_m$ are disjoint subsets of $E - X$,

(i) each X_i has cardinality one or two;
(ii) $\lambda(X \cup X_1 \cup X_2 \cup \cdots \cup X_j) \leq 3$, for all j in $[m]$; and
(iii) the sequence $(X_i)_{i=1}^m$ has maximal length with respect to properties (i) and (ii).

We note that the full 2-span operator is a generalization of the full closure operator, since if $|X_i| = 1$, then $X_i \subseteq \mathrm{cl}^{(*)}(X \cup X_1 \cup X_2 \cup \cdots \cup X_{i-1})$. We call $(X_i)_{i=1}^m$ a 4-*sequence for* $\mathrm{fs}_2(X)$ if it satisfies (i), (ii), and (iii). The next two lemmas show that the full 2-span operator, defined on 4-separating sets, is a well-defined closure operator.

Lemma 2.4. *The full* 2*-span operator is well-defined on* 4*-separating subsets of* $E(M)$ *having at least three elements.*

Proof. Let X be a 4-separating subset of $E(M)$ having at least three elements. Let $(X_i)_{i=1}^m$ and $(Y_i)_{i=1}^k$ be 4-sequences for $\mathrm{fs}_2(X)$. As the lengths are maximal, $k = m$. For each j in $[m]$, since $X \cup \bigcup_{i=1}^m X_i$ and $X \cup \bigcup_{i=1}^j Y_i$ are both 4-separating and their intersection contains X, by uncrossing, their union, $X \cup (\bigcup_{i=1}^m X_i) \cup (\bigcup_{i=1}^j Y_i)$, is 4-separating. Suppose that $X \cup (\bigcup_{i=1}^m X_i) \cup (\bigcup_{i=1}^j Y_i)$ properly contains $X \cup (\bigcup_{i=1}^m X_i)$ for some j and choose the least such j. Let $Y'_j = Y_j - (X \cup (\bigcup_{i=1}^m X_i))$. Then $|Y'_j| \in \{1, 2\}$ and $(X_1, X_2, \ldots, X_m, Y'_j)$ is a 4-sequence for $\mathrm{fs}_2(X)$; a contradiction. Thus $X \cup (\bigcup_{i=1}^m X_i)$ contains $\bigcup_{i=1}^m Y_i$ and, by symmetry, $X \cup (\bigcup_{i=1}^m Y_i)$ contains $\bigcup_{i=1}^m X_i$, so $\mathrm{fs}_2(X)$ is well-defined. □

By a similar argument, it is not difficult to establish the following result.

Lemma 2.5. *In a* 4*-connected matroid* M*, the full* 2*-span operator is a closure operator for* 4*-separating sets having at least three elements.*

Let M be a 4-connected matroid. We say that two exactly 4-separating partitions (X, Y) and (X', Y') of $E(M)$ are 2-*equivalent* if $(\mathrm{fs}_2(X), \mathrm{fs}_2(Y)) = (\mathrm{fs}_2(X'), \mathrm{fs}_2(Y'))$. Thus if two exactly 4-separating partitions are 2-equivalent, then one can be obtained from the other by moving a sequence of 1- and 2-element sets across the partition in such a way that, at any intermediate step, the result is a 4-separating partition. Two exactly 4-separating sets Y and Z in M are 2-*equivalent* if $\mathrm{fs}_2(Y) = \mathrm{fs}_2(Z)$. We say that a 4-separating subset X of $E(M)$ is *sequential* if $\mathrm{fs}_2(E - X) = E(M)$. Notice that, in a 4-connected matroid, when a set X contains at most four elements, it is automatically sequential. We call an exactly 4-separating partition (X, Y) *sequential* if either X or Y is sequential. Since, in a 4-connected matroid, any set having at most three elements is 4-separating, we regard sequential 4-separations as being trivial and we make no attempt to display them.

The next lemma, which generalizes [5, Lemma 2.7], follows easily from Lemma 2.5.

Lemma 2.6. *In a* 4*-connected matroid* M*, let* X *and* Y *be* 4*-separating sets such that* $|E(M) - X| \geq 3$ *and* $Y \subseteq X$*. If* X *is sequential, then so is* Y*.*

The next lemma is used in the proof of the main result, Theorem 5.1.

Lemma 2.7. *Let* A *and* B *be disjoint non-sequential, exactly* 4*-separating sets in a* 4*-connected matroid* M*. If* $\mathrm{fs}_2(A)$ *does not contain* B*, and* $\mathrm{fs}_2(A)$

is not 2*-equivalent to* $E-B$*, then* $\mathrm{fs}_2(A)-B$ *is* 4*-separating and* $\mathrm{fs}_2(\mathrm{fs}_2(A)-B)=\mathrm{fs}_2(A)$.

Proof. First, we show that $\mathrm{fs}_2(A)-B$ is 4-separating. Consider the set $\mathrm{fs}_2(A)\cup(E-B)=E-(B-\mathrm{fs}_2(A))$. Suppose $|B-\mathrm{fs}_2(A)|\leq 2$ and let $B-\mathrm{fs}_2(A)=Y$. Then, since $\mathrm{fs}_2(A)$ is not 2-equivalent to $E-B$, we know that $|(E-B)-\mathrm{fs}_2(A)|=|E-(\mathrm{fs}_2(A)\cup B)|\geq 3$. Thus, by uncrossing the sets $\mathrm{fs}_2(A)$ and B, we see that their intersection is 4-separating. But $\mathrm{fs}_2(A)\cap B=B-Y$. If $|B-Y|\leq 2$, then $|B|\leq 4$ and B is a sequential 4-separating set, which is a contradiction. Therefore, $|B-Y|\geq 3$. Then, by uncrossing, $\mathrm{fs}_2(A)\cup B$ is 4-separating, which implies that $B\subseteq\mathrm{fs}_2(A)$; a contradiction. It follows that $|B-\mathrm{fs}_2(A)|\geq 3$. We know that $\mathrm{fs}_2(A)-B=\mathrm{fs}_2(A)\cap(E-B)$, and both $\mathrm{fs}_2(A)$ and $(E-B)$ are 4-separating. So, by uncrossing the sets $\mathrm{fs}_2(A)$ and $E-B$, we see that $\mathrm{fs}_2(A)-B$, is 4-separating. Moreover, since $A\subseteq\mathrm{fs}_2(A)-B\subseteq\mathrm{fs}_2(A)$, we have $\mathrm{fs}_2(A)\subseteq\mathrm{fs}_2((\mathrm{fs}_2(A)-B))\subseteq\mathrm{fs}_2(\mathrm{fs}_2(A))=\mathrm{fs}_2(A)$, and the lemma holds. $\square$

Let M be a 4-connected matroid. A 2-element set $\{a,b\}$ in M is called a *pod* if there is a partition $(X,\{a,b\},Y)$ of $E(M)$ such that both X and Y are 4-separating but neither $X\cup a$ nor $X\cup b$ is 4-separating. Since any set in a 4-connected matroid of size at most three is 4-separating, for such a partition to occur, it must be that $|X|,|Y|\geq 3$ so X and Y are exactly 4-separating. The partition $(X,\{a,b\},Y)$ of $E(M)$ is called a *pod partition*. Amongst pods, we distinguish two different types. A pod $\{a,b\}$ is called *weak* if there is a non-sequential 4-separation (A,B) of M with $a\in A$ and $b\in B$. Such a 4-separation is said to *divide the pod* $\{a,b\}$. If a pod is not weak, then it is called *strong*. Hence strong pods cannot be divided by non-sequential 4-separations. Let X be a 4-separating set in M. We say that $\{a,b\}$ is a pod with respect to X if $X\cap\{a,b\}=\emptyset$ and $(X,\{a,b\},E-(X\cup\{a,b\}))$ is a pod partition. Evidently $\{a,b\}$ is a pod with respect to X if and only if it is a pod with respect to $E-(X\cup\{a,b\})$. Let (X,Y) be a 4-separation of M and suppose that $\{a,b\}\subseteq Y$. Then $\{a,b\}$ is a *pod with respect to the* 4*-separation* (X,Y) if $(X,\{a,b\},Y-\{a,b\})$ is a pod partition. The next two lemmas give some basic properties of pods in 4-connected matroids. We omit the routine proof of the first.

Lemma 2.8. *Let* M *be a* 4*-connected matroid and let* X *be a* 4*-separating subset of* $E(M)$ *with* $|X|\geq 3$*. If* Z *is a pod with respect to* X*, then* $r(X\cup Z)-r(X)=r^*(X\cup Z)-r^*(X)=1$ *and* $\sqcap(Z,X)=\sqcap^*(Z,X)=\sqcap(Z,E-(X\cup Z))=\sqcap^*(Z,E-(X\cup Z))=1$.

Lemma 2.9. *Let* M *be a* 4*-connected matroid with* $|E(M)|\geq 11$*. If* $(X,\{a,b\},Y)$ *is a pod partition of* $E(M)$ *with* $|X|,|Y|\geq 5$*, then* $\{a,b\}$ *is a strong pod.*

Proof. Suppose there is a non-sequential 4-separation (A,B) with $a\in A$ and $b\in B$. Then, since (A,B) is non-sequential, $|A|,|B|\geq 5$. We represent this situation in Figure 4.

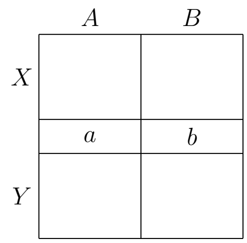

FIGURE 4. The pod $\{a, b\}$ split by the non-sequential 4-separation (A, B).

2.9.1. *Each of* $|X \cap A|$, $|X \cap B|$, $|Y \cap A|$, *and* $|Y \cap B|$ *is at least two.*

To prove (2.9.1), suppose that $|Y \cap B| \leq 1$. Then, since $|Y|, |B| \geq 5$, we see that $|B \cap X| \geq 3$ and $|A \cap Y| \geq 3$. By uncrossing A and Y, we have $\lambda(Y \cap A) = 3$. Also, by uncrossing A and $Y \cup \{a, b\}$, we see that $\lambda((Y \cap A) \cup a) = 3$. Hence $a \in \mathrm{cl}^{(*)}(Y \cap A)$, so $a \in \mathrm{cl}^{(*)}(Y)$ contradicting the fact that $(X, \{a, b\}, Y)$ is a pod partition. The rest of (2.9.1) now follows by symmetry.

Now, if each of $|X \cap A|$, $|X \cap B|$, $|Y \cap A|$ and $|Y \cap B|$ is exactly two, then $|E(M)| = 10$, a contradiction. So, we may assume without loss of generality that $|X \cap A| \geq 3$. If $|B \cap Y| \geq 3$, uncrossing B and Y gives $\lambda(B \cap Y) = 3$. Then, by uncrossing B and $Y \cup \{a, b\}$, we see that $\lambda((B \cap Y) \cup b) = 3$. Thus $b \in \mathrm{cl}^{(*)}(B \cap Y)$, so $b \in \mathrm{cl}^{(*)}(Y)$, a contradiction. Hence we may further assume that $|B \cap Y| = 2$. Then $|A \cap Y| \geq 3$, since $|Y| \geq 5$. So, by symmetry, $|B \cap X| = 2$.

Let $B \cap X = \{x_1, x_2\}$. Now, A is 4-separating, and so is $B - \{x_1, x_2\}$ since $|B - \{x_1, x_2\}| = 3$. Hence, by uncrossing, $A \cup X$ is 4-separating. That is, $(A \cup \{x_1, x_2\}, B - \{x_1, x_2\})$ is an exactly 4-separating partition. Thus B is sequential, a contradiction. Therefore, the lemma holds. □

The following is a straightforward consequence of the last lemma.

Corollary 2.10. *Let M be a 4-connected matroid with $|E(M)| \geq 11$. If $\{a, b\}$ is a pod with respect to a non-sequential 4-separation of M, then $\{a, b\}$ is a strong pod.*

Corollary 2.11. *Let M be a 4-connected matroid with $|E(M)| \geq 11$. If, for each i in $\{1, 2\}$, the set Z_i is a pod with respect to some non-sequential 4-separation of M, then $Z_1 \cap Z_2 = \emptyset$, or $Z_1 = Z_2$.*

Proof. Suppose Z_1 and Z_2 are distinct. Then $|Z_1 \cap Z_2| \leq 1$. Assume that $|Z_1 \cap Z_2| = 1$. Since Z_1 is a pod with respect to a non-sequential 4-separation of M, there is a pod partition (X, Z_1, Y) of $E(M)$ such that X and Y are non-sequential 4-separating sets. Let $Z_2 - Z_1 = \{z\}$. Either $z \in X$ or $z \in Y$. Thus either $(Y \cup Z_1, X)$ or $(X \cup Z_1, Y)$ is a non-sequential 4-separation that divides the pod Z_2, contradicting Corollary 2.10. □

3. Equivalence of 4-Flowers

For the remainder of this paper, unless otherwise stated, we will assume that we are working in a 4-connected matroid M. Moreover, whenever we say that Φ is a flower, it is to be understood that Φ is a 4-flower for M. For simplicity, we will also abbreviate 4-anemones and 4-daisies to anemones and daisies, respectively. Let Φ be a flower. Recall that Φ displays a 4-separation (X, Y) of M if X is a union of petals of Φ. Now let Φ_1 and Φ_2 be flowers. Then $\Phi_1 \preccurlyeq \Phi_2$ if every non-sequential 4-separation displayed by Φ_1 is 2-equivalent to one displayed by Φ_2. We say that two flowers, Φ_1 and Φ_2, are 2-*equivalent* if $\Phi_1 \preccurlyeq \Phi_2$ and $\Phi_2 \preccurlyeq \Phi_1$. Thus 2-equivalent flowers display, up to 2-equivalence of 4-separations, exactly the same non-sequential 4-separations. The *order* of a flower Φ is the minimum number of petals needed to display, up to 2-equivalence of 4-separations, the same non-sequential 4-separations as Φ. A flower has order 1 if it does not display any non-sequential 4-separations. If a flower has order 2, then it displays exactly one non-sequential 4-separation. Clearly, a flower of order n has at least n petals.

In a 4-connected matroid M, let $(P_1, P_2, \dots, P_n)$ be a flower Φ. We call Φ *irredundant* if, for all i in $[n]$, there is a non-sequential 4-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_{i+1} \subseteq Y$. If a flower is not irredundant, it is called *redundant*. Since we are interested in displaying only non-sequential 4-separations, it is inefficient to do so using flowers that are redundant. Therefore, in what follows, we will commonly assume that the flowers we are dealing with are irredundant, and we will always assume that their order is at least two. The reader familiar with [4] may be surprised to see the notion of irredundance used with 4-flowers since the same notion is not explicitly used when treating 3-flowers in 3-connected matroids. At the end of this section, we shall briefly discuss this difference.

We begin this section by defining an *elementary move for* 2*-equivalence*. Let $(P_1, P_2, \dots, P_n)$ be a 4-flower Φ. We say that Φ' is obtained from Φ by an *elementary move* if one of the following holds:

(0) Φ' is obtained by an arbitrary permutation of the petals of Φ in the case that Φ is an anemone, or is obtained from Φ by a cyclic shift or a reversal of the order of the petals of Φ in the case that Φ is a daisy.

(1) There exists $Y \subseteq P_2$ with $|Y| \in \{1, 2\}$ and $|P_2 - Y| \geq 3$ such that $\lambda(P_1 \cup Y) = 3$, and
$$\Phi' = (P_1 \cup Y, P_2 - Y, P_3, \dots, P_n).$$

(2) There exists $Y \subseteq P_2$ with $|Y| \in \{1, 2\}$ and $|P_2 - Y| \leq 2$ such that $\lambda(P_1 \cup Y) = 3$, and
$$\Phi' = (P_1 \cup P_2, P_3, \dots, P_n).$$

(3) There exist Y_1 and Y_2 contained in P_1 with $|Y_1|, |Y_2| \in \{1, 2\}$ and $|Y_1 \cup Y_2|, |P_1 - (Y_1 \cup Y_2)| \geq 3$ such that $\lambda(P_2 \cup Y_1) = 3 = \lambda(P_2 \cup Y_1 \cup Y_2)$,

and

$$\Phi' = (P_1 - (Y_1 \cup Y_2), Y_1 \cup Y_2, P_2, P_3, \dots, P_n).$$

We refer to these as *Type*-0, -1, -2, and -3 moves, respectively. The goal of this section is to prove the following result.

Theorem 3.1. *Two irredundant flowers of order at least three are* 2-*equivalent if and only if one can be obtained from the other by a sequence of elementary moves.*

To prove this theorem, we will need some preliminaries. Let $\Phi = (P_1, P_2, \dots, P_n)$ be a flower. An element e of M is *loose* in the flower Φ if $e \in \mathrm{fs}_2(P_i) - P_i$ for some petal P_i of Φ. An element that is not loose is *tight*. A set X in M is *loose* in the flower Φ if all of the elements in X are loose. A set that is not loose is *tight*. The petal P_i is *loose* if it is a loose set. A *tight* petal is one that is not loose. A flower of order at least three is *tight* if all of its petals are tight. A flower of order two or one is *tight* if it has two petals or one petal, respectively.

Lemma 3.2. *Let $\Phi = (P_1, P_2, \dots, P_n)$ be a flower and suppose that Φ' is obtained from Φ by an elementary move. Then Φ and Φ' are* 2*-equivalent flowers and an element is loose in Φ if and only if it is loose in Φ'.*

Proof. Evidently $n \geq 2$. It is clear that the lemma holds if Φ' is obtained from Φ by a Type-0 move. Now consider performing a Type-1 move on Φ. Let $Y \subseteq P_2$ with $|Y| \in \{1, 2\}$, $|P_2 - Y| \geq 3$, and $\lambda(P_1 \cup Y) = 3$. Let $\Phi' = (P_1 \cup Y, P_2 - Y, P_3, \dots, P_n)$. We first show that Φ' is a flower. If $n = 2$, this is immediate. If $n = 3$, since $P_1 \cup Y$ and $P_3 \cup P_1$ are 4-separating and their intersection is P_1, by uncrossing, their union, $P_3 \cup P_1 \cup Y$, is 4-separating. Hence $P_2 - Y$ is also 4-separating. It follows that Φ' is a flower when $n = 3$. Suppose $n \geq 4$. Consider consecutive pairs of petals of Φ'. The only unions of such pairs that are not unions of consecutive petals of Φ are $(P_2 - Y) \cup P_3$ and $P_n \cup (P_1 \cup Y)$. By repeating a similar uncrossing argument as in the case of $n = 3$, we see that $(P_1 \cup Y) \cup P_n$, is 4-separating. By uncrossing $P_1 \cup Y$ and $P_4 \cup P_5 \cup \cdots \cup P_n \cup P_1$, we see that $(P_2 - Y) \cup P_3$ is 4-separating. Similarly, uncrossing $P_1 \cup Y$ and $P_3 \cup P_4 \cup P_5 \cup \cdots \cup P_n \cup P_1$ shows that $P_2 - Y$ is 4-separating. Therefore, Φ' is a flower. Moreover, $Y \subseteq \mathrm{fs}_2(P_2 - Y)$.

Next, we show that Φ and Φ' are 2-equivalent. Let (R, G) be a non-sequential 4-separation in M. Suppose that (R, G) is displayed by Φ, where $P_1 \subseteq R$. If $P_2 \subseteq R$, then (R, G) is displayed by Φ', hence we may assume that $P_2 \subseteq G$. We know that $\lambda(P_1 \cup Y) = 3$ so, by uncrossing R and $P_1 \cup Y$, we see that $\lambda(R \cup Y) = 3$. Thus $(R \cup Y, G - Y)$ is a 4-separation that is 2-equivalent to (R, G) and is displayed by Φ'. A symmetric argument shows that if (R, G) is displayed by Φ', then it is 2-equivalent to a 4-separation displayed by Φ. Therefore, Φ and Φ' are 2-equivalent.

We now consider the loose elements. Since $\lambda(P_1 \cup Y) = 3$, we have $Y \subseteq \mathrm{fs}_2(P_1)$. Thus $\mathrm{fs}_2(P_1 \cup Y) = \mathrm{fs}_2(P_1)$. Similarly, $Y \subseteq \mathrm{fs}_2(P_2 - Y)$ so that

$\mathrm{fs}_2(P_2) = \mathrm{fs}_2(P_2 - Y)$. It now follows easily that the loose elements in Φ and Φ' are the same.

Consider a Type-2 move. Let $Y \subseteq P_2$ with $|Y| \in \{1,2\}$, $|P_2 - Y| \leq 2$, and $\lambda(P_1 \cup Y) = 3$. Let $\Phi' = (P_1 \cup P_2, P_3, \dots, P_n)$. It easily follows in this case that Φ' is a flower. We show that Φ and Φ' are 2-equivalent. Let (R, G) be a non-sequential 4-separation of M. Since the underlying partition for Φ refines that of Φ', it is immediate that if (R, G) is displayed by Φ', then it is displayed by Φ. Assume that (R, G) is displayed by Φ, where $P_1 \subseteq R$. If $P_2 \subseteq R$, then (R, G) is displayed by Φ'. Assume that $P_2 \subseteq G$. Since $P_1 \cup Y$ and $P_1 \cup P_2$ are 4-separating and $|P_2 - Y| \leq 2$, it follows that $P_2 \subseteq \mathrm{fs}_2(P_1)$. Hence (R, G) is 2-equivalent to the 4-separation $(R \cup P_2, G - P_2)$, which is displayed by Φ'. It now follows that Φ and Φ' are 2-equivalent.

Consider the loose elements of Φ and Φ'. Since Φ and Φ' are 2-equivalent and Φ has order at least 2, it must be that $n \geq 3$. Since $P_2 \subseteq \mathrm{fs}_2(P_1)$, it follows that Φ and Φ' have the same loose elements as long as all of the elements of $\mathrm{fs}_2(P_2)$ are loose in Φ'. Clearly, elements of $\mathrm{fs}_2(P_2)$ that are not in P_1 are loose in Φ'. But, it is easily seen that $P_2 \subseteq \mathrm{fs}_2(P_3)$, hence $\mathrm{fs}_2(P_2) \cap P_1 \subseteq \mathrm{fs}_2(P_3) \cap P_1$. Therefore, the elements of $\mathrm{fs}_2(P_2)$ are loose in Φ' as required.

Finally, consider a Type-3 move. Let Y_1 and Y_2 be contained in P_1 with $|Y_1|, |Y_2| \in \{1,2\}$ and $|Y_1 \cup Y_2| \geq 3$. Suppose that $\lambda(P_2 \cup Y_1) = 3$, $\lambda(P_2 \cup Y_1 \cup Y_2) = 3$, and $|P_1 - (Y_1 \cup Y_2)| \geq 3$. Let $\Phi' = (P_1 - (Y_1 \cup Y_2), Y_1 \cup Y_2, P_2, P_3, \dots, P_n)$. By uncrossing $E - (P_2 \cup Y_1 \cup Y_2)$ and P_1, we see that the intersection, $P_1 - (Y_1 \cup Y_2)$, is 4-separating. Similarly, $P_n \cup (P_1 - (Y_1 \cup Y_2))$ is 4-separating. It follows that every union of consecutive petals is 4-separating. Hence Φ' is a flower. Observe that Φ is obtained from Φ' by a Type-2 move. Thus Φ and Φ' are 2-equivalent and have the same loose elements. □

We will say that the flower Φ_1 is *move-equivalent* to the flower Φ_2 if one can be obtained from the other by a sequence of elementary moves. We omit the elementary proof of the next result.

Lemma 3.3. *Move-equivalence is an equivalence relation on the set of flowers of order at least two.*

We now work toward showing that every flower of order at least 3 is move-equivalent to a tight flower.

Lemma 3.4. *Let $(P_1, P_2, \dots, P_n)$ be a tight flower Φ of order at least 3. Let $(Y_i)_{i=1}^m$ be a 4-sequence for $\mathrm{fs}_2(P_1 \cup P_2 \cup \cdots \cup P_j)$ where $j \leq n-2$. Let d be the largest member of $[m+1]$ such that, for all i in $[d-1]$, the set Y_i is contained in one of $P_{j+1}, P_{j+2}, \dots, P_n$.*

(i) *If $d \leq m$, then*
  (a) *$j = n-2$;*
  (b) *Y_d meets both P_{n-1} and P_n;*
  (c) *each of $P_{n-1} - (Y_1 \cup Y_2 \cup \cdots \cup Y_d)$ and $P_n - (Y_1 \cup Y_2 \cup \cdots \cup Y_d)$ has exactly two elements;*

(d) $(P_n \cup P_{n-1}) - (Y_1 \cup Y_2 \cup \cdots \cup Y_d)$ *is a* 4*-circuit or a* 4*-cocircuit of* M*; and*

(e) $\mathrm{fs}_2(P_1 \cup P_2 \cup \cdots \cup P_j) = E(M)$.

(ii) *When* $i \leq d-1$,

(a) *if* Y_i *is contained in* P_n*, then* $Y_i \subseteq \mathrm{fs}_2(P_1) - P_1$*; and*

(b) *if* Y_i *is not contained in* P_n*, then* $Y_i \subseteq \mathrm{fs}_2(P_j) - P_j$.

(iii) *For* $Y' = Y_1 \cup Y_2 \cup \cdots \cup Y_{d-1}$*, the flower* Φ *is* 2*-equivalent to*

$$(P_1 \cup (Y' \cap P_n), P_2, \ldots, P_{j-1}, P_j \cup (Y' - P_n), P_{j+1} - Y', \ldots, P_n - Y').$$

Proof. We begin by establishing (ii) and (iii). Assume that $1 \leq d-1$. Suppose $Y_1 \subseteq P_n$. Then $P_1 \cup P_2 \cup \cdots \cup P_j \cup Y_1$ and $P_1 \cup P_n$ are 4-separating. Their union avoids P_{n-1}, so their intersection, $P_1 \cup Y_1$, is 4-separating. Thus $Y_1 \subseteq \mathrm{fs}_2(P_1)$ if $Y_1 \subseteq P_n$. Now assume Y_1 is not contained in P_n. Then $P_n \cap Y_1 = \emptyset$ and $P_1 \cup P_2 \cup \cdots \cup P_j \cup Y_1$ and $P_j \cup P_{j+1} \cup \cdots \cup P_{n-1}$ are 4-separating. Their union avoids P_n, so their intersection, $P_j \cup Y_1$, is 4-separating; that is, $Y_1 \subseteq \mathrm{fs}_2(P_j)$.

If $Y_1 \subseteq P_n$, then we replace $(P_1, P_2, \ldots, P_n)$ by $(P_1 \cup Y_1, P_2, P_3, \ldots, P_{n-1}, P_n - Y_1)$. If $Y_1 \subseteq P_t$, for $j+1 \leq t \leq n-1$, then we replace P_j by $P_j \cup Y_1$ and replace P_t by $P_t - Y_1$. In each case, as Φ is tight, we get a new tight flower $(P'_1, P'_2, \ldots, P'_n)$ where $\mathrm{fs}_2(P'_1 \cup P'_2 \cup \cdots \cup P'_j) = \mathrm{fs}_2(P_1 \cup P_2 \cup \cdots \cup P_j)$ and $(Y_i)_{i=2}^m$ is a 4-sequence for $\mathrm{fs}_2(P'_1 \cup P'_2 \cup \cdots \cup P'_j)$. Provided $2 \leq d-1$, we can repeat this process using Y_2 rather than Y_1 in our new flower, and we will get that Y_2 is contained in one of $P'_{j+1}, P'_{j+2}, \ldots, P'_n$. Hence Y_2 is contained in one of $P_{j+1}, P_{j+2}, \ldots, P_n$. Then, either $P'_1 \cup Y_2$ or $P'_j \cup Y_2$ is 4-separating. Continuing in this way, we obtain (ii) and (iii) without difficulty.

To prove (i), let $\Phi'' = (P''_1, P''_2, \ldots, P''_n)$ where

$$\Phi'' = (P_1 \cup (Y' \cap P_n), P_2, \ldots, P_{j-1}, P_j \cup (Y' - P_n), P_{j+1} - Y', \ldots, P_n - Y').$$

Since Y_d is not contained in any of $P_{j+1}, P_{j+2}, \ldots, P_n$, we may assume $Y_d = \{a, b\}$ where $a \in P_s$ and $b \in P_t$ with $j+1 \leq s < t \leq n$. Then $P''_1 \cup P''_2 \cup \cdots \cup P''_j \cup Y_d$ is 4-separating and so is $P''_1 \cup P''_2 \cup \cdots \cup P''_s$. Their intersection is $P''_1 \cup P''_2 \cup \cdots \cup P''_j \cup \{a\}$. If their union avoids at least 3 elements, we get that $P''_1 \cup P''_2 \cup \cdots \cup P''_j \cup \{a\}$ is 4-separating, contradicting the maximality of the 4-sequence. Thus we may assume that $|(P''_{s+1} \cup P''_{s+2} \cup \cdots \cup P''_n) - Y_d| = 2$, so $t = s+1 = n$ and $|P''_t| = 3$. By symmetry, $s = j+1$ and $|P''_s| = 3$. Hence $j = n-2$ and $|P''_{n-1}| = 3 = |P''_n|$. Thus $|(P''_{n-1} \cup P''_n) - Y_d| = 4$ and $(P''_{n-1} \cup P''_n) - Y_d$ is 4-separating, so it is either a circuit or a cocircuit. Since $P''_{n-1} \cup P''_n$ is also 4-separating, we deduce that $\mathrm{fs}_2(P_1 \cup P_2 \cup \cdots \cup P_j) = E(M)$. Thus (i) holds. □

The following is a useful consequence of the last lemma.

Corollary 3.5. *Let* $(P_1, P_2, \ldots, P_n)$ *be a tight flower* Φ *of order at least* 3*. Then no union of at least three consecutive petals of* Φ *is a sequential set.*

The proofs of the next two lemmas are obtained by making straightforward modifications to the proofs of Lemmas 5.6 and 5.7 of [4].

Lemma 3.6. *Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ. Let $Y \subseteq P_i$ for some $i > 1$ with $|Y| \in \{1, 2\}$ and suppose that $\lambda(P_1 \cup Y) = 3$.*

(i) *If $|P_i - Y| \geq 3$, then $(P_1 \cup Y, P_2, \ldots, P_{i-1}, P_i - Y, P_{i+1}, \ldots, P_n)$ is a flower Φ' that is move-equivalent to Φ via a sequence of Type-1 moves. Moreover, $\mathrm{fs}_2(P'_j) = \mathrm{fs}_2(P_j)$ for every petal P'_j of Φ'.*

(ii) *If $|P_i - Y| \leq 2$, then Φ is move equivalent to*

$$(P_1 \cup P_2, P_3, \ldots, P_n) \text{ when } i = 2\text{; and to}$$

$$(P_1 \cup Y, P_2, \ldots, P_{i-1} \cup (P_i - Y), P_{i+1}, \ldots, P_n) \text{ when } i \geq 3.$$

We call the moves described in the last lemma *moves of Type-1a and Type-2a*, respectively.

Lemma 3.7. *Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ of order at least three. Then Φ is move-equivalent to a tight flower.*

Lemma 3.8. *Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ of order at least three, and let T be the set of tight elements of Φ.*

(i) *If Φ' is a flower that is move-equivalent to Φ, then there is a bijection α between the tight petals of Φ and those of Φ' such that $P \cap T = \alpha(P) \cap T$ for every tight petal P of Φ.*

(ii) *If P is a petal of Φ, then $|P \cap T| \notin \{1, 2\}$.*

(iii) *If P is a tight petal of Φ, then $\mathrm{fs}_2(P \cap T) = \mathrm{fs}_2(P)$.*

Proof. Suppose that Φ' is obtained from Φ by a single Type-1, Type-2, or Type-3 move. Since such a move does not involve tight elements, it follows that (i) holds for a single Type-1, -2, or -3 move. Hence it holds for a sequence of such moves. To prove (ii), it suffices to show that $|P_n \cap T| \notin \{1, 2\}$. Assume the contrary. By a sequence of moves of Type-1a and -2a, we add elements to P_1 transforming it into the form $(P_1^1, P_2^1, \ldots, P_n^1)$ where $P_1^1 = \mathrm{fs}_2(P_1)$, and $P_i^1 = P_i - \mathrm{fs}_2(P_1)$ for all $i > 1$. If any P_i^1 has at most two elements, we absorb it into P_{i-1}^1 via a move of Type-2a. This results in a new flower Φ_1. Moreover, for each petal P_i^1 of Φ_1, we have $\mathrm{fs}_2(P_i^1) = \mathrm{fs}_2(P_i)$. Let P_j^1 be the first petal of Φ_1 after P_1^1. By a sequence of moves of Type-1a and -2a, we add elements to P_j^1 transforming it into the form $(P_1^2, P_j^2, \ldots, P_n^1)$ where $P_1^2 = P_1^1$ and $P_j^2 = \mathrm{fs}_2(P_j^1) - \mathrm{fs}_2(P_1^1)$, while $P_i^2 = P_i^1 - \mathrm{fs}_2(P_j^1)$ for all $i > j$. Again if any set P_i^2 has at most two elements, it is absorbed into the previous petal. Let the resulting flower be Φ_2. By repeating this process with successive petals, since $|P_n \cap T| \in \{1, 2\}$, we eventually remove all but at most two elements from P_n. At that stage, the remaining elements of P_n are absorbed into the preceding petal contradicting the fact that P_n has at least two tight elements. We conclude that (ii) holds.

Consider (iii). It suffices to prove this when $P = P_n$. Clearly $\mathrm{fs}_2(P_n \cap T) \subseteq \mathrm{fs}_2(P_n)$. By (ii), since P is tight, $|P_n \cap T| \geq 3$. Thus the flower Φ_{n-1} obtained at the conclusion of the process in (ii) has $P_n \cap T$ as its last petal. By reversing the moves used in (ii), we get a sequence of elementary moves that transforms Φ_{n-1} into Φ. If, for some i, a set Y of cardinality one or two is added to P_n^i in going from Φ_i to Φ_{i-1}, then Y was added via move-equivalence. Thus $P_n \subseteq \mathrm{fs}_2(P_n \cap T)$. So $\mathrm{fs}_2(P_n) \subseteq \mathrm{fs}_2(\mathrm{fs}_2(P_n \cap T)) = \mathrm{fs}_2(P_n \cap T)$ and (iii) holds. $\square$

Next we observe that performing a Type-1a move on a tight irredundant flower produces another tight irredundant flower. The proof, which follows easily from Lemmas 3.6 and 3.2, is omitted.

Lemma 3.9. *Let $(P_1, P_2, \dots, P_n)$ be a tight irredundant flower Φ. Let $Y \subseteq P_i$ for some $i > 1$ with $|Y| \in \{1, 2\}$ and suppose that $\lambda(P_1 \cup Y) = 3$. Then $(P_1 \cup Y, P_2, \dots, P_{i-1}, P_i - Y, P_{i+1}, \dots, P_n)$ is a tight irredundant flower that is 2-equivalent to Φ.*

Extending this, we have the following.

Lemma 3.10. *If $\Phi = (P_1, P_2, \dots, P_n)$ is a tight irredundant flower, and P_1' is a 4-separating set that contains and is 2-equivalent to P_1, then $(P_1', P_2 - P_1', \dots, P_n - P_1')$ is a tight irredundant flower that is 2-equivalent to Φ.*

Proof. Let $(Y_i)_{i=1}^m$ be a 4-sequence for $\mathrm{fs}_2(P_1)$. Then, for all i in $[m]$, the set $P_1 \cup Y_1 \cup Y_2 \cup \dots \cup Y_i$ is exactly 4-separating, as is P_1'. Their union avoids at least three elements as Φ is tight, so their intersection is 4-separating. Hence $(P_1', P_2 - P_1', \dots, P_n - P_1')$ can be obtained from Φ by a sequence of moves of Type-1a. The result follows from the previous lemma. $\square$

Lemma 3.11. *Let Φ and Ψ be 2-equivalent tight flowers $(P_1, P_2, \dots, P_n)$ and $(O_1, O_2, \dots, O_m)$ in a 4-connected matroid where $n \geq m$. Let T be the set of tight elements of Φ and suppose that, for all 2-element subsets $\{i, j\}$ of $[n]$, there is a non-sequential 4-separation displayed by Φ that has at least two petals on each side and has P_i and P_j on opposite sides. Then, for all such i and j, the sets $P_i \cap T$ and $P_j \cap T$ are contained in different petals of Ψ, so $n = m$. Moreover, the set of tight elements of Ψ is T.*

Proof. Let (X, Y) be a non-sequential 4-separation displayed by Φ having at least two petals on each side, where $P_i \subseteq X$ and $P_j \subseteq Y$. By Lemma 3.4, every 4-separation (X_1, Y_1) 2-equivalent to (X, Y) has $P_i \cap T$ and $P_j \cap T$ contained in X_1 and Y_1, respectively. It follows that $P_i \cap T$ and $P_j \cap T$ are contained in different petals of Ψ. Thus, since $n \geq m$, the flower Ψ has exactly n petals. Let σ be a permutation of $\{1, 2, \dots, n\}$ such that $P_i \cap T = O_{\sigma(i)} \cap T$. Now suppose $x \in T$, but x is not tight in Ψ. Then $x \in O_{\sigma(i)}$ for some i and $x \in \mathrm{fs}_2(O_{\sigma(j)})$ for some $j \neq i$. We have $x \in P_i \cap T$, since $P_i \cap T = O_{\sigma(i)} \cap T$. Say $X = P_1 \cup P_2 \cup \dots \cup P_s$, and $Y = P_{s+1} \cup P_{s+2} \cup \dots \cup P_n$. Then $\mathrm{fs}_2(X) - X \subseteq (\mathrm{fs}_2(P_1) - P_1) \cup (\mathrm{fs}_2(P_s) - P_s)$, and $\mathrm{fs}_2(O_{\sigma(j)}) \subseteq \mathrm{fs}_2(Y)$. As $x \in \mathrm{fs}_2(Y) - Y$, we have $x \in (\mathrm{fs}_2(P_{s+1}) - P_{s+1}) \cup (\mathrm{fs}_2(P_n) - P_n)$ so we deduce that x is not tight in Φ; a contradiction. Now we take the non-sequential 4-separation (X', Y') displayed by Ψ that is 2-equivalent to (X, Y). Then, arguing as above, we get that an element that is tight in Ψ is tight in Φ. $\square$

Lemma 3.12. *Let $(P_1, P_2, \dots, P_n)$ and $(O_1, O_2, \dots, O_n)$ be 2-equivalent tight flowers Φ and Ψ in a 4-connected matroid where Φ and Ψ have the same set T of tight elements and $P_i \cap T = O_i \cap T$ for all i. Then Ψ can be obtained from Φ by a sequence of elementary moves.*

Proof. This follows from the same argument as in the last paragraph of the proof of Theorem 5.1 of [4] using Lemma 3.8 to get $\mathrm{fs}_2(P_i) = \mathrm{fs}_2(P_i \cap T) = \mathrm{fs}_2(O_i \cap T) = \mathrm{fs}_2(O_i)$ for all i. □

We are now ready to prove the main result of this section. Most of the effort in this proof goes into dealing with flowers with three, four, or five petals, with the last of these cases being the most difficult.

Proof of Theorem 3.1. By Lemma 3.2, two irredundant flowers of order at least 3 are 2-equivalent if one can be obtained from the other by a sequence of elementary moves. To prove the converse, let $(P_1, P_2, \dots, P_n)$ and $(O_1, O_2, \dots, O_m)$ be irredundant 2-equivalent flowers Φ and Ψ of order at least 3. By Lemma 3.7, we may assume that Φ and Ψ are both tight flowers. We may also assume that $n \geq m$. Let T be the set of tight elements of Φ.

First, suppose that $n = 3$. Then $m = 3$ as well, otherwise Φ displays only one non-sequential 4-separation and is redundant; a contradiction. Now Φ must display at least two non-sequential 4-separations. Hence we may assume that $(P_1, P_2 \cup P_3)$ and $(P_2, P_3 \cup P_1)$ are non-sequential. The flower Φ can be transformed by elementary moves into $(\mathrm{fs}_2(P_1), P_2 - \mathrm{fs}_2(P_1), P_3 - \mathrm{fs}_2(P_1))$. Call this new flower (P'_1, P'_2, P'_3). By Lemma 3.6, $\mathrm{fs}_2(P'_1) = \mathrm{fs}_2(P_1)$ and $\mathrm{fs}_2(P'_2) = \mathrm{fs}_2(P_2)$. Now consider $(P'_1, \mathrm{fs}_2(P'_2) - P'_1, P'_3 - \mathrm{fs}_2(P'_2))$. We show next that we can transform (P'_1, P'_2, P'_3) into the last flower by a sequence of elementary moves. Let $(Y_i)_{i=1}^k$ be a 4-sequence for $\mathrm{fs}_2(P'_2)$. Then, by uncrossing the 4-separating sets $P'_2 \cup Y_1 \cup Y_2 \cup \dots \cup Y_j$ and $P'_2 \cup P'_3$ for all j in $[k]$, we see that we can indeed do the desired transformation via elementary moves. It follows that we may assume that, via elementary moves, we can transform Φ into a flower (P''_1, P''_2, P''_3) which equals $(\mathrm{fs}_2(P_1), \mathrm{fs}_2(P_2) - \mathrm{fs}_2(P_1), P_3 - \mathrm{fs}_2(P_1) - \mathrm{fs}_2(P_2))$. Now Ψ must display 4-separations 2-equivalent to $(P_1, P_2 \cup P_3)$ and $(P_2, P_3 \cup P_1)$. Suppose $(P_1, P_2 \cup P_3)$ is 2-equivalent to $(O_1 \cup O_2, O_3)$. Then, without loss in generality, $(P_2, P_3 \cup P_1)$ is 2-equivalent to either $(O_1 \cup O_3, O_2)$ or $(O_1, O_2 \cup O_3)$. In the first case, $\mathrm{fs}_2(O_2) = \mathrm{fs}_2(P_3 \cup P_1) \supseteq \mathrm{fs}_2(P_1) = \mathrm{fs}_2(O_1 \cup O_2)$, so O_1 is loose; a contradiction. In the second case, $\mathrm{fs}_2(O_3) = \mathrm{fs}_2(P_2 \cup P_3) \supseteq \mathrm{fs}_2(O_1)$, so O_1 is again loose. Hence we may assume that $(P_1, P_2 \cup P_3)$ is 2-equivalent to $(O_1, O_2 \cup O_3)$, and $(P_2, P_3 \cup P_1)$ is 2-equivalent to $(O_2, O_3 \cup O_1)$. As above, via elementary moves, we can transform (O_1, O_2, O_3) into a flower (O''_1, O''_2, O''_3) which equals $(\mathrm{fs}_2(O_1), \mathrm{fs}_2(O_2) - \mathrm{fs}_2(O_1), O_3 - \mathrm{fs}_2(O_1) - \mathrm{fs}_2(O_2))$. Thus $P''_1 = \mathrm{fs}_2(P_1) = \mathrm{fs}_2(O_1) = O''_1$; $P''_2 = \mathrm{fs}_2(P_2) - \mathrm{fs}_2(P_1) = \mathrm{fs}_2(O_2) - \mathrm{fs}_2(O_1) = O''_2$; and $P''_3 = E(M) - (P''_1 \cup P''_2) = E(M) - (O''_1 \cup O''_2) = O''_3$. We conclude that when Φ has exactly three petals, Φ can be transformed into Ψ by a sequence of elementary moves.

Now suppose that $n = 4$. First we show the following.

3.13.1. *At most one of* $(P_1 \cup P_2, P_3 \cup P_4)$, $(P_2 \cup P_3, P_1 \cup P_4)$, *and* $(P_1 \cup P_3, P_2 \cup P_4)$ *is a sequential* 4*-separation of* M.

Assume that at least two of $(P_1 \cup P_2, P_3 \cup P_4)$, $(P_2 \cup P_3, P_1 \cup P_4)$, and $(P_1 \cup P_3, P_2 \cup P_4)$ are sequential 4-separations. If the third arises, then the flower is an anemone, so we can reorder the petals so that we have the first two being sequential. Then, without loss in generality, $P_3 \cup P_4$ and $P_1 \cup P_4$ are sequential. Hence so are P_1, P_3, and P_4. Thus the only possible non-sequential 4-separations displayed by Φ are $(P_2, P_3 \cup P_4 \cup P_1)$ and $(P_1 \cup P_3, P_2 \cup P_4)$. This implies that Φ is a redundant flower; a contradiction. We deduce that (3.13.1) holds.

We may now assume that $(P_1 \cup P_2, P_3 \cup P_4)$ is non-sequential. First we consider the case when $(P_2 \cup P_3, P_1 \cup P_4)$ is also non-sequential. Recall that T is the set of tight elements of Φ. Then, by Lemma 3.11, T is also the set of tight elements of Ψ. Moreover, Ψ has exactly four petals $O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)}$ where σ is a permutation of $\{1, 2, 3, 4\}$, and $O_{\sigma(i)} \cap T = P_i \cap T$ for all i. Now $(O_{\sigma(1)} \cup O_{\sigma(2)}, O_{\sigma(3)} \cup O_{\sigma(4)})$ and $(O_{\sigma(2)} \cup O_{\sigma(3)}, O_{\sigma(4)} \cup O_{\sigma(1)})$ are 4-separations displayed by Ψ. If Ψ is an anemone, we may reorder its petals to get that $\Psi = (O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)})$. Thus we may assume Ψ is a daisy. But the 4-separations we have mean that the following pairs are consecutive in the cyclic order on $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4)\}$ imposed by Ψ: $(\sigma(1), \sigma(2))$, $(\sigma(3), \sigma(4))$, $(\sigma(2), \sigma(3))$, $(\sigma(4), \sigma(1))$. Again we get that $\Psi = (O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)})$. When the last equation holds, we may assume that $\sigma(i) = i$ for all i. In this case, the theorem follows by Lemma 3.12.

When $n = 4$, it remains to consider the case when $(P_2 \cup P_3, P_1 \cup P_4)$ is sequential. This will require a more careful analysis. Without loss in generality, $P_2 \cup P_3$, P_2, and P_3 are sequential sets. We may assume that neither $(P_1 \cup P_4, P_2 \cup P_3)$ nor $(P_1 \cup P_3, P_2 \cup P_4)$ is a non-sequential 4-separation, otherwise the theorem follows from the previous paragraph. Thus if Φ is an anemone, then $(P_1 \cup P_3, P_2 \cup P_4)$ is a sequential 4-separation, which contradicts (3.13.1). Therefore Φ is a daisy. Moreover, the only possible non-sequential 4-separations in M are $(P_1 \cup P_2, P_3 \cup P_4)$, $(P_1, P_2 \cup P_3 \cup P_4)$, and $(P_4, P_1 \cup P_2 \cup P_3)$. If either of the last two is sequential, then we get a contradiction to Φ being irredundant. Thus we may assume that all of the indicated 4-separations are non-sequential. Now choose s_1, s_4, s_2, s_3 in, respectively, $P_1 - \mathrm{fs}_2(P_2 \cup P_3 \cup P_4)$, $P_4 - \mathrm{fs}_2(P_1 \cup P_2 \cup P_3)$, $P_2 - \mathrm{fs}_2(P_3) - \mathrm{fs}_2(P_4) - \mathrm{fs}_2(P_1)$, and $P_3 - \mathrm{fs}_2(P_1) - \mathrm{fs}_2(P_2) - \mathrm{fs}_2(P_4)$. Using Lemma 3.4, it follows that, for each distinct pair $\{s_i, s_j\}$ of elements of $\{s_1, s_2, s_3, s_4\}$, there is a non-sequential 4-separation displayed by Ψ so that s_i and s_j are on opposite sides. Hence Ψ has four petals $O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)}$, where σ is a permutation of $\{1, 2, 3, 4\}$, and $s_i \in O_{\sigma(i)}$ for all i. Moreover, $(P_1, E - P_1)$ and $(P_4, E - P_4)$ are 2-equivalent to $(O_{\sigma(1)}, E - O_{\sigma(1)})$ and $(O_{\sigma(4)}, E - O_{\sigma(4)})$, respectively.

We show next that

3.13.2. $\Psi = (O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)})$.

Because $(P_1 \cup P_2, P_3 \cup P_4)$ is a non-sequential 4-separation, $(\sigma(1), \sigma(2))$ and $(\sigma(3), \sigma(4))$ are consecutive pairs in the cyclic order imposed by Ψ. Thus Ψ is either $(O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)})$ or $(O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(4)}, O_{\sigma(3)})$.

Since $(P_1 \cup P_2, P_3 \cup P_4)$ is non-sequential, $\mathrm{fs}_2(P_1 \cup P_2) = \mathrm{fs}_2(P_1) \cup \mathrm{fs}_2(P_2)$. Now, by elementary moves, we can transform Φ into the form (P_1, P_2, P_3, P_4) where $\mathrm{fs}_2(P_1) = P_1$ and $P_2 = \mathrm{fs}_2(P_1 \cup P_2) - P_1$. Next, consider Ψ. We may assume that $(\sigma(1), \sigma(2)) = (1, 2)$, and $(\sigma(3), \sigma(4)) \in \{(3, 4), (4, 3)\}$. We have that $(O_1 \cup O_2, O_3 \cup O_4)$ is 2-equivalent to $(P_1 \cup P_2, P_3 \cup P_4)$. Now consider $(O_2 \cup O_3, O_4 \cup O_1)$. If this is non-sequential, then Φ displays a non-sequential 4-separation (X, Y) with $\mathrm{fs}_2(X) = \mathrm{fs}_2(O_2 \cup O_3)$ and $\mathrm{fs}_2(Y) = \mathrm{fs}_2(O_4 \cup O_1)$. We have $s_i \in O_{\sigma(i)}$ so $\{s_2, s_{\sigma^{-1}(3)}\} \subseteq \mathrm{fs}_2(X)$ and $\{s_1, s_{\sigma^{-1}(4)}\} \subseteq \mathrm{fs}_2(Y)$. Thus $P_{\sigma^{-1}(4)} \cup P_1 \subseteq Y$ and $P_2 \cup P_{\sigma^{-1}(3)} \subseteq X$ so either $(P_4 \cup P_1, P_2 \cup P_3)$ or $(P_3 \cup P_1, P_2 \cup P_4)$ is a non-sequential 4-separation; a contradiction. Therefore, $(O_2 \cup O_3, O_4 \cup O_1)$ must be sequential. A similar argument establishes that $(O_1, O_2 \cup O_3 \cup O_4)$ and $(O_{\sigma(4)}, O_1 \cup O_2 \cup O_{\sigma(3)})$ are 2-equivalent to $(P_1, P_2 \cup P_3 \cup P_4)$ and $(P_4, P_1 \cup P_2 \cup P_3)$, respectively.

By elementary moves, we can transform Ψ into (O_1, O_2, O_3, O_4) such that $O_1 = \mathrm{fs}_2(O_1)$ and $O_2 = \mathrm{fs}_2(O_1 \cup O_2) - O_1$. Thus $O_1 = \mathrm{fs}_2(O_1) = \mathrm{fs}_2(P_1) = P_1$ and $O_2 = \mathrm{fs}_2(O_1 \cup O_2) - O_1 = \mathrm{fs}_2(P_1 \cup P_2) - P_1 = P_2$. Now $(O_2 \cup O_3, O_4 \cup O_1)$ is sequential. Thus one of the sets $O_2 \cup O_3$ or $O_4 \cup O_1$ is sequential. In the first case, O_2 and O_3 are sequential, so $O_3 \neq O_{\sigma(4)}$. In the second case, O_1 and O_4 are sequential; a contradiction as O_1 is not sequential. Thus $O_3 = O_{\sigma(3)}$, and (3.13.2) holds.

We may now assume that $\sigma(i) = i$ for all i. We modify (P_1, P_2, P_3, P_4) again making $P_4 = \mathrm{fs}_2(P_4) - (P_1 \cup P_2)$. We can do this via elementary moves by repeatedly uncrossing the 4-separating sets $P_4 \cup P_3$ and $P_4 \cup Y_1 \cup Y_2 \cup \cdots \cup Y_i$, where $(Y_i)_{i=1}^m$ is a 4-sequence for $\mathrm{fs}_2(P_4)$. Similarly, we can modify (O_1, O_2, O_3, O_4) by elementary moves so that $O_4 = \mathrm{fs}_2(O_4) - (O_1 \cup O_2)$. Then, as $\mathrm{fs}_2(O_4) = \mathrm{fs}_2(P_4)$, we have $P_4 = O_4$. As $P_3 = E(M) - (P_1 \cup P_2 \cup P_4)$, we deduce $O_3 = P_3$. Hence Φ can be transformed into Ψ by a sequence of elementary moves. We conclude that the theorem holds when $n = 4$.

Next, consider the case when $n = 5$. First we prove the following.

3.13.3. *If $n = 5$, then either*

- *(i) for some i in $\{1, 2, 3, 4, 5\}$, all of $(P_i \cup P_{i+1}, E - (P_i \cup P_{i+1}))$, $(P_{i+1} \cup P_{i+2}, E - (P_{i+1} \cup P_{i+2}))$, and $(P_{i+2} \cup P_{i+3}, E - (P_{i+2} \cup P_{i+3}))$ are non-sequential; or*
- *(ii) Φ is an anemone and, for some i in $\{1, 2, 3, 4, 5\}$, all of $(P_i \cup P_{i+1}, E - (P_i \cup P_{i+1}))$, $(P_{i+1} \cup P_{i+2}, E - (P_{i+1} \cup P_{i+2}))$, and $(P_{i+1} \cup P_{i+3}, E - (P_{i+1} \cup P_{i+3}))$ are non-sequential; or*
- *(iii) Φ is an anemone and its petals can be reordered so that (i) or (ii) hold.*

Assume that (3.13.3) fails.

3.13.4. *Φ displays a non-sequential 4-separation with two petals on one side and three on the other.*

Assume that the last assertion fails. Then we may assume that P_4 and P_5 are sequential. Thus there is no non-sequential 4-separation displayed by Φ with P_4 and P_5 on opposite sides. Hence Φ is redundant. This contradiction establishes that (3.13.4) holds.

Next we show the following.

3.13.5. *For all i in $\{1, 2, 3, 4, 5\}$, at least one of $(P_i \cup P_{i+1}, E - (P_i \cup P_{i+1}))$ and $(P_{i+1} \cup P_{i+2}, E - (P_{i+1} \cup P_{i+2}))$ is sequential.*

To prove this, suppose that both $(P_1 \cup P_2, E - (P_1 \cup P_2))$ and $(P_2 \cup P_3, E - (P_2 \cup P_3))$ are non-sequential. Then, as (3.13.3) fails, both $(P_3 \cup P_4, E - (P_3 \cup P_4))$ and $(P_5 \cup P_1, E - (P_5 \cup P_1))$ are sequential. It follows, by Corollary 3.5, that $P_3 \cup P_4, P_5 \cup P_1, P_3, P_4, P_5$, and P_1 are sequential. As Φ is irredundant, it displays some non-sequential 4-separation with P_4 and P_5 on opposite sides. By the symmetry between P_4 and P_5, we may assume that one of $(P_1 \cup P_4, E - (P_1 \cup P_4))$ and $(P_2 \cup P_4, E - (P_2 \cup P_4))$ is a non-sequential 4-separation of M. Hence Φ is an anemone. Moreover, if $(P_2 \cup P_4, E - (P_2 \cup P_4))$ is a non-sequential 4-separation, then (ii) holds; a contradiction. On the other hand, if $(P_1 \cup P_4, E - (P_1 \cup P_4))$ is a non-sequential 4-separation, then we can reorder the petals of Φ so that (i) holds, again obtaining a contradiction. Thus (3.13.5) holds.

By (3.13.4), we may assume that $(P_1 \cup P_2, E - (P_1 \cup P_2))$ is non-sequential. Then, by (3.13.5), $(P_2 \cup P_3, E - (P_2 \cup P_3))$ and $(P_5 \cup P_1, E - (P_5 \cup P_1))$ are sequential. Hence, by Corollary 3.5, P_2, P_3, P_5, and P_1 are sequential. Thus, if Φ is a daisy, we get the contradiction that it does not display a non-sequential 4-separation with P_1 and P_2 on opposite sides. Therefore Φ is an anemone.

As $(P_1 \cup P_2, E - (P_1 \cup P_2))$ is non-sequential, by (3.13.5), $(P_i \cup P_j, E - (P_i \cup P_j))$ is sequential for all i in $\{1, 2\}$ and all j in $\{3, 4, 5\}$. By (3.13.5) again, at least two of $(P_3 \cup P_4, E - (P_3 \cup P_4))$, $(P_4 \cup P_5, E - (P_4 \cup P_5))$, and $(P_3 \cup P_5, E - (P_3 \cup P_5))$ are sequential. By symmetry, we may assume the first two are sequential. Then P_4 is sequential, so all of P_1, P_2, P_3, P_4, and P_5 are sequential. Hence there is no non-sequential 4-separation displayed by Φ with P_3 and P_5 on opposite sides. This contradiction completes the proof of (3.13.3).

On combining (3.13.3) and Lemma 3.11, we deduce that if $n = 5$, then $m = 5$, and Φ and Ψ have the same set T of tight elements. Next we establish the following.

3.13.6. *There is a permutation σ of $\{1, 2, 3, 4, 5\}$ such that $P_i \cap T = O_{\sigma(i)} \cap T$ for all i.*

By symmetry, it suffices to show that T does not contain two elements that are in the same petal in Φ but in different petals in Ψ. Suppose that x and y are elements of some $P_j \cap T$, but x and y are in distinct petals,

O_s and O_t, of Ψ. If all 4-separations (X, Y) displayed by Ψ with $O_s \subseteq X$ and $O_t \subseteq Y$ are sequential, then Ψ is redundant; a contradiction. Thus there must be such a separation (X, Y) that is non-sequential. If X and Y can be chosen so that each contains at least two petals of Ψ, then, as x and y are tight, it follows from Lemma 3.4 that there is no 4-separation 2-equivalent to (X, Y) having x and y on the same side. Now Φ must display some non-sequential 4-separation (X', Y') 2-equivalent to (X, Y). But x and y cannot be on different sides of (X', Y') as they are in the same petal in Φ. Thus we may assume that $X = O_s$, so O_s is non-sequential. Let O_k be an adjacent petal to O_s that is different from O_t. Then the 4-separation $(O_s \cup O_k, E - (O_s \cup O_k))$ is non-sequential by Corollary 3.5; a contradiction. It now follows that, for all j in $[n]$ and any two elements x and y in $P_j \cap T$, both x and y must be contained in the same petal of Ψ. Hence (3.13.6) holds.

Clearly we may assume that $\sigma(1) = 1$. We show next that we may assume that $P_i \cap T = O_i \cap T$ for all i in $\{1, 2, 3, 4, 5\}$. This follows immediately if Φ or Ψ is an anemone. Thus we assume that Φ and Ψ are daisies. By (3.13.3), we may assume that $(P_1 \cup P_2, E - (P_1 \cup P_2)), (P_2 \cup P_3, E - (P_2 \cup P_3))$, and $(P_3 \cup P_4, E - (P_3 \cup P_4))$ are non-sequential. It follows that $O_{\sigma(1)}$ and $O_{\sigma(2)}$ are consecutive petals of Ψ. Likewise, $O_{\sigma(2)}$ and $O_{\sigma(3)}$ occur consecutively, as do $O_{\sigma(3)}$ and $O_{\sigma(4)}$. Hence $O_{\sigma(1)}, O_{\sigma(2)}, O_{\sigma(3)}, O_{\sigma(4)}$ is a sequence of consecutive petals of Ψ. Thus, we may assume that $\sigma(i) = i$ for all i in $\{1, 2, 3, 4, 5\}$. Then, by Lemma 3.12, the theorem holds when $n = 5$.

Finally, suppose $n \geq 6$. Then, by Corollary 3.5, $(P_i \cup P_{i+1} \cup P_{i+2}, E - (P_i \cup P_{i+1} \cup P_{i+2}))$ is a non-sequential 4-separation for all i in $[n]$. Then, by Lemma 3.11, Ψ has n petals, Φ and Ψ have the same set T of tight elements, and there is a permutation σ of $[n]$ such that $P_i \cap T = O_{\sigma(i)} \cap T$ for all i.

If Φ or Ψ is an anemone, then we may assume that $\sigma(i) = i$ for all i, and the theorem follows by Lemma 3.12. Hence we may assume that Φ and Ψ are both daisies. Now, for all i in $[n]$, there is a set of three consecutive petals of Ψ containing $\sigma(i), \sigma(i+1)$, and $\sigma(i+2)$. It is straightforward to see that Ψ can be written as $(O_{\sigma(1)}, O_{\sigma(2)}, \dots, O_{\sigma(n)})$, so we may take $\sigma(i) = i$ for all i in $[n]$, and the theorem again follows by Lemma 3.12. $\square$

Next we briefly discuss the need to make the notion of irredundance explicit when dealing with 4-flowers in 4-connected matroids even though it is not used with 3-flowers in 3-connected matroids. Following the terminology of [4] for the moment, we note that it is straightforward to prove that if Φ is a tight flower of order at least three in a 3-connected matroid N, then, for all distinct i and j in $[n]$, the 3-flower Φ displays a non-sequential 3-separation in which P_i and P_j are on opposite sides. Hence, for 3-flowers of order at least three, tightness implies irredundance. The next example shows that, with the definitions of this paper, tightness of 4-flowers does not imply irredundance.

Example 3.14. Let (P_1, P_2, P_3, P_4) be a paddle in a 3-connected matroid N where P_1 consists of 8 points freely placed in rank 4 and, for each i in $\{2, 3, 4\}$, the set P_i is a triad $\{x_i, y_i, z_i\}$. In addition, it is not difficult to see that we can arrange the lines $\{x_2, y_2\}, \{x_4, y_4\}, \{x_4, z_4\}$, and $\{x_3, z_3\}$ so that each of $\{x_2, y_2, x_4, y_4\}$ and $\{x_4, z_4, x_3, z_3\}$ is a circuit of N. Then (P_1, P_2, P_3, P_4) is a tight 3-flower in N of order 4. Moreover, (P_1, P_2, P_3, P_4) is a tight 4-flower Φ in $T(N)$. The only non-sequential 4-separations displayed by the 4-flower Φ are $(P_1, P_2 \cup P_3 \cup P_4)$ and $(P_1 \cup P_4, P_2 \cup P_3)$, so Φ is 2-equivalent to $(P_1, P_2 \cup P_3, P_4)$. Hence Φ has order three. Moreover, it is redundant, in spite of the fact that it is tight.

The next result is a straightforward consequence of the proof of Theorem 3.1 and the definition of tightness for flowers of order one or two.

Corollary 3.15. *The order of a tight irredundant flower is equal to its number of petals.*

We shall use the next lemma in the proof of the main result of the paper.

Lemma 3.16. *Let $(P_1, P_2, \dots, P_n)$ be a tight irredundant flower Φ. If $2 \leq i \leq n-2$ and (X, Y) is a non-sequential 4-separation that is 2-equivalent to $(P_1 \cup P_2 \cup \dots \cup P_i, P_{i+1} \cup P_{i+2} \cup \dots \cup P_n)$, then there is a tight irredundant flower 2-equivalent to Φ that displays (X, Y).*

Proof. Since $n \geq 4$, it follows by Corollary 3.15 that Φ has order at least 4, so $|E(M)| \geq 12$. We may assume that $\mathrm{fs}_2(X) = \mathrm{fs}_2(P_1 \cup P_2 \cup \dots \cup P_i)$ and $\mathrm{fs}_2(Y) = \mathrm{fs}_2(P_{i+1} \cup P_{i+2} \cup \dots \cup P_n)$. Then all of the tight elements of Φ contained in $P_1 \cup P_2 \cup \cdots P_i$ and $P_{i+1} \cup P_{i+2} \cup \dots \cup P_n$ are contained in X and Y, respectively. We argue by induction on $|X - (P_1 \cup P_2 \cup \dots \cup P_i)| + |Y - (P_{i+1} \cup P_{i+2} \cup \dots \cup P_n)|$ noting that if this sum is zero, then the result is immediate. Thus we may assume that there is an element x in $X - (P_1 \cup P_2 \cup \dots \cup P_i)$. Then $x \in \mathrm{fs}_2(P_1 \cup P_2 \cup \dots \cup P_i) - (P_1 \cup P_2 \cup \dots \cup P_i)$. We know that $(P_1 \cup P_2 \cup \dots \cup P_i, P_{i+1} \cup P_{i+2} \cup \dots \cup P_n)$ is non-sequential since it is 2-equivalent to the non-sequential 4-separation (X, Y). Hence, by Lemma 3.4, we may assume that $x \in \mathrm{fs}_2(P_i) - P_i$. Let $(Y_j)_{j=1}^m$ be a 4-sequence for $\mathrm{fs}_2(P_i)$. Then x is in Y_k for some k in $[m]$.

Let h be the smallest index for which $Y_h \cap X \neq \emptyset$. Then, by Lemma 2.9, $Y_h \subseteq X$. Let T be the set of tight elements of Φ in $P_1 \cup P_2 \cup \dots \cup P_i$. Then $T \subseteq X$ and, by Lemma 3.8 (ii) and the fact that Φ is tight, we have $|T \cap P_i| \geq 3$. Hence $|X \cap P_i| \geq 3$. Also, since (X, Y) is non-sequential, $|E - (X \cup P_i)| \geq 3$. Thus, by uncrossing, we see that $X \cap P_i$ is 4-separating. Similarly, by uncrossing X and $P_i \cup (Y_1 \cup Y_2 \cup \dots \cup Y_h)$, we see that $(X \cap P_i) \cup Y_h$ is 4-separating. It follows that $\lambda(P_i \cup Y_h) = 3$.

By performing a Type-1a move, we see that Φ is 2-equivalent to the tight irredundant flower Φ' that is obtained by adjoining Y_h to P_i and removing it from its original petal. The result now follows by applying the induction assumption to Φ'. □

4. Maximal 4-flowers

A flower Φ is *maximal* if Φ is 2-equivalent to Φ' whenever $\Phi \preccurlyeq \Phi'$. Let (X, Y) be a 4-separation of M. We say that (X, Y) *conforms* with the flower Φ if either (X, Y) is 2-equivalent to a 4-separation that is displayed by Φ, or (X, Y) is 2-equivalent to a 4-separation (X', Y') with the property that either X' or Y' is contained in a petal of Φ.

The aim of this section is to prove the following result, which will be crucial in proving the main result of the paper.

Theorem 4.1. *Let M be a 4-connected matroid with at least 17 elements and let Φ be a tight irredundant maximal flower for M. Then every non-sequential 4-separation of M conforms with Φ.*

A flower Φ is a *refinement* of a flower Φ' if the ordered partition corresponding to Φ refines that of Φ', that is, Φ can be obtained from Φ' by a sequence of moves each consisting of replacing a petal P by an ordered partition of P. Clearly if Φ is a refinement of Φ', then $\Phi' \preccurlyeq \Phi$. A partition (X, Y) of $E(M)$ *crosses* a petal P of a flower Φ if P meets both X and Y.

The proof of the next lemma is a straightforward modification of the proof of [4, Lemma 8.2] and we omit the details.

Lemma 4.2. *Let $(P_1, P_2, \ldots, P_n)$ be a flower Φ for M, and let (R, G) be a 4-separation such that*

(i) *neither R nor G is contained in a petal of Φ; and*
(ii) *if (R, G) crosses a petal P of Φ, then $|R \cap P| \geq 3$ and $|G \cap P| \geq 3$.*

Then there is a flower that refines Φ and displays (R, G).

Proof of Theorem 4.1. Let $\Phi = (P_1, P_2, \ldots, P_n)$. Assume that the theorem fails, and that (X, Y) is a non-sequential 4-separation that does not conform with Φ. Let (R, G) be a 4-separation 2-equivalent to Φ that is chosen so that it crosses the smallest possible number of petals. Since (R, G) is non-sequential, it follows that $|R|, |G| \geq 5$.

4.3.1. *If $|R \cap P_i| \leq 2$, then $|G \cap P_i| \leq 2$.*

Suppose that there is an element i in $[n]$ such that $|R \cap P_i| \leq 2$ and $|G \cap P_i| \geq 3$. Then, by uncrossing, $G \cup P_i$ is 4-separating. But $G \cup P_i = G \cup (R \cap P_i)$ and, since $|R \cap P_i| \leq 2$, the 4-separation $(R - (R \cap P_i), G \cup (R \cap P_i))$ is 2-equivalent to (R, G). But $(R - (R \cap P_i), G \cup (R \cap P_i))$ crosses fewer petals than (R, G), contradicting the choice of (R, G). This contradiction establishes (4.3.1).

4.3.2. *There is no petal P_i with $|R \cap P_i| \leq 2$.*

Assume that $|R \cap P_1| \leq 2$. Then by (4.3.1), $|G \cap P_1| \leq 2$. Certainly Φ has at least two petals. If Φ has exactly two petals, then Φ displays no non-sequential 4-separation, so Φ is 2-equivalent to the trivial flower and not tight; a contradiction. We may now assume that Φ has at least three petals. Next, we define a partition (P^+, P^-) of $E(M) - P_1$ into 4-separating sets P^+ and P^- such that

$$(4.1)\quad \lambda(P^+ \cup P_1) = 3; \quad |P^-| \geq 5; \quad \text{and } |R \cap P^+| \geq 3 \text{ or } |G \cap P^+| \geq 3.$$

Assume first that Φ has exactly three petals. If $|P_2| \leq 4$, then Φ displays at most one non-sequential 4-separation, contradicting the fact that Φ is irredundant. Thus $|P_2|, |P_3| \geq 5$. In this case, set $P^+ = P_2$ and $P^- = P_3$. Then (4.1) clearly holds. Next, assume that Φ has four petals. Since $|E(M)| \geq 17$, one of the petals of Φ has at least 5 elements. This means we can assume that, amongst the petals of Φ crossed by (R, G) and containing at most two elements from each of R and G, the petal P_1 is chosen so that $|P_2 \cap R| \geq 3$ or $|P_2 \cap G| \geq 3$. In this case, set $P^+ = P_2$ and $P^- = P_3 \cup P_4$. Again, (4.1) holds. If Φ has at least five petals, set $P^+ = P_2 \cup P_3$ and $P^- = P_4 \cup P_5 \cup \cdots \cup P_n$. Then (4.1) holds again. Hence it holds in general.

Next, we assert that we may assume, by possibly interchanging R and G, that

$$(4.2)\qquad |P^+ \cap R| \geq 3 \text{ and } |P^- \cap G| \geq 3.$$

By (4.1), either $|P^+ \cap R| \geq 3$ or $|P^+ \cap G| \geq 3$. If both of the last two inequalities hold, then, since $|P^-| \geq 5$, either $|P^- \cap R| \geq 3$ or $|P^- \cap G| \geq 3$. Hence (4.2) holds. Now, by symmetry, we may assume that $|P^+ \cap R| \geq 3$ and $|P^+ \cap G| \leq 2$. Then, as $|G| \geq 5$, we have $P^- \cap G \neq \emptyset$. If $|P^- \cap G| \geq 3$, then (4.2) holds. Thus we may assume that $|P^- \cap G| \leq 2$. Then $|P^- \cap R| \geq 3$ and, as $P^+ \cup P_1$ is 4-separating, we can uncross $P^+ \cup P_1$ and G to see that $(P^+ \cup P_1) \cup G$ is 4-separating. Hence, so is the complement, $P^- \cap R$. Then, by uncrossing R and P^-, we see that their union, $R \cup (P^- \cap G)$, is 4-separating. But the complement of this union is $(P_1 \cup P^+) \cap G$, which contains at most four elements. This implies that (R, G) is sequential; a contradiction. We conclude that (4.2) holds.

As $(P^+ \cup P_1) \cup R$ avoids $P^- \cap G$, it follows by uncrossing that $(P^+ \cup P_1) \cap R$ is 4-separating. Similarly, by uncrossing P^+ and R, we get that $P^+ \cap R$ is 4-separating. It follows that $P_1 \cap R \subseteq \mathrm{fs}_2(P^+ \cap R)$. Hence $P_1 \cap R \subseteq \mathrm{fs}_2(P^+)$. But P^+ is the union of at most $n-2$ petals of Φ. If P^+ is the union of at most $n-3$ petals, then, by Lemma 3.4, $P_1 \cap R$ is a loose set of cardinality one or two. We deduce that, since $|P_1| \leq 4$, the petal P_1 is loose; a contradiction. If P^+ is the union of exactly $n-2$ petals, then, by the definition of P^+, we see that $n = 3$ and $(P^+, P^-) = (P_2, P_3)$. By uncrossing the 4-separating sets $R \cap (P^+ \cup P_1)$ and $P_1 \cup P_2$, we see that, as their union avoids P_3, their intersection, $(P_1 \cap R) \cup P_2$, is 4-separating. Thus, $P_1 \cap R \subseteq \mathrm{fs}_2(P_2)$, so $P_1 \subseteq \mathrm{fs}_2(P_2)$; a contradiction. Therefore, (4.3.2) holds.

From (4.3.2), we see that (R, G) satisfies the hypothesis of Lemma 4.2. Thus, by that lemma, there is a flower that refines Φ and displays (R, G), contradicting the fact that Φ is maximal. □

The requirement for M to have at least 17 elements in Theorem 4.1 is indeed necessary. This is not surprising since in [4], when considering 3-connected matroids, there is a similar requirement to have at least 9 elements in order to have every non-sequential 3-separation conform with a tight maximal 3-flower. The 8-element example in [4] that demonstrates the need for this constraint is R_8, the rank-4 real cube. The cube of one greater rank provides an example of a 16-element matroid for which Theorem 4.1 fails.

Let H_{16} be the 16-element binary affine hypercube of rank 5 pictured in Figure 5. Note that the illustration is slightly deceiving since H_{16} appears to be in rank 4. We have also provided a binary matrix representation for H_{16} in Figure 6, where the column labels correspond to the labels on the points in Figure 5. It is easily checked that the partition $\Phi = (\{1,2,3,4\}, \{5,6,7,8\}, \{9,10,11,12\}, \{13,14,15,16\})$ is an irredundant tight maximal flower having four petals. On the other hand, the non-sequential 4-separation $(\{1,2,7,8,9,10,15,16\}, \{3,4,5,6,11,12,13,14\})$ does not conform with Φ.

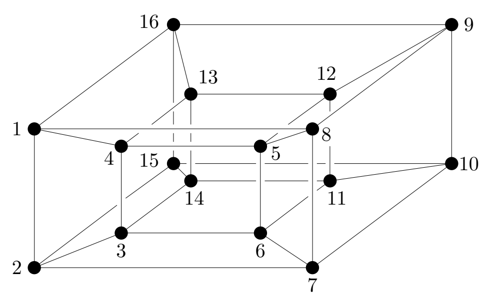

FIGURE 5. The binary affine hypercube of rank 5.

$$
\begin{array}{c}
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\end{array}
$$

FIGURE 6. A matrix representing the binary affine hypercube of rank 5.

5. The Main Result

In this section, we prove the main result of the paper showing that there is a labelled tree that displays, up to 2-equivalence, all non-sequential 4-separations of a 4-connected matroid having at least seventeen elements.

In [4], the type of tree that was used to display the non-sequential 3-separations in a 3-connected matroid, up to equivalence, was a partial 3-tree. We develop an analogous structure here that will be used to display the non-sequential 4-separations in a 4-connected matroid up to 2-equivalence. The definitions that follow are identical to those in [4], with the exception that they are stated in terms of 2-equivalence.

Let π be a partition of a finite set E. Let T be a tree such that every member of π labels a vertex of T; some vertices may be unlabelled and no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*.

Let T' be a subtree of T. The union of those bags that label vertices of T' is the subset of E *displayed* by T'. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by the components of $T\backslash e$. Let v be a vertex that is not a bag vertex. Then the *partition of E displayed by v* is the partition displayed by the components of $T-v$. The edges incident with v are in natural one-to-one correspondence with the components of $T-v$, and so with the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \dots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a 4-connected matroid with ground set E. An *almost partial 4-tree* T for M is a π-labelled tree, where π is a partition of E such that the following conditions hold:

(i) For each edge e of T, the partition (X, Y) of E displayed by e is 4-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential 4-separation.
(ii) Every non-bag vertex v is labelled either A or D. Moreover, if v is labelled D, then there is a cyclic order on the edges incident with v.
(iii) If a vertex v is labelled A, then the partition of E displayed by v is a tight irredundant maximal anemone of order at least three.
(iv) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic ordering induced by the cyclic ordering on the edges incident with v, is a tight irredundant maximal daisy of order at least three.

By conditions (iii) and (iv), a vertex v labelled A or D corresponds to a flower for M. The 4-separations displayed by this flower are the 4-separations *displayed by v*. A vertex of an almost partial 4-tree is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. A 4-separation is *displayed by an almost partial 4-tree T* if it is displayed by some edge or some flower vertex of T. We remark here that, as is the case with almost partial 3-trees in [4], it is possible for a bag vertex to be labelled by the empty set although this cannot occur if the bag vertex is a leaf.

A 4-separation (R, G) of M *conforms* with an almost partial 4-tree T if either (R, G) is 2-equivalent to a 4-separation that is displayed by a flower vertex or an edge of T, or (R, G) is 2-equivalent to a 4-separation (R', G') with the property that either R' or G' is contained in a bag of T.

An almost partial 4-tree for M is a *partial* 4-*tree* if

(v) every non-sequential 4-separation of M conforms with T.

We define a quasi order on the set of partial 4-trees for M just as the quasi order was defined for partial 3-trees in [4]. If T_1 and T_2 are partial 4-trees for M, then $T_1 \preccurlyeq T_2$ if all of the non-sequential 4-separations displayed by T_1 are displayed by T_2. If $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$, then T_1 is 2-*equivalent* to T_2. A partial 4-tree is *maximal* if it is maximal with respect to this quasi order. Following [5], we call a maximal partial 4-tree for M a 4-*tree* for M.

The next theorem is the main result of the paper.

Theorem 5.1. *Let M be a 4-connected matroid having at least 17 elements, and let T be a 4-tree for M. Then every non-sequential 4-separation of M is 2-equivalent to a 4-separation displayed by T.*

We can associate a π-labelled tree T with a flower $\Phi = (P_1, P_2, \dots, P_n)$. If Φ consists of a single petal, then T consists of a single bag vertex labelled by P_1. If Φ consists of two petals, then T consists of two adjacent bag vertices labelled by P_1 and P_2. When Φ has at least three petals, we let $\{v, v_1, v_2, \dots, v_n\}$ be the vertices of T, where v is adjacent to each v_i and each v_i is labelled by the bag P_i. The vertex v is then labelled by either an A or a D depending on whether Φ is an anemone or a daisy, respectively. If Φ has exactly three petals, then we are free to label v by either A or D. This association of a flower to a π-labelled tree gives us the following immediate consequence of Theorem 4.1.

Corollary 5.2. *Tight irredundant maximal flowers for 4-connected matroids on at least 17 elements are partial 4-trees.*

We will use the following lemma in the proof of Theorem 5.1.

Lemma 5.3. *If $(X, E-X)$ is a non-sequential 4-separation of a 4-connected matroid M, then there is a tight irredundant maximal flower that displays a 4-separation 2-equivalent to $(X, E-X)$.*

Proof. It is clear that $(X, E-X)$ is a tight flower Φ_0 that displays $(X, E-X)$. If Φ_0 is not maximal, then there is a maximal flower $\Phi_1 \succcurlyeq \Phi_0$. Since Φ_1 must display some non-sequential 4-separation that is not 2-equivalent to one displayed by Φ_0, it must be that Φ_1 has order at least three. Thus, by Lemmas 3.7 and 3.2, the flower Φ_1 is 2-equivalent to a tight maximal flower Φ_2. The lemma holds if Φ_2 is irredundant. Thus assume that Φ_2 is redundant. Then there are petals P_i and P_{i+1} of Φ_2 that are contained on the same side of every non-sequential 4-separation displayed by Φ_2. In this case, we concatenate the petals P_i and P_{i+1} into a single petal $P_i' = P_i \cup P_{i+1}$

letting the resulting flower be Φ_3. Clearly, $\Phi_3 \preccurlyeq \Phi_2$. If (A, B) is a non-sequential 4-separation displayed by Φ_2, then P_i' is contained on one side, say $P_i' \subseteq A$. Thus (A, B) is displayed by Φ_3. Hence Φ_2 and Φ_3 are 2-equivalent flowers. If Φ_3 is irredundant, the lemma follows. Thus we may assume that Φ_3 is redundant and repeat the argument above replacing Φ_2 by Φ_3. Continuing in this way, we eventually obtain a tight irredundant maximal flower Φ_t that is 2-equivalent to Φ_2. As $\Phi_t \succcurlyeq \Phi_0$, there is a 4-separation 2-equivalent to $(X, E - X)$ that is displayed by Φ_t. □

To prove Theorem 5.1, we mimic the technique used to prove Theorem 9.1 in [4] by first proving the following lemma that generalizes Lemma 9.4 in [4]. The core of the proof of Theorem 5.1 is contained in the proof of this lemma. As in the proof of Theorem 3.1, extra care is needed when dealing with 4-petal flowers. Recall that two exactly 4-separating sets Y and Z are 2-equivalent if $\mathrm{fs}_2(Y) = \mathrm{fs}_2(Z)$.

Lemma 5.4. *Let M be a 4-connected matroid with $|E(M)| \geq 17$ and let T be a partial 4-tree for M having at least one edge. If M has a non-sequential 4-separation $(W, E-W)$ that is not 2-equivalent to any 4-separation displayed by T, then there is a partial 4-tree T' such that $T' \succcurlyeq T$ and T' displays some non-sequential 4-separation that is not 2-equivalent to any 4-separation displayed by T.*

Proof. Assume that the lemma fails. By the definition of a partial 4-tree, $(W, E-W)$ conforms with T and so is 2-equivalent to a 4-separation $(X, E-X)$, where X is contained in a bag B of T. Since $(W, E - W)$ is non-sequential, so is $(X, E - X)$. Let u be the vertex of T labelled by B. We note that since $E-B \subseteq E-X$ and $(X, E-X)$ is non-sequential, $\mathrm{fs}_2(E-B) \neq E(M)$. Hence B is non-sequential.

We proceed by breaking the argument into the following two cases:

(I) u is a leaf of T; and

(II) u is not a leaf of T.

We first consider Case (I). In that case, we assert the following.

5.4.1. $(B, E - B)$ *is non-sequential.*

In the event that u is adjacent to a bag vertex, (5.4.1) follows from the definition of a partial 4-tree. Suppose that u is adjacent to a flower vertex v. As noted above, $\mathrm{fs}_2(E-B) \neq E(M)$, so we need only show that $\mathrm{fs}_2(B) \neq E(M)$. Let Ψ be the flower for M given by the partition displayed by v. Then Ψ is tight and B is a petal of Ψ. Suppose that $\mathrm{fs}_2(B) = E(M)$. Then Ψ displays no non-sequential 4-separations so it has order one; a contradiction. Thus $\mathrm{fs}_2(B) \neq E(M)$ and (5.4.1) holds.

5.4.2. *There is a partial 4-tree 2-equivalent to T in which u is labelled by* $\mathrm{fs}_2(B)$.

Clearly, if $\mathrm{fs}_2(B) = B$, then (5.4.2) holds. Suppose that $\mathrm{fs}_2(B) \neq B$ and let $(Y_i)_{i=1}^m$ be a 4-sequence for $\mathrm{fs}_2(B)$. To prove (5.4.2), we begin by showing that Y_i is contained in a bag for all i in $[m]$. Suppose, to the contrary, that there is a j in $[m]$ such that Y_j is not contained in a bag. Then $Y_j = \{a, b\}$ and there are bags B_1 and B_2 such that $a \in B_1$ and $b \in B_2$. Let u_1 and u_2 be the vertices labelled by B_1 and B_2, respectively, in T. Since $(B, E - B)$ is non-sequential, $\min\{|B|, |E - \mathrm{fs}_2(B)|\} \geq 5$, and $(B \cup (Y_1 \cup Y_2 \cup \cdots \cup Y_{j-1}), E - (B \cup (Y_1 \cup Y_2 \cup \cdots \cup Y_{j-1})))$ is a non-sequential 4-separation. Thus, by Lemma 2.9, there is no non-sequential 4-separation displayed by T having B_1 and B_2 on opposite sides. Since every edge of T incident with two bag vertices displays a non-sequential 4-separation, we deduce that u_1 and u_2 are adjacent to the same flower vertex v. Let $(O_1, O_2, \ldots O_n)$ be the flower Ψ displayed by v. We may assume that $B_1 \subseteq O_1$ and $B_2 \subseteq O_k$ for some k in $\{2, 3, \ldots, n\}$. Since there is no non-sequential 4-separation displayed by T having B_1 and B_2 on opposite sides, it follows that every 4-separation displayed by Ψ having O_1 and O_k on opposite sides must be sequential. It is not difficult to see that this implies that Ψ is a redundant flower; a contradiction. Therefore, Y_i is contained in a bag for all i in $[m]$.

Now, let T' be the tree obtained from T by adjoining each Y_i to the bag B and removing it from its original bag in T. We show that T' is a partial 4-tree 2-equivalent to T. Let e be an edge of T' and let $(A', E-A')$ be the partition displayed by e in T'. Furthermore, let $(A, E - A)$ be the non-sequential 4-separation displayed by e in T. Without loss in generality, we may assume that $B \subseteq A$. By Lemma 2.9, and the fact that $(B, E - B)$ is non-sequential, each Y_i is either contained in $A - B$ or in $E - A$. If all Y_i are contained in $A - B$, then $(A', E - A') = (A, E - A)$. Thus we may assume that there is some Y_j contained in $E-A$ and that among all such sets, Y_j has the smallest index. Then $A = A \cup (Y_1 \cup Y_2 \cup \cdots \cup Y_{j-1})$. By uncrossing $B \cup (Y_1 \cup Y_2 \cup \cdots \cup Y_j)$ and A, we see that since their intersection contains B, their union, $A \cup Y_j$, is 4-separating. Hence $(A \cup Y_j, E - (A \cup Y_j))$ is a non-sequential 4-separation 2-equivalent to $(A, E - A)$. By repeating this argument, continuing with the next lowest indexed set from $Y_{j+1}, Y_{j+2}, \ldots, Y_m$ that is contained in $E - (A \cup Y_j)$, we eventually arrive at the set $A' = A \cup \mathrm{fs}_2(B)$. It follows that $(A', E - A')$ is a non-sequential 4-separation 2-equivalent to $(A, E - A)$.

It remains to see that the flower vertices of T' display flowers that are 2-equivalent to the flowers displayed by the corresponding vertices in T. Let v be a flower vertex in T'. Then v displays a flower Ψ in T where $\Psi = (O_1, O_2, \ldots, O_n)$. Moreover, B is contained in a petal, say O_1. By adjoining the sets $Y_1, Y_2, \ldots, Y_m$ to the bag B and removing them from the bags of T that contained them, we are performing a sequence of elementary moves in Ψ. At the conclusion of this process, we have transformed Ψ into the flower $(O_1 \cup \mathrm{fs}_2(B), O_2 - \mathrm{fs}_2(B), \ldots, O_n - \mathrm{fs}_2(B))$, which, by Theorem 3.1, is 2-equivalent to Ψ. As all non-sequential 4-separations of M conform with T, they also conform with T'. Hence T' is a partial 4-tree 2-equivalent to T that has $\mathrm{fs}_2(B)$ as a bag labelling the vertex u, and (5.4.2) holds.

It follows from (5.4.2) that we may assume $\mathrm{fs}_2(B) = B$. Now, X is a 4-separating set that is contained in, but is not 2-equivalent to B. Let Y be a 4-separating set with $X \subseteq Y \subseteq B$ such that Y is not 2-equivalent to B and $\mathrm{fs}_2(Y)$ is maximal among all such sets. As $X \subseteq Y$ and X is non-sequential, so is Y. By Lemma 5.3, there is a tight irredundant maximal flower Φ in M that displays a 4-separation $(Z, E - Z)$ that is 2-equivalent to $(Y, E - Y)$. Since $\mathrm{fs}_2(B) = B$, we have $Z \subseteq B$. Moreover,

5.4.3. *Z is non-sequential.*

Next we show the following.

5.4.4. *There is a tight irredundant maximal flower 2-equivalent to Φ that has a petal containing $E - B$.*

To establish (5.4.4), we note that, by Theorem 4.1, $(E - B, B)$ conforms with Φ. Thus either

(i) $E - B$ is 2-equivalent to a 4-separating set Q' contained in a petal Q of Φ; or
(ii) $E - B$ is 2-equivalent to a union of petals of Φ.

Suppose that (i) holds. Since $E - B$ is non-sequential, so is Q'. Hence, as $Q' \subseteq Q$, by Lemma 2.6, Q is a non-sequential set. As Z is also non-sequential, we may assume that $E - Z$ is not a single petal of Φ otherwise (5.4.4) holds. Since Φ is tight, $\mathrm{fs}_2(Q)$ does not contain Z, and $\mathrm{fs}_2(Q) \neq \mathrm{fs}_2(E - Z)$. Thus we may apply Lemma 2.7 to see that $\mathrm{fs}_2(Q) - Z$ is 2-equivalent to Q. Also, $E - B \subseteq \mathrm{fs}_2(Q) - Z$, so by Corollary 3.10, there is a flower 2-equivalent to Φ that displays Z such that $E - B$ is contained in a petal. Thus, when (i) holds, so does (5.4.4).

Now suppose that (ii) holds. Let $\Phi = (Q_1, Q_2, \dots, Q_n)$. Then we may assume that $E-B$ is 2-equivalent to $Q_1 \cup Q_2 \cup \cdots \cup Q_k$ for some $k \geq 2$. As Z is displayed by Φ but is not 2-equivalent to B, we have $n - k \geq 2$. By Lemmas 3.16 and 3.8 (i), there is a tight irredundant flower $\Phi' = (Q'_1, Q'_2, \dots, Q'_n)$ that is 2-equivalent to Φ, where $(Q'_1 \cup Q'_2 \cup \cdots \cup Q'_k, Q'_{k+1} \cup Q'_{k+2} \cup \cdots \cup Q'_n) = (E - B, B)$. Evidently, $n \geq 4$. First we show that

5.4.5. *$n \leq 5$.*

Assume that $n \geq 6$. If $k = 2$, let $P' = Q'_{n-1} \cup Q'_n \cup Q'_1$. If $k > 2$, let $P' = Q'_n \cup Q'_1 \cup Q'_2$. By Corollary 3.5, $(P', E - P')$ is a non-sequential 4-separation. Also, by Lemma 3.4, neither $\mathrm{fs}_2(P')$ nor $\mathrm{fs}_2(E - P')$ contains either B or $E-B$. Thus every 4-separation that is 2-equivalent to $(P', E-P')$ crosses both B and $E - B$. Therefore $(P', E - P')$ does not conform with T, contradicting the fact that T is a partial 4-tree. Hence (5.4.5) holds.

5.4.6. *$n = 4$.*

Assume that $n = 5$. Then $k \in \{2, 3\}$. By relabelling the petals if necessary, we may assume that $\{B, E - B\} = \{Q'_1 \cup Q'_2, Q'_3 \cup Q'_4 \cup Q'_5\}$. Suppose $P' \in \{Q'_4 \cup Q'_5 \cup Q'_1, Q'_2 \cup Q'_3 \cup Q'_4\}$. Then P' is non-sequential.

If $E-P'$ is non-sequential, then $(P', E-P')$ is a non-sequential 4-separation that does not conform with T; a contradiction. We deduce that $Q_2' \cup Q_3'$ and $Q_1' \cup Q_5'$ are sequential, as are Q_1', Q_2', Q_3', and Q_5'. If Φ' is a daisy, then the only possible non-sequential 4-separations it displays are $(Q_4', E-Q_4'), (Q_1' \cup Q_2', E-(Q_1' \cup Q_2')), (Q_3' \cup Q_4', E-(Q_3' \cup Q_4'))$, and $(Q_4' \cup Q_5', E-(Q_4' \cup Q_5'))$. Each of these has Q_1' and Q_2' on the same side, so Φ' is redundant; a contradiction. Thus Φ' is an anemone. Then, by symmetry with the above, each of $Q_2' \cup Q_3', Q_2' \cup Q_4', Q_2' \cup Q_5', Q_1' \cup Q_3', Q_1' \cup Q_4'$, and $Q_1' \cup Q_5'$ is sequential. Once again, we find that all possible non-sequential 4-separations displayed by Φ' have Q_1' and Q_2' on the same side, so Φ' is redundant. This contradiction implies that (5.4.6) holds.

We may now assume that $E-B = Q_1' \cup Q_2'$ and $B = Q_3' \cup Q_4'$. Next we show that we may also assume that

5.4.7. *Φ' is a daisy, Q_3' is 2-equivalent to Z, and the only non-sequential 4-separations displayed by Φ are $(Q_1' \cup Q_2', Q_3' \cup Q_4'), (Q_3', Q_1' \cup Q_2' \cup Q_4')$, and $(Q_2', Q_1' \cup Q_3' \cup Q_4')$.*

If either $(Q_1' \cup Q_4', Q_2' \cup Q_3')$ or $(Q_1' \cup Q_3', Q_2' \cup Q_4')$ is a non-sequential 4-separation, then it does not conform with T; a contradiction. Hence $(Q_1' \cup Q_4', Q_2' \cup Q_3')$ is sequential. Moreover, if Φ' is an anemone, then $(Q_1' \cup Q_3', Q_2' \cup Q_4')$ is sequential. Now $Z \subseteq B = Q_3' \cup Q_4'$. By the choice of Z, we deduce that Z is 2-equivalent to Q_3' or Q_4', so we may assume the former. Then Q_3' is non-sequential, so $Q_2' \cup Q_3'$ is non-sequential. Hence each of $Q_1' \cup Q_4', Q_1'$, and Q_4' is sequential. By symmetry, if Φ' is an anemone, then $Q_2' \cup Q_4', Q_2'$, and Q_4' are sequential, and so Φ' is redundant; a contradiction. We deduce that (5.4.7) holds.

Now consider $Q_2' \cup Q_3'$. We know that $\mathrm{fs}_2(Q_2' \cup Q_3') = E(M)$ and that each of Q_1' and Q_4' is tight. Let $(Y_j)_{j=1}^m$ be a 4-sequence for $\mathrm{fs}_2(Q_2' \cup Q_3')$ and let Y_d be the smallest indexed Y_j that meets both Q_1' and Q_4'. By Lemma 3.4, each of $Y_1, Y_2, \dots, Y_{d-1}$ is contained in Q_1' or Q_4'. One by one we move these sets from Q_1' or Q_4' into Q_2' or Q_3', respectively, maintaining a 2-equivalent flower. Let $(Q_1'', Q_2'', Q_3'', Q_4'')$ be the flower obtained at the conclusion of this process. By Lemma 3.4, $|Q_1''| = |Q_4''| = 3$. Moreover, by construction, $Q_1'' \cup Q_2'' = Q_1' \cup Q_2' = E-B$ and $Q_3'' \cup Q_4'' = Q_3' \cup Q_4' = B$. In addition, Z is 2-equivalent to Q_3''.

Let v be the vertex of T that is adjacent to the leaf vertex u that is labelled by B. Next we show that

5.4.8. *v does not label a bag vertex of T.*

Assume the contrary. Form T' from T by adding a new leaf vertex z adjacent to u and labelled by Q_3'', and relabel u by $B-Q_3''$. It is easily verified that T' satisfies the first four properties of a partial 4-tree. Suppose that it does not satisfy (v). Then there is a non-sequential 4-separation $(U, E-U)$ that does not conform with T'. We may assume, by possibly replacing $(U, E-U)$ with a 2-equivalent 4-separation, that $U \subseteq B$ and both

$U \cap Q_3''$ and $U \cap (B - Q_3'')$ are non-empty. Suppose that $|U \cap Q_3''| \leq 2$. Since Q_3'' is non-sequential, $|Q_3''| \geq 5$, so $|Q_3'' - (U \cap Q_3'')| \geq 3$. But $Q_3'' - (U \cap Q_3'') = E - (U \cup (E - Q_3''))$. Thus, by uncrossing, $U \cap (E - Q_3'')$ is 4-separating. But $|U \cap (E - Q_3'')| \leq 3$, so U is sequential; a contradiction. We deduce that $|U \cap Q_3''| \geq 3$. By uncrossing, as Q_3'' is not 2-equivalent to B, it follows that $U \supseteq B - Q_3'' = Q_4''$.

By Lemma 4.2, $(Q_1'', Q_2'', Q_3'' - U, Q_3'' \cap U, Q_4'')$ is a flower. Moreover, this flower is easily seen to be tight and irredundant. Since it displays $(U, E - U)$ but no 2-equivalent 4-separation is displayed by $(Q_1'', Q_2'', Q_3'', Q_4'')$, we contradict the fact that the latter flower is maximal. We deduce that there is no non-sequential 4-separation that does not conform with T'. Hence T' is a partial 4-tree displaying a non-sequential 4-separation that is not 2-equivalent to one displayed by T; a contradiction. Hence (5.4.8) holds.

By (5.4.8), the vertex v of T that is adjacent to u labels a flower vertex for which the corresponding flower Ψ is tight, irredundant, and maximal, and has B as a petal. The non-sequential 4-separation $(Q_2'', E - Q_2'')$ conforms with Ψ. Suppose there is a non-sequential 4-separation (R, G) that is 2-equivalent to $(Q_2'', E - Q_2'')$ and has R or G contained in a petal P of Ψ. Then $\mathrm{fs}_2(P)$ contains $\mathrm{fs}_2(Q_2'')$ or $\mathrm{fs}_2(Q_1'' \cup Q_3'' \cup Q_4'')$. If $\mathrm{fs}_2(P) \supseteq \mathrm{fs}_2(Q_1'' \cup Q_3'' \cup Q_4'')$, then $\mathrm{fs}_2(P) \supseteq B$. Hence $P = B$. But $\mathrm{fs}_2(B) = B$; a contradiction. Thus we may assume that $\mathrm{fs}_2(P) \supseteq \mathrm{fs}_2(Q_2'')$. As $E - (B \cup Q_2'') = Q_1''$, and the last set has only three elements, it follows that Ψ has exactly three petals and is 2-equivalent to (Q_1'', Q_2'', B). This contradicts the maximality of Ψ.

We may now assume that $(Q_2'', E - Q_2'')$ is 2-equivalent to a 4-separation (R, G) that is displayed by Ψ. By construction $Q_2'' = \mathrm{fs}_2(Q_2'') - B$. Moreover, $R \cap B = \emptyset$. As $\mathrm{fs}_2(R) = \mathrm{fs}_2(Q_2'') \neq E(M)$, it follows by Lemma 3.4 and uncrossing with $E - B$ that if $(Y_j)_{j=1}^m$ is a 4-sequence for $\mathrm{fs}_2(R)$, we can move each Y_j that is not contained in B into one of the petals whose union is R always maintaining a 2-equivalent flower. At the conclusion of this process, the resulting flower Ψ' is $(B, Q_1'', Q_{2,1}'', Q_{2,2}'', \ldots, Q_{2,k}'')$ say, where $Q_2'' = Q_{2,1}'' \cup Q_{2,2}'' \cup \cdots \cup Q_{2,k}''$. As Ψ' is maximal, $k \geq 2$. Let $\Psi'' = (Q_3'', Q_4'', Q_1'', Q_{2,1}'', Q_{2,2}'', \ldots, Q_{2,k}'')$. We know that $Q_4'' \cup Q_1''$ is 4-separating. Moreover, so are $Q_{2,k}'' \cup Q_3'' \cup Q_4''$ and $Q_{2,1}'' \cup Q_{2,2}'' \cup \cdots \cup Q_{2,k}'' \cup Q_3''$. By uncrossing, so is the intersection of the last two sets, $Q_{2,k}'' \cup Q_3''$. It follows that Ψ'' is a flower. Moreover, it is tight and irredundant. As Ψ'' displays $(Q_3'', E - Q_3'')$, but Ψ' does not display a 2-equivalent 4-separation, the maximality of Ψ' is contradicted. We conclude that $n \neq 4$. This contradiction to (5.4.6) completes the proof of (5.4.4).

By (5.4.4), we may assume that $\Phi = (P_1, P_2, \ldots, P_n)$ where $E - B \subseteq P_n$ and Z is some union of consecutive petals from $\{P_1, P_2, \ldots, P_{n-1}\}$. Since Z is not 2-equivalent to B, we must have that $n \geq 3$. To construct T' from T, first adjoin a new flower vertex v adjacent to u, labelling v either A or D, depending on whether Φ is an anemone or a daisy, respectively; then adjoin bag vertices $v_1, v_2, \ldots, v_{n-1}$ adjacent to v, labelling these by $P_1, P_2 \ldots, P_{n-1}$;

finally, relabel the vertex u by $B - (P_1 \cup P_2 \cup \cdots \cup P_{n-1})$. To verify that T' is a partial 4-tree, it suffices to consider the non-sequential 4-separations (R, G) with $R \subseteq B$. The argument here is the same as that given in [4, p. 292] so we omit the details. Clearly, $T' \succcurlyeq T$. Moreover, $(Z, E - Z)$ is a non-sequential 4-separation displayed by T' for which there is no 2-equivalent 4-separation displayed by T. Thus the lemma holds in Case (I).

Consider Case (II). Let Z be a 4-separating set of M that is maximal with the property that $X \subseteq Z \subseteq B$. Let T' be the tree obtained from T by adjoining a new leaf vertex v adjacent to u such that v is a bag vertex labelled by Z, and u is relabelled by $B - Z$. Once again, to show that T' is a partial 4-tree, we follow [4, p. 292].

Clearly, $T \preccurlyeq T'$ and $(Z, E - Z)$ is a non-sequential 4-separation. Thus either the lemma holds in this case, or Z is 2-equivalent to a 4-separating set displayed by T. As X is not 2-equivalent to such a 4-separating set, X and Z are not 2-equivalent. We may assume that $(X, E - X)$ is not 2-equivalent to any 4-separation displayed by T', otherwise the lemma holds. As X is contained in the bag Z of T', and this bag is a leaf, it follows from Case (I) that there is a partial 4-tree $T'' \succcurlyeq T'$ such that T'' displays some non-sequential 4-separation that is not 2-equivalent to any 4-separation displayed by T', and hence is not 2-equivalent to any 4-separation displayed by T. $\square$

Proof of Theorem 5.1. If M has no non-sequential 4-separations, then the tree T consisting of a single bag vertex labelled by $E(M)$ satisfies the theorem. If M has a non-sequential 4-separation (R, G), then, by Lemma 5.3, there is a tight irredundant maximal flower that displays a 4-separation 2-equivalent to (R, G). Hence, by Corollary 5.2, there is a partial 4-tree that displays a 4-separation 2-equivalent to (R, G). Let T be a maximal partial 4-tree for M. Clearly, T has at least one edge. The theorem follows by applying Lemma 5.4 to T. $\square$

6. Acknowledgements

The authors thank Geoff Whittle for his useful insight. The second author was partially supported by a grant from the National Security Agency.

References

[1] J. Aikin, J. Oxley, The structure of crossing separations in matroids, Adv. Appl. Math. 41 (2008) 10–26.
[2] W. H. Cunningham, J. Edmonds, A combinatorial decomposition theory, Canad. J. Math. 32 (1980) 734–765.
[3] J. Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.
[4] J. Oxley, C. Semple, G. Whittle, The structure of 3-separations of 3-connected matroids, J. Combin. Theory Ser. B 92 (2004) 257–293.
[5] J. Oxley, C. Semple, G. Whittle, An upgraded Wheels-and-Whirls Theorem for 3-connected matroids, submitted.
[6] W. T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.

Mathematics Department, Macon State College, Macon, Georgia, 31206
E-mail address: jeremy.aikin@maconstate.edu

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana, 70803-4918
E-mail address: oxley@math.lsu.edu