UNAVOIDABLE MINORS OF MATROIDS WITH MINIMUM COCIRCUIT SIZE FOUR

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ABSTRACT. In 1963, Halin and Jung proved that every simple graph with minimum degree at least four has K_5 or $K_{2,2,2}$ as a minor. Mills and Turner proved an analog of this theorem by showing that every 3-connected binary matroid in which every cocircuit has size at least four has $F_7, M^*(K_{3,3}), M(K_5)$, or $M(K_{2,2,2})$ as a minor. Generalizing these results, this paper proves that every simple matroid in which all cocircuits have at least four elements has as a minor one of nine matroids, seven of which are well known. All nine of these special matroids have rank at most five and have at most twelve elements.

1. Introduction

The purpose of this paper is to prove a matroid analog of the following result of Halin and Jung [14].

Theorem 1.1. *Every simple graph with minimum degree at least four has K_5 or $K_{2,2,2}$ as a minor.*

We have followed Bollobás [6, p.373] and Maharry [19, p.96] in attributing Theorem 1.1 to Halin and Jung. Fijavž and Wood [10, Corollary A.4] give a short proof of that theorem and briefly discuss its origins.

When G is a 2-connected loopless graph, the set of edges that meet a fixed vertex of G is a bond of G and a cocircuit of its cycle matroid $M(G)$. Because of this, it is common in matroid theory to take minimum cocircuit size as a matroid analog of minimum vertex degree in a graph. Moreover, the minimum cocircuit size $M(G)$ is precisely the edge connectivity of G.

The next theorem is the main result of the paper. Most of the matroids appearing in it are familiar. Geometric representations of the rank-3 matroids P_7 and O_7 are shown in Figure 1. We define H_{12} to be the 12-element rank-5 matroid $O_7 \oplus_2 O_7$ where the basepoint of the 2-sum in each copy of O_7 is the point p denoted in Figure 1. We recall that $M^*(K_{3,3})$, the bond matroid of $K_{3,3}$, is the rank-4 matroid that can be obtained from a twisted 3×3 grid (see [23, p.652]). The matroid Q_9, for which a geometric representation is shown in Figure 2, is a 9-element rank-4 matroid that is obtained

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(a) P_7 (b) O_7

FIGURE 1. The matroids P_7 and O_7.

from a twisted 3×4 grid by removing the three boxed elements as shown. It is straightforward to check that this matroid is representable over a field $\mathbb{F}$ if and only if $|\mathbb{F}| \geq 3$. The matroid Q_9 is represented by $[I_4|A]$ for the ternary matrix A in Figure 2. One can also show that this matroid is affine over $GF(3)$.

Theorem 1.2. *Every simple matroid in which every cocircuit has at least four elements has $U_{2,5}, F_7, F_7^-, P_7, M^*(K_{3,3}), Q_9, M(K_5), M(K_{2,2,2})$, or H_{12} as a minor.*

Guoli Ding (private communication) asked how the list of matroids in the last theorem changes when we replace "simple" in the hypothesis by "3-connected" and, in addition, we require that every matroid in the list is 3-connected. The following theorem answers this question.

Theorem 1.3. *Every 3-connected matroid in which every cocircuit has at least four elements has $U_{2,5}, F_7, F_7^-, P_7, M^*(K_{3,3}), Q_9, M(K_5)$, or $M(K_{2,2,2})$ as a minor.*

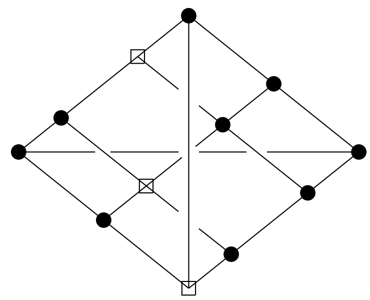
It is straightforward to check that each of the nine matroids listed in Theorem 1.2 is a minor-minimal simple matroid in which every cocircuit has size at least four. As consequences of this theorem, we have the next two results. The first of these was proved by Mills and Turner [21]. The second is the key step in the proof of Theorem 1.2 and its proof occupies most of the paper.

Theorem 1.4. *Let M be a simple binary matroid in which every cocircuit has at least four elements. Then M has $F_7, M^*(K_{3,3}), M(K_5)$, or $M(K_{2,2,2})$ as a minor.*

Theorem 1.5. *Let M be a simple ternary matroid in which every cocircuit has at least four elements. Then M has $F_7^-, P_7, M^*(K_{3,3}), Q_9, M(K_5)$, $M(K_{2,2,2})$, or H_{12} as a minor.*

Recall that $K_{2,2,2}$ is the octahedron, so its planar dual is the cube. Using this, we see that the next result is the dual of Theorem 1.2.

Corollary 1.6. *Every matroid in which all cocircuits have at least three elements and all circuits have at least four elements has, as a minor, $U_{3,5}, F_7^*$, $(F_7^-)^*, P_7^*, Q_9^*, M^*(K_5), H_{12}^*$, or the cycle matroid of the cube or $K_{3,3}$.*



$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

FIGURE 2. The rank-4 ternary affine matroid Q_9 and the matrix A, where $[I_4|A]$ is a ternary representation of Q_9.

Applying this result to graphs, we immediately obtain the following well-known result.

Corollary 1.7. *Every 3-edge-connected graph with girth at least four has the cube or $K_{3,3}$ as a minor.*

The study of matroids with many small circuits and cocircuits started with Tutte [28] when in his Wheels-and-Whirls Theorem, he proved that the only 3-connected matroids in which every element is in a 3-circuit and a 3-cocircuit are wheels and whirls. Miller [20] found all the matroids with at least thirteen elements such that every pair of elements is in a 4-circuit and a 4-cocircuit. Motivated by these results, Pfeil, Oxley, Semple, and Whittle [24] found the 3-connected matroids with the property that every pair of elements is in a 4-circuit and every element is in a 3-cocircuit. The nine matroids listed in Theorem 1.2 have the property that every element is in a 3-circuit and in a 4-cocircuit.

To provide a context for our main theorem, we note the following result, which is an immediate consequence of the excluded-minor characterizations of the classes of binary [29] and series-parallel matroids [1, 8, 9].

Proposition 1.8. *Let M be a simple matroid in which every cocircuit has at least three elements. Then M has $U_{2,4}$ or $M(K_4)$ as a minor.*

After a section of preliminary results, the core of the paper is in Section 3, which proves a sequence of structural results that are used in the proof of Theorem 1.5. The strategy for that proof is outlined at the start of the section. Section 4 is a brief section in which we complete the proof of Theorem 1.5 and then prove Theorem 1.2. We conclude this paper in Section 5 by proving Theorem 1.3.

2. Preliminaries

Throughout this paper, we will follow the notation and terminology of [23]. We denote by $\mathcal{M}_4$ the class of simple matroids in which every cocircuit has at least four elements. The *connectivity function* λ_M of a matroid M is

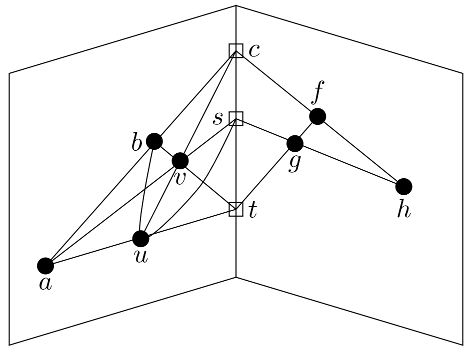


FIGURE 3. The labels of F_7^* used in Theorem 2.3.

defined, for all subsets X of $E(M)$, by

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

When it is clear which matroid we are referring to, we will use $\lambda(X)$ in place of $\lambda_M(X)$. For disjoint subsets X and Y of $E(M)$, let $\kappa_M(X,Y) = \min\{\lambda_M(S) : X \subseteq S \subseteq E(M) - Y\}$. If S is a set for which this minimum is attained, then $\kappa_M(X,Y) = \lambda_M(S) = \kappa_M(S, E(M) - S)$. In many of our proofs we will use Geelen, Gerards, and Whittle's extension [12] (see also, for example, [23, Theorem 8.5.7]) of Tutte's Linking Theorem [27].

Theorem 2.1. *Let X and Y be disjoint subsets of the ground set of a matroid M. Then M has a minor N with $E(N) = X \cup Y$ for which $\kappa_N(X,Y) = \kappa_M(X,Y)$ such that $N|X = M|X$ and $N|Y = M|Y$.*

The following result of D. W. Hall [15] will be used in the proof of Theorem 1.4.

Theorem 2.2. *If G is a 3-connected graph, then G has no $K_{3,3}$-minor if and only if G is planar or its associated simple graph is K_5.*

The next theorem for ternary matroids is reminiscent of the last result. Hall, Mayhew, and van Zwam [16] considered similar such results.

Theorem 2.3. *Let M be a 3-connected matroid having rank and corank at least three. Then M is ternary if and only if M has no $U_{2,5}$- or F_7-minor and $M \not\cong F_7^*$.*

Proof. If M is ternary, then M has no $U_{2,5}$- or F_7-minor and $M \not\cong F_7^*$. Conversely, assume M has no $U_{2,5}$- or F_7-minor and $M \not\cong F_7^*$. As M is 3-connected having rank and corank at least three, M does not have $U_{3,5}$ as a minor by [22] (see also [23, Proposition 12.2.5]). Then by [3, 25] (see also [23, Theorem 6.5.7]), M is ternary unless M has F_7^* as a minor. Since $M \not\cong F_7^*$, by the Splitter Theorem, M has a 3-connected single-element extension or coextension of F_7^* as a minor. Now, a 3-connected extension N of F_7 by the element e either adds e freely to a line of F_7, or adds e freely

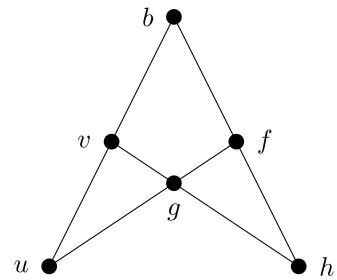


FIGURE 4. The matroid $N\backslash e/a$ in Theorem 2.3.

to F_7 itself. In each case, N/e has a $U_{2,5}$-minor, so N^* has a $U_{3,5}$-minor, a contradiction.

We now need only consider the 3-connected single-element extensions of F_7^*. A geometric representation of F_7^* is shown in Figure 3 where c, s, and t are not elements of F_7^*, but show how F_7^* can be obtained from F_7 by a ΔY-exchange. Let N be a 3-connected single-element extension of F_7^* by the element e and let $\mathcal{M}$ be the corresponding modular cut. Then $\mathcal{M}$ does not contain any rank-one flats. If $\mathcal{M} = \{E(F_7^*)\}$, then e is freely added to F_7^*, and N/e has $U_{3,5}$ as a minor, a contradiction. Thus $\mathcal{M}$ must contain some line or some hyperplane of F_7^*. Assume that $\mathcal{M}$ contains a line of F_7^*. As F_7^* has a doubly transitive automorphism group [23, p.643], we may assume that $\{a, b\} \in \mathcal{M}$. Then $\{a, b, u, v\}$ and $\{a, b, f, h\}$ are in $\mathcal{M}$. Assume $\{f, h\} \in \mathcal{M}$. Then $\{f, h, u, v\} \in \mathcal{M}$. As $\{a, b, u, v\}$ and $\{f, h, u, v\}$ form a modular pair, $\{u, v\} \in \mathcal{M}$. Similarly, if $\{u, v\} \in \mathcal{M}$, then $\{f, h\} \in \mathcal{M}$. We deduce that if we have $\{a, b\} \in \mathcal{M}$ and at least one of $\{f, h\}$ and $\{u, v\}$ is in $\mathcal{M}$, then the point we added corresponding to $\mathcal{M}$ is c and the extension is isomorphic to S_8.

If e is added freely on $\{a, b\}$, then contracting e and g gives $U_{2,5}$ as a minor, a contradiction. We may now assume that e is not added to any 2-point line of F_7^*. We now know that the smallest flat in $\mathcal{M}$ has rank three. Because the hyperplanes of F_7^* are of two types, complements of triangles in F_7 and complements of 4-circuits in F_7, by symmetry, we may assume that e is added on one of the planes spanned by $\{f, g, h\}$ or $\{a, b, u, v\}$.

If e is placed on the plane $\{a, b, u, v\}$, then, since it is not on any 2-point lines, we see that $N|\{a, b, u, v, e\} \cong U_{3,5}$, a contradiction. We deduce that e is placed on the plane spanned by $\{f, g, h\}$ but not on any of the lines spanned by $\{f, g\}, \{f, h\}$, or $\{g, h\}$. We know that $\{a, b, u, v\} \notin \mathcal{M}$. Moreover, if M is the principal modular cut generated by $\{f, g, h\}$, then $N/a/e$ has $U_{2,5}$ as a restriction, a contradiction. We deduce that $\mathcal{M}$ contains a flat other than $\{f, g, h\}$ and $E(F_7^*)$. Now $N\backslash e/a$ is isomorphic to the copy of $M(K_4)$ labeled as in Figure 4. By symmetry, we may assume that $\{a, b, e, g\}$ is a circuit of N. We see that $\{e, f, v\}$ and $\{e, h, u\}$ are not both circuits of N/a otherwise $N/a \cong F_7$. By symmetry, we may assume that $\{e, h, u\}$ is not a circuit of N/a. If $\{e, f, v\}$ is not a circuit of N/a, then contracting e from N/a gives a matroid with $U_{2,5}$ as a restriction. Thus we may assume that $\{e, f, v\}$ is a circuit of N/a, so $\{a, e, f, v\}$ is a circuit of N.

We now know that N has $\{a, b, e, g\}$ and $\{a, e, f, v\}$ as circuits. The matroid N/h has e on the line spanned by $\{f, g\}$. But $\{a, e, u\}$ is not a triangle of N/h. If $\{b, e, v\}$ is not a triangle of N/h, then $(N/h)|\{a, b, e, u, v\} \cong U_{3,5}$. Thus $\{b, e, v\}$ is a triangle of N/h. Hence $\{b, e, h, v\}$ is a circuit of N.

Consider N/v. We know that $N/v\backslash e \cong M(K_4)$. Also N/v has $\{a, e, f\}$ and $\{b, e, h\}$ as triangles. Since N/v is not isomorphic to F_7, we deduce that $\{e, g, u, v\}$ is not a circuit of N. Now consider N/u. We know that $\{a, e, h\}$ and $\{e, g, v\}$ are not triangles of this matroid. Then $N/u/e$ has $U_{2,5}$ as a restriction, a contradiction. □

The next lemma identifies a key property of the minor-minimal members of $\mathcal{M}_4$ that will be used repeatedly throughout the paper.

Lemma 2.4. *Let M be a minor-minimal matroid in $\mathcal{M}_4$. Let e be an element of M. Then e is in a triangle and a 4-cocircuit.*

Proof. Assume e is not in a triangle. Consider M/e. Then every cocircuit in M/e has size at least four and is simple. Thus M/e contradicts the minimality of M. Therefore, e is in a triangle. Now assume that e is not in a 4-cocircuit. Since $M\backslash e$ has a cocircuit C^* of size less than four, we see that $C^* \cup e$ is a cocircuit of M having size four. □

Bixby [5], Cunningham [7], and Seymour [26] independently proved the following result.

Theorem 2.5. *A 2-connected matroid M is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which has at least three elements and is isomorphic to a proper minor of M.*

The previous theorem implies that a matroid that is 2-connected but not 3-connected can be decomposed into two matroids M_1 and M_2. If either of these matroids fails to be 3-connected, then it too can be written as a 2-sum, and this process can continue. In order to keep track of this decomposition, we introduce the following concept. A *matroid-labeled tree* is a tree T with vertex set $\{M_1, M_2, \dots, M_k\}$ for some positive integer k such that

(i) each M_i is a matroid;
(ii) if M_{j_1} and M_{j_2} are joined by an edge e_i of T, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$, and $\{e_i\}$ is not a separator for M_{j_1} or M_{j_2}; and
(iii) if M_{j_1} and M_{j_2} are non-adjacent, then $E(M_{j_1}) \cap E(M_{j_2})$ is empty.

Furthermore, a *tree decomposition* of a 2-connected matroid M is a matroid-labeled tree such that whenever $V(T) = \{M_1, M_2, \dots, M_k\}$ and $E(T) = \{e_1, e_2, \dots, e_{k-1}\}$, then

(i) $E(M) = (E(M_1) \cup E(M_2) \cup \dots \cup E(M_k)) - \{e_1, e_2, \dots, e_{k-1}\}$;
(ii) $|E(M_i)| \geq 3$ for all i unless $|E(M)| < 3$, in which case $k = 1$ and $M_1 = M$; and

(iii) M is the matroid that labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$.

We call $M_1, M_2, \ldots, M_k$ the *vertex labels* of T. Additionally, Cunningham and Edmonds [7] were able to guarantee the uniqueness of the decomposition of a 2-connected matroid.

Theorem 2.6. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label is 3-connected, a circuit, or a cocircuit, and there are no two adjacent vertices that are both labeled by circuits or are both labeled by cocircuits. Moreover, T is unique to within relabeling of its edges.*

The tree decomposition of M whose existence and uniqueness are asserted in the last theorem will be called the *canonical tree decomposition* for M. Observe in the previous theorem that when a leaf of a canonical tree decomposition is a circuit, then matroid has a 2-cocircuit.

Lemma 2.7. *Let M be a minor-minimal matroid in $\mathcal{M}_4$. If M is not 3-connected, then $M = M_1 \oplus_2 M_2$ where M_1 and M_2 are 3-connected and each has rank at least three.*

Proof. The minimality of M implies that M is 2-connected. Let T be the canonical tree decomposition of M (see, for example, [23, Section 8.3]). Consider a matroid L that labels a leaf of T. If L is a circuit, then M has a 2-cocircuit, a contradiction. Moreover, since M is simple, L is not a cocircuit. Thus L is a 3-connected matroid with at least four elements. Hence if M_1 and M_2 label distinct leaves of T, then M has $M_1 \oplus_2 M_2$ as a minor. Therefore, as every cocircuit of $M_1 \oplus_2 M_2$ has size at least four, we deduce that $M = M_1 \oplus_2 M_2$. If $r(M_i) = 2$ for some i, then, as every cocircuit of M has at least four elements, we deduce that $|E(M_i)| \geq 5$, so M has $U_{2,5}$ as a minor, a contradiction. $\square$

Theorem 1.4 was originally proved by Mills and Turner [21]; we provide their short proof for the sake of completeness. Note that the proof of Theorem 1.2 does not rely on Theorem 1.4. Instead, Theorem 1.4 can be deduced as an immediate corollary of Theorem 1.2.

Proof of Theorem 1.4. Assume that M has none of $F_7, M^*(K_{3,3}), M(K_5)$, or $M(K_{2,2,2})$ as a minor. Then, as M does not have an F_7-minor, it follows by [26] (see also [23, Proposition 12.2.3]) that M is regular otherwise $M \cong F_7^*$, which is a contradiction since F_7^* has a triad. We show next that

2.7.1. *M is not 3-connected.*

Assume that M is 3-connected. Then, by a result of Seymour [26] (see also [23, Theorem 13.1.2]), M is graphic or cographic, or M has a minor isomorphic to R_{10} or R_{12}. By Theorem 1.1, M is not graphic. Suppose M is cographic. Then, as M is not graphic, M is the bond matroid of a nonplanar graph G. Since M does not have $M^*(K_{3,3})$ as a minor, it follows, by Theorem 2.2, that $M \cong M^*(K_5)$. Thus M has a triad, a contradiction.

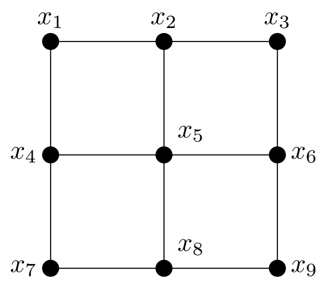


FIGURE 5. Some 3-point lines in a rank-4 matroid.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_3 & x_7 & x_8 & x_6 & x_9 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & u_2 \\ 0 & 0 & 1 & u_1 & u_1u_2 \end{pmatrix} \end{array}$$

FIGURE 6. The matrix A_4 in the proof of Lemma 2.8.

We conclude that M is not cographic. Finally, if M has R_{10} or R_{12} as a minor, then M has $M^*(K_{3,3})$ as a minor, a contradiction. Thus 2.7.1 holds.

By Lemma 2.7, $M = M_1 \oplus_2 M_2$ where M_1 and M_2 are 3-connected. As M_1 has none of $F_7, M^*(K_{3,3}), M(K_5)$, or $M(K_{2,2,2})$ as a minor, 2.7.1 implies that M_1 is not 3-connected, a contradiction. □

The following lemmas show that one can uniquely determine the matroid whose triangles form the configurations shown in Figures 5 and 7.

Lemma 2.8. *Let M be a 9-element rank-4 matroid that has each of the 3-point lines in Figure 5 as a triangle. Then M is isomorphic to $M^*(K_{3,3})$.*

Proof. We observe that, as $r(M) = 4$, every set of four points that form the vertices of a rectangle in Figure 5 is a cocircuit of M. As M has rank four, it has no other triangles apart from the six shown. Now, for all i in $\{x_1, x_2, \dots, x_9\}$, the matroid M/x_i is ternary since it can be obtained from $M(K_4)$ by adding parallel elements. Thus M does not have $U_{2,5}, U_{3,5}$, or F_7 as a minor. Moreover, M does not have F_7^* as a minor since we cannot eliminate all of the triangles of M by deleting two elements. We conclude that M is ternary [3, 25].

Now M has $\{x_1, x_2, x_4, x_5\}$ as a basis B otherwise $r(M) = 3$. Let $[I_4|A_4]$ be a ternary representation of M with respect to B. Scaling the rows and columns of A_4 so that the first non-zero entry of each is a one, we get that A_4 is as shown in Figure 6 by using the fundamental circuits with respect to B along with the circuit $\{x_3, x_6, x_9\}$, where u_1 and u_2 are non-zero. Finally, by using the circuit $\{x_7, x_8, x_9\}$, we deduce that $u_2 = 1$ and $u_1 = 1$. Thus

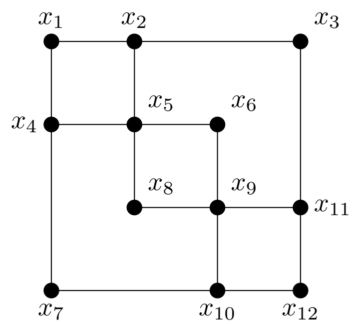


FIGURE 7. Some 3-point lines in a rank-5 matroid.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_9 \end{array}\begin{array}{c} \begin{array}{ccccccc} x_3 & x_7 & x_8 & x_6 & x_{10} & x_{11} & x_{12} \end{array} \\ \left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & u_1 & u_2 & u_3 & u_4 \\ 0 & 0 & 0 & 0 & u_5 & u_6 & u_7 \end{array}\right) \end{array}$$

FIGURE 8. Building a ternary representation for M.

M is represented by the matrix $[I_4|A_4]$ over $GF(3)$ where A_4 is as shown in Figure 6 with $u_2 = 1 = u_1$. As $M^*(K_{3,3})$ is a rank-4 ternary matroid that has the six triangles indicated in the figure, we deduce that $M \cong M^*(K_{3,3})$. $\square$

Lemma 2.9. *Let M be a 12-element rank-5 simple ternary matroid for which each of the 3-point lines in Figure 7 is a triangle of M. Then M is isomorphic to $M(K_{2,2,2})$.*

Proof. Since $r(M) = 5$, the triangles of M imply that $\{x_1, x_2, x_4, x_5\}$, $\{x_1, x_3, x_7, x_{12}\}$, $\{x_2, x_3, x_8, x_{11}\}$, $\{x_4, x_6, x_7, x_{10}\}$, $\{x_5, x_6, x_8, x_9\}$, and $\{x_9, x_{10}, x_{11}, x_{12}\}$ are cocircuits. Observe that these six sets coincide with the sets of corners of rectangles in Figure 7. Moreover each such set must be an independent set in M. We now construct a ternary representation $[I_5|A_5]$ for M. Let $\{x_1, x_2, x_4, x_5, x_9\}$ be the basis B of M.

We shall scale the matrix A_5 so that the first non-zero entry of each column is a one. We also scale rows 2–5 so that each has its first non-zero entry equal to one. The fundamental circuits of x_3, x_7, x_8, and x_6 with respect to B imply that we may assume the first four columns are as shown where $u_1 \neq 0$. The cocircuits $\{x_1, x_3, x_7, x_{12}\}, \{x_2, x_3, x_8, x_{11}\}$, and $\{x_4, x_6, x_7, x_{10}\}$ determine the first three rows of x_{10}, x_{11}, and x_{12}. The remaining two rows of x_{10}, x_{11}, and x_{12} are unknown. We label their entries as $u_2, u_3, \ldots, u_6$, and u_7 noting that these entries may be zero. The triangle $\{x_6, x_9, x_{10}\}$ implies that $u_1 = u_2$. The triangle $\{x_8, x_9, x_{11}\}$ implies that

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_9 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_3 & x_7 & x_8 & x_6 & x_{10} & x_{11} & x_{12} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

FIGURE 9. A ternary representation for M.

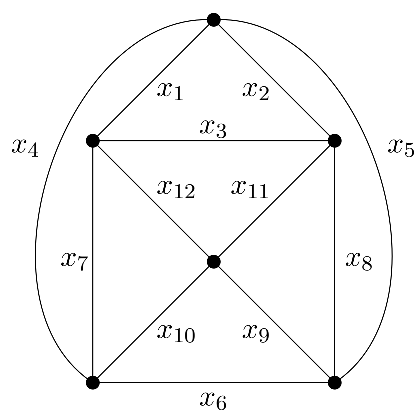


FIGURE 10. $K_{2,2,2}$

$u_3 = 1$. The triangle $\{x_7, x_{10}, x_{12}\}$ implies that $u_4 = -u_2$ and $u_7 = -u_5$. The triangle $\{x_3, x_{11}, x_{12}\}$ implies that $u_2 = 1$ and $u_6 = u_5$. Because the first non-zero entry of row 5 is one, the matrix A_5 is as shown in Figure 9.

Let $K_{2,2,2}$ be labeled as in Figure 10. We see that the eight triangles in this graph coincide with the eight 3-point lines in Figure 7. Since $M(K_{2,2,2})$ is a simple ternary 12-element rank-5 matroid and such a matroid with the specified eight triangles has the ternary representation $[I_5|A_5]$ where A_5 is as shown in Figure 9. We deduce that $M(K_{2,2,2}) \cong M[I_5|A_5]$, and the lemma is proved. $\square$

3. STRUCTURAL LEMMAS FOR TERNARY MATROIDS

This section contains the core of the proof of Theorem 1.5. Recall that $\mathcal{M}_4$ is the class of simple matroids in which every cocircuit has size at least four.

Because this section is long, we begin with an outline of the proof of Theorem 1.5. Let M be a ternary minor-minimal matroid in $\mathcal{M}_4$ that is not isomorphic to $F_7^-, P_7, M^*(K_{3,3}), Q_9, M(K_5), M(K_{2,2,2})$, or H_{12}. By Lemma 2.4, every element of M is in a triangle and a 4-cocircuit. The proof strategy for Theorem 1.5 involves analyzing how the many triangles and

4-cocircuits of M interact. We begin by considering what happens when M has five of the six triangles in Figure 5 as circuits. We then show, in Lemma 3.2, that M must be 3-connected. Lemma 3.3 shows that M has no 4-point lines, and Lemmas 3.4 and 3.5 show that every 4-cocircuit is independent. Next we show that M cannot have two 4-cocircuits contained in the union of two disjoint triangles. Lemmas 3.7 and 3.8 show that no element of M is in more than two triangles. Lemmas 3.10 and 3.11 show that M must have an element that is in more than one triangle. Lemmas 3.13–3.15 identify and analyze an infinite family of matroids in $\mathcal{M}_4$ the first two members of which are $M(K_5)$ and $M(K_{2,2,2})$ and subsequent members of which have one of these two matroids as a minor. Lemmas 3.12, 3.16, 3.17, and 3.18 build from a particular 4-cocircuit containing an element that is two triangles to get one of the forbidden possibilities for M. We now implement this strategy.

Lemma 3.1. *Let M be a rank-4 simple matroid having ground set $\{x_1, x_2, \dots, x_9\}$. Suppose that M has $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}$, $\{x_7, x_8, x_9\}$, $\{x_1, x_4, x_7\}$, and $\{x_3, x_6, x_9\}$ as triangles. Then $M \cong M^*(K_{3,3})$ or M has $U_{2,5}$ or P_7 as a minor.*

Proof. Observe that if $\{x_2, x_5, x_8\}$ is a triangle of M then, by Lemma 2.8, $M \cong M^*(K_{3,3})$. Thus may assume that each of x_2, x_5, and x_8 is in a unique triangle of M. Now consider $M/x_5\backslash x_6$. This has $\{x_1, x_2, x_3\}, \{x_7, x_8, x_9\}$, $\{x_1, x_4, x_7\}$, and $\{x_3, x_4, x_9\}$ as triangles. We may assume that $\{x_2, x_4, x_8\}$ is not a triangle, otherwise this minor is isomorphic to P_7. It follows that $M/x_2, x_5\backslash x_3, x_6 \cong U_{2,5}$. □

Lemma 3.2. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. Then M is 3-connected or $M \cong H_{12}$.*

Proof. Assume that the result fails. By Lemma 2.7, $M = M_1 \oplus_2 M_2$, where M_1 and M_2 are 3-connected matroids and p is the basepoint of the 2-sum. We may assume that M_1 or M_2, say M_1, is not isomorphic to O_7 having p in its triad otherwise $M \cong H_{12}$. Observe that if M_1 has no triads, then the minimality of M is contradicted. Thus every triad of M_1 contains p otherwise M has a triad. Let $\{x_1, x_3, p\}$ be a triad T^* of M_1. Thus M has a cocircuit C^* that meets $E(M_1)$ in $\{x_1, x_3\}$. As Lemma 2.4 implies that every element of M is in a triangle, M has a triangle containing x_1. By orthogonality with C^*, this triangle contains x_3 and an element x_2 of $E(M_1) - T^*$. Observe that $\{x_1, x_3\}$ is a cocircuit of $M_1\backslash p$, so $\mathrm{co}(M_1\backslash p)$ has a 2-circuit. Thus $\mathrm{co}(M_1\backslash p)$ is not 3-connected, so, by Bixby's Lemma [4] (see also [23, Lemma 8.7.3]), $\mathrm{si}(M_1/p)$ is 3-connected. If neither x_1 nor x_3 is in a triangle of M_1 other than $\{x_1, x_2, x_3\}$, then M_1/p is 3-connected and has no cocircuit of size less than four. This is a contradiction since M_1/p is a minor of M. Therefore, we may assume that M_1 has $\{p, x_1, x_4\}$ as a triangle for some element x_4. Then, either the only triangle containing p in M_1 is $\{p, x_1, x_4\}$, or M_1 has another triangle containing p. In the latter case, by orthogonality, this second triangle must be $\{p, x_3, x_5\}$ for some element x_5.

We first assume that $\{p, x_3, x_5\}$ is a triangle of M_1. Since the intersection of $\mathrm{cl}_{M_1}(\{p, x_1, x_3\})$ and $E(M) - \{p, x_1, x_3\}$ contains $\{x_2, x_4, x_5\}$, this set is a triangle of M. By Lemma 2.4, M has a 4-cocircuit C^* containing x_2. By symmetry and orthogonality, $x_4 \in C^*$. Furthermore, by orthogonality with a circuit of M that meets $E(M_1)$ in $\{x_1, x_4\}$, we deduce that $x_1 \in C^*$. Now suppose that M_1 has a point in $\mathrm{cl}_{M_1}(\{x_2, x_4\}) - \{x_2, x_4, x_5\}$. Then M_1 has an O_7-minor M_1' having p in the unique triad. Since $M_1' \oplus_2 M_2$ has no cocircuits of size less than four, we deduce that $M_1 \cong M_1' \cong O_7$, a contradiction. We may now assume that $C^* = \{x_1, x_2, x_4, x_6\}$ where $x_6 \notin \mathrm{cl}_{M_1}(\{x_1, x_2, x_4\})$. By Lemma 2.4, x_5 is in a 4-cocircuit D^* in M. By orthogonality and symmetry with C^*, we deduce that $x_3 \in D^*$. If $x_2 \notin D^*$, then $x_4 \in D^*$ and $x_1 \in D^*$. Thus $D^* = \{x_1, x_3, x_4, x_5\}$. Then, because $\{p, x_1, x_2, \dots, x_5\}$ contains the cocircuits D^* and $\{p, x_1, x_3\}$ of M_1, we deduce that $\lambda_{M_1}(\{p, x_1, x_2, x_3, x_4, x_5\}) \leq 3 + (6-2) - 6 = 1$. This is a contradiction as it implies that x_6 is a coloop of M_1. Therefore, we may assume that $x_2 \in D^*$ and neither x_1 nor x_4 is in D^*. By Lemma 2.4, x_6 is in a triangle of M_1, so $\{x_2, x_6\}$ or $\{x_4, x_6\}$ is a triangle of M_1.

3.2.1. *When $\{p, x_3, x_5\}$ is a triangle, M_1 has no triangle containing $\{x_2, x_6\}$.*

Assume that there is a triangle containing $\{x_2, x_6\}$. Thus $\{x_2, x_6, x_7\}$ is a triangle for some new element x_7. By orthogonality between this triangle and D^*, we see that $D^* = \{x_2, x_3, x_5, x_6\}$ or $D^* = \{x_2, x_3, x_5, x_7\}$. Since M_1 has D^*, C^*, and $\{p, x_1, x_3\}$ as cocircuits,

$$\lambda_{M_1}(\{p, x_1, x_2, \dots, x_7\}) \leq 4 + (8-3) - 8 = 1.$$

As M is 3-connected, $|E(M) - \{p, x_1, x_2, \dots, x_7\}| \leq 1$, so M_1 has a cocircuit containing $\{x_6, x_7\}$ that has size at most three and that does not contain p, a contradiction. Thus 3.2.1 holds.

3.2.2. *When $\{p, x_3, x_5\}$ is a triangle, M_1 has no triangle containing $\{x_4, x_6\}$.*

Assume that there is a triangle containing $\{x_4, x_6\}$. Then $\{x_4, x_6, x_7\}$ is a triangle of M_1 for some element x_7. Then $D^* = \{x_2, x_3, x_5, x_8\}$ for some element x_8. By orthogonality, x_8 is neither x_6 nor x_7. Thus $x_8 \notin \{p, x_1, x_2, \dots, x_7\}$. Consider $M/x_6\backslash x_7$. By 3.2.1 and orthogonality, we see that $\{x_4, x_6, x_7\}$ is the only triangle of M containing x_6, so $M/x_6\backslash x_7$ is simple. Therefore, it has a triad T^*, and M has $T^* \cup x_7$ as a cocircuit that avoids x_6. Then $x_4 \in T^*$ so, by orthogonality, $x_1 \in T^*$. As $\{x_1, x_2, x_3\}$ and $\{x_2, x_4, x_5\}$ are triangles, it follows by orthogonality that $\{x_1, x_2, x_4, x_7\}$ is a cocircuit. However, as $\{x_1, x_2, x_4, x_6\}$ is a cocircuit, $M^*|\{x_1, x_2, x_4, x_6, x_7\} \cong U_{3,5}$, a contradiction. Thus 3.2.2 holds.

By 3.2.1 and 3.2.2, $\{p, x_3, x_5\}$ is not a triangle of M_1. Thus the only triangle containing p in M_1 is $\{p, x_1, x_4\}$. As x_4 is in a triangle of M, it follows that

3.2.3. *M has $\{x_2, x_4, x_5\}$ as a triangle for some x_5 in $\mathrm{cl}_{M_1}(\{x_1, x_2, x_4\})$, or M has $\{x_4, x_5, x_6\}$ as a triangle for some x_5 and x_6 not in $\mathrm{cl}_{M_1}(\{x_1, x_2, x_4\})$.*

Next we eliminate the first possibility.

3.2.4. *$\{x_2, x_4, x_5\}$ is not a triangle of M.*

Assume that $\{x_2, x_4, x_5\}$ is a triangle. Let C^* be a 4-cocircuit of M containing x_4. Then, by orthogonality, C^* contains x_2 or x_5. Moreover, as $\{p, x_1, x_4\}$ is a circuit of M_1, it follows that M has a circuit that meets $E(M_1)$ in $\{x_1, x_4\}$. We deduce, by orthogonality, that $x_1 \in C^*$. Thus C^* contains $\{x_1, x_2, x_4\}$ or $\{x_1, x_4, x_5\}$. Suppose $C^* \subseteq \{x_1, x_2, x_3, x_4, x_5\}$. Then

$$\lambda_{M_1}(\{p, x_1, x_2, \dots, x_5\}) \leq 3 + (6 - 2) - 6 = 1.$$

Since M has no triads, $|E(M_1)| \geq 7$. Thus, as M_1 is 3-connected, it has exactly one element, say e, not in $\{p, x_1, x_2, \dots, x_5\}$. Since M_1 has a $\{p, x_1, x_3\}$ as a triad, $e \in \mathrm{cl}_{M_1}(\{x_2, x_4\})$. Thus M_1 and hence M has a $U_{2,5}$-minor, a contradiction. We deduce that C^* contains an element x_6 that is not in $\{p, x_1, x_2, \dots, x_5\}$. Then C^* does not contain $\{x_1, x_4, x_5\}$ by orthogonality with $\{x_1, x_2, x_3\}$, so $C^* = \{x_1, x_2, x_4, x_6\}$. Now $C^* \cap \{p, x_1, x_2, \dots, x_5\}$ is a union of cocircuits and hence is a cocircuit of $M_1|\{p, x_1, x_2, \dots, x_5\}$. But $\{x_1, x_2, x_4\}$ is not a cocircuit of the last matroid otherwise M_1 has $\{p, x_3, x_5\}$ as a triangle, a contradiction. Thus 3.2.4 holds.

Following 3.2.3, the rest of the proof of this lemma will be devoted to proving the following.

3.2.5. *M has no triangle of the form $\{x_4, x_5, x_6\}$ where x_5 and x_6 are not in $\mathrm{cl}_{M_1}(\{x_1, x_2, x_4\})$.*

Assume that M does have such a triangle. Then, as x_4 is in a 4-cocircuit of M, we may assume that either $\{x_1, x_2, x_4, x_5\}$ or $\{x_1, x_3, x_4, x_5\}$ is a cocircuit of M. Thus

$$\lambda_{M_1 \backslash x_5}(\{p, x_1, x_2, x_3, x_4\}) \leq 3 + (5 - 2) - 5 \leq 1.$$

Suppose $\mathrm{co}(M_1 \backslash x_5)$ is 3-connected. Then $\{p, x_1, x_2, x_3, x_4\}$ or $E(M_1 \backslash x_5) - \{p, x_1, x_2, x_3, x_4\}$ is a series class of $M_1 \backslash x_5$. In each case, since we must have that $|E(M_1)| \geq 9$, we get that M_1 has a triad avoiding p, a contradiction. Thus, by Bixby's Lemma, $\mathrm{si}(M/x_5)$ is 3-connected. Observe that $\mathrm{si}(M_1/x_5)$ is $M_1/x_5 \backslash x_4$ or is $M_1/x_5 \backslash x_2, x_4$ where the latter occurs when M_1 has $\{x_2, x_5, x_7\}$ as a triangle and $\{x_1, x_2, x_4, x_5\}$ as a cocircuit where $x_7 \notin \{p, x_1, x_2, \dots, x_6\}$.

Continuing the proof of 3.2.5, we show next that

3.2.6. *$M_1/x_5 \backslash x_4$ is not simple.*

Assume that $M_1/x_5 \backslash x_4$ is simple. Then $M/x_5 \backslash x_4$ has a triad T^*, otherwise the choice of M is contradicted. Then M has $T^* \cup x_4$ as a 4-cocircuit that avoids x_5. Thus, by orthogonality, $x_1 \in T^*$, $x_6 \in T^*$, and x_2 or $x_3 \in T^*$. Hence $\{x_1, x_2, x_4, x_6\}$ or $\{x_1, x_3, x_4, x_6\}$ is a cocircuit of M. We cannot have

$\{x_1, x_2, x_4, x_6\}$ and $\{x_1, x_2, x_4, x_5\}$ as cocircuits of M, or both $\{x_1, x_3, x_4, x_5\}$ and $\{x_1, x_3, x_4, x_6\}$ as cocircuits of M, otherwise M^* has $U_{3,5}$ as a restriction, a contradiction. Thus either both $\{x_1, x_2, x_4, x_5\}$ and $\{x_1, x_3, x_4, x_6\}$ are cocircuits of M, or both $\{x_1, x_2, x_4, x_6\}$ and $\{x_1, x_3, x_4, x_5\}$ are cocircuits of M. Using either of these pairs of 4-cocircuits, we eliminate x_4 to get a cocircuit contained in $\{x_1, x_2, x_3, x_5, x_6\}$. By orthogonality with a circuit of M containing $\{x_1, x_4\}$ and elements of $E(M_2) - p$, we deduce that $\{x_2, x_3, x_5, x_6\}$ is a cocircuit of M. As M_1 also has $\{p, x_1, x_3\}$ and either $\{x_1, x_2, x_4, x_5\}$ or $\{x_1, x_3, x_4, x_5\}$ as a cocircuit,

$$\lambda_{M_1}(\{p, x_1, x_2, \ldots, x_6\}) \leq 4 + (7 - 3) - 7 = 1.$$

This is a contradiction since it implies that M_1 has a triad containing $\{x_5, x_6\}$ that avoids p. We conclude that 3.2.6 holds.

We now know that $\text{si}(M_1/x_5)$ is $M_1/x_5 \backslash x_2, x_4$, and M_1 has $\{x_2, x_5, x_7\}$ as a triangle and $\{x_1, x_2, x_4, x_5\}$ as a cocircuit. Now $M_1|\{p, x_1, x_2, \ldots, x_7\}$ has rank four and has $\{x_1, x_2, x_4, x_5\}$ as a cocircuit. Thus $r(\{p, x_3, x_6, x_7\}) = 3$, so

3.2.7. *$\{p, x_3, x_6, x_7\}$ is a cocircuit of M_1.*

We show next that

3.2.8. *M_1 has no triangle containing $\{x_6, x_7\}$.*

Assume that M_1 has a triangle $\{x_6, x_7, e\}$. Since M_1 has no triangle containing $\{p, x_3\}$, we deduce that $e \notin \text{cl}_{M_1}(\{p, x_1, x_3\})$. Then the simplification of $(M|\{p, x_1, x_2, \ldots, x_7, e\})/e$ is O_7 and has $\{p, x_1, x_3\}$ as a triad. Replacing M_1 by this copy of O_7 contradicts the minimality of M. Thus 3.2.8 holds.

We now show that

3.2.9. *$r(M_1) \geq 5$.*

Suppose that $r(M_1) \leq 4$. Then $r(M_1) = 4$. The cocircuits $\{p, x_1, x_3\}$ and $\{x_1, x_2, x_4, x_5\}$ of M_1 imply that every element of M_1 not in these cocircuits is in $\text{cl}_{M_1}(\{x_6, x_7\})$. As $\{x_2, x_3, x_7\}$ is not a triad of M_1, there is at least one element in $\text{cl}_{M_1}(\{x_6, x_7\}) - \{x_6, x_7\}$, a contradiction to 3.2.8. Thus 3.2.9 holds.

Now M has a 4-cocircuit C_6^* containing x_6. We next show that

3.2.10. *$C_6^* = \{x_5, x_6, x_7, x_8\}$ for some element $x_8 \notin \{p, x_1, x_2, \ldots, x_7\}$.*

By orthogonality, C_6^* contains $\{x_2, x_5, x_6\}, \{x_5, x_6, x_7\}, \{x_1, x_2, x_4, x_6\}$, or $\{x_1, x_3, x_4, x_6\}$. The third possibility is eliminated because $\{x_1, x_2, x_4, x_5\}$ is a cocircuit. If the fourth possibility holds, then

$$\lambda_{M_1}(\{p, x_1, x_2, \ldots, x_6\}) \leq 4 + (7 - 3) - 7 = 1,$$

so $r(M_1) = 4$, a contradiction to 3.2.9. Suppose $C_6^* \supseteq \{x_2, x_5, x_6\}$. Then, by orthogonality, $C_6^* = \{x_2, x_3, x_5, x_6\}$. Using the last lambda calculation, we again obtain the contradiction that $r(M_1) = 4$. We conclude that

$C_6^* = \{x_5, x_6, x_7, x_8\}$ for some element x_8. Moreover, by orthogonality, $x_8 \notin \{p, x_1, x_2, \dots, x_7\}$. Thus 3.2.10 holds.

Recall that $M/x_5\backslash x_2, x_4$ is 3-connected. By the minimality of M, it follows that $M/x_5\backslash x_2, x_4$ has a triad T^*. Then $T^* \cup x_4, T^* \cup x_2$, or $T^* \cup \{x_2, x_4\}$ is a cocircuit of M where $\{x_2, x_4, x_5\} \cap T^* = \emptyset$. Suppose $T^* \cup x_4$ is a cocircuit of M. Then, by orthogonality, $x_1 \in T^*$ and $x_3 \in T^*$. Moreover, $x_6 \in T^*$, so $\{x_1, x_3, x_4, x_6\}$ is a cocircuit of M. Then

$$\lambda_{M_1}(\{p, x_1, x_2, \dots, x_6\}) \leq 4 + (7 - 3) - 7 = 1.$$

Since $\{x_5, x_6, x_7\}$ is not a triad of M_1, we deduce that M_1 has an element in $\mathrm{cl}_{M_1}(\{x_6, x_7\}) - \{x_6, x_7\}$, a contradiction to 3.2.8. Thus $T^* \cup x_4$ is not a cocircuit of M.

Next assume that $T^* \cup x_2$ is a cocircuit of M. Then $x_7 \in T^*$. As $x_4 \notin T^*$, we see that $x_1 \notin T^*$, so $x_3 \in T^*$. Hence $\{x_2, x_3, x_7\}$ is in a 4-cocircuit F^* of M that avoids $\{x_1, x_4, x_5, x_6\}$. Let $F^* = \{x_2, x_3, x_7, x_9\}$, for some element x_9 that is not in $\{p, x_1, x_2, \dots, x_7\}$. Now, by Lemma 2.4, x_9 is in a triangle T. By orthogonality, T contains $\{x_2, x_9\}$, $\{x_3, x_9\}$, or $\{x_7, x_9\}$. In the first case, by orthogonality, T is $\{x_1, x_2, x_9\}$, $\{x_2, x_4, x_9\}$, or $\{x_2, x_5, x_9\}$. By orthogonality with the triad $\{p, x_1, x_3\}$ in M_1, it follows that $T \neq \{x_1, x_2, x_9\}$. If T is $\{x_2, x_4, x_9\}$ or $\{x_2, x_5, x_9\}$, then M/x_6 has an O_7-minor having p in its triad, contradicting the minimality of M_1. If T contains $\{x_3, x_9\}$, then, by orthogonality with the triad $\{p, x_1, x_3\}$ in M_1, we get that $T = \{p, x_3, x_9\}$, a contradiction. We conclude that T contains $\{x_7, x_9\}$.

Next we show that

3.2.11. $x_8 \neq x_9$.

Assume that $x_8 = x_9$. Then M_1 has a cocircuit J^* such that

$$J^* \subseteq (\{x_2, x_3, x_7, x_8\} \cup \{x_5, x_6, x_7, x_8\}) - \{x_8\}.$$

If $x_7 \in J^*$, then, by orthogonality, T contains an element of $\{x_2, x_3, x_5, x_6\}$. Thus $x_8 \in \mathrm{cl}_{M_1}(\{p, x_1, x_3, x_6\})$, so

$$\lambda_{M_1}(\{p, x_1, x_2, \dots, x_8\}) \leq 4 + (9 - 4) - 9 = 0.$$

Hence $E(M_1) = \{p, x_1, x_2, \dots, x_8\})$. Thus $r(M_1) = 4$, a contradiction to 3.2.9. We conclude that 3.2.11 holds.

By orthogonality and 3.2.8, T is $\{x_7, x_8, x_9\}$. Then

$$\lambda_{M_1}(\{p, x_1, x_2, \dots, x_9\}) \leq 5 + (10 - 4) - 10 = 1.$$

Therefore $|E(M) - \{p, x_1, x_2, \dots, x_9\}| \leq 1$. By 3.2.9, $r(M_1) \geq 5$. As $r(\{p, x_1, x_2, \dots, x_7\}) = 4$, we deduce that $E(M_1) - \{p, x_1, x_2, \dots, x_7\}$ is a triad, a contradiction. We deduce, when $T^* \cup x_2$ is not a cocircuit of M.

Therefore we now know that $T^* \cup \{x_2, x_4\}$ is a cocircuit of M where $T^* \cap \{x_2, x_4, x_5\} = \emptyset$. Then, by orthogonality, $x_1 \in T^*$, $x_6 \in T^*$, and $x_7 \in T^*$. Thus M has $\{x_1, x_2, x_4, x_6, x_7\}$ as a cocircuit. As $\{x_1, x_2, x_4, x_5\}$ is also a cocircuit, by eliminating x_1, from the union of these two cocircuits, we get a cocircuit contained is $\{x_2, x_4, x_5, x_6, x_7\}$. The triangles $\{x_1, x_2, x_3\}$ and

$\{p, x_1, x_4\}$ of M_1 mean that $\{x_5, x_6, x_7\}$ is a cocircuit of M_1, a contradiction. Thus 3.2.5 holds and, by 3.2.3, the lemma follows. □

We may now focus on a 3-connected minor-minimal simple ternary matroid M in $\mathcal{M}_4$.

Lemma 3.3. *Let M be a 3-connected minor-minimal ternary matroid in $\mathcal{M}_4$. Then M does not have $U_{2,4}$ as a restriction.*

Proof. Assume that $M|\{x_1, x_2, x_3, x_4\} \cong U_{2,4}$. Since every cocircuit of M has at least four elements, $|E(M)| \geq 8$. By Lemma 2.4, we may assume that M has 4-cocircuits C_1^* and C_2^* such that $\{x_1, x_2, x_3\} \subseteq C_1^*$ and $\{x_2, x_3, x_4\} \subseteq C_2^*$. Observe that $C_1^* \neq C_2^*$ otherwise

$$\lambda(\{x_1, x_2, x_3, x_4\}) \leq 2 + (4 - 1) - 4 = 1,$$

so $|E(M)| \leq 5$, a contradiction. Now suppose M has a point x_5 such that $C_1^* = \{x_1, x_2, x_3, x_5\}$ and $C_2^* = \{x_2, x_3, x_4, x_5\}$. Then

$$\lambda(\{x_1, x_2, x_3, x_4, x_5\}) \leq 3 + (5 - 2) - 5 = 1.$$

However, this implies that $|E(M)| \leq 6$, a contradiction. Therefore $C_1^* = \{x_1, x_2, x_3, x_5\}$ and $C_2^* = \{x_2, x_3, x_4, x_6\}$ for distinct elements x_5 and x_6 not in $\{x_1, x_2, x_3, x_4\}$. Now, by Lemma 2.4, x_5 is in a triangle T. If $x_2 \in T$, then $T = \{x_2, x_5, x_6\}$. Thus $\lambda(\{x_1, x_2, x_3, x_4, x_5, x_6\}) \leq 3 + (6 - 2) - 6 = 1$, so $|E(M)| \leq 7$, a contradiction. Therefore neither $\{x_2, x_5\}$ nor $\{x_3, x_5\}$ is in a triangle and we may assume that $\{x_1, x_5, x_7\}$ is a triangle for some element x_7 that is not in $\{x_1, x_2, \ldots, x_6\}$. Similarly, we may assume that $\{x_4, x_6, x_8\}$ is a triangle for some element x_8 that is not in $\{x_1, x_2, \ldots, x_6\}$. Suppose $x_7 = x_8$. Then $\lambda(\{x_1, x_2, \ldots, x_7\}) \leq 1$. As M has no triads, it follows that $E(M) - \{x_1, x_2, \ldots, x_7\} = \{e\}$ for some element e, and $r(M) = 3$. As $E(M) - (C_1^* \cup C_2^*)$ is a rank-one set containing $\{x_7, e\}$, we have a contradiction. Thus x_7 and x_8 are distinct.

Now, $\lambda_{M\backslash x_5}(\{x_1, x_2, x_3, x_4, x_6\}) \leq 3 + (5 - 2) - 5 = 1$. As $M\backslash x_5$ has no 2-cocircuits, $\text{co}(M\backslash x_5) = M\backslash x_5$ and this matroid is not 3-connected. Thus, by Bixby's Lemma [4], $\text{si}(M/x_5)$ is 3-connected. Observe that $\text{si}(M/x_5) = M/x_5\backslash x_1$. By the minimality of M, there is a triad T^* in $M/x_5\backslash x_1$ and $T^* \cup x_1$ is a cocircuit of M that avoids x_5. Then $x_7 \in T^*$ and two of x_2, x_3, and x_4 are in T^*. Observe that $x_4 \notin T^*$, otherwise T^* also contains x_6 or x_8, so $|T^*| \geq 4$, a contradiction. Thus $T^* = \{x_2, x_3, x_7\}$, so $T^* \cup x_1 = \{x_1, x_2, x_3, x_7\}$ is a cocircuit of M. However, this implies that $M^*|\{x_1, x_2, x_3, x_5, x_7\} \cong U_{3,5}$. As M is ternary, this is a contradiction. □

Lemma 3.4. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. If M has a 4-cocircuit D^* that contains a triangle, then M is isomorphic to P_7 or H_{12}.*

Proof. Assume that M is not isomorphic to P_7 or H_{12}. Let $\{x_1, x_2, x_3, x_4\}$ be D^* and let $\{x_1, x_2, x_3\}$ be a triangle. Since $M \ncong H_{12}$, Lemma 3.2

implies that M is 3-connected. Then $\mathrm{co}(M\backslash x_4)$ is not 3-connected because $M\backslash x_4$ has $\{x_1, x_2, x_3\}$ as a circuit and a cocircuit. Thus $\mathrm{si}(M/x_4)$ is 3-connected. Now, by Lemma 2.4, x_4 is in a triangle of M. We may assume that $\{x_3, x_4, x_5\}$ is a triangle, for some element x_5 that is not in $\{x_1, x_2, x_3, x_4\}$.

3.4.1. *x_4 is in a triangle other than $\{x_3, x_4, x_5\}$.*

Assume that x_4 is in a unique triangle. Consider $M/x_4\backslash x_3$. This matroid is simple, so, by the minimality of M, it has a triad T_3^*. Thus $T_3^* \cup x_3$ is a cocircuit of M that does not contain x_4. By orthogonality, $x_5 \in T^*$. Observe that if $T_3^* \cup x_3 = \{x_1, x_2, x_3, x_5\}$, then $M^*|\{x_1, x_2, x_3, x_4, x_5\} \cong U_{3,5}$, a contradiction. Thus, by symmetry between x_1 and x_2, we may assume that $T^* \cup x_3 = \{x_2, x_3, x_5, x_6\}$ for some element x_6 not in $\{x_1, x_2, \dots, x_5\}$. Then, by Lemma 2.4, x_6 is in a triangle C_6. Suppose x_2 or x_3 is in C_6. As $x_4 \notin C_6$, by orthogonality with $\{x_1, x_2, x_3, x_4\}$, we deduce that C_6 contains two elements of $\{x_1, x_2, x_3\}$. Thus M has a 4-point line, a contradiction to Lemma 3.3. We now know that $C_6 = \{x_5, x_6, x_7\}$ for some element x_7 not in $\{x_1, x_2, \dots, x_7\}$.

Consider $M/x_2\backslash x_1$. Because $\{x_3, x_4, x_5\}$ is the unique triangle of M containing x_4 and M has no 4-point lines, $M/x_2\backslash x_1$ is simple. By the minimality of M, it follows that $M/x_2\backslash x_1$ has a triad T_1^*. Thus $T_1^* \cup x_1$ is a cocircuit of M avoiding x_2. By orthogonality, $x_3 \in T_1^*$, and x_4 or x_5 is in T_1^*. Suppose $T_1^* \cup x_1 = \{x_1, x_3, x_4, e\}$ for some element e. Then $e \neq x_2$ and $M^*|\{x_1, x_2, x_3, x_4, e\} \cong U_{3,5}$, a contradiction. Thus $T_1^* \cup x_1$ contains $\{x_1, x_3, x_5\}$ so, by orthogonality, $T_1^* \cup x_1$ is $\{x_1, x_3, x_5, x_6\}$ or $\{x_1, x_3, x_5, x_7\}$. In the first case, we get the contradiction that $M^*|\{x_1, x_2, x_3, x_5, x_6\} \cong U_{3,5}$. Hence $T_1^* \cup x_1 = \{x_1, x_3, x_5, x_7\}$. Thus

$$\lambda(\{x_1, x_2, \dots, x_7\}) \leq r(\{x_1, x_2, \dots, x_7\}) + (7-3) - 7 = r(\{x_1, x_2, \dots, x_7\}) - 3.$$

If $r(\{x_1, x_2, \dots, x_7\}) = 3$, then $E(M) = \{x_1, x_2, \dots, x_7\}$ and, from the known circuits and cocircuits, we obtain the contradiction that $M \cong P_7$. Thus $r(\{x_1, x_2, \dots, x_7\}) = 4$, so M has at most one element not in $\{x_1, x_2, \dots, x_7\}$. As $r(\{x_1, x_2, x_3, x_4, x_5\}) = 3$, we deduce that M has a cocircuit of size less than four, a contradiction. We conclude that 3.4.1 holds.

Now, by orthogonality, we may assume that M has $\{x_2, x_4, x_6\}$ as a triangle for some element x_6 not in $\{x_1, x_2, \dots, x_5\}$. Assume that x_1 is in a triangle T_1 other than $\{x_1, x_2, x_3\}$. Then, by Lemma 3.3 and orthogonality, $T_1 = \{x_1, x_4, x_7\}$ for some element x_7 not in $\{x_1, x_2, \dots, x_6\}$. Now M has a 4-cocircuit D_5^* containing x_5. By orthogonality, $D_5^* \subseteq \{x_1, x_2, \dots, x_7\}$. Thus $\lambda(\{x_1, x_2, \dots, x_7\}) \leq 3 + (7-2) - 7 = 1$. Hence M has at most one element not in $\{x_1, x_2, \dots, x_7\}$, and $r(M) = 3$. As $\{x_1, x_2, x_3, x_4\}$ is a cocircuit of M, it follows that $E(M) - \{x_1, x_2, x_3, x_4\}$ is a line of M. By Lemma 3.3, this line has exactly three points, that is, $\{x_5, x_6, x_7\}$ is a triangle. Hence $M \cong P_7$, a contradiction. We deduce that $\{x_1, x_2, x_3\}$ is the only triangle containing x_1. Then $M/x_1\backslash x_2$ is simple. Hence this matroid has a triad T_2^* avoiding x_1, so $T_2^* \cup x_2$ is a cocircuit of M. By orthogonality, $x_3 \in T_2^*$. If $x_4 \in T_2^*$, then $M^*|(\{x_1, x_2, x_3, x_4\} \cup T_2^*) \cong U_{3,5}$, a contradiction. Thus $x_4 \notin T_2^*$. Then, by orthogonality, $T_2^* \cup x_2 = \{x_2, x_3, x_5, x_6\}$. Thus $\lambda(\{x_1, x_2, \ldots, x_6\}) \leq 3 + (6-2) - 6 = 1$. Hence $r(M) = 3$ and $|E(M)| = 7$. Let x_7 be the element of $E(M) - \{x_1, x_2, \ldots, x_6\}$. Then $\{x_1, x_4, x_7\}$ is the complement in M of the cocircuit $\{x_2, x_3, x_5, x_6\}$. Thus $\{x_1, x_4, x_7\}$ is a triangle, a contradiction. □

Lemma 3.5. *Let M be a 3-connected minor-minimal ternary matroid in $\mathcal{M}_4$. If M has a 4-cocircuit that is also a 4-circuit, then M is isomorphic to F_7^- or P_7.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$ and assume that X is a circuit and a cocircuit of M. Assume that $r(M) \geq 4$. Take y in $E(M) - \mathrm{cl}(X)$. Then y is in a triangle Y of M. Since $y \notin \mathrm{cl}(X)$, by orthogonality, $X \cap Y = \emptyset$. Observe that $\lambda(X) = 3 + 3 - 4 = 2 = \kappa_M(X, Y)$. By Theorem 2.1, M has a minor N such that $\kappa_N(X, Y) = 2$ and $M|X = N|X$ while $M|Y = N|Y$. Then $r_N(Y) = r_M(Y) = 2$ and $r_N(X) = r_M(X) = 3$. As $2 = r_N(X) + r_N(Y) - r(N)$, we deduce that $r(N) = 3$. Thus N is a rank-3 simple ternary matroid having X as a circuit and Y as a hyperplane. As N has no 4-point lines, it follows that N is isomorphic to F_7^- or P_7. By the minimality of M, we obtain a contradiction.

We now know that $r(M) = 3$. Then $E(M) - X$ is a hyperplane Y of M. By Lemma 3.3, Y cannot have more than three elements. Suppose Y has exactly two elements. Then M has no triangles, otherwise it has a triad. Thus $M \cong U_{3,6}$, a contradiction. We deduce that M is a rank-3 ternary matroid whose ground set is the disjoint union of a triangle Y and a set that is both a circuit and a cocircuit. Thus M is isomorphic to F_7^- or P_7. □

For the rest of the section, every time we consider a 4-cocircuit C^* of M, we may assume that $r(C^*) = 4$. The following lemma shows that we cannot have two 4-cocircuits contained in the disjoint union of two triangles.

Lemma 3.6. *Let M be a 3-connected minor-minimal ternary matroid in $\mathcal{M}_4$ and suppose that M is not isomorphic to $P_7, M^*(K_{3,3})$, or Q_9. Let $C^* = \{x_1, x_2, x_4, x_5\}$ be a cocircuit of M. Let $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$ be disjoint triangles of M. Then there is no 4-cocircuit other than C^* that meets both $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$.*

Proof. Assume there is such a 4-cocircuit D^*. By Lemma 3.5, $r(D^*) = 4$. Assume that $\{x_1, x_2\} \subseteq D^*$. As D^* contains two members of $\{x_4, x_5, x_6\}$, it follows that $M^*|(C^* \cup D^*) \cong U_{3,5}$, a contradiction. Therefore, $\{x_1, x_3\} \subseteq D^*$ or $\{x_2, x_3\} \subseteq D^*$. Furthermore, by symmetry, one of $\{x_4, x_6\}$ or $\{x_5, x_6\}$ is contained in D^*. Thus we may assume that $D^* = \{x_2, x_3, x_5, x_6\}$. Let $X = \{x_1, x_2, \ldots, x_6\}$. By Lemma 3.5, $r(X) \neq 3$, so $r(X) = 4$. Thus $\lambda(X) \leq 4 + (6-2) - 6 = 2$. We next show that

3.6.1. $r(M) = 4$.

Assume that $r(M) \geq 5$ and take $y \in E(M) - \mathrm{cl}(X)$. Then y is in a triangle Y of M, and Y avoids X. We have that $\kappa_M(X, Y) = \lambda(X) = 2$. Thus, by Theorem 2.1, M has a minor N with ground set $X \cup Y$ such that $\kappa_N(X, Y) = 2$ while $N|X = M|X$ and $N|Y = M|Y$. Thus $r_N(X) = 4$ and $r_N(Y) = 2$, so $r(N) = 4$. Let $Y = \{x_7, x_8, x_9\}$. Then $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$ are triangles of N. Moreover, since $\{x_1, x_2, x_4, x_5\}$ and $\{x_2, x_3, x_5, x_6\}$ are cocircuits of M, each of these sets is a union of cocircuits of N. Because $r_N(\{x_7, x_8, x_9\}) = 2$, it follows that $\{x_1, x_2, x_4, x_5\}$ and $\{x_2, x_3, x_5, x_6\}$ are cocircuits of N. Because N is a simple minor of M, it follows that N has a cocircuit of size less than four. Suppose N has a 2-cocircuit Z. Then Z is contained in one of $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}$, or $\{x_7, x_8, x_9\}$, and $N \backslash Z$ is a 7-point plane of N. As $r(\{x_1, x_2, \ldots, x_6\}) = 4$, we may assume that $Z \subseteq \{x_1, x_2, x_3\}$. Then $N \backslash (Z \cup \{x_1, x_2, x_4, x_5\})$ is a line, $\{x_6, x_7, x_8, x_9\}$, of N, a contradiction to orthogonality with the cocircuit $\{x_2, x_3, x_5, x_6\}$. Hence N is cosimple. Thus N has a triad. By orthogonality and the fact that $r_N(X) = 4$, we may assume that $\{x_1, x_2, x_3\}$ is a triad of N. Then deleting $\{x_1, x_2, \ldots, x_6\}$ from N produces a rank-one matroid with ground set $\{x_7, x_8, x_9\}$, a contradiction. We conclude that 3.6.1 holds.

As $r(M) = 4$ and M has a plane with at least four points, it follows that $|E(M)| \geq 8$. Let x_7 be an element of $E(M) - \{x_1, x_2, \ldots, x_6\}$. Then x_7 is in a triangle T of M. If T avoids $\{x_1, x_2, \ldots, x_6\}$, then $|E(M)| \geq 9$. If T meets $\{x_1, x_2, \ldots, x_6\}$, then M has a plane with at least five points, so $|E(M)| \geq 9$. Let $E(M) - \{x_1, x_2, \ldots, x_6\} = \{x_7, x_8, \ldots, x_n\}$ for some $n \geq 9$. As $E(M) - (\{x_1, x_2, x_4, x_5\} \cup \{x_2, x_3, x_5, x_6\})$ is a flat of M of rank at most two and this flat contains $\{x_7, x_8, \ldots, x_n\}$, we deduce, by Lemma 3.3, that $n = 9$ and $\{x_7, x_8, x_9\}$ is a rank-2 flat of M. Since M has no triads, $r(\{x_7, x_8, x_9\} \cup L) = 4$ for each L in $\{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$.

Since $\{x_1, x_2, x_4, x_5\}$ and $\{x_2, x_3, x_5, x_6\}$ are cocircuits of M, the sets $\{x_1, x_4, x_7, x_8, x_9\}$ and $\{x_3, x_6, x_7, x_8, x_9\}$ are planes of M. As $r(X) = 4$, we see that $\{x_1, x_3, x_4, x_6\}$ is not a circuit of M. Since M is ternary, in the rank-4 ternary projective geometry P of which M is a restriction, $\mathrm{cl}_P(\{x_7, x_8, x_9\})$ has a single point z_3 that is not in $\{x_7, x_8, x_9\}$. Thus, by symmetry, we may assume that $\{x_3, x_6, x_9\}$ is a triangle of M. Moreover, by symmetry again, $\{x_1, x_4, z_3\}$ or $\{x_1, x_4, x_7\}$ is a triangle of P.

We may assume that $\{x_1, x_4, z_3\}$ is a triangle of P, otherwise, by Lemma 3.1, we obtain the contradiction that $M \cong M^*(K_{3,3})$ or M has P_7 as a minor. Lemma 3.1 also implies that $\{x_7, x_8, x_9\}$ is the unique triangle of M containing e for each e in $\{x_7, x_8\}$. Now $M/x_7 \backslash x_8$ has rank three and has $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}$, $\{x_3, x_6, x_9\}$ and $\{x_1, x_4, x_9\}$ as triangles. Since $P_7 \not\cong M/x_7 \backslash x_8$, the last matroid, which is ternary, has $\{x_1, x_2, x_3, x_5\}$ or $\{x_2, x_4, x_5, x_6\}$ as a line. Thus M has $\{x_1, x_2, x_3, x_5, x_7\}$ or $\{x_2, x_4, x_5, x_6, x_7\}$ as a rank-3 set. Then, by considering $M/x_8 \backslash x_7$ instead of $M/x_7 \backslash x_8$, we deduce that M has $\{x_1, x_2, x_3, x_5, x_8\}$ or $\{x_2, x_4, x_5, x_6, x_8\}$ as a plane. Now M cannot have both $\{x_1, x_2, x_3, x_5, x_7\}$ and $\{x_1, x_2, x_3, x_5, x_8\}$ as planes or M has a triad. Therefore either both $\{x_1, x_2, x_3, x_5, x_7\}$ and $\{x_2, x_4, x_5, x_6, x_8\}$ are planes of M or both $\{x_1, x_2, x_3, x_5, x_8\}$ and $\{x_2, x_4, x_5, x_6, x_7\}$ are planes of M. The first of these possibilities gives that P has $\{x_5, x_7, z_1\}$ and $\{x_2, x_8, z_2\}$ as triangles where $\{x_1, x_2, x_3, z_1\}$ and $\{x_4, x_5, x_6, z_2\}$ are lines of P for some points z_1 and z_2 of P. The second case is symmetric. In both cases, $M \cong Q_9$, a contradiction. □

Lemma 3.7. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. Let $x_1, x_2, \dots, x_6$, and x_7 be distinct elements of $E(M)$ such that $\{x_1, x_2, x_3\}$, $\{x_1, x_4, x_6\}$, and $\{x_1, x_5, x_7\}$ are triangles of M, and $\{x_1, x_2, x_4, x_5\}$ and $\{x_1, x_3, x_6, x_7\}$ are cocircuits of M. Then M is isomorphic to F_7^-, P_7, or $M(K_5)$.*

Proof. Let $X = \{x_1, x_2, \dots, x_7\}$. We may assume that $r(X) \geq 4$, otherwise $M \cong P_7$. Assume $r(X) = 4$. Then $|E(M) - X| \geq 2$ as $M \in \mathcal{M}_4$ and X contains a plane with at least five points. Suppose $|E(M) - X| = 2$ and $y \in E(M) - X$. Then y is in a triangle that, by symmetry, has x_3 and x_6 as its other elements. Then M has a plane with at least six elements, so $|E(M) - X| \geq 4$, a contradiction. We may now assume that $|E(M) - X| \geq 3$. As $\lambda(X) = 2$, we see that $r(E(M) - X) = 2$. Then, by Lemma 3.3, $E(M) - X$ is a triangle of M. When $r(M) = 4$, we call this triangle Y. When $r(M) \geq 5$, take z to be an element of $E(M) - \mathrm{cl}(X)$. In this case, we let Y be this triangle. Thus, whenever $r(M) \geq 4$, we have a triangle of M that avoids X. Now $\kappa_M(X, Y) = \lambda(Y) = 2$, so, by Theorem 2.1, M has a minor N with $E(N) = X \cup Y$ and $\kappa_N(X, Y) = 2$ such that $M|X = N|X$ and $M|Y = N|Y$. Thus $r_N(Y) = 2$ and $r_N(X) = r(N) = 4$. Now each of $\{x_1, x_2, x_4, x_5\}$ and $\{x_1, x_3, x_6, x_7\}$ is a cocircuit of N, otherwise $r_N(Y) \leq 1$, a contradiction.

Let P be the rank-4 ternary projective space of which N is a restriction. Let $\mathrm{cl}_P(Y) = \{a, b, c, d\}$. Now N has $Y \cup \{x_2, x_4, x_5\}$ and $Y \cup \{x_3, x_6, x_7\}$ as planes that meet on the line Y. Let $\{x_2, x_4, a\}, \{x_2, x_5, b\}$, and $\{x_4, x_5, c\}$ be triangles of P. Note that $r(\{x_2, x_3, x_4, x_6\}) = 3$. Now the projective line $\mathrm{cl}_P(Y)$ meets the projective plane $\mathrm{cl}_P(\{x_2, x_3, x_4, x_6\})$ in the point a because $\mathrm{cl}_P(Y)$ is not contained in the projective plane. Thus $\{x_3, x_6, a\}$ is a triangle of P. Similarly, $\{x_3, x_7, b\}$ and $\{x_6, x_7, c\}$ are triangles of P. If all of a, b, and c are in Y, then $N \backslash x_1 \cong M(K_5 \backslash e)$. The triangles containing x_1 imply that $N \cong M(K_5)$. We may now assume that exactly two of a, b, and c are in Y. Thus $d \in Y$ and N/d is a rank-3 matroid having three 3-point lines containing x_1 and having $\{x_2, x_4, x_5\}$ and $\{x_3, x_6, x_7\}$ as triangles. Thus N/d has a P_7-minor, a contradiction. □

Lemma 3.8. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. If M is 3-connected, then no element of M is in at least three triangles unless M is isomorphic to F_7^-, P_7, or $M(K_5)$.*

Proof. Assume that M is not isomorphic to F_7^-, P_7, or $M(K_5)$. Recall that, by Lemma 3.5, every 4-cocircuit is independent. Let $\{x_1, x_2, x_4, x_5\}$ be a

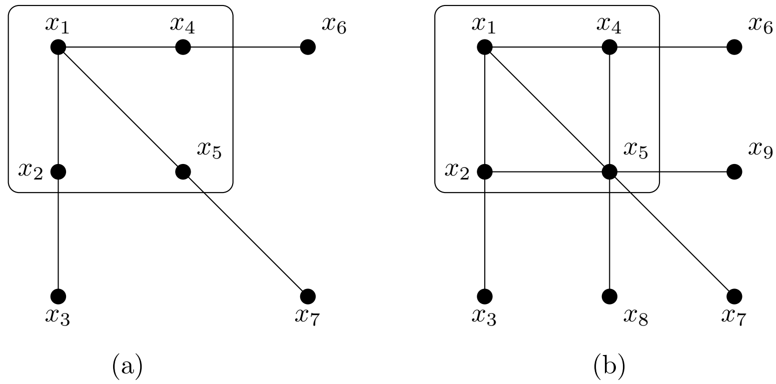


FIGURE 11. The triangles in Lemma 3.8. In each case, $\{x_1, x_2, x_4, x_5\}$ is a 4-cocircuit.

cocircuit C^* of M. Assume that x_1 is in more than three triangles. Let T_1, T_2, T_3, and T_4 be four such triangles. Then, by Lemma 3.4, $|C^* \cap T_i| = 2$ for all i in $\{1, 2, 3, 4\}$. However, this would imply that x_1 is contained in a four-point line, a contradiction to Lemma 3.3. Thus we may assume that x_1 is in exactly three triangles, say, $\{x_1, x_2, x_3\}$, $\{x_1, x_4, x_6\}$, and $\{x_1, x_5, x_7\}$. We first show the following.

3.8.1. *x_4 is in a triangle other than $\{x_1, x_4, x_6\}$.*

Assume that this fails. Observe that the given triangles form the configuration shown in Figure 11(a). Then $M/x_4\backslash x_1$ is simple. By the minimality of M, it follows that $M/x_4\backslash x_1$ has a triad T^*. Thus $T^* \cup x_1$ is a cocircuit of M where $x_4 \notin T^*$. Then $x_6 \in T^*$ and, by orthogonality, T^* contains x_2 or x_3, and T^* contains x_5 or x_7. If $\{x_1, x_2, x_5, x_6\}$ is a cocircuit, then $M^*|\{x_1, x_2, x_4, x_5, x_6\} \cong U_{3,5}$, a contradiction. If $\{x_1, x_3, x_5, x_6\}$ is a cocircuit, then, as $\{x_1, x_2, x_4, x_5\}$ is a cocircuit, by circuit elimination, we deduce that M has a cocircuit contained in $\{x_2, x_3, x_4, x_5, x_6\}$. By orthogonality with the triangle $\{x_1, x_5, x_7\}$, we see that $\{x_2, x_3, x_4, x_6\}$ is a cocircuit. However, $\{x_2, x_3, x_4, x_6\}$ is also a circuit, a contradiction to Lemma 3.5. Now assume that $\{x_1, x_2, x_6, x_7\}$ is a cocircuit. Then, again, as $\{x_1, x_2, x_4, x_5\}$ is a cocircuit, there is a cocircuit contained in $\{x_2, x_4, x_5, x_6, x_7\}$. By orthogonality with the triangle $\{x_1, x_2, x_3\}$, we deduce that $\{x_4, x_5, x_6, x_7\}$ is a cocircuit. As $\{x_4, x_5, x_6, x_7\}$ is also a circuit, we get a contradiction to Lemma 3.5. We conclude that $\{x_1, x_3, x_6, x_7\}$ is a cocircuit of M. By Lemma 3.7, this yields a contradiction. Thus 3.8.1 holds.

By 3.8.1 and symmetry, $\{x_4, x_5, x_8\}$ is a triangle of M for some element x_8 not in $E(M) - \{x_1, x_2, \ldots, x_7\}$. By symmetry again, x_2 is in a triangle other than $\{x_1, x_2, x_3\}$, so M has $\{x_2, x_4, x_9\}$ or $\{x_2, x_5, x_9\}$ as a triangle for some element x_9. By applying the permutation $(x_4, x_5)(x_6, x_7)$ to $E(M)$, we deduce that these two cases are symmetric, so it suffices to consider the former. Observe that if $x_8 = x_9$, then, as $\{x_2, x_5, x_9\}$ and $\{x_4, x_5, x_8\}$ are triangles, we deduce that $\{x_2, x_4, x_5, x_8\}$ is a 4-point line, a contradiction to Lemma 3.3. Thus, the known triangles form the configuration shown in Figure 11(b).

Suppose first that $x_1, x_2, \dots, x_8$, and x_9 are distinct. Now M has a cocircuit C_6^* that contains x_6. By orthogonality, x_1 or x_4 is in C_6^*, so C_6^* is $\{x_1, x_3, x_6, x_7\}$ or $\{x_4, x_6, x_8, x_9\}$. The first case gives a contradiction by Lemma 3.7. In the second case, M has $\{x_1, x_4, x_6\}$, $\{x_2, x_4, x_9\}$, and $\{x_4, x_5, x_8\}$ as triangles and has $\{x_1, x_2, x_4, x_5\}$ and $\{x_4, x_6, x_8, x_9\}$ as cocircuits and again we get a contradiction by Lemma 3.7.

We may now assume that $x_1, x_2, \dots, x_8$, and x_9 are not distinct. Since $x_1, x_2, \dots, x_6$, and x_7 are distinct, x_8 or x_9 is in $\{x_1, x_2, \dots, x_7\}$. But one easily checks that each possibility implies that $r(\{x_1, x_2, \dots, x_7\}) = 3$, a contradiction. We conclude that Lemma 3.8 holds. □

Lemma 3.9. *Let M be a minor-minimal matroid in $\mathcal{M}_4$. If M has $\{x_1, x_2, x_3\}$ as the unique triangle containing x_1, then M has a 4-cocircuit that meets $\{x_1, x_2, x_3\}$ in $\{x_2, x_3\}$.*

Proof. Consider $M/x_1\backslash x_3$. Since it is simple, by the minimality of M, the matroid $M/x_1\backslash x_3$ has a triad T^*. Thus $T^* \cup x_3$ is a cocircuit of M. By orthogonality, $x_2 \in T^*$ since $x_1 \notin T_1^*$. Hence the 4-cocircuit $T^* \cup x_3$ meets $\{x_1, x_2, x_3\}$ in $\{x_2, x_3\}$. □

Lemma 3.10. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$ that is not isomorphic to $M^*(K_{3,3})$ or Q_9. Assume that M has $\{x_1, x_2, x_3\}$, $\{x_4, x_5, x_6\}$, and $\{x_7, x_8, x_9\}$ as disjoint triangles and has no other triangles meeting $\{x_1, x_2, \dots, x_9\}$. Then M does not have all of $\{x_1, x_2, x_4, x_5\}$, $\{x_2, x_3, x_7, x_8\}$, and $\{x_5, x_6, x_8, x_9\}$ as cocircuits.*

Proof. Assume that M does have the three specified sets as cocircuits. By Lemma 3.9, M has a 4-cocircuit C^* containing $\{x_1, x_3\}$. Thus, by Lemmas 3.4 and 3.6, $C^* = \{x_1, x_3, x_{10}, x_{11}\}$ for some elements x_{10} and x_{11} in $E(M) - \{x_1, x_2, \dots, x_9\}$.

3.10.1. *M has no 4-circuit containing $\{x_2, x_3\}$.*

Assume M has such a circuit C. As $\{x_1, x_2, x_4, x_5\}$ is a cocircuit, C contains x_4 or x_5. By Lemma 3.5, C does not contain $\{x_4, x_5\}$ otherwise $r(\{x_1, x_2, x_3, x_4, x_5, x_6\}) = 3$ and M has $\{x_1, x_2, x_4, x_5\}$ as a circuit and a cocircuit. Thus C is $\{x_2, x_3, x_4, \alpha\}$ or $\{x_2, x_3, x_5, \beta\}$ for some elements α and β. Suppose $C = \{x_2, x_3, x_5, \beta\}$. Then $\beta \in \{x_6, x_8, x_9\}$ and we obtain a contradiction to orthogonality with the cocircuit $\{x_1, x_3, x_{10}, x_{11}\}$. Thus $C = \{x_2, x_3, x_4, \alpha\}$ and, by symmetry, we may assume that $\alpha = x_{10}$.

As $\{x_2, x_3, x_4, x_{10}\}$ and $\{x_1, x_2, x_3\}$ are circuits of M, we can eliminate x_2 to obtain that $\{x_1, x_3, x_4, x_{10}\}$ is a circuit of M since each of x_1, x_3, and x_4 is in just one triangle. But this circuit meets the cocircuit $\{x_2, x_3, x_7, x_8\}$ in a single element, a contradiction. Hence 3.10.1 holds.

Consider $M/x_2, x_3\backslash x_1$. By 3.10.1 and the fact that each of x_2 and x_3 is in a single triangle, we deduce that this matroid is simple. Now $M/x_2, x_3\backslash x_1$

has no triad T^* otherwise M has $T^* \cup x_1$ as a cocircuit that meets $\{x_1, x_2, x_3\}$ in a single element. We deduce that $M/x_2, x_3 \backslash x_1$ contradicts the minimality of M. □

Lemma 3.11. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. Assume that no element of M is more than one triangle. Then there is no integer n such that M has $T_1, T_2, \dots, T_n$ as disjoint triangles and has $C_1^*, C_2^*, \dots, C_n^*$ as 4-cocircuits such that, interpreting all subscripts modulo n, the sets $C_i^* \cap T_i$ and $C_{i-1}^* \cap T_i$ are distinct and $|C_i^* \cap T_i| = 2 = |C_i^* \cap T_{i+1}|$ for all i.*

Proof. Assume that there is such an integer. Choose a least such integer n. By Lemmas 3.5 and 3.10, $n \geq 4$. Let $T_i = \{x_{i1}, x_{i2}, x_{i3}\}$ for all i. We can shuffle the labels within each T_i such that, for $j < n$, the cocircuit C_j^* is $\{x_{j2}, x_{j3}, x_{(j+1)2}, x_{(j+1)3}\}$ when j is odd and $\{x_{j1}, x_{j2}, x_{(j+1)1}, x_{(j+1)2}\}$ when j is even. Moreover, we can take C_n^* to be $\{x_{n2}, x_{n3}, x_{11}, x_{12}\}$ when n is odd and $\{x_{n1}, x_{n2}, x_{11}, x_{12}\}$ when n is even. Now consider $M/x_{11}, x_{13} \backslash x_{12}$. Suppose it has a triad T^*. Then $T^* \cup x_{12}$ is a cocircuit of M that meets T_1 in a single element, a contradiction. By the minimality of M, it has a 4-circuit C that contains $\{x_{11}, x_{13}\}$. By orthogonality, C contains x_{22} or x_{23} and C also contains a member of $\{x_{n2}, x_{n3}\}$ when n is odd and a member of $\{x_{n1}, x_{n2}\}$ when n is even. The cocircuit $\{x_{21}, x_{22}, x_{31}, x_{32}\}$ implies that $x_{22} \notin C$, so $x_{23} \in C$. Thus $C = \{x_{11}, x_{13}, x_{23}, x_{n\alpha}\}$ for some α in $\{1, 2, 3\}$. Now, by Lemma 3.9, M has a 4-cocircuit D^* that contains $\{x_{21}, x_{23}\}$. By Lemma 3.5, this cocircuit does not meet T_1, Thus $x_{n\alpha} \in D^*$ so $|D^* \cap T_n| = 2$. The minimality of n is contradicted unless $D^* \cap T_n = T_n \cap C_{n-1}^*$. In the exceptional case, $D^* \Delta C_{n-1}^*$ is a 4-cocircuit D_{n-1}^* of M that meets T_{n-1} in exactly two elements. Then the triangles $T_2, T_3, \dots, T_{n-2}$, and T_{n-1} and the cocircuits $C_2^*, C_3^*, \dots, C_{n-2}^*$, and D_{n-1}^* violate the minimality of n. □

Lemma 3.12. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. Suppose that M has $\{x_1, x_2, x_4, x_5\}$ as a cocircuit and has $\{x_1, x_2, x_3\}, \{x_1, x_4, x_6\}$, and $\{x_2, x_4, x_7\}$ as triangles. Then M is isomorphic to F_7^-, P_7, or $M(K_5)$.*

Proof. Assume that M is not isomorphic to F_7^-, P_7 or $M(K_5)$. Since M has a triangle containing x_5, by orthogonality, this triangle contains x_1, x_2, or x_4. Thus one of those elements is in at least three triangles, a contradiction to Lemma 3.8. □

For $r \geq 4$, let M_r be the matroid that is represented over $GF(3)$ by the matrix $[I_r|A_r]$ where A_r is the ternary matrix shown in Figure 12. We omit the routine proof of the following result.

Lemma 3.13. *For all $r \geq 6$,*

$$M_r/a_1, a_2 \backslash b_1, b_2 \cong M_{r-2}.$$

For all i in $[r+1]$, we see that $\{b_i, a_i, b_{i+1}\}$ is a triangle of M_r, and $\{a_i, b_{i+1}, b_{i+2}, a_{i+2}\}$ is a cocircuit where all subscripts are interpreted modulo $r+1$.

$$
\begin{array}{c}
 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{r-1} \\ b_r
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
b_{r+1} & a_1 & a_2 & a_3 & \dots & a_{r-1} & a_r & a_{r+1}
\end{array} \\
\left(\begin{array}{cccccccc}
1 & 1 & 0 & 0 & & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & & 0 & 1 & 1 \\
 & & & & & & & \\
1 & 0 & 0 & 0 & & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & & 1 & 0 & 1
\end{array}\right)
\end{array}
$$

FIGURE 12. The matrix A_r.

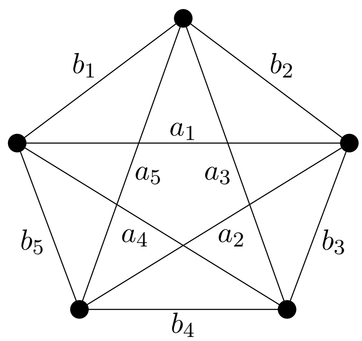


FIGURE 13. K_5.

Lemma 3.14. *$M_4 \cong M(K_5)$ and $M_5 \cong M(K_{2,2,2})$.*

Proof. It is straightforward to check that each of the triangles in the graph K_5 in Figure 13 is a circuit in M_4. It follows that M_4/e is binary for all e. Thus M_4 is binary as $r(M_4) = 4$. Therefore, as M_4 is ternary, M_4 is regular. Since $|E(M_4)| = 10 = \binom{r(M_4)+1}{2}$, it follows by a result of Heller [17] that $M_4 \cong M(K_5)$. It is straightforward to check that the triangles of M_5 have the structure of Figure 7. Thus, by Lemma 2.9, $M_5 \cong M(K_{2,2,2})$. □

Lemma 3.15. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. Assume that M is not isomorphic to $M(K_5)$ or $M(K_{2,2,2})$. For some $k \geq 4$, let $y_1, z_1, y_2, z_2, \dots, y_k, z_k, y_{k+1}$ be a sequence of distinct elements of M such that $\{y_i, z_i, y_{i+1}\}$ is a triangle for all i in $[k]$ and $\{z_j, y_{j+1}, y_{j+2}, z_{j+2}\}$ is a cocircuit for all j in $[k-2]$. Assume that none of $z_1, z_2, \dots, z_{k-1}$, or z_k is in more than one triangle of M. Then M has elements z_{k+1} and y_{k+2} that are not in $\{y_1, z_1, y_2, \dots, y_k, z_k, y_{k+1}\}$ such that $\{y_{k+1}, z_{k+1}, y_{k+2}\}$ is a triangle, $\{z_{k-1}, y_k, y_{k+1}, z_{k+1}\}$ is a cocircuit, and z_{k+1} is in only one triangle of M.*

Proof. As $M/z_k\backslash y_{k+1}$ is simple, it has a triad T^*_{k+1}. Then $T^*_{k+1} \cup y_{k+1}$ is a cocircuit of M and $y_k \in T^*_{k+1}$. By orthogonality, z_{k-1} or y_{k-1} is in

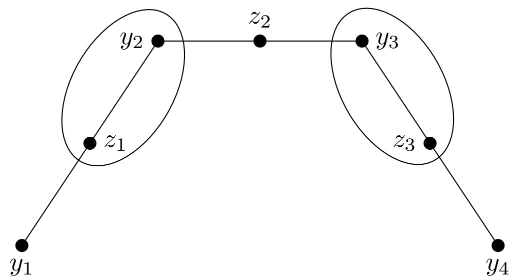


FIGURE 14. $\{z_1, y_2, y_3, z_3\}$ is a 4-cocircuit in Lemma 3.15 and this pattern continues.

T^*_{k+1}. Suppose $y_{k-1} \in T^*_{k+1}$. Then z_{k-2} or y_{k-2} is in T^*_{k+1}. In the latter case, the triangle $\{y_{k-3}, z_{k-3}, y_{k-2}\}$ contradicts orthogonality. In the former case, $M^*|\{z_{k-2}, y_{k-1}, y_k, z_k, y_{k+1}\} \cong U_{3,5}$, a contradiction. Then $y_{k-1} \notin T^*_{k+1}$, so $z_{k-1} \in T^*_{k+1}$. Let $T^*_{k+1} - \{z_{k-1}, y_k\} = \{z_{k+1}\}$. By orthogonality and Lemma 3.3, $z_{k+1} \notin \{y_1, z_1, y_2, \dots, z_k, y_{k+1}\}$. Now z_{k+1} is in a triangle T of M. By Lemma 3.8, y_k is not in three triangles, and, by assumption, z_{k-1} is in only one triangle, so z_{k+1} is in only one triangle and this triangle avoids $\{z_{k-1}, y_k\}$. Thus $y_{k+1} \in T$. Let $T - \{y_{k+1}, z_{k+1}\} = \{y_{k+2}\}$. From the known 4-cocircuits of M, when $y_{k+2} \neq y_1$, we have that $y_{k+2} \notin \{z_1, y_2, z_2, \dots, y_{k+1}, z_{k+1}\}$.

3.15.1. *If $y_{k+2} = y_1$, then M is isomorphic to $M(K_5)$ or $M(K_{2,2,2})$.*

Assume $y_{k+2} = y_1$. Now $M/z_{k+1}\backslash y_{k+2}$ is simple and so has triad T^*_{k+2}. Then $T^*_{k+2} \cup y_{k+2}$ is a cocircuit of M that contains y_{k+1}. Thus z_k or y_k is in T^*_{k+2}, and z_1 or y_2 is in T^*_{k+2}. The triangles $\{y_{k-1}, z_{k-1}, y_k\}$ and $\{y_2, z_2, y_3\}$ imply that z_k and z_1 are in T^*_{k+2}, so $\{z_k, y_{k+1}, y_1, z_1\}$ is a cocircuit of M.

Let $V = \{y_1, z_1, \dots, y_{k+1}, z_{k+1}\}$. Then V is spanned by $\{y_1, y_2, \dots, y_{k+1}\}$. Moreover, we get $\{z_1, y_2, y_3, z_3\}, \{z_2, y_3, y_4, z_4\}, \dots, \{z_{k-1}, y_k, y_{k+1}, z_{k+1}\}$, and $\{z_k, y_{k+1}, y_1, z_1\}$ as cocircuits of V, where each of which contains an element that is not in the union of its predecessors, so $r^*(V) \leq 2(k+1) - k$. Since $r(V) \leq k+1$, it follows that $\lambda(V) \leq 1$. Thus $|E(M) - V| \leq 1$. Assume $E(M) - V = \{w\}$. Then w is in a triangle. But each of $y_1, y_2, \dots, y_{k+1}$ is already in two triangles, while each of $z_1, z_2, \dots, z_{k+1}$ is already in one triangle. Thus w is not in a triangle, a contradiction. We deduce that $E(M) - V = \emptyset$, so $\lambda(V) = 0$. Hence $0 = r(V) + r^*(V) - 2(k+1)$. If $r(V) = k+1$, then $E(M) - \mathrm{cl}(\{y_1, y_2, \dots, y_k\}) \subseteq \{z_k, y_{k+1}, z_{k+1}\}$, so M has a triad, a contradiction. Hence $r(V) \leq k$. The k cocircuits listed above imply that $r(V) \geq k$. Hence $r(V) = k = r(M)$.

We now construct a representation of M. Let $B \cup y_{k+1} = \{y_1, y_2, \dots, y_{k+1}\}$ where B is the basis $\{y_1, y_2, \dots, y_k\}$. Then $B \cup y_{k+1}$ contains a circuit containing y_{k+1}. By orthogonality with the known 4-cocircuits of M, we deduce that $\{y_1, y_2, \dots, y_{k+1}\}$ is a circuit of M.

$$
\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{k-1} \\ y_k \end{array}
\begin{array}{c}
\begin{array}{cccccccc} y_{k+1} & z_1 & z_2 & z_3 & \cdots & z_{k-1} & z_k & z_{k+1} \end{array} \\
\left(\begin{array}{cccccccc}
1 & 1 & 0 & 0 & & 0 & 1 & w_{k+1} \\
1 & v_2 & 1 & 0 & & 0 & 1 & 1 \\
1 & 0 & v_3 & 1 & & 0 & 1 & 1 \\
1 & 0 & 0 & v_4 & & 0 & 1 & 1 \\
 & & & & & & & \\
1 & 0 & 0 & 0 & & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & & v_k & w_k & 1
\end{array}\right)
\end{array}
$$

FIGURE 15. A ternary representation for M.

$$
\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{k-1} \\ y_k \end{array}
\begin{array}{c}
\begin{array}{cccccccc} y_{k+1} & z_1 & z_2 & z_3 & \cdots & z_{k-1} & z_k & z_{k+1} \end{array} \\
\left(\begin{array}{cccccccc}
1 & 1 & 0 & 0 & & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & & 0 & 1 & 1 \\
 & & & & & & & \\
1 & 0 & 0 & 0 & & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & & 1 & 0 & 1
\end{array}\right)
\end{array}
$$

FIGURE 16. The matrix A_r in Lemma 3.15.

From this representation, we now make it a ternary representation for M. By using the fundamental circuits with respect to B and scaling the rows so that y_{k+1} consists of all ones, we see that the matrix in Figure 15 is a representation for M where $v_2, v_3, \dots, v_k$ are all non-zero. The circuits $\{y_k, z_k, y_{k+1}\}$ and $\{y_{k+1}, z_{k+1}, y_1\}$ imply we may assume that the columns z_k and z_{k+1} are as shown, where w_k and w_{k+1} may be zero. Now, the cocircuits $\{z_{k-1}, y_k, y_{k+1}, z_{k+1}\}$ and $\{z_k, y_{k+1}, y_1, z_1\}$ imply that $w_k = 0 = w_{k+1}$. The cocircuit $\{z_1, y_2, y_3, z_3\}$ implies $v_3 = 1$. By symmetry, each of $v_4, v_5, \dots, v_k$ is 1. Finally, the cocircuit $\{z_{k+1}, y_1, y_2, z_2\}$ implies that $v_2 = 1$. Hence M is represented as shown in Figure 16.

By Lemma 3.14, if k is 4 or 5, then M is isomorphic to $M(K_5)$ or $M(K_{2,2,2})$, respectively. By Lemma 3.13, if $k \geq 6$, then M has $M(K_5)$ or $M(K_{2,2,2})$ as a proper minor. The minimality of M implies that 3.15.1 holds. Hence so does the lemma. □

Lemma 3.16. *Let M be a 3-connected minor-minimal ternary matroid in $\mathcal{M}_4$. Assume that M has $\{x_1, x_2, x_4, x_5\}$ as a cocircuit and has $\{x_1, x_2, x_3\}$, $\{x_1, x_4, x_6\}$, and $\{x_2, x_5, x_7\}$ as triangles, but M has no triangle containing $\{x_4, x_5\}$. Then M is isomorphic to $P_7, M^*(K_{3,3}), Q_9, M(K_5)$, or $M(K_{2,2,2})$.*

Proof. Assume that M is not isomorphic to $P_7, Q_9, M^*(K_{3,3})$, or $M(K_{2,2,2})$. By Lemma 3.5, $r(\{x_1, x_2, x_4, x_5\}) = 4$. Moreover, by Lemma 3.3, M has no 4-point line. Thus $x_7 \neq x_3$. Also $x_7 \neq x_6$ otherwise $r(\{x_1, x_2, x_4, x_5\}) = 3$, a contradiction. Thus $x_1, x_2, \ldots, x_6$, and x_7 are distinct.

By Lemma 3.12, $M/x_4\backslash x_6$ is simple. By the minimality of M, the matroid $M/x_4\backslash x_6$ has a triad T_6^*, and $T_6^* \cup x_6$ is a cocircuit of M that avoids x_4. Thus M has $\{x_1, x_2, x_5, x_6\}$, $\{x_1, x_2, x_6, x_7\}$, or $\{x_1, x_3, x_6, x_8\}$ as a cocircuit for some element x_8 not in $\{x_1, x_2, \ldots, x_7\}$. The first two cases violate Lemma 3.6. Thus M has $\{x_1, x_3, x_6, x_8\}$ as a cocircuit. Now x_8 is in a triangle T of M. By Lemma 3.8, $x_1 \notin T$. Thus $\{x_3, x_8\}$ or $\{x_3, x_6\}$ is in T.

3.16.1. *M has no triangle containing $\{x_3, x_8\}$.*

Assume that $\{x_3, x_8\} \subseteq T$ and let α be the third element of T. We show next that $\alpha \notin \{x_1, x_2, \ldots, x_8\}$. Assume $\alpha \in \{x_1, x_2, \ldots, x_8\}$. Then $\alpha \in \{x_6, x_7\}$. By Lemma 3.4, $\alpha \neq x_6$. Thus $\alpha = x_7$. Now x_7 is in a 4-cocircuit D^*. By Lemma 3.12, $\{x_2, x_3, x_7\} \not\subseteq D^*$. By orthogonality, x_2 or x_5 is in D^*, and x_3 or x_8 is in D^*. Moreover, $x_1 \notin D^*$. Thus if x_2 or x_3 is in D^*, then both are, a contradiction. Hence $\{x_5, x_7, x_8\} \subseteq D^*$. Let $D^* = \{x_5, x_7, x_8, \beta\}$. The triangles $\{x_1, x_2, x_3\}$ and $\{x_1, x_4, x_6\}$ imply that $\beta \notin \{x_1, x_2, \ldots, x_8\}$. Now β is in a triangle T'. By Lemma 3.8 and the lemma's hypothesis, $x_7 \notin T'$ and $x_5 \notin T'$. Thus $x_8 \in T'$. Let $T' = \{\beta, x_8, \gamma\}$. The cocircuit $\{x_1, x_3, x_6, x_8\}$ implies, by Lemma 3.8, that $\gamma = x_6$, that is, $\{\beta, x_6, x_8\}$ is a triangle. Let $X = \{x_1, x_2, \ldots, x_8, \beta\}$. Then $r(X) = 4$ and $\lambda(X) \leq 4 + (9 - 3) - 9 = 1$, so $|E(M) - X| \leq 1$. As $r(\{x_1, x_2, x_3, x_5, x_7, x_8\}) = 3$, we deduce that $|E(M) - X| = 1$ otherwise M has a triad. Let $E(M) - X = \{\delta\}$. Deleting the cocircuits $\{x_1, x_2, x_4, x_5\}$ and $\{x_1, x_3, x_6.x_8\}$ from M leaves $\{x_7, \beta, \delta\}$, which must be a triangle. Thus x_7 is in three triangles, a contradiction to Lemma 3.8. We conclude that $\alpha \notin \{x_1, x_2, \ldots, x_8\}$.

By the minimality of M, the simple matroid $M/x_5\backslash x_7$ has a triad T_7^* and $T_7^* \cup x_7$ is a cocircuit of M that avoids x_5, so it contains x_2. Now $x_1 \notin T_7^*$, otherwise $\{x_1, x_2, x_4, x_7\}$ or $\{x_1, x_2, x_6, x_7\}$ is a cocircuit of M. The first case gives the contradiction that M^* has a $U_{3,5}$-minor, while the second case violates Lemma 3.6. We deduce that $\{x_2, x_3, x_7, x_8\}$ or $\{x_2, x_3, x_7, \alpha\}$ is a cocircuit. Suppose $\{x_2, x_3, x_7, x_8\}$ is a cocircuit. As $\{x_1, x_3, x_6, x_8\}$ is a cocircuit, eliminating x_8 gives a cocircuit contained in $\{x_1, x_2, x_3, x_6, x_7\}$. The triangle $\{x_3, x_8, \alpha\}$ implies that the cocircuit avoids x_3, so it must be $\{x_1, x_2, x_6, x_7\}$. This gives a contradiction to Lemma 3.6. We deduce that $\{x_2, x_3, x_7, \alpha\}$ is a cocircuit of M.

Let $X = \{x_1, x_2, \ldots, x_8, \alpha\}$. Then $r(X) \leq 5$ and $\lambda(X) \leq r(X) - 3$, so $\lambda(X) \leq 2$. Suppose $r(X) = 4$. Then $\lambda(X) \leq 1$, so M has at most one element that is not in X. Assume that $E(M) - X = \{e\}$. Then $r(M) = 4$. As M has $\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_6, x_8\}$, and $\{x_2, x_3, x_7, \alpha\}$ as cocircuits, by considering the set obtained from $E(M)$ by deleting each two of these three cocircuits, we deduce that M has $\{x_6, x_8, e\}$, $\{x_7, \alpha, e\}$, and $\{x_4, x_5, e\}$ as triangles. Thus e is in at least three triangles, a contradiction to Lemma 3.8. We conclude that $X = E(M)$. Then $\{x_1, x_2, x_3, x_8, \alpha\}$ is a hyperplane of M, so M has $\{x_4, x_5, x_6, x_7\}$ as a cocircuit. As $\{x_4, x_5\}$ is not in a triangle, we obtain the contradiction that $M \cong Q_9$. We conclude that $r(X) > 4$, so $r(X) = 5$.

Assume that $r(M) = 5$. If $\lambda(X) \leq 1$, then M has at most one element not in X. As $r(\{x_1, x_2, \dots, x_7\}) = 4$, it follows that M has a cocircuit of size less than four. Then $\lambda(X) = 2$, so $r(X) = 5$ and $r(E(M) - X) = 2$. Hence $E(M) - X$ is a line containing exactly two or exactly three points. Assume $E(M) - X = \{y_1, y_2\}$. Then each y_i is in a triangle T_i. Because X is a union of cocircuits, $\{y_1, y_2\}$ is not contained in a triangle. By Lemma 3.8, none of x_1, x_2, or x_3 is in T_i. By assumption, T_i avoids $\{x_4, x_5\}$. Thus $T_i - y_i \subseteq \{x_6, x_7, x_8, \alpha\}$ for each i. By Lemma 3.3, M has no line with more than three points, so, by using the cocircuits $\{x_1, x_3, x_6, x_8\}$ and $\{x_2, x_3, x_7, \alpha\}$, we may assume that $\{y_1, x_6, x_8\}$ and $\{y_2, x_7, \alpha\}$ are triangles. Then $r(\{x_1, x_2, x_3, x_4, x_6, x_8, y_1, \alpha\}) \leq 4$ so M has a cocircuit of size less than four, a contradiction. We conclude that $E(M) - X$ is a triangle when $r(M) = 5$.

Assume $r(M) > 5$. Then, as $r(X) = 5$, we must have $\lambda(X) = 2$ otherwise M has a cocircuit of size less than four. Now, M has a triangle Y disjoint from $\text{cl}(X)$. By Theorem 2.1, M has a 12-element rank-5 minor N having ground set $X \cup Y$ with $M|X = N|X$ and $M|Y = N|Y$ such that $\lambda_N(Y) = 2$. Then $r(N) = 5$. Observe that $\{x_1, x_4, x_6\}, \{x_2, x_5, x_7\}, \{x_3, x_8, \alpha\}$, and $\{e, f, g\}$ are disjoint triangles in N. Therefore, $E(N)$ can be written as a disjoint union of triangles, so N has no triads. By the minimality of M, we see that N must have a 2-element cocircuit S^*. As $r(X) = 5$, and Y is a triangle of N, we see that $S^* \subseteq X$. The triangles in X imply that S^* is $\{x_4, x_6\}$,$\{x_5, x_7\}$ or $\{x_8, \alpha\}$. Since $r_N(Y) = 2$, each of the cocircuits $\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_6, x_8\}$ and $\{x_2, x_3, x_7, \alpha\}$ of M is also a cocircuit of N. The additional cocircuit S^* gives a contradiction. We conclude that $r(M) = 5$.

Let $E(M) - X = \{e, f, g\}$, which is a triangle of M. The cocircuits of M imply that M has $\{e, f, g, x_7, \alpha\}, \{e, f, g, x_4, x_5\}$ and $\{e, f, g, x_6, x_8\}$ as planes. We may view M as a restriction of a rank-5 ternary projective space P. Observe that $\text{cl}_P(\{e, f, g\})$ is a 4-point line. Thus there is a unique point h in $\text{cl}_P(\{e, f, g\}) - \{e, f, g\}$. Since $\{x_4, x_5\}$ is not in a triangle of M, we deduce that $\{x_4, x_5, h\}$ is a triangle of P. If $\{x_4, x_5, x_7, \alpha\}, \{x_4, x_5, x_6, x_8\}$, or $\{x_6, x_7, x_8, \alpha\}$ is a circuit of M, we obtain the contradiction that $r(X) = 4$. Thus each of $\text{cl}_P(\{x_7, \alpha\}), \text{cl}_P(\{x_4, x_5\})$, and $\text{cl}_P(\{x_6, x_8\})$ meets $\{e, f, g, h\}$ in a distinct point. Hence we may assume that $\{x_6, x_8, e\}$ and $\{x_7, \alpha, f\}$ are triangles of M.

Next we construct a ternary representation for the matroid M' that is obtained from M by adjoining the element h. Then $\{x_1, x_2, x_4, x_5, x_8\}$ is a basis B for this matrix and the representation is shown in Figure 17. To verify that this is indeed a representation, we use the fundamental circuits

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_8 \end{array}\begin{array}{c} \begin{array}{cccccccc} x_3 & x_6 & x_7 & \alpha & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & u_5 & 1 \\ 0 & 0 & 1 & 0 & 0 & u_3 & u_6 & u_1 \\ 0 & 0 & 0 & 1 & u_2 & u_4 & u_7 & 0 \end{pmatrix} \end{array}$$

FIGURE 17. Building a representation for M'.

of x_3, x_6, and x_7 with respect to B and the circuit $\{x_3, x_8, \alpha\}$. Scaling rows 2–5 so that their first non-zero entries are 1, we deduce that the columns corresponding to x_3, x_6, x_7, and α are as indicated. Moreover, the column corresponding to h is as indicated, where $u_1 \neq 0$. The circuit $\{x_6, x_8, e\}$ implies that the column e is as indicated with $u_2 \neq 0$. Because $\{e, f, g, h\}$ is a 4-point line, the second coordinates of columns f and g are zero. The triangle $\{x_7, \alpha, f\}$ implies that the third coordinate of column f is zero while we may take the first coordinate of f to be 1. The fourth and fifth coordinates of column f are u_3 and u_4 both of which are non-zero. The triangle $\{e, g, h\}$ implies that we may take the first coordinate of column g to be 1. The last three coordinates of g are non-zero, u_5, u_6, and u_7, respectively.

The circuit $\{x_7, \alpha, f\}$ implies that $u_3 = -1$ and $u_4 = 1$. The circuit $\{e, f, g\}$ implies that $u_5 = -1, u_6 = 1$, and $u_7 = -u_2 - 1$. Finally, the circuit $\{f, g, h\}$ implies that $u_1 = 1$ and $u_2 = 1$, so Figure 18 is a representation for the extension of M by the element h. One can now check that $M'/g\backslash e, f, h \cong Q_9$, a contradiction. We conclude that $\{x_3, x_8\} \not\subseteq T$. Thus 3.16.1 holds.

We now know that $\{x_6, x_8\}$ is contained in a triangle T. By Lemma 3.12 applied to the cocircuit $\{x_1, x_3, x_6, x_8\}$, we deduce that x_3 is not in a triangle apart from $\{x_1, x_2, x_3\}$, otherwise this triangle contains x_8, which we eliminated above. Let $T = \{x_6, x_8, \alpha_1\}$. Then, by Lemma 3.12, the

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_8 \end{array}\begin{array}{c} \begin{array}{cccccccc} x_3 & x_6 & x_7 & \alpha & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

FIGURE 18. A matrix A such that $M' \cong M[I_5|A]$.

cocircuit $\{x_1, x_3, x_6, x_8\}$ implies that M has no additional triangles containing x_8. Observe that $\alpha_1 \notin \{x_1, x_2, \dots, x_8\}$ unless $\alpha_1 = x_7$. Now $M/x_8\backslash\alpha_1$ is simple and so has a triad T_1^* avoiding x_8. Then $x_6 \in T_1^*$, and $T_1^* \cup \alpha_1$ is a cocircuit of M. By orthogonality and Lemma 3.4, exactly one x_1 and x_4 is in T_1^*. Suppose $x_1 \in T_1^*$. Then either $\{x_1, x_3, x_6, \alpha_1\}$ is a cocircuit, or $\alpha_1 = x_7$ and $\{x_1, x_2, x_6, x_7\}$ is a cocircuit. In the first case, $M^*|\{x_1, x_3, x_6, x_8, \alpha_1\} \cong U_{3,5}$, a contradiction. In the second case, the union of the two triangles $\{x_1, x_2, x_3\}$ and $\{x_6, x_7, x_8\}$ contains two 4-circuits, a contradiction to Lemma 3.6. We conclude that $x_1 \notin T_1^*$. Thus $x_4 \in T_1^*$. Again if $\alpha_1 = x_7$, then $T_1^* \cup \alpha_1 = \{x_4, x_5, x_6, x_7\}$, and this cocircuit together with $\{x_1, x_2, x_4, x_5\}$ gives a contradiction to Lemma 3.6. Thus $\alpha_1 \notin \{x_1, x_2, \dots, x_8\}$. Letting

$$(x_7, x_5, x_2, x_3, x_1, x_4, x_6, x_8, \alpha_1) = (y_1, z_1, y_2, z_2, y_3, z_3, y_4, z_4, y_5)$$

in Lemma 3.15, we deduce, by repeated applications of that lemma, that M is isomorphic to $M(K_5)$ or $M(K_{2,2,2})$, otherwise $|E(M)|$ is infinite. □

Lemma 3.17. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. If M has $\{x_1, x_2, x_4, x_5\}$ as a cocircuit and has $\{x_1, x_2, x_3\}, \{x_1, x_4, x_7\}$, $\{x_4, x_5, x_6\}$, and $\{x_2, x_5, x_8\}$ as triangles, then $x_1, x_2, \dots, x_7$, and x_8 are distinct, and M has a 4-cocircuit $\{x_2, x_3, x_8, \sigma\}$ or $\{x_4, x_6, x_7, \tau\}$ where neither σ nor τ is in $\{x_1, x_2, \dots, x_8\}$.*

Proof. By Lemma 3.5, $x_3 \neq x_6$ otherwise $r(\{x_1, x_2, x_4, x_5\}) = 3$. By symmetry, $x_7 \neq x_8$. Similarly, $x_3 \neq x_8$ and, by Lemma 3.3, $x_3 \neq x_7$. Thus $x_1, x_2, \dots, x_7$, and x_8 are distinct.

Consider $M/x_5\backslash x_2, x_4$. This matroid is simple. By Lemma 3.6, M does not have a 4-cocircuit containing $\{x_2, x_4\}$ and avoiding x_5. Thus $M/x_5\backslash x_2, x_4$ is cosimple. By the minimality of M, the matroid $M/x_5\backslash x_2, x_4$ has a triad T^*. Thus one of $T^* \cup x_2, T^* \cup x_4$, or $T^* \cup \{x_2, x_4\}$ is a cocircuit of M where $x_5 \notin T^*$.

Assume that $T^* \cup \{x_2, x_4\}$ is a cocircuit of M. Then, by orthogonality. $\{x_6, x_8\} \subseteq T^*$ and also $x_1 \in T^*$. Thus $\{x_1, x_2, x_4, x_6, x_8\}$ is a cocircuit of M. Eliminating x_1 from the union of $\{x_1, x_2, x_4, x_5\}$ and $\{x_1, x_2, x_4, x_6, x_8\}$, we deduce that M has a cocircuit D^* that is contained in $\{x_2, x_4, x_5, x_6, x_8\}$. By orthogonality, $x_4 \notin D^*$ and $x_2 \notin D^*$, so $|D^*| \leq 3$, a contradiction. Hence $T^* \cup \{x_2, x_4\}$ is not a cocircuit of M.

Next suppose that $T^* \cup x_2$ is a cocircuit of M. As $x_5 \notin T^*$, we deduce that $x_8 \in T^*$. Also x_1 or x_3 is in T^*. If $x_1 \in T^*$, then, by orthogonality, we deduce that $T^* \cup x_2 = \{x_1, x_2, x_7, x_8\}$. This 4-cocircuit gives a contradiction to Lemma 3.6. Thus $x_3 \in T^*$, so $T^* \cup x_2 = \{x_2, x_3, x_8, \sigma\}$ for some element σ that, by orthogonality, is not in $\{x_1, x_2, \dots, x_8\}$. By symmetry, if $T^* \cup x_4$ is a cocircuit of M, then $T^* \cup x_4 = \{x_4, x_6, x_7, \tau\}$ for some element τ not in $\{x_1, x_2, \dots, x_8\}$. □

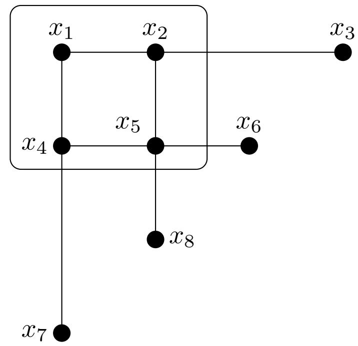


FIGURE 19. The triangles of M at the beginning of the proof of Lemma 3.18.

Lemma 3.18. *Let M be a minor-minimal ternary matroid in $\mathcal{M}_4$. Assume that M has $\{x_1, x_2, x_4, x_5\}$ as a cocircuit and has $\{x_1, x_2, x_3\}$ and $\{x_1, x_4, x_7\}$, as triangles. Then M is isomorphic to $P_7, M^*(K_{3,3}), Q_9, M(K_5), M(K_{2,2,2})$, or H_{12}.*

Proof. As x_5 is in a triangle of M, by Lemmas 3.8 and 3.16, we may assume that $\{x_4, x_5, x_6\}$ and $\{x_2, x_5, x_8\}$ are triangles of M. Moreover, by Lemma 3.17, $x_1, x_2, \dots, x_7$, and x_8 are distinct. We shall use the diagram in Figure 19 to expose the symmetries that arise in the argument. The ring around the set $\{x_1, x_2, x_4, x_5\}$ is to indicate that this set is a cocircuit of M. By Lemma 3.17, M has $\{x_2, x_3, x_8, y_2\}$ or $\{x_4, x_6, x_7, y_4\}$ as a cocircuit for some elements y_2 and y_4 not in $\{x_1, x_2, \dots, x_8\}$. Redrawing Figure 19 as Figure 20 corresponding to the two possibilities above, we see that these two cases are symmetric. We may assume that M has $\{x_2, x_3, x_8, y_2\}$ as a cocircuit. By Lemma 3.16, M has triangles $\{x_3, y_2, y_3\}$ and $\{x_8, y_2, y_8\}$. As M has no 4-point lines, $y_3 \neq y_8$. Thus the known triangles form the configuration shown in Figure 21. Moreover, by orthogonality, $\{y_3, y_8\} \cap \{x_1, x_2, x_3, x_4, x_5, x_8, y_2\} = \emptyset$.

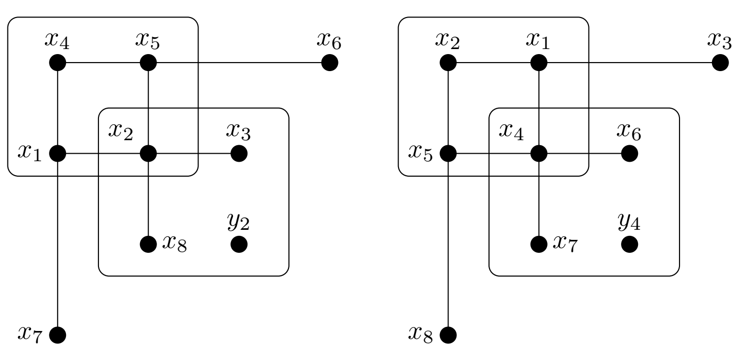


FIGURE 20. The ringed sets correspond to cocircuits.

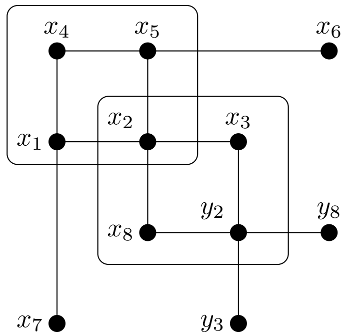


FIGURE 21. The ringed sets correspond to cocircuits.

Suppose $x_6 = y_3$. There is a 4-cocircuit C_6^* containing x_6. Then, by Lemma 3.4, C_6^* contains exactly one of x_4 and x_5 and exactly one of x_3 and y_2. By Lemma 3.6, $x_3 \notin C_6^*$ and $x_5 \notin C_6^*$. Thus $\{x_4, x_6, y_2\} \subseteq C_6^*$ so, by orthogonality, C_6^* must contain x_7 and y_8, so $x_7 = y_8$. It follows by Lemma 2.8 that M has $M^*(K_{3,3})$ as a minor, so we deduce that $x_6 \neq y_3$. By symmetry, $x_7 \neq y_8$.

Next suppose that $x_6 = y_8$. Let D_6^* be a 4-cocircuit containing x_6. Then, by Lemma 3.4, D_6^* contains exactly one of x_4 and x_5 and contains exactly one of x_8 and y_2. Now D_6^* cannot contain $\{x_4, x_6, x_8\}$ otherwise, by orthogonality, $|D_6^*| \geq 5$, a contradiction. Moreover, by Lemma 3.12, $\{x_5, x_6, x_8\} \not\subseteq D_6^*$. We deduce that $\{x_6, y_2\} \subseteq D_6^*$.

Suppose $\{x_4, x_6, y_2\} \subseteq D_6^*$. Then, by Lemma 3.4, D_6^* contains x_1 or x_7 and contains x_3 or y_3. As $|D_6^*| = 4$, it follows that $y_3 = x_7$ and $D_6^* = \{x_4, x_6, x_7, y_2\}$. The cocircuits $\{x_1, x_2, x_4, x_5\}, \{x_2, x_3, x_8, y_2\}$, and $\{x_4, x_6, x_7, y_2\}$ imply that $\lambda(\{x_1, x_2, \dots, x_8, y_2\}) \leq 4 + (9 - 3) - 9 = 1$, so $|E(M) - \{x_1, x_2, \dots, x_8, y_2\}| \leq 1$. As $r(\{x_2, x_4, x_5, x_6, x_8, y_2\}) = 3$, to avoid M having a cocircuit of size less than four, we must have that $E(M) - \{x_1, x_2, \dots, x_8, y_2\}$ contains a single element, say γ. The complement of the hyperplane $\{x_2, x_4, x_5, x_6, x_8, y_2\}$ is $\{x_1, x_3, x_7, \gamma\}$. As $\{x_1, x_2, x_3\}$, $\{x_3, x_7, y_2\}$, and $\{x_1, x_4, x_7\}$ are triangles, we get a contradiction to Lemma 3.12. We conclude that $\{x_4, x_6, y_2\} \not\subseteq D_6^*$.

We now know that $\{x_5, x_6, y_2\} \subseteq D_6^*$. Then x_2 or x_8 is in D_6^*, and x_3 or y_3 is in D_6^*. Then, we obtain the contradiction that $|D_6^*| \geq 5$ unless x_3 or y_3 is in $\{x_2, x_5, x_6, x_8, y_2\}$. Consider the exceptional case. As $x_1, x_2, \dots, x_8$, and y_2 are distinct, we must have that $y_3 \in \{x_2, x_5, x_6, x_8, y_2\}$. But we showed that $y_3 \neq x_6$. Also $y_3 \neq y_2$. By orthogonality between the triangle $\{x_3, y_2, y_3\}$ and the cocircuit $\{x_1, x_2, x_4, x_5\}$, we see that $y_3 \notin \{x_2, x_5\}$. Finally, $y_3 \neq x_8$ or else the cocircuit $\{x_2, x_3, x_8, y_2\}$ contains a triangle, a contradiction to Lemma 3.4. We conclude that $x_6 \neq y_8$. By symmetry, $x_7 \neq y_3$. Thus

3.18.1. *$x_1, x_2, \ldots, x_8, y_2, y_3$, and y_8 are distinct.*

Let $Z = \{x_1, x_2, \ldots, x_8, y_2, y_3, y_8\}$. Now $\{x_5, x_6\}$ or $\{x_4, x_6\}$ is in a 4-cocircuit S_6^* of M. By Lemma 3.6, S_6^* avoids $\{x_1, x_2, x_3\}$. Thus S_6^* is $\{x_5, x_6, x_8, y_8\}$ or $\{x_4, x_6, x_7, \beta_6\}$ for some element β_6 not in Z. By symmetry, M has a 4-cocircuit S_7^* containing x_7 and x_1 or x_4. Then S_7^* is $\{x_1, x_3, x_7, y_3\}$ or $\{x_4, x_6, x_7, \beta_7\}$ for some element β_7 not in Z. By symmetry, M has 4-cocircuits S_3^* and S_8^* where S_3^* contains $\{y_2, y_3\}$ or $\{x_3, y_3\}$, while S_8^* contains $\{y_2, y_8\}$ or $\{x_8, y_8\}$. Then S_3^* is $\{y_2, y_3, y_8, \beta_3\}$ or $\{x_1, x_3, x_7, y_3\}$, and S_8^* is $\{y_2, y_3, y_8, \beta_8\}$ or $\{x_5, x_6, x_8, y_8\}$ where neither β_3 nor β_8 is in Z.

3.18.2. *If M has both $\{x_5, x_6, x_8, y_8\}$ and $\{x_1, x_3, x_7, y_3\}$ as cocircuits, then $M \cong M(K_{2,2,2})$.*

Observe that as M has $\{x_5, x_6, x_8, y_8\}$ and $\{x_1, x_3, x_7, y_3\}$ as cocircuits, $\lambda(Z) \leq 5 + (11 - 4) - 11 = 1$, so $|E(M) - Z| \leq 1$. If $r(M) = 4$, then deleting the cocircuits $\{x_1, x_2, x_4, x_5\}$ and $\{x_2, x_3, x_8, y_2\}$ from $E(M)$ gives a rank-2 flat that contains $\{x_6, x_7, y_3, y_8\}$, a contradiction. Therefore $r(M) \geq 5$. As $r(\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}) = 4$, to avoid M having a triad, we must have that $|E(M) - Z| = 1$ and $r(M) = 5$. Let $E(M) - Z = \{\delta\}$. Deleting the union of the three cocircuits $\{x_1, x_2, x_4, x_5\}, \{x_2, x_3, x_8, y_2\}$, and $\{x_5, x_6, x_8, y_8\}$ from $E(M)$ leaves $\{x_7, y_3, \delta\}$, so this set is a triangle of M. By symmetry, $\{x_6, y_8, \delta\}$ is a triangle of M. It follows by Lemma 2.9 that $M \cong M(K_{2,2,2})$.

3.18.3. *$\{x_5, x_6, x_8, y_8\}$ or $\{x_1, x_3, x_7, y_3\}$ is a cocircuit, or $M \cong M(K_{2,2,2})$.*

Assume that neither $\{x_5, x_6, x_8, y_8\}$ nor $\{x_1, x_3, x_7, y_3\}$ is a cocircuit of M. Then both $\{x_4, x_6, x_7, \beta_6\}$ and $\{y_2, y_3, y_8, \beta_3\}$ are cocircuits of M where neither β_6 nor β_3 is in Z, although β_6 and β_3 may be equal. By Lemma 3.16, M has distinct triangles containing $\{x_6, \beta_6\}$ and $\{x_7, \beta_6\}$, and M has distinct triangles containing $\{y_3, \beta_3\}$ and $\{y_8, \beta_3\}$.

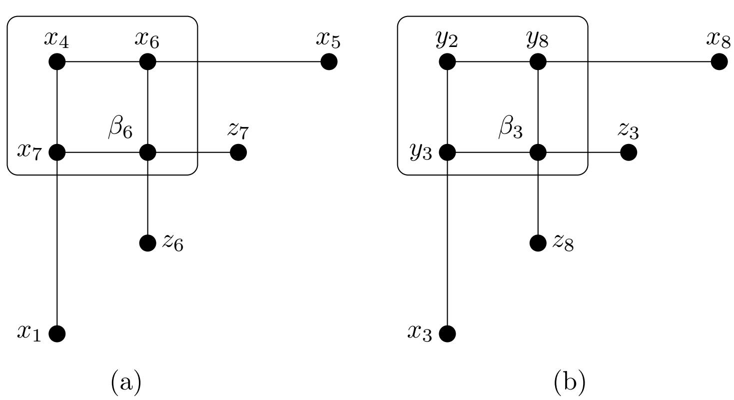


FIGURE 22. The symmetry between two cases in Lemma 3.18.

Suppose $\beta_6 = \beta_3$. Then M has $\{x_6, y_3, \beta_6\}$ and $\{x_7, y_8, \beta_6\}$ as triangles, or M has $\{x_6, y_8, \beta_6\}$ and $\{x_7, y_3, \beta_6\}$ as triangles. Then

$$\lambda(Z \cup \beta_6) \leq 5 + (12 - 4) - 12 = 1.$$

Thus $|E(M) - (Z \cup \beta_6\}| \leq 1$. If $|E(M) - (Z \cup \beta_6)| = 1$, let ν be the element in $E(M) - (Z \cup \beta_6)$. Then ν is in a triangle of M. But the other elements of these triangles are in $Z \cup \beta_6$ and each element of this set is already in two triangles. Thus we have a contradiction to Lemma 3.8. We conclude $E(M) = Z \cup \beta_6$. Moreover, when M has $\{x_6, y_8, \beta_6\}$ and $\{x_7, y_3, \beta_6\}$ as triangles, by Lemma 2.9, $M \cong M(K_{2,2,2})$ because $r(M) = 5$. Finally, when M has $\{x_6, y_3, \beta_6\}$ and $\{x_7, y_8, \beta_6\}$ as triangles, the matroid $M/\beta_6 \backslash y_3, y_8$ has rank 4 and has six triangles as in Figure 5. Thus, by Lemma 2.8, this matroid is isomorphic to $M^*(K_{3,3})$, a contradiction. We may now assume that $\beta_6 \neq \beta_3$.

As $\{x_4, x_6, x_7, \beta_6\}$ is a cocircuit of M, by Lemma 3.16, $\{x_7, z_7, \beta_6\}$ and $\{x_6, z_6, \beta_6\}$ are triangles for some elements z_6 and z_7. By Lemma 3.17, $x_1, x_4, x_5, x_6, x_7, \beta_6, z_6$, and z_7 are distinct. Moreover, M has a 4-cocircuit D_6^* that contains $\{x_5, x_6, z_6\}$ or $\{x_1, x_7, z_7\}$. By symmetry, as $\{y_2, y_3, y_8, \beta_3\}$ is a cocircuit, M has triangles $\{y_3, z_3, \beta_3\}$ and $\{y_8, z_8, \beta_3\}$ for some elements z_3 and z_8, where the elements $x_3, x_8, y_2, y_3, y_8, z_3, z_8$, and β_3 are distinct. Moreover, M has a 4-cocircuit D_3^* that contains $\{x_8, y_8, z_8\}$ or $\{x_3, y_3, z_3\}$.

Suppose $\{x_5, x_6, z_6\} \subseteq D_6^*$. The triangle $\{x_2, x_5, x_8\}$ implies that x_2 or x_8 is in D_6^*. Now $x_2 \neq z_6$ otherwise x_2 is in three triangles, a contradiction to Lemma 3.8. If $x_2 \in D_6^*$, then x_1 or x_3 is in D_6^*. Thus $z_6 \in \{x_1, x_3\}$, so z_6 is in three triangles, a contradiction to Lemma 3.9. Thus $x_2 \notin D_6^*$, so $x_8 \in D_6^*$. Now $x_8 \neq z_6$, otherwise x_8 is in three triangles. By orthogonality, y_2 or y_8 is in D_6^*. Thus $z_6 \in \{y_2, y_8\}$. If $z_6 = y_2$, then z_6 is in the triangles $\{x_3, y_3, z_6\}, \{x_8, y_8, z_6\}$, and $\{x_6, z_6, \beta_6\}$, a contradiction. Thus $z_6 = y_8$, so z_6 is in the triangles $\{x_8, y_2, z_6\}, \{z_6, z_8, \beta_3\}$, and $\{x_6, z_6, \beta_6\}$, a contradiction. We deduce that $\{x_5, x_6, z_6\} \not\subseteq D_6^*$. Because (a) and (b) in Figure 23 have the same configuration of triangle with one 4-cocircuit, the argument for $(x_4, x_6, x_5, x_7, \beta_6, z_7, z_6, x_1)$ may also be applied to $(y_2, y_8, x_8, y_3, \beta_3, z_3, z_8, x_3)$. Thus, by the symmetry shown in Figure 22, $\{x_1, x_7, z_7\} \not\subseteq D_6^*$. We conclude that 3.18.3 holds.

We may now assume that M has as cocircuits

(i) both $\{x_5, x_6, x_8, y_8\}$ and $\{y_2, y_3, y_8, \beta_3\}$, or
(ii) both $\{x_1, x_3, x_7, y_3\}$ and $\{x_4, x_6, x_7, \beta_6\}$.

To see that these two cases are symmetric, recall that we began knowing that M has $\{x_1, x_2, x_4, x_5\}$ as a 4-cocircuit. Then we assumed, by symmetry, that M has $\{x_2, x_3, x_8, y_2\}$ as a cocircuit. The two cases noted above can be represented as in Figure 23.

By symmetry, we may assume that Figure 23(a) holds. By 3.18.1, $x_1, x_2, \ldots, x_8, y_2, y_3$, and y_8 are distinct. Moreover, we noted when β_3 was introduced that it is not equal to any of these eleven elements. By Lemmas 3.16 and 3.17, M has triangles $\{y_3, z_3, \beta_3\}$ and $\{y_8, z_8, \beta_3\}$ for some

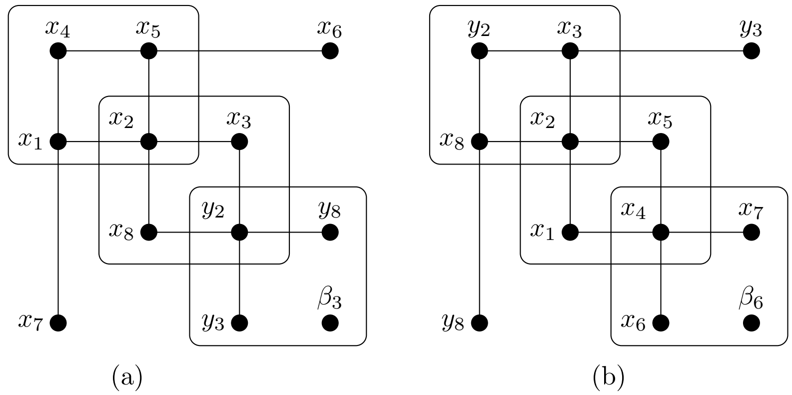


FIGURE 23. The cocircuits include the ringed sets along with $\{x_5, x_6, x_8, y_8\}$ or $\{x_1, x_3, x_7, y_3\}$, respectively.

elements z_3 and z_8, where $x_3, x_8, y_2, y_3, y_8, z_3, z_8$, and β_3 are distinct. Moreover, M has a 4-cocircuit D^* that contains $\{x_8, y_8, z_8\}$ or $\{x_3, y_3, z_3\}$. If $\{x_8, y_8, z_8\} \subseteq D^*$, then x_2 or x_5 is in D^*. If $x_2 \in D^*$, then $z_8 \in \{x_1, x_3\}$, so z_8 is in at least three triangles, a contradiction to Lemma 3.8. Thus $x_5 \in D^*$, so $z_8 \in \{x_4, x_6\}$. To avoid having z_8 in more than two triangles, we must have that $z_8 = x_6$, so $D^* = \{x_5, x_6, x_8, y_8\}$. Similarly, if $\{x_3, y_3, z_3\} \subseteq D^*$, then $z_3 = x_7$ and $D^* = \{x_1, x_3, x_7, y_3\}$.

Recall that $Z = \{x_1, x_2, \dots, x_8, y_2, y_3, y_8\}$ and β_3 is not in Z. Assume that $\{x_8, y_8, z_8\} \subseteq D^*$. Then $z_8 = x_6$ and $D^* = \{x_5, x_6, x_8, y_8\}$. Thus $\lambda((Z - x_7) \cup \beta_3) \leq 5 + (11-4) - 11 = 1$. Therefore $|E(M) - ((Z - x_7) \cup \beta_3)| \leq 1$. Thus $E(M) = Z \cup \beta_3$, so $z_3 \in Z \cup \beta_3$. As each element of $Z \cup \beta_3$ except x_7 is in two triangles, we deduce that $z_3 = x_7$. Then, by Lemma 2.9, we get the contradiction that $M \cong M(K_{2,2,2})$ when $x_6 = z_8$. We may now assume that $\{x_3, y_3, z_3\} \subseteq D^*$. Then $z_3 = x_7$ and $D^* = \{x_1, x_3, x_7, y_3\}$. Thus $\lambda((Z - x_6) \cup \beta_3) \leq 1$, so $E(M) = Z \cup \beta_3$ and $z_8 \in Z \cup \beta_3$. By symmetry with the previous case, $z_8 = x_6$ and so $M \cong M(K_{2,2,2})$ a contradiction. □

4. Proof of the Main Theorem

In this section, we prove Theorem 1.5 and then use that to prove Theorem 1.2.

Proof of Theorem 1.5. Assume that every cocircuit of M has size at least four, but M does not have $F_7^-, P_7, M^*(K_{3,3}), Q_9, M(K_5), M(K_{2,2,2})$, or H_{12} as a minor. If M is not 3-connected, then, by Lemma 3.2, $M \cong H_{12}$. Thus we may assume that M is 3-connected. By Lemmas 3.4 and 3.5, we may assume that every 4-cocircuit of M is independent. By Lemma 3.8, every element of M is in at most two triangles. If every element of M is in at most one triangle, then, as every element of M is in a triangle, $E(M)$ is a disjoint union of triangles. This contradicts Lemmas 3.10 and 3.11.

We now know that M has an element x_1 that is in two triangles $\{x_1, x_2, x_3\}$ and $\{x_1, x_4, x_7\}$. By Lemma 2.4, there is a 4-cocircuit C^* containing x_1. As C^* is independent, we may assume that $C^* = \{x_1, x_2, x_4, x_5\}$ for a new element x_5. By Lemma 3.12, there is not a triangle containing $\{x_2, x_4\}$. Now, by Lemma 2.4, x_5 is in a triangle. Then either there is a triangle containing $\{x_2, x_5\}$ but not $\{x_4, x_5\}$, a triangle containing $\{x_4, x_5\}$ but not $\{x_2, x_5\}$, or there are two triangles, one containing $\{x_2, x_5\}$ and the other containing $\{x_4, x_5\}$. The first two cases are symmetric, so we may assume there is a triangle containing $\{x_2, x_5\}$ but not $\{x_4, x_5\}$. Then, by Lemma 3.16, M is isomorphic to $P_7, M^*(K_{3,3}), Q_9$, or $M(K_{2,2,2})$, a contradiction. Therefore there is a triangle containing $\{x_2, x_5\}$ and a triangle containing $\{x_4, x_5\}$. As M has no 4-point lines, these two triangles are distinct. Therefore, by Lemma 3.18, M is isomorphic to $P_7, M^*(K_{3,3}), Q_9, M(K_5), M(K_{2,2,2})$, or H_{12}, a contradiction. This contradiction completes the theorem. $\square$

Proof of Theorem 1.2. Assume that every cocircuit of M has size at least four, but M does not have $U_{2,5}, F_7, F_7^-, P_7, M^*(K_{3,3}), Q_9, M(K_5), M(K_{2,2,2})$, or H_{12} as a minor. We first assume that M is not 3-connected. Then, by Lemma 2.7, $M = M_1 \oplus_2 M_2$ for some 3-connected matroids M_1 and M_2 each of which has rank at least three. Assume that some M_i is not ternary. If $r^*(M_i) = 2$, then M has a cocircuit of size less than four, a contradiction. Thus we may assume that the rank and corank of each M_i are at least three. Hence, by Theorem 2.3, as M does not have $U_{2,5}$ or F_7 as a minor, $M_i \cong F_7^*$. Then M has a triad, a contradiction. We deduce that both M_1 and M_2 are ternary. Thus, by Lemma 3.2, M has an H_{12}-minor, a contradiction.

We now know that M is 3-connected. Then, by Theorem 2.3, as M does not have F_7 or $U_{2,5}$ as a minor and $M \not\cong F_7^*$, we deduce that M is ternary. Therefore, by Theorem 1.5, the theorem holds. $\square$

5. When M is 3-connected

In this section, we answer a question of Guoli Ding (private communication) by finding the unavoidable minors for the class of 3-connected matroids in which every cocircuit has size at least four. In particular, we prove Theorem 1.3. This proof relies on the use of blocking sequences, which were introduced by Bouchet, Cunningham, and Geelen [2] (see also [11, 13] and [23, Chapter 13.3]). We will follow the treatment in Oxley [23, Chapter 13.3]. Let M be a matroid and D and C be disjoint subsets of $E(M)$ such that D is coindependent and C is independent. Assume $N = M\backslash D/C$. Then, for $Z \subseteq E(M)$, we define $M[D, C; Z] = M\backslash(D - Z)/(C - Z)$. Let k be a positive integer and (X, Y) be an exact k-separation of N. Then a sequence $(v_1, v_2, \dots, v_t)$ of elements of $E(M) - E(N)$ is a *blocking sequence* for the k-separation (X, Y) of N with respect to D and C, if

(1) (i) $(X, Y \cup v_1)$ is not a k-separation of $M[D, C; \{v_1\}]$;
(ii) $(X \cup v_t, Y)$ is not a k-separation of $M[D, C; \{v_t\}]$; and

(iii) for all i in $\{1, 2, \ldots, t-1\}$, the pair $(X \cup v_i, Y \cup v_{i+1})$ is not a k-separation of $M[D, C; \{v_i, v_{i+1}\}]$; and

(2) no proper subsequence of $(v_1, v_2, \ldots, v_t)$ satisfies (i).

We say that (X, Y) *induces* a k-separation of M if M has a k-separation (X', Y') such that $X \subseteq X'$ and $Y \subseteq Y'$. If (X, Y) does not induce a k-separation of M, we call M a *bridge* for the k-separation (X, Y) and also say that M *bridges* (X, Y). To prove Theorem 1.3, we will use Theorem 1.2 and the following.

Theorem 5.1. *Let M be a ternary matroid that is minor minimal with the properties of being 3-connected and having an H_{12}-minor. Then M has F_7^-, P_7, or Q_9 as a minor.*

The proof of this theorem will rely on the following consequence of a theorem of Lemos and Oxley [18, Theorem 1.1].

Lemma 5.2. *Let N be a connected matroid having a unique 2-separation. Let M be a minor-minimal 3-connected matroid having N as a minor. Then $|E(M)| - |E(N)| \leq 5$.*

The proof of Theorem 5.1 relies heavily on computer computations to find all 3-connected ternary matroids with an H_{12}-minor that can be constructed via a blocking sequence. The following lemmas (see, for example, [23, Lemma 13.3.8] and [23, Lemma 13.3.9]) will help us construct such matroids. Assume that $(v_1, v_2, \ldots, v_t)$ is a blocking sequence, with respect to D and C, of the k-separation (X, Y) of N. Let $X_0 = X$ and $Y_{t+1} = Y$. For each i in $\{1, 2, \ldots, t\}$, let $X_i = X \cup \{v_1, v_2, \ldots, v_i\}$ and $Y_i = Y \cup \{v_i, v_{i+1}, \ldots, v_t\}$.

Lemma 5.3. *Let i and j be integers with $0 \leq i < j - 1 \leq t$. Then (X_i, Y_j) is an exact k-separation in $M[D, C; X_i \cup Y_j]$ and $(v_{i+1}, v_{i+2}, \ldots, v_{j-1})$ is a blocking sequence for this k-separation.*

Lemma 5.4. *The sequence $(v_1, v_2, \ldots, v_t)$ contains no two consecutive members of C and no two consecutive members of D.*

We now provide the proof of Theorem 5.1. Most of the proof describes the algorithm used to verify the computations.

Proof of Theorem 5.1. As M is 3-connected and has an H_{12}-minor, by Lemma 5.2, we deduce that $13 \leq |E(M)| \leq 17$. Therefore, $H_{12} = M[D, C; Z] = M \backslash (D - Z)/(C - Z)$ for disjoint subsets D and C of $E(M)$, where D is coindependent and C is independent.

We need to check that each 3-connected ternary matroid M that is minor minimal with the properties of being 3-connected and having H_{12} as a minor has F_7^-, P_7, or Q_9 as a minor. If $|E(M)| = 13$, then M is a single-element extension or coextension of H_{12}. If $|E(M)| = 14$, then M has a minor N that is a single-element extension or coextension of H_{12} and that is not 3-connected. By Lemma 5.4, if N is a single-element extension of H_{12}, then every 3-connected single-element coextension of N is a possible choice for

Require: A rank-r matroid M' that does not have F_7^-, P_7, or Q_9 as a minor
Ensure: A list of extensions of M' that do not contain F_7^-, P_7, or Q_9 as a minor

Initialize empty list $\mathcal{L}_{EMA}$
for each vector v in $\mathrm{PG}(r-1,3)$ **do**
  Extend M' by v to get N_{EMA}
  if $|E(N_{EMA})| < 12$ **then**
    if N_{EMA} has no minor isomorphic to P_7, F_7^-, or Q_9 **then**
      Add N_{EMA} to $\mathcal{L}_{EMA}$
    end if
  else if $|E(N_{EMA})| > 12$ and N_{EMA} is 3-connected **then**
    if N_{EMA} has no minor isomorphic to P_7, F_7^-, or Q_9 **then**
      Add N_{EMA} to $\mathcal{L}_{EMA}$
    end if
  end if
end for
return $\mathcal{L}_{EMA}$

Figure 24. Extend Matroid Algorithm (EMA)

M. Similarly, by Lemma 5.4, if N is a single-element coextension of H_{12}, then every 3-connected extension of N is a possible choice for M.

Recall that H_{12} is the 2-sum of two copies of O_7. Therefore, in the construction of M, we will first perform a 2-sum on matroids N_1 and N_2 that are produced by repeated single-element extensions or coextensions to the two copies of O_7. Each possibility for N created by the 2-sum of N_1 and N_2 will have H_{12} as a minor. Then we will look at a 3-connected single-element extension or coextension of N. This will be our candidate for M. We now describe the algorithms used to prove this theorem. There are three main algorithms in the program: the Extend Matroid Algorithm (EMA), the Coextend Matroid Algorithm (CMA), and the Two-Sum Algorithm (TSA).

The Extend Matroid Algorithm (Figure 24) takes a $GF(3)$-representation of a matroid M' and extends M' by a single element to get N_{EMA}. Depending on the size of the ground set of N_{EMA}, there are two different checks. Initially, we take M' to be a $GF(3)$-representation of O_7. If $|E(N_{EMA})| < 12$, then we are successively performing single-element extensions either on O_7 or on a matroid that was created by performing single-element extensions and coextensions beginning with O_7. We check to see if N_{EMA} has F_7^-, P_7, or Q_9 as a minor. The choices for N_{EMA} that do not have F_7^-, P_7, or Q_9 as a minor are added to a list $\mathcal{L}_{EMA}$ of matroids for iterative use in subsequent steps. If $|E(N_{EMA})| > 12$, then we are in the final stages of the construction of M. In this case, the matroid M' being provided as input to EMA contains an H_{12}-minor and is not 3-connected. We will see that M' will be a matroid from a list created by TSA. When we extend M' to get

Require: A rank-r matroid M'^* that does not have $(F_7^-)^*, P_7^*$, or Q_9^* as a minor
Ensure: A list of coextensions of M' that do not contain F_7^-, P_7, or Q_9 as a minor

Initialize empty list $\mathcal{L}_{CMA}$
for each vector v in PG$(r-1, 3)$ **do**
  Extend M'^* by v to obtain N_{CMA}^*
  Let $N_{CMA} = (N_{CMA}^*)^*$
  if $|E(N_{CMA})| < 12$ **then**
    if N_{CMA} has no minor isomorphic to F_7^-,P_7, or Q_9 **then**
      Add N_{CMA} to $\mathcal{L}_{CMA}$
    end if
  else if $|E(N_{CMA})| > 12$ and N_{CMA} is 3-connected **then**
    if N_{CMA} has no minor isomorphic to F_7^-,P_7, or Q_9 **then**
      Add N_{CMA} to $\mathcal{L}_{CMA}$
    end if
  end if
end for
return $\mathcal{L}_{CMA}$

FIGURE 25. Coextend Matroid Algorithm (CMA)

the matroid N_{EMA}, we check to see if N_{EMA} does not have F_7^-, P_7, or Q_9 as a minor and whether N_{EMA} is 3-connected. If N_{EMA} is 3-connected and does not have F_7^-, P_7, or Q_9 as a minor, then N_{EMA} is added to the list $\mathcal{L}_{EMA}$. We note that, in this case, the matroid N_{EMA} is a candidate for the matroid M described in the theorem. As M has an H_{12}-minor, the case when $|E(N_{EMA})| = 12$ does not arise in the program. Recall that EMA adds N_{EMA} to $\mathcal{L}_{EMA}$ if N_{EMA} does not have F_7^-, P_7, or Q_9 as a minor. Our program has determined that when $|E(N_{EMA})| > 12$, the list $\mathcal{L}_{EMA}$ is empty. This implies if N_{EMA} is 3-connected and $|E(N_{EMA})| > 12$, then N_{EMA} has F_7^-, P_7, or Q_9 as a minor. The Coextend Matroid Algorithm (Figure 25) has an analogous purpose, but it performs single-element coextensions instead. When using CMA, it adds the single-element coextension N_{CMA} of M' to a list $\mathcal{L}_{CMA}$.

The final algorithm is the Two-Sum Algorithm (Figure 26). This algorithm takes, as input, two lists $\mathcal{X}$ and $\mathcal{Y}$ of matroids. Each list will consist of O_7 along with with matroids produced by repeated uses of the EMA and CMA on O_7. Let $M_X \in \mathcal{X}$ and $M_Y \in \mathcal{Y}$. Recall the geometric representation of O_7 from Figure 1 with the distinguished point p. Then TSA finds the point p in both M_X and M_Y, and adds the matroid $M_X \oplus_2 M_Y$, with basepoint p, to a list $\mathcal{L}_{TSA}$. We note that the size of the ground set of every matroid added to $\mathcal{L}_{TSA}$ will be at least 12. In the case that the ground set

Require: Lists $\mathcal{X}$ and $\mathcal{Y}$ of matroids
Ensure: A list of matroids containing an H_{12}-minor

Initialize empty list $\mathcal{L}_{TSA}$
for each $M_X \in \mathcal{X}$ **do**
  Obtain O_7 as a minor N_X of M_X
  Find $e_X \in E(N_X)$ such that $N_X \setminus e_X \cong O_7 \backslash p$
  Turn e_X into a standard basis vector using row and column operations on the representation of N_X
  Move the 1 in e_X to the bottom right corner of the representation of N_X.
  for each $M_Y \in \mathcal{Y}$ **do**
    Obtain O_7 as a minor N_Y of M_Y
    Find $e_Y \in E(N_Y)$ such that $N_Y \setminus e_Y \cong O_7 \backslash p$
    Turn e_Y into a standard basis vector using row and column operations on the representation of N_Y
    Move the 1 in e_Y to the top left corner of the representation of N_Y.
    Form the 2-sum $N_{TSA} := M_X \oplus_2 M_Y$ across the point $e_X = e_Y$
    if N_{TSA} has an H_{12}-minor **then**
      Add N_{TSA} to $\mathcal{L}_{TSA}$
    end if
  end for
end for
return $\mathcal{L}_{TSA}$

FIGURE 26. Two-Sum Algorithm (TSA)

has size 12, the matroid will be H_{12}. Hence, any use of EMA or CMA on a matroid from $\mathcal{L}_{TSA}$ will produce a 3-connected matroid with an H_{12}-minor.

We now go over a few cases of the algorithm. Recall that, when $|E(M)| = 13$, the matroid M is a 3-connected single-element extension or coextension of H_{12}. Let $\mathcal{X} = \{O_7\}$ and $\mathcal{Y} = \{O_7\}$. Inputting $\mathcal{X}$ and $\mathcal{Y}$ into TSA, we get the matroid H_{12}. The code determines that inputting the matrix representation of H_{12} into EMA or CMA will produce a list that is empty, that is, every 3-connected single-element extension or coextension of H_{12} has F_7^-, P_7, or Q_9 as a minor.

When $|E(M)| = 14$, the matroid M has a minor N that is a single-element extension or coextension of H_{12} and that is not 3-connected. We now construct the matroid N. Recall that EMA is an algorithm that has a matroid representation as its input. Thus we provide EMA with a $GF(3)$-representation of O_7. This algorithm produces a list $\mathcal{L}_{EMA}$ of single-element extensions of O_7 without F_7^-, P_7, or Q_9 as a minor. Using TSA on the lists $\mathcal{X} = \mathcal{L}_{EMA}$ and $\mathcal{Y} = \{O_7\}$, we form the matroid $N_{TSA} = M_X \oplus_2 O_7$ for each matroid M_X in $\mathcal{X}$. Add every matroid N_{TSA} to a list $\mathcal{L}_{TSA}$. We note that TSA guarantees that N_{TSA} has an H_{12}-minor. Using CMA on every matroid in $\mathcal{L}_{TSA}$, we get a candidate for the matroid M. In every instance, CMA determines that M has F_7^-, P_7, or Q_9 as a minor. Swapping the order of EMA and CMA covers the alternative scenario where N is a single-element coextension of H_{12} that is not 3-connected. In this case, EMA determines that every resulting 3-connected matroid M has F_7^-, P_7, or Q_9 as a minor.

Now assume that $|E(M)| = 15$. In this case, M can be constructed via the blocking sequence (v_1, v_2, v_3) for the unique 2-separation of H_{12}. Then, by Lemma 5.4, either $H_{12} \cong M/v_1\backslash v_2/v_3$, or $H_{12} \cong M\backslash v_1/v_2\backslash v_3$. First assume that $H_{12} \cong M/v_1\backslash v_2/v_3$. We now construct the sets $X_1 = E(O_7) \cup \{v_1\}$ and $Y_3 = E(O_7) \cup \{v_3\}$. As v_1 is a contracted element, we will use CMA on O_7 to get a list $\mathcal{L}_{CMA}$ of all single-element coextensions N_{CMA} of O_7 without F_7^-, P_7, or Q_9 as a minor. Having constructed the candidates for X_1, we let $\mathcal{X} = \mathcal{L}_{CMA}$. Similarly, as v_3 is a contracted element, we will use CMA on a $GF(3)$-representation of O_7 to get a list of all possible matroid representations for Y_3. We denote this list as $\mathcal{Y}$. For each M_X in $\mathcal{X}$ and M_Y in $\mathcal{Y}$, TSA produces the matroid $N_{TSA} = M_X \oplus_2 M_Y$ and adds N_{TSA} to the list $\mathcal{L}_{TSA}$. Note that TSA guarantees that N_{TSA} has an H_{12}-minor. Since v_2 is a deleted element, we apply EMA on every matroid N_{TSA} in $\mathcal{L}_{TSA}$. As $|E(N_{TSA})| = 14$, EMA will produce a 3-connected matroid. Furthermore, EMA determines that every single-element extension of N_{TSA} has F_7^-, P_7, or Q_9 as a minor. Given that these matroids are minor minimal with the properties of being 3-connected and having an H_{12}-minor, we deduce that every 15-element candidate for M contains one of the excluded minors. A symmetric process applies to the case where $H_{12} \cong M\backslash v_1/v_2\backslash v_3$, with the roles of the EMA and CMA being interchanged accordingly.

We now consider the case where $|E(M)| = 16$. Assume (v_1, v_2, v_3, v_4) is the blocking sequence and $H_{12} = M/v_1\backslash v_2/v_3\backslash v_4$. We first construct all possible matroids with ground sets $X_2 = E(O_7) \cup \{v_1, v_2\}$ and $Y_4 = E(O_7) \cup \{v_4\}$. As v_1 is a contracted element, we use CMA on a $GF(3)$-representation of O_7 to get a list $\mathcal{L}_{CMA}$ of all coextensions of O_7 that do not contain F_7^- or P_7 as a minor. We then use EMA on every matroid N_{CMA} in $\mathcal{L}_{CMA}$ to produce the list $\mathcal{L}_{EMA}$ of matroids corresponding to every possible matroid with ground set X_2. We now construct the possibilities for the matroid with ground set Y_4. As v_4 is a deleted element, we use EMA on a $GF(3)$-representation of O_7 to get the list $\mathcal{L}'_{EMA}$ of such matroids. We apply TSA to the lists $\mathcal{X} = \mathcal{L}_{EMA}$ and $\mathcal{Y} = \mathcal{L}'_{EMA}$. Then, for every M_X in $\mathcal{X}$ and M_Y in $\mathcal{Y}$, TSA adds the matroid $N_{TSA} = M_X \oplus_2 M_Y$ to the list $\mathcal{L}_{TSA}$. Finally, as v_3 is a contracted element, we coextend every matroid in $\mathcal{L}_{TSA}$ using CMA. As $|E(N_{TSA})| = 15$, CMA produces a 3-connected matroid. Furthermore, the CMA determines that every 3-connected single-element coextension of N_{TSA} has F_7^-, P_7, or Q_9 as a minor.

We provide an outline of the other cases that are verified by the program. Note that the matroids M_X and M_Y with ground sets X_i and Y_j listed in each case are constructed by a similar process to that described above

(i) Assume (v_1, v_2, v_3, v_4) is the blocking sequence for the unique 2-separation of H_{12} and $H_{12} = M\backslash v_1/v_2\backslash v_3/v_4$. Let $X_1 = E(O_7) \cup \{v_1\}$ and $Y_3 = E(O_7) \cup \{v_3, v_4\}$. Let M_X and M_Y be the matroids constructed with ground sets X_1 and Y_3, respectively. Construct all possible matroids $N_{TSA} = M_X \oplus_2 M_Y$ using TSA. By coextending N_{TSA} by v_2, we get the matroid M.

(ii) Assume $(v_1, v_2, v_3, v_4, v_5)$ is the blocking sequence for the unique 2-separation of H_{12} and $H_{12} = M/v_1\backslash v_2/v_3\backslash v_4/v_5$. Let $X_1 = E(O_7) \cup \{v_1\}$ and $Y_3 = E(O_7) \cup \{v_3, v_4, v_5\}$. Let M_X and M_Y be the matroids constructed with ground sets X_1 and Y_3, respectively. Construct all possible matroids $N_{TSA} = M_X \oplus_2 M_Y$ using TSA. By extending N_{TSA} by v_2, we get the matroid M.

(iii) Assume $(v_1, v_2, v_3, v_4, v_5)$ is the blocking sequence for the unique 2-separation of H_{12} and $H_{12} = M\backslash v_1/v_2\backslash v_3/v_4\backslash v_5$. Let $X_2 = E(O_7) \cup \{v_1, v_2\}$ and $Y_4 = E(O_7) \cup \{v_4, v_5\}$. Let M_X and M_Y be the matroids constructed with ground sets X_2 and Y_4, respectively. Construct all possible matroids $N_{TSA} = M_X \oplus_2 M_Y$ using TSA. By extending N_{TSA} by v_3, we get the matroid M.

The remaining cases listed above were verified computationally. Therefore, we have determined that every ternary matroid that is minor minimal with the properties of being 3-connected and having an H_{12}-minor contains F_7^-, P_7, or Q_9 as a minor. This completes the proof of Theorem 5.1. □

We are now able to complete the proof of the main result of this section.

Proof of Theorem 1.3. Assume that M is 3-connected, that every cocircuit of M has size at least four, and that M has none of $U_{2,5}, F_7, F_7^-, P_7, M^*(K_{3,3})$, $Q_9, M(K_5)$, or $M(K_{2,2,2})$ as a minor. By Theorem 2.3, as M does not have F_7 or $U_{2,5}$ as a minor and $M \not\cong F_7^*$, we see that M is ternary. By Theorem 1.2, M has H_{12} as a minor. Then, by Theorem 5.1, M has F_7^-, P_7, or Q_9 as a minor, a contradiction. □

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