

# THE EXCLUDED MINORS FOR THE MATROIDS THAT ARE BINARY OR TERNARY

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ABSTRACT. We show that a matroid is binary or ternary if and only if it has no minor isomorphic to  $U_{2,5}$ ,  $U_{3,5}$ ,  $U_{2,4} \oplus F_7$ ,  $U_{2,4} \oplus F_7^*$ ,  $U_{2,4} \oplus_2 F_7$ ,  $U_{2,4} \oplus_2 F_7^*$ , or the unique matroids obtained by relaxing a circuit-hyperplane in either  $AG(3, 2)$  or  $T_{12}$ . The proof makes essential use of results obtained by Truemper on the structure of almost-regular matroids.

## 1. INTRODUCTION

In [5], Brylawski considered certain natural operations on minor-closed classes of matroids, and examined how they effect the set of excluded minors for those classes. In particular, he invited the reader to explore the excluded minors for the union of two minor-closed classes. We do so in one special case, and determine the excluded minors for the union of the classes of binary and ternary matroids. This solves Problem 14.1.8 in Oxley's list [19].

**Theorem 1.1.** *The excluded minors for the class of matroids that are binary or ternary are  $U_{2,5}$ ,  $U_{3,5}$ ,  $U_{2,4} \oplus F_7$ ,  $U_{2,4} \oplus F_7^*$ ,  $U_{2,4} \oplus_2 F_7$ ,  $U_{2,4} \oplus_2 F_7^*$ , and the unique matroids that are obtained by relaxing a circuit-hyperplane in either  $AG(3, 2)$  or  $T_{12}$ .*

Recall that the matroid  $AG(3, 2)$  is a binary affine space, and is produced by deleting a hyperplane from  $PG(3, 2)$ . Up to isomorphism, there is a unique matroid produced by relaxing a circuit-hyperplane in  $AG(3, 2)$ . We shall use  $AG(3, 2)'$  to denote this unique matroid.

The matroid  $T_{12}$  was introduced by Kingan [13]. It is represented over  $GF(2)$  by the matrix displayed in Figure 1. It is clear that  $T_{12}$  is self-dual. Moreover,  $T_{12}$  has a transitive automorphism group and a unique pair of circuit-hyperplanes. These two circuit-hyperplanes are disjoint. Up to isomorphism, there is a unique matroid produced by relaxing a circuit-hyperplane in  $T_{12}$ . We denote this matroid by  $T'_{12}$ .

A result due to Semple and Whittle [22] can be interpreted as showing that  $U_{2,5}$  and  $U_{3,5}$  are the only 3-connected excluded minors for the class in Theorem 1.1 that are representable over at least one field. We complete the characterization by finding the non-representable excluded minors and the excluded minors that are not 3-connected.

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$$I_6 \left[ \begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right]$$

FIGURE 1. A representation of  $T_{12}$ .

The binary matroids and the ternary matroids are well known to have, respectively, one excluded minor and four excluded minors. In this case, the union of two classes with finitely many excluded minors itself has only finitely many excluded minors. Brylawski [5] asked whether this is always true in the case that the two classes have a single excluded minor each. In unpublished work, Vertigan answered this question in the negative (see [7, Section 5]).

Vertigan's examples indicate that Brylawski's project of finding the excluded minors of the union of minor-closed classes is a difficult one. However, in some special cases it may be more tractable. Matroids that are representable over a fixed finite field have received considerable research attention. Indeed, the most famous unsolved problem in matroid theory is Rota's conjecture that there is only a finite number of excluded minors for representability over any fixed finite field [21]. This would stand in contrast to general minor-closed classes. Rota's conjecture is currently known to hold for the fields  $\text{GF}(2)$ ,  $\text{GF}(3)$ , and  $\text{GF}(4)$  [3, 9, 24, 29].

For a collection,  $\mathcal{F}$ , of fields, let  $\mathcal{M}_\cup(\mathcal{F})$  be the set of matroids that are representable over at least one field in  $\mathcal{F}$ . We believe that the following is true.

**Conjecture 1.2.** *Let  $\mathcal{F}$  be a finite family of finite fields. There is only a finite number of excluded minors for  $\mathcal{M}_\cup(\mathcal{F})$ .*

Conjecture 1.2 is true if and only if it is true for all families  $\mathcal{F}$  for which  $\mathcal{F}$  is an antichain in the sub-field order. Until now, Conjecture 1.2 was known to hold for only three families satisfying this condition, namely  $\{\text{GF}(2)\}$ ,  $\{\text{GF}(3)\}$ , and  $\{\text{GF}(4)\}$ . Thus Theorem 1.1 proves the first case of Conjecture 1.2 that does not reduce to Rota's conjecture.

We note that Conjecture 1.2 is false if we relax the constraint that  $\mathcal{F}$  is a finite collection. The authors of [14] construct an infinite number of excluded minors for real-representability that are not representable over any field. Rado [20] shows that any real-representable matroid is representable over at least one finite field. Thus, if  $\mathcal{F}$  is the collection of all finite fields, then  $\mathcal{M}_\cup(\mathcal{F})$  has an infinite number of excluded minors.

We remark also that although an affirmative answer to Conjecture 1.2 would imply that Rota's conjecture is true, it is conceivable that Conjecture 1.2 fails while Rota's conjecture holds.

Next we note a conjecture of Kelly and Rota [12] that is a natural companion to Conjecture 1.2. Suppose that  $\mathcal{F}$  is a family of fields. Let  $\mathcal{M}_\cap(\mathcal{F})$  be the class of matroids that are representable over every field in  $\mathcal{F}$ .

**Conjecture 1.3.** *Let  $\mathcal{F}$  be a family of finite fields. There is only a finite number of excluded minors for  $\mathcal{M}_\cap(\mathcal{F})$ .*

It is easy to see that this conjecture holds when  $\mathcal{F}$  is finite and contains only fields for which Rota's conjecture holds. Thus Conjecture 1.3 is known to hold if  $\mathcal{F}$  contains no field other than  $\text{GF}(2)$ ,  $\text{GF}(3)$ , or  $\text{GF}(4)$ . Moreover, unpublished work of Geelen shows that the conjecture holds if  $\mathcal{F} = \{\text{GF}(3), \text{GF}(4), \text{GF}(5)\}$ , in which case  $\mathcal{M}_\cap(\mathcal{F})$  is Whittle's class of "near-regular" matroids (see [10, 30, 31]).

It seems likely that the Matroid Minors Project of Geelen, Gerards, and Whittle will affirm both Rota's conjecture and Conjecture 1.3 (see [8]).

The proof of Theorem 1.1 relies heavily upon results due to Truemper [27]. If a matrix is not totally unimodular, but each of its proper submatrices is totally unimodular, then it is called a *minimal violation matrix* for total unimodularity. Truemper studied such matrices, and related them to a class of binary matroids which he called "almost-regular". An almost-regular matroid is not regular, but every element has the property that either its deletion or its contraction produces a regular matroid. Truemper gives a characterization of almost-regular matroids, by showing that they can all be produced from the Fano plane or an eleven-element matroid called  $N_{11}$ , using only  $\Delta$ - $Y$  and  $Y$ - $\Delta$  operations, along with series and parallel extensions.

Truemper's characterization of almost-regular matroids is deep, and perhaps not sufficiently appreciated within the matroid theory community. He does much more than simply provide a  $\Delta$ - $Y$  reduction theorem. In the process of obtaining this characterization, he obtains specific detailed information about the structure of almost-regular matroids. Without access to these structural insights, we would not have been able to obtain Theorem 1.1. We define almost-regular matroids and discuss Truemper's result in Section 2.6.

In the first half of our proof, we establish that every excluded minor for the class of binary or ternary matroids is a relaxation of an excluded minor for the class of almost-regular matroids, or more precisely the class consisting of the almost-regular matroids and their minors. (Here we are assuming certain conditions on the rank, corank, and connectivity of the excluded minor.) Having done this, we perform a case analysis that bounds the size of the excluded minor.

Now we give a more detailed description of the article. Section 2 establishes some fundamental notions and results that we use throughout the rest of the proof. In Section 2.9 we prove that each of the matroids listed in

Theorem 1.1 is indeed an excluded minor for the class of matroids that are binary or ternary. Section 3 contains a discussion of the excluded minors that have low rank, corank, or connectivity. Specifically, we show that any excluded minor that has rank or corank at most three, or that fails to be 3-connected, must be one of those listed in Theorem 1.1. In Section 4 we examine the excluded minors on eight or nine elements, and we show that there is precisely one such matroid:  $\text{AG}(3, 2)'$ .

The results of Sections 3 and 4 show that we can restrict our attention to 3-connected excluded minors with rank and corank at least four, and with at least ten elements. We do so in Section 5 where Theorem 5.1 shows that if  $M$  is such an excluded minor, then  $M$  can be produced by relaxing a circuit-hyperplane in a binary matroid, which we call  $M_B$ . Section 6 shows that every proper minor of  $M_B$  is either regular, or belongs to Truemper's class of almost-regular matroids.

In Section 7 we use Truemper's structural results on almost-regular matroids and perform a case analysis that reduces the problem of finding the remaining excluded minors to a finite task. We consider three cases:  $M_B$  has an  $R_{10}$ -minor;  $M_B$  has an  $R_{12}$ -minor; and  $M_B$  has neither an  $R_{10}$ - nor an  $R_{12}$ -minor. In the first case we show that  $|E(M_B)| = 12$ . Next we show that the second case cannot arise, and finally we show that if  $M_B$  has no minor isomorphic to  $R_{10}$  or  $R_{12}$ , then  $|E(M_B)| \leq 16$ . Having reduced the problem to a finite case-check, we complete the proof of Theorem 1.1 in Section 8.

## 2. PRELIMINARIES

Throughout the article,  $\mathcal{M}$  will denote the class of matroids that are either binary or ternary; that is,  $\mathcal{M} = \mathcal{M}_{\cup}(\{\text{GF}(2), \text{GF}(3)\})$ . The matroid terminology used throughout will follow Oxley [19], except that  $\text{si}(M)$  and  $\text{co}(M)$  respectively are used to denote the simple and cosimple matroids associated with the matroid  $M$ . A *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. We shall occasionally refer to a rank-2 flat as a *line*. Suppose that a binary matroid is represented over  $\text{GF}(2)$  by  $[I_r|A]$ . We shall say that  $A$  is a *reduced representation* of  $M$ .

We start by stating the well-known excluded-minor characterizations of binary and ternary matroids.

**Theorem 2.1.** (Tutte [29]). *A matroid is binary if and only if it has no minor isomorphic to  $U_{2,4}$ .*

**Theorem 2.2.** (Reid, Bixby [3], Seymour [24]). *A matroid is ternary if and only if it has no minor isomorphic to  $U_{2,5}$ ,  $U_{3,5}$ ,  $F_7$ , or  $F_7^*$ .*

**2.1. Connectivity.** Suppose that  $M$  is a matroid on the ground set  $E$ . If  $X \subseteq E$ , then  $\lambda_M(X)$  (or just  $\lambda(X)$ ) is defined to be  $r_M(X) + r_M(E - X) - r(M)$ . Note that  $\lambda(X) = \lambda(E - X)$  and  $\lambda_M(X) = \lambda_{M^*}(X)$  for all subsets  $X \subseteq E$ . A *k-separation* of  $M$  is a partition  $(X_1, X_2)$  of  $E$  such

that  $|X_1|, |X_2| \geq k$ , and  $\lambda_M(X_1) < k$ . A  $k$ -separation  $(X_1, X_2)$  is *exact* if  $\lambda_M(X_1) = k - 1$ . We say that  $M$  is  $n$ -*connected* if it has no  $k$ -separations where  $k < n$ . A 2-connected matroid is often said to be *connected*. We say that  $M$  is *internally 4-connected* if  $M$  is 3-connected, and, whenever  $(X_1, X_2)$  is a 3-separation,  $\min\{|X_1|, |X_2|\} = 3$ .

**Proposition 2.3.** *Suppose that  $N$  is a minor of a matroid  $M$ , and that  $X$  is a subset of  $E(N)$ . Then  $\lambda_N(X) \leq \lambda_M(X)$ .*

Suppose that  $M_1$  and  $M_2$  are matroids on the ground sets  $E_1$  and  $E_2$  respectively, and that  $\mathcal{C}_i$  is the collection of circuits of  $M_i$  for  $i = 1, 2$ . If  $E_1 \cap E_2 = \emptyset$ , then the 1-sum of  $M_1$  and  $M_2$ , denoted by  $M_1 \oplus M_2$ , is defined to be the matroid with  $E_1 \cup E_2$  as its ground set and  $\mathcal{C}_1 \cup \mathcal{C}_2$  as its collection of circuits.

If  $E_1 \cap E_2 = \{p\}$  and neither  $M_1$  nor  $M_2$  has  $p$  as a loop or a coloop, then we can define the 2-sum of  $M_1$  and  $M_2$ , denoted by  $M_1 \oplus_2 M_2$ . The ground set of  $M_1 \oplus_2 M_2$  is  $(E_1 \cup E_2) - p$ , and its circuits are the members of

$$\begin{aligned} & \{C \in \mathcal{C}_1 \mid p \notin C\} \cup \{C \in \mathcal{C}_2 \mid p \notin C\} \cup \\ & \{(C_1 \cup C_2) - p \mid C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2, p \in C_1 \cap C_2\}. \end{aligned}$$

We say that  $p$  is the *basepoint* of the 2-sum.

The next results follow from [25, (2.6)] and [19, Proposition 7.1.15 (v)] respectively.

**Proposition 2.4.** *If  $(X_1, X_2)$  is an exact 2-separation of a matroid  $M$ , then there are matroids  $M_1$  and  $M_2$  on the ground sets  $X_1 \cup p$  and  $X_2 \cup p$  respectively, where  $p$  is in neither  $X_1$  nor  $X_2$ , such that  $M$  is equal to  $M_1 \oplus_2 M_2$ . Moreover,  $M$  has proper minors isomorphic to both  $M_1$  and  $M_2$ .*

**Proposition 2.5.** *Suppose that  $M_1$  and  $M_2$  are matroids and that the 2-sum of  $M_1$  and  $M_2$  along the basepoint  $p$  is defined. If  $N_i$  is a minor of  $M_i$  such that  $p \in E(N_i)$  for  $i = 1, 2$ , and  $p$  is a loop or coloop in neither  $N_1$  nor  $N_2$ , then  $N_1 \oplus_2 N_2$  is a minor of  $M_1 \oplus_2 M_2$ .*

**2.2. Relaxations.** Suppose that  $M_1$  and  $M_2$  are matroids sharing a common ground set, and that the collections of bases of  $M_1$  and  $M_2$  agree with the exception of a single set  $Z$  that is a circuit-hyperplane in  $M_1$  and a basis in  $M_2$ . In this case we say that  $M_2$  is obtained from  $M_1$  by *relaxing* the circuit-hyperplane  $Z$ .

Next we list some well-known properties of relaxation.

**Proposition 2.6.** *Suppose that  $M_2$  is obtained from  $M_1$  by relaxing the circuit-hyperplane  $Z$ . If  $e \in Z$  then  $M_1 \setminus e = M_2 \setminus e$ . Moreover,  $Z - e$  is a hyperplane of  $M_1/e$ , and  $M_2/e$  is obtained from  $M_1/e$  by relaxing  $Z - e$ . Similarly, if  $e \notin Z$  then  $M_1/e = M_2/e$  and  $Z$  is a hyperplane of  $M_1 \setminus e$ . Then  $M_2 \setminus e$  is obtained from  $M_1 \setminus e$  by relaxing  $Z$ .*

If  $M_1$  and  $M_2$  are matroids on the same set such that  $M_1 \neq M_2$ , then there is a some set that is independent in exactly one of  $M_1$  and  $M_2$ . We shall call such a set a *distinguishing set*. The next result is obvious.

**Proposition 2.7.** *Suppose that  $M_1$  and  $M_2$  are two matroids on the same ground set and that  $Z$  is a minimal distinguishing set for  $M_1$  and  $M_2$ . Then  $Z$  is a circuit in one of  $M_1$  and  $M_2$ , and independent in the other.*

**Proposition 2.8.** *Let  $M_1$  and  $M_2$  be loopless matroids such that  $E(M_1) = E(M_2)$  and  $r(M_1) = r(M_2)$ . Suppose that  $M_1$  and  $M_2$  have a unique distinguishing set  $Z$ , and that  $Z$  is independent in  $M_2$ . Then  $Z$  is a circuit-hyperplane of  $M_1$  and a basis of  $M_2$ , and  $M_2$  is obtained from  $M_1$  by relaxing  $Z$ . Furthermore,  $Z \cup e$  is a circuit of  $M_2$  for all  $e \in E(M_2) - Z$ .*

*Proof.* As  $Z$  is the unique distinguishing set, it is also a minimal distinguishing set. Therefore  $Z$  is a circuit of  $M_1$  by Proposition 2.7. If  $Z$  is not a basis of  $M_2$ , then  $Z$  is properly contained in a basis  $B$  of  $M_2$ . Since  $Z \subset B$ , we deduce that  $B$  is dependent in  $M_1$ , and we have a contradiction to the uniqueness of  $Z$ . Thus  $Z$  is a basis of  $M_2$ .

Suppose that there is an element  $y$  in  $\text{cl}_{M_1}(Z) - Z$ . Then there is a circuit  $C$  of  $M_1$  such that  $y \in C$  and  $C \subseteq Z \cup \{y\}$ . Since  $C \neq Z$  and  $C$  is dependent in  $M_1$ , it follows that  $C$  is dependent in  $M_2$ . But  $Z \cup \{y\}$  contains a unique circuit  $C_{M_2}(y, Z)$  of  $M_2$ . Therefore  $C_{M_2}(y, Z) \subseteq C$ . As  $y$  is not a loop, it follows that there is an element  $e$  in  $C_{M_2}(y, Z) - \{y\}$ . By circuit elimination in  $M_1$  using the circuits  $C$  and  $Z$  and the common element  $e$ , we deduce that there is a circuit  $C'$  of  $M_1$  such that  $y \in C'$  and  $C' \subseteq (Z \cup \{y\}) - \{e\}$ . Now  $C' \neq Z$ , so  $C'$  is dependent in  $M_2$ . We can again conclude that  $C_{M_2}(y, Z) \subseteq C'$ . But this is a contradiction as  $e \notin C'$ . Therefore  $Z$  is a flat of  $M_1$ . As  $|Z| = r(M_2) = r(M_1)$ , it follows that  $Z$  is a circuit-hyperplane of  $M_1$ .

The independent sets of the matroid obtained from  $M_1$  by relaxing  $Z$  are precisely the independent sets of  $M_1$ , along with  $Z$ . This is exactly the collection of independent sets of  $M_2$ , so  $M_2$  is obtained from  $M_1$  by relaxing  $Z$ . Suppose that  $e \in E(M_2) - Z$ . As  $Z$  is a basis of  $M_2$ , there is a circuit  $C$  of  $M_2$  such that  $e \in C$  and  $C \subseteq Z \cup e$ . Since  $C \neq Z$ , the set  $C$  cannot be distinguishing. Therefore  $C$  is dependent in  $M_1$ . But the only circuit of  $M_1$  that is contained in  $Z \cup e$  is  $Z$  itself. Therefore  $C$  contains  $Z$ , so  $C = Z \cup e$ . This completes the proof.  $\square$

Recall that  $\mathcal{W}_n$  is the graph obtained from the cycle on  $n$  vertices by adding a new vertex adjacent to all other vertices. The edges adjacent to the new vertex are known as *spoke* edges, and all other edges are known as *rim* edges. We refer to  $M(\mathcal{W}_n)$  as the *rank- $n$  wheel*. The rim edges form a circuit-hyperplane of the rank- $n$  wheel. The matroid produced by relaxing this circuit-hyperplane is the *rank- $n$  whirl*, denoted by  $\mathcal{W}^n$ .

An *enlarged wheel* is obtained by adding parallel elements to spoke edges, and adding series elements to rim edges by subdividing them. The rim

edges of the original graph, along with all the added series elements, form a circuit-hyperplane of the enlarged wheel, and this set of edges is called the *rim* of the enlarged wheel.

The following result of Oxley and Whittle characterizes when a relaxation of a ternary matroid is ternary.

**Lemma 2.9.** [16, Theorem 5.3]. *Suppose that  $M$  is a ternary matroid and that  $Z$  is a circuit-hyperplane of  $M$ . Let  $M'$  be the matroid obtained from  $M$  by relaxing  $Z$  in  $M$ . If  $M'$  is ternary, then there is an enlarged wheel  $G$  such that  $M = M(G)$  and  $Z$  is the rim of  $G$ .*

**2.3. The Splitter Theorem.** Suppose that  $\mathcal{N}$  is a class of matroids that is closed under taking minors. A *splitter* of  $\mathcal{N}$  is a matroid  $N \in \mathcal{N}$  such that if  $N'$  is a 3-connected member of  $\mathcal{N}$  and  $N'$  has an  $N$ -minor, then  $N'$  is isomorphic to  $N$ .

Seymour's Splitter Theorem [25] reduces the problem of identifying splitters to a finite case check (see [19, Theorem 11.1.2]).

**Theorem 2.10.** *Let  $N$  be a 3-connected proper minor of a 3-connected matroid  $M$  and suppose that  $|E(N)| \geq 4$ . Assume also that if  $N$  is a wheel, then  $M$  has no larger wheel as a minor, while if  $N$  is a whirl, then  $M$  has no larger whirl as a minor. Then  $M$  has an element  $e$  such that  $M \setminus e$  or  $M/e$  is 3-connected and has an  $N$ -minor.*

**2.4. The  $\Delta$ -Y operation.** Suppose that  $M$  is a matroid and that  $T$  is a coindependent triangle of  $M$ . Let  $N$  be an isomorphic copy of  $M(K_4)$ , where  $E(N) \cap E(M) = T$  and  $T$  is a triangle of  $N$ . Then  $P_T(N, M)$ , the *generalized parallel connection* of  $N$  and  $M$ , is defined [4]. It is the matroid on the ground set  $E(M) \cup E(N)$  with flats being all sets  $F$  such that  $F \cap E(M)$  and  $F \cap E(N)$  are flats of  $M$  and  $N$  respectively. Then  $P_T(N, M) \setminus T$  is said to be obtained from  $M$  by performing a  $\Delta$ -Y operation upon  $M$ . We denote this matroid by  $\Delta_T(M)$ . If  $T$  is an independent triad of  $M$ , then  $(\Delta_T(M^*))^*$  is defined, and is said to be obtained from  $M$  by a Y- $\Delta$  operation. The resulting matroid is denoted by  $\nabla_T(M)$ .

**2.5. Regular decomposition.** We shall make use of some of the intermediate results proved by Seymour [25] as part of his decomposition theorem for regular matroids.

**Theorem 2.11.** *Every regular matroid can be constructed using 1-, 2-, and 3-sums, starting from matroids that are graphic, cographic, or isomorphic copies of  $R_{10}$ .*

The following matrix is a reduced representation of  $R_{10}$ .

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Any single-element deletion of  $R_{10}$  is isomorphic to  $M(K_{3,3})$  and any single-element contraction is isomorphic to  $M^*(K_{3,3})$ . Moreover, the automorphism group of  $R_{10}$  acts transitively upon pairs of elements, and  $R_{10}$  is isomorphic to its dual [25, p. 328].

**Proposition 2.12.** [25, (7.4)]. *The matroid  $R_{10}$  is a splitter for the class of regular matroids.*

The proof of the decomposition theorem features another important binary matroid,  $R_{12}$ . The following matrix,  $A$ , is a reduced representation of  $R_{12}$ .

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Clearly  $R_{12}$  is self-dual. Suppose that the columns of  $[I_6|A]$  are labeled  $1, \dots, 12$ . Then  $X_1 = \{1, 2, 5, 6, 9, 10\}$  is a union of two triangles, and if we let  $X_2$  be the complement of  $X_1$ , then  $(X_1, X_2)$  is a 3-separation of  $R_{12}$ . Moreover, if  $M$  is a regular matroid and  $R_{12}$  is a minor of  $M$ , then there is a 3-separation  $(Y_1, Y_2)$  of  $M$  such that  $X_i \subseteq Y_i$  for  $i = 1, 2$  (see [25, (9.2)]).

One of the important steps in the decomposition theorem is to prove the following result.

**Lemma 2.13.** [25, (14.2)]. *If a 3-connected regular matroid has no minor isomorphic to  $R_{10}$  or  $R_{12}$ , then it is either graphic or cographic.*

**2.6. Almost-regular matroids.** Next we discuss Truemper's class of almost-regular matroids [27]. Recall that a matroid is regular if and only if it can be represented by a matrix over the real numbers with the property that every subdeterminant belongs to  $\{0, 1, -1\}$ . Such a matrix is said to be *totally unimodular*. If a matrix is not totally unimodular, but removing any row or column produces a totally unimodular matrix, then it is said to be a *minimal violation matrix* for total unimodularity. The study of this class of matrices motivated Truemper to make the following definition.

**Definition 2.14.** A matroid  $M$  is *almost-regular* if it is binary but not regular, and  $E(M)$  can be partitioned into non-empty sets  $del$  and  $con$ , such that

- (i) if  $e \in del$  then  $M \setminus e$  is regular;
- (ii) if  $e \in con$  then  $M/e$  is regular;
- (iii) the intersection of any circuit with  $con$  has even cardinality; and
- (iv) the intersection of any cocircuit with  $del$  has even cardinality.

Truemper shows that the study of minimal violation matrices for total unimodularity is essentially reduced to the study of almost-regular matroids



(see [28, Section 12.4]). Any such matrix that does not represent an almost-regular matroid (over  $\text{GF}(2)$ ) belongs to one of two simple classes.

**Proposition 2.15.** [27, Theorem 21.4 (ii)]. *The class of almost-regular matroids is closed under duality.*

**Proposition 2.16.** [27, Theorem 21.4 (iii)]. *Suppose that  $M$  is an almost-regular matroid. Then every minor of  $M$  is either regular or almost-regular.*

The focus of Truemper's investigation into almost-regular matroids is the class of almost-regular matroids that are irreducible. An almost-regular matroid  $M$  is *irreducible* if  $M$  cannot be reduced in size by performing a sequence of the following operations: (i)  $\Delta$ - $Y$  and  $Y$ - $\Delta$  operations; and (ii) replacing a parallel (series) class with a non-empty parallel (series) class of a different size. (Note that certain restrictions are placed upon these operations. The restrictions depend upon the partition of the ground set into *del* and *con*.) An irreducible almost-regular matroid is necessarily internally 4-connected [27, Theorem 22.1].

The main result of [27] shows that every almost-regular matroid can be constructed using a sequence of the operations listed above, starting from one of two matroids:  $F_7$  and  $N_{11}$ . The second of these matroids is defined in Section 7.1.

**2.7. Grafts.** Suppose that  $G$  is a graph and that  $D$  is a set of vertices of  $G$ . We say that the pair  $(G, D)$  is a *graft*. Let  $A$  be the vertex-edge incidence matrix describing  $G$ , so that the rows of  $A$  correspond to vertices of  $G$ , and columns of  $A$  correspond to edges. Then  $M(G) = M[A]$ , where  $A$  is considered as a matrix over  $\text{GF}(2)$ . Let  $A'$  be the matrix obtained from  $A$  by adding a column with entries from  $\text{GF}(2)$ , so that an entry in the new column is non-zero if and only if it appears in a row corresponding to a vertex in  $D$ . Let  $M(G, D)$  be the binary matroid  $M[A']$ . We abuse terminology slightly by calling any binary matroid of the form  $M(G, D)$  a *graft*. We shall call the element of  $M(G, D)$  that corresponds to the new column of  $A'$  the *graft element*. Clearly a binary matroid is a graft if and only if it is a single-element extension of a graphic matroid.

The next result is easy to verify.

**Proposition 2.17.** *Suppose that  $(G, D)$  is a graft. Let  $e$  be an edge of  $G$  with end-vertices  $u$  and  $v$ . Then  $M(G, D) \setminus e = M(G \setminus e, D)$ . Furthermore, suppose that  $w$  is the vertex of  $G/e$  produced by identifying  $u$  and  $v$ . Then  $M(G, D)/e = M(G/e, D')$ , where:*

- (i)  $D' = D$  if  $|\{u, v\} \cap D| = 0$ ;
- (ii)  $D' = (D - \{u, v\}) \cup w$  if  $|\{u, v\} \cap D| = 1$ ; and
- (iii)  $D' = D - \{u, v\}$  if  $|\{u, v\} \cap D| = 2$ .

Let  $(G, D)$  be a graft. Suppose that  $v$  is a vertex of degree two in  $G$  and that  $v \in D$ . Suppose that  $v$  is adjacent to the two vertices  $u$  and  $w$ . Let  $a$  be the edge between  $v$  and  $u$ , and let  $b$  be the edge between  $v$  and  $w$ .

Consider the graph  $G'$  with the following properties:  $G'$  has the same edge set as  $G$ , and  $a$  joins  $v$  to  $w$  in  $G'$ , while  $b$  joins  $v$  to  $u$ . All other edges have the same incidences as they do in  $G$ . Let  $D'$  be the symmetric difference of  $D$  and  $\{u, w\}$ . Then  $M(G', D') = M(G, D)$ . We say that  $(G', D')$  is obtained from  $(G, D)$  by *switching*.

**2.8. Truemper graphs.** In this section we introduce a family of graphs that provide an important tool for studying almost-regular matroids.

**Definition 2.18.** A graph  $G$  is a *Truemper graph* if it contains two vertex-disjoint paths  $R$  and  $S$ , such that every vertex of  $G$  is in either  $R$  or  $S$ , and any edge not in either  $R$  or  $S$  joins a vertex of  $R$  to a vertex of  $S$ .

We shall use the notation  $G = (R, S)$  to indicate that  $G$  is a Truemper graph, and that  $R$  and  $S$  are the vertex-disjoint paths described in Definition 2.18. In this case we shall say that an edge in either  $R$  or  $S$  is a *path edge*, and any other edge is a *cross edge*. We shall say that the end-vertices of  $R$  and  $S$  are *terminal vertices*. All other vertices will be known as *internal vertices*. Often we are interested in a graft  $(G, D)$ , where  $G$  is a Truemper graph, and  $D$  consists of the four terminal vertices of  $G$ . However, much of our argument will focus on structure in the underlying Truemper graph.

Let  $G = (R, S)$  be a Truemper graph. We say that  $G$  has an XX-minor if we can obtain the graph shown in Figure 2 by contracting path edges and deleting cross edges from  $G$ . The remaining path edges of  $G$  are the horizontal edges in the diagram.

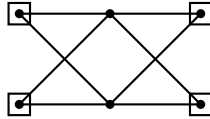


FIGURE 2. An XX-minor.

**Proposition 2.19.** [27, 23.50]. *Suppose that  $G = (R, S)$  is a Truemper graph. Let  $D$  be the set of terminal vertices of  $G$ . If the graft  $M(G, D)$  is almost-regular, then  $G$  does not have an XX-minor.*

*Proof.* Assume that  $G$  does have an XX-minor. Proposition 2.17 implies that  $M(G, D)$  has  $M(G', D)$  as a minor, where  $G'$  is the graph shown in Figure 2, and  $D$  is the set of vertices marked by squares. But  $M(G', D)$  has a minor isomorphic to  $AG(3, 2)$ . Certainly  $AG(3, 2)$  is not regular, and every single-element deletion or contraction of  $AG(3, 2)$  is isomorphic to  $F_7^*$  or  $F_7$  respectively. Therefore  $AG(3, 2)$  is not almost-regular. Proposition 2.16 implies that  $M(G, D)$  cannot be almost-regular.  $\square$

The next result is easy to prove.

**Proposition 2.20.** *Let  $G = (R, S)$  be a Truemper graph with no XX-minor such that both  $R$  and  $S$  contain at least two vertices. Suppose that  $F$  is a set*

of four cross edges such that every terminal vertex of  $G$  is incident with at least one edge in  $F$ . Then at least one edge in  $F$  joins two terminal vertices.

**Corollary 2.21.** *Let  $G = (R, S)$  be a Truemper graph with no XX-minor such that both  $R$  and  $S$  have at least two vertices. Suppose that the cross edges of  $G$  form a spanning cycle. Then one of the following holds:*

- (i) *One of the end-vertices of  $R$  is adjacent to both of the end-vertices of  $S$ .*
- (ii) *One of the end-vertices of  $S$  is adjacent to both of the end-vertices of  $R$ .*

*Proof.* Suppose that the result fails. Since every vertex in  $G$  is incident with exactly two cross edges, this means that for each terminal vertex  $v$ , we can find a cross edge which joins  $v$  to an internal vertex. This provides a contradiction to Proposition 2.20.  $\square$

**2.9. Excluded minors.** We end this preliminary section by proving one direction of our main theorem.

**Lemma 2.22.** *The matroids  $U_{2,5}$ ,  $U_{3,5}$ ,  $U_{2,4} \oplus F_7$ ,  $U_{2,4} \oplus F_7^*$ ,  $U_{2,4} \oplus_2 F_7$ ,  $U_{2,4} \oplus_2 F_7^*$ ,  $AG(3, 2)'$ , and  $T'_{12}$  are all excluded minors for  $\mathcal{M}$ .*

*Proof.* The only matroids listed here for which the result is not obvious are  $AG(3, 2)'$  and  $T'_{12}$ . Let  $M_1$  be a matroid such that  $M_1 \cong AG(3, 2)$  and let  $Z$  be a circuit-hyperplane of  $M_1$ . Let  $M_2$  be the matroid obtained from  $M_1$  by relaxing  $Z$ . Suppose that  $e \in Z$ . By Proposition 2.6, we see that  $M_2 \setminus e \cong AG(3, 2) \setminus e \cong F_7^*$  and that  $M_2/e$  can be obtained from  $AG(3, 2)/e \cong F_7$  by relaxing a circuit-hyperplane. Therefore  $M_2/e \cong F_7^-$ , where  $F_7^-$  is illustrated in Figure 3. Since  $F_7^-$  is non-binary, these facts show that  $M_2$  is neither binary nor ternary.

On the other hand, if  $e \notin Z$  then  $M_2/e = M_1/e \cong F_7$ , and  $M_2 \setminus e$  is isomorphic to the matroid obtained from  $AG(3, 2) \setminus e \cong F_7^*$  by relaxing a circuit-hyperplane. Thus  $M_2 \setminus e \cong (F_7^-)^*$ , so every single-element deletion or contraction of  $M_2$  is either binary or ternary, and we are done.

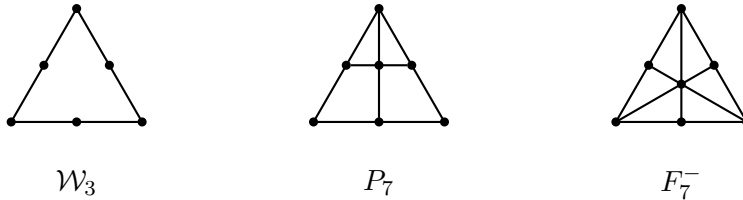


FIGURE 3.  $W_3$ ,  $P_7$ , and  $F_7^-$ .

Now we will assume that  $M_1$  is isomorphic to  $T_{12}$ . Assume that the columns of the matrix in Figure 1 are labeled  $\{1, \dots, 12\}$ . Then  $Z = \{2, 4, 6, 8, 10, 12\}$  is a circuit-hyperplane. Let  $M_2$  be the matroid obtained

by relaxing  $Z$ . Certainly  $M_2$  is not binary, for  $Z \cup \{1\}$  and  $Z \cup \{3\}$  are circuits of  $M_2$ . If  $M_2$  were binary, then the symmetric difference of these sets, that is  $\{1, 3\}$ , would be a union of circuits, which is clearly not true.

By pivoting on the entry in column 7 and row 2, we see that  $M_1/\{1, 3, 7\} \setminus \{2, 12\}$  is isomorphic to  $F_7$ , so  $M_1 \setminus 12$  has an  $F_7$ -minor. Therefore  $M_2 \setminus 12$  has an  $F_7$ -minor, by Proposition 2.6, so  $M_2$  is not ternary.

Proposition 2.6 implies that  $M_2 \setminus 6 = M_1 \setminus 6$ , so  $M_2 \setminus 6$  is binary. Consider  $M_2/6$ . It is not difficult to show that this matroid is represented over  $\text{GF}(3)$  by the matrix produced by deleting row 6 from the matrix in Figure 1. Thus  $M_2/6$  is ternary. Now suppose that  $e$  is any element in  $\{1, \dots, 12\}$ . Since the automorphism group of  $T_{12}$  is transitive, there is an automorphism which takes  $e$  to 6. Thus  $M_2 \setminus e$  and  $M_2/e$  are isomorphic to  $M_2 \setminus 6$  and  $M_2/6$ , and are therefore binary and ternary respectively. It follows that  $M_2$  is an excluded minor for  $\mathcal{M}$ , as desired.  $\square$

### 3. EXCLUDED MINORS WITH LOW RANK, CORANK, OR CONNECTIVITY

In this section we find all the excluded minors for  $\mathcal{M}$  that have rank or corank at most three, or that fail to be 3-connected.

**Proposition 3.1.** *If  $M$  is an excluded minor for  $\mathcal{M}$ , then  $M$  cannot have as a minor either a simple connected single-element extension of  $F_7$  or a cosimple connected single-element coextension of  $F_7^*$ .*

*Proof.* It follows from the fact that  $F_7$  is a projective plane that it has exactly two simple connected single-element extensions; one is obtained by adding an element freely to  $F_7$ , and the other is obtained by adding an element freely on a line of  $F_7$ . In either case, on contracting the newly added element, we obtain a matroid with a  $U_{2,5}$ -restriction, a contradiction as  $U_{2,5}$  is an excluded minor for  $\mathcal{M}$ . Hence  $M$  has no simple connected single-element extension of  $F_7$  as a minor. The second part of the lemma follows by duality.  $\square$

**Lemma 3.2.** *The only excluded minors for  $\mathcal{M}$  that have rank or corank less than four are  $U_{2,5}$  and  $U_{3,5}$ .*

*Proof.* It is clear that  $U_{2,5}$  is the only rank-2 excluded minor for  $\mathcal{M}$ . By duality,  $U_{3,5}$  is the unique excluded minor for  $\mathcal{M}$  with corank two. Now let  $M$  be a rank-3 excluded minor for  $\mathcal{M}$  that is not isomorphic to  $U_{3,5}$ . Since  $M$  is non-ternary and has rank three, it follows from Theorem 2.2 that  $M$  has  $F_7$  as a minor. But  $M$  is non-binary and simple, and so has a simple connected single-element extension of  $F_7$  as a restriction. This contradiction to Proposition 3.1 implies that  $U_{3,5}$  is the unique rank-3 excluded minor for  $\mathcal{M}$  and, by duality,  $U_{2,5}$  is the unique excluded minor for  $\mathcal{M}$  with corank three.  $\square$

**Lemma 3.3.** *The only excluded minors for  $\mathcal{M}$  that are not 3-connected are  $U_{2,4} \oplus F_7$ ,  $U_{2,4} \oplus F_7^*$ ,  $U_{2,4} \oplus_2 F_7$ , and  $U_{2,4} \oplus_2 F_7^*$ .*

*Proof.* We shall show that the excluded minors for  $\mathcal{M}$  that are connected but not 3-connected are  $U_{2,4} \oplus_2 F_7$  and  $U_{2,4} \oplus_2 F_7^*$ . A similar, but simpler, argument shows that the disconnected excluded minors for  $\mathcal{M}$  are precisely  $U_{2,4} \oplus F_7$  and  $U_{2,4} \oplus F_7^*$ .

Let  $M$  be an excluded minor for  $\mathcal{M}$  that is connected but not 3-connected. It follows from Proposition 2.4 that  $M$  is the 2-sum of matroids  $M_1$  and  $M_2$  along the basepoint  $p$ . Then  $M_1$  and  $M_2$  are connected, for otherwise  $M$  is not connected. Each of  $M_1$  and  $M_2$ , being isomorphic to a proper minor of  $M$ , is either binary or ternary. Moreover, since the property of being representable over a particular field is closed under 2-sums, it follows that at least one of  $M_1$  and  $M_2$  is non-binary, and at least one is non-ternary. Thus we may assume that  $M_1$  is ternary but non-binary, and that  $M_2$  is binary but non-ternary. Thus  $M_1$  has a  $U_{2,4}$ -minor, and  $M_2$  has a minor isomorphic to one of  $U_{2,5}$ ,  $U_{3,5}$ ,  $F_7$ , or  $F_7^*$ . Both  $U_{2,5}$  and  $U_{3,5}$  are excluded minors for  $\mathcal{M}$ . Thus neither is a minor of  $M_2$ . Hence  $M_2$  has a minor isomorphic to one of  $F_7$  and  $F_7^*$ . It follows from roundedness results of Seymour [23] and Bixby [2] (or see [19, p. 374]), that  $M_2$  has an  $F_7$ - or  $F_7^*$ -minor using  $p$ , and  $M_1$  has a  $U_{2,4}$ -minor using  $p$ . Thus  $M$  has a minor isomorphic to one of  $U_{2,4} \oplus_2 F_7$  or  $U_{2,4} \oplus_2 F_7^*$  by Proposition 2.5. Since these two matroids are excluded minors for  $\mathcal{M}$ , it follows that  $M$  is isomorphic to either  $U_{2,4} \oplus_2 F_7$  or  $U_{2,4} \oplus_2 F_7^*$ . This completes the proof.  $\square$

#### 4. EXCLUDED MINORS WITH AT MOST NINE ELEMENTS

In this section we find those excluded minors for  $\mathcal{M}$  that have at most nine elements.

**Lemma 4.1.** *There is a unique 8-element excluded minor for  $\mathcal{M}$ , namely  $\text{AG}(3, 2)'$ .*

*Proof.* Let  $M$  be an 8-element excluded minor for  $\mathcal{M}$ . Thus  $M$  has no  $U_{2,5}$ -minor and no  $U_{3,5}$ -minor. Moreover,  $M$  must be 3-connected by Lemma 3.3. It follows from Lemma 3.2 that  $r(M) \geq 4$  and  $r^*(M) \geq 4$ , so in fact  $r(M) = r^*(M) = 4$ . We shall show next that  $M$  has no triangles and no triads. By duality, it suffices to show that  $M$  has no triads.

Assume that  $M$  has a triad,  $T$ . Certainly  $T$  is independent, for  $M$  is 3-connected. Suppose that  $T = \{a, b, c\}$ . Note that  $T$  is a triangle in  $\nabla_T(M)$ . Now  $\nabla_T(M)$  has rank three (see [15, Lemma 2.6]). Moreover, since  $M$  is neither binary nor ternary, it follows by the proof of Theorem 6.1 in [1] that  $\nabla_T(M)$  is neither binary nor ternary. Lemma 3.2 implies that  $\nabla_T(M)$  has a minor isomorphic to either  $U_{2,5}$  or  $U_{3,5}$ . If  $\nabla_T(M)$  has a  $U_{3,5}$ -minor, then, as  $U_{3,5}$  has no triangles, we can assume by relabeling if necessary that  $\nabla_T(M) \setminus a$  has a  $U_{3,5}$ -minor. It follows that  $M/a$  has a  $U_{3,5}$ -minor ([15, Corollary 2.14]). This is a contradiction, so  $\nabla_T(M)$  has no  $U_{3,5}$ -minor but it does have a  $U_{2,5}$ -minor. Note that  $\text{si}(\nabla_T(M))$  has rank three. Suppose that the corank of  $\text{si}(\nabla_T(M))$  is at most two. Then  $\text{si}(\nabla_T(M))$  contains at most five elements. Since we can assume that  $T$  is a triangle of  $\text{si}(\nabla_T(M))$ ,

it follows that  $\text{si}(\nabla_T(M))$  is not 3-connected. If  $\text{si}(\nabla_T(M))$  has rank and corank at least three, then it follows from [19, Proposition 11.2.16] that  $\text{si}(\nabla_T(M))$  is not 3-connected, so  $\text{si}(\nabla_T(M))$  is not 3-connected in either case. Thus  $\nabla_T(M)$  is the union of two rank-2 flats, one of which contains  $\{a, b, c\}$ . Since  $M = \Delta_T(\nabla_T(M))$  [15, Corollary 2.12], it is easy to see that  $M$  also fails to be 3-connected, a contradiction. We conclude that  $M$  has no triads.

Now  $M$  is non-ternary but has no  $U_{2,5}$ - or  $U_{3,5}$ -minor. Thus  $M$  has  $F_7$  or  $F_7^*$  as a minor. By duality, we may assume that  $M$  has an  $F_7^*$ -minor. Let us assume that  $E(M) = \{1, \dots, 8\}$  and that  $M \setminus 8 \cong F_7^*$ . Consider  $M/8$ . Since  $M$  is non-binary and 3-connected, and  $M \setminus 8$  is binary, it follows from [18, Corollary 3.9] that if  $M/8$  is binary, then  $M \cong U_{2,4}$ , which is impossible. Therefore  $M/8$  is non-binary and hence ternary. Since  $M$  has no triangles and no  $U_{3,5}$ -minors, we see that  $M/8$  is simple and has no rank-2 flat containing more than three points. This implies that  $M/8$  is 3-connected. Since  $M/8$  has  $\mathcal{W}^2$  (that is,  $U_{2,4}$ ) as a minor but has no  $U_{2,5}$ - or  $U_{3,5}$ -minor, we deduce from Theorem 2.10 that  $M/8$  has a  $\mathcal{W}^3$ -minor. Thus  $M/8$  is a 3-connected and ternary single-element extension of  $\mathcal{W}^3$  and  $M/8$  has no lines with more than three points. We will show that  $M/8$  is isomorphic to either  $P_7$  or  $F_7^-$ , where these matroids are illustrated in Figure 3.

Let us suppose that  $M/8 \setminus 7 \cong \mathcal{W}^3$ . Since matroid representations over  $\text{GF}(3)$  are unique [6], we can assume that  $M/8 \setminus 7$  has the following reduced representation.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

By adjoining a single column to this matrix, we can obtain a representation over  $\text{GF}(3)$  of  $M/8$ . This new column must contain three non-zero elements, for  $M/8$  is 3-connected and has no four-element lines. By scaling we may assume that the first entry is 1. If the new column is  $[1 \ 1 \ 1]^T$ , then  $M/8$  is isomorphic to  $F_7^-$ . In all other cases,  $M/8 \cong P_7$ .

Suppose that  $M/8 \cong P_7$ . Since  $M/8$  has two disjoint triangles,  $M$  has two 4-element circuits meeting in  $\{8\}$ . These circuits must also be hyperplanes of  $M$ , as  $M$  has no triangle and no  $U_{3,5}$ -minor. Deleting 8 from each of these two circuit-hyperplanes produces two disjoint hyperplanes of  $F_7^*$  of size three. Thus  $F_7$  has two 4-element circuits whose union covers its ground set, and this is easily seen to be false. Hence  $M/8 \cong F_7^-$ .

Since  $F_7^-$  has exactly six non-trivial lines, there are exactly six 4-element circuits of  $M$  that contain 8. Each of these must also be a hyperplane of  $M$ . Thus  $M$  has exactly six 4-element cocircuits that avoid 8. Each of these cocircuits is also a 4-element cocircuit of  $M \setminus 8 \cong F_7^*$ . But  $F_7^*$  has exactly seven 4-element cocircuits. Thus precisely one of the 4-element cocircuits of  $M \setminus 8$  arises by deleting 8 from a 5-element cocircuit of  $M$ . We may assume,

without loss of generality, that  $\{4, 5, 6, 7, 8\}$  is a cocircuit of  $M$ . As the complement of a 4-element cocircuit is independent in  $F_7^*$ , we can assume that  $\{1, 2, 3, 4\}$  is a basis of  $M$ .

When  $B$  is a basis of a matroid  $N$ , consider a matrix  $[I_{r(N)}|A]$ , where the columns of  $I_{r(N)}$  and of  $A$  are labeled by the elements of  $B$  and by the elements of  $E(N) - B$ , respectively. We call  $[I_{r(N)}|A]$  a *partial representation* of  $N$  with respect to  $B$  if, for each  $x$  in  $B$  and each  $y$  in  $E(N) - B$ , the entry in row  $x$  and column  $y$  of  $A$  is a one if  $(B - x) \cup y$  is a basis of  $N$ , and a zero otherwise.

Let  $[I_4|A]$  be a partial representation of  $M$  with respect to  $\{1, 2, 3, 4\}$ . The fact that  $\{4, 5, 6, 7, 8\}$  is a cocircuit of  $M \setminus 8$  means that each of the entries in  $A$  in the row associated with 4 and in a column labeled by 5, 6, 7 or 8 must be one. Note that, as  $M \setminus 8$  is binary, the matrix produced by deleting the column labeled by 8 from  $[I_4|A]$  actually represents  $M \setminus 8$  over  $\text{GF}(2)$ . Each column labeled by 5, 6, or 7 must contain at least three ones, as  $M \setminus 8 \cong F_7^*$  has no triangles. However,  $M \setminus 8$  has no circuits of size five, so each of these columns contains exactly three ones. Now we can assume that  $[I_4|A]$  is the matrix shown in Figure 4. As  $M$  has no triads, each of  $x_1, x_2$ , and  $x_3$  must be equal to one.

$$\begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \left[ \begin{array}{cccc|cccc}
 & & & & & 0 & 1 & 1 & x_1 \\
 & & & & & 1 & 0 & 1 & x_2 \\
 & & & & & 1 & 1 & 0 & x_3 \\
 & & & & & 1 & 1 & 1 & x_4 \\
 \end{array} \right.
 \end{array}$$

FIGURE 4. A partial representation of  $M$ .

For each  $e$  in  $\{1, 2, 3\}$ , the matroid  $M \setminus 8/e \cong M(K_4)$ . Thus  $M/e$  is a binary or ternary extension of  $M(K_4)$  with no 4-element lines, so  $M/e$  is isomorphic to  $F_7$  or  $F_7^-$ . Because  $\{1, 2, 3, 8\}$  is not a circuit of  $M$ , it follows that  $M/e \cong F_7^-$  for each  $e \in \{1, 2, 3\}$ . Using this, one easily checks that the following six sets must be circuits of  $M$ :

$\{1, 4, 5, 8\}, \{1, 6, 7, 8\}, \{2, 4, 6, 8\}, \{2, 5, 7, 8\}, \{3, 4, 7, 8\}, \{3, 5, 6, 8\}$ .

In addition, all seven 4-element circuits of  $M \setminus 8$  are also circuits of  $M$ . We have now described thirteen 4-element circuits of  $M$ . If this is the complete list of 4-element circuits of  $M \setminus 8$ , then it is easy to see that  $M \cong \text{AG}(3, 2)'$ . Therefore assume that  $C$  is a 4-element circuit of  $M$  that is not one of the thirteen circuits we have described. Obviously  $8 \in C$ . If our only choice of  $C$  is  $C = \{1, 2, 3, 8\}$ , then  $M$  must be isomorphic to  $\text{AG}(3, 2)$ , a contradiction. Therefore we assume that  $C \neq \{1, 2, 3, 8\}$ . Now  $|C \cap \{1, 2, 3, 8\}| \neq 3$ , otherwise  $M$  has a  $U_{3,5}$ -restriction. Similarly,  $|C \cap \{4, 5, 6, 7\}| \neq 3$ . Thus  $C$  contains 8, a single element from  $\{1, 2, 3\}$ , and a single element from  $\{4, 5, 6, 7\}$ . We can again find a 4-element circuit that meets  $C$  in three

elements, and deduce the presence of a  $U_{3,5}$ -minor. This contradiction completes the proof.  $\square$

Our next task is to eliminate the excluded minors on nine elements. We need some preliminary facts.

**Proposition 4.2.** *Suppose that  $M$  is a 3-connected excluded minor for  $\mathcal{M}$ . For every element  $e \in E(M)$ , either  $M \setminus e$  or  $M/e$  is ternary.*

*Proof.* Suppose that, for some element  $e$  of  $M$ , neither  $M \setminus e$  nor  $M/e$  is ternary. Then both  $M \setminus e$  and  $M/e$  are binary. Thus  $M$  is isomorphic to  $U_{2,4}$  by a result of Oxley's [18, Corollary 3.9]. This contradiction completes the proof.  $\square$

**Proposition 4.3.** *Suppose that  $M$  is a 3-connected excluded minor for  $\mathcal{M}$ . Then  $M$  has no minor isomorphic to  $\text{AG}(3, 2)$ .*

*Proof.* For every element  $e$  of  $\text{AG}(3, 2)$ , the matroids  $\text{AG}(3, 2) \setminus e$  and  $\text{AG}(3, 2)/e$  are isomorphic to  $F_7^*$  and  $F_7$ , respectively. As neither of the last two matroids is ternary, the result follows by Proposition 4.2.  $\square$

The binary matroid  $S_8$  is represented over  $\text{GF}(2)$  by the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Clearly  $S_8$  is self dual. Seymour [26] proved the following result.

**Proposition 4.4.** *The only 3-connected binary single-element coextensions of  $F_7$  are  $\text{AG}(3, 2)$  and  $S_8$ .*

**Proposition 4.5.** *Suppose that  $M$  is a 3-connected excluded minor for  $\mathcal{M}$  and that  $|E(M)| \geq 9$ . Then  $M$  has  $S_8$  as a minor.*

*Proof.* The hypotheses imply that  $M$  has no minor isomorphic to  $U_{2,5}$  or  $U_{3,5}$ . As  $M$  is non-ternary it must have either an  $F_7$ -minor or a  $F_7^*$ -minor. The Splitter Theorem (2.10) implies that  $M$  has a minor  $M_1$  such that  $M_1$  is a 3-connected single-element extension or coextension of either  $F_7$  or  $F_7^*$ . Proposition 3.1 implies that  $M_1$  is an extension of  $F_7^*$  or a coextension of  $F_7$ . If  $M_1$  is non-binary, then  $M_1$  is both non-binary and non-ternary, so  $M_1 = M$  and hence  $|E(M)| = 8$ , contradicting our assumption. Therefore  $M_1$  is binary and so, by Propositions 4.4 and 4.3,  $M_1$  is isomorphic to  $S_8$ .  $\square$

**Lemma 4.6.** *Suppose that  $M$  is a 3-connected excluded minor for  $\mathcal{M}$ . Then  $|E(M)| \neq 9$ .*

*Proof.* Assume that  $E(M) = \{1, \dots, 9\}$ . Lemma 3.2 implies that the rank and corank of  $M$  both exceed three. By duality we may assume that  $r(M) = 4$ . Proposition 4.5 implies that  $M$  has an  $S_8$ -minor, so assume that  $M \setminus 9 \cong S_8$ . Thus  $M$  has the partial representation shown in Figure 5.



$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\left[ \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & x_1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & x_2 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & x_3 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & x_4
\end{array} \right]
\end{array}$$

FIGURE 5. A partial representation for  $M$ .

Let  $M_B$  be the binary matroid for which this partial representation is a  $\text{GF}(2)$ -representation. Clearly  $M \setminus 9 = M_B \setminus 9$ . Furthermore,  $M \setminus 8 \setminus 9 = M_B \setminus 8 \setminus 9 \cong F_7^*$ , so  $M \setminus 8$  is non-ternary. Thus  $M \setminus 8$  is binary, so  $M \setminus 8 = M_B \setminus 8$ . Moreover,  $M \setminus 9 / 1 = M_B \setminus 9 / 1 \cong F_7$ . Therefore  $M / 1$  is non-ternary, and hence binary, so  $M / 1 = M_B / 1$ .

Recall that a distinguishing set for  $M$  and  $M_B$  is some set  $Z \subseteq \{1, \dots, 9\}$  such that  $Z$  is independent in one of  $M$  and  $M_B$  and dependent in the other. Let  $Z$  be such a distinguishing set. The arguments above show that

$$(4.1) \quad \{8, 9\} \subseteq Z \subseteq E(M) - \{1\}.$$

Suppose that  $M_B$  is not simple. As 9 is not a loop of  $M$ , it follows that 9 is in a parallel pair  $P$  in  $M_B$ . As  $M$  contains no parallel pairs, we deduce that  $P$  is a distinguishing set for  $M$  and  $M_B$ , so (4.1) implies that  $P = \{8, 9\}$ . Thus  $(x_1, x_2, x_3, x_4)^T = (1, 1, 1, 1)$ . Now  $\{2, 7, 9\}$  and  $\{3, 6, 9\}$  are triangles of  $M_B \setminus 8 = M \setminus 8$ . Moreover,  $\{2, 7, 8\}$  and  $\{3, 6, 8\}$  are triangles of  $M_B \setminus 9 = M \setminus 9$ . Let  $A = \{2, 7, 8, 9\}$  and let  $B = \{3, 6, 8, 9\}$ . Then  $r_M(A) = r_M(B) = 2$ . Moreover,  $r_M(A \cup B) > 2$ , otherwise  $M|(A \cup B) \cong U_{2,6}$ . Now

$$r_M(\{8, 9\}) = r_M(A \cap B) \leq r_M(A) + r_M(B) - r_M(A \cup B) \leq 1,$$

so  $M$  contains a parallel pair, a contradiction.

We may now assume that  $M_B$  is simple. Let  $Z$  be a minimal distinguishing set for  $M$  and  $M_B$ . By symmetry, there are three possibilities for  $(x_1, x_2, x_3, x_4)^T$ :

- (i)  $(0, 1, 1, 1)$ ;
- (ii)  $(0, 0, 1, 1)$ ; and
- (iii)  $(1, 1, 0, 0)$ .

In the first case,  $M_B \setminus 8 = M \setminus 8$  is isomorphic to  $\text{AG}(3, 2)$ , contradicting Proposition 4.3. Suppose that case (ii) holds. Note that  $\{2, 7, 8\}$  is a circuit of  $M_B$ , and as it avoids 9, it is also a circuit of  $M$ . Hence

$$M / 2 \setminus 7 \cong M / 2 \setminus 8 = M_B / 2 \setminus 8 \cong F_7.$$

Thus  $M / 2 \setminus 7$  is non-ternary, so  $M / 2$  and  $M \setminus 7$  are non-ternary and hence binary. Therefore  $M / 2 = M_B / 2$  and  $M \setminus 7 = M_B \setminus 7$ . Hence  $7 \in Z$  but  $2 \notin Z$ , so  $\{7, 8, 9\} \subseteq Z \subseteq \{3, \dots, 9\}$ . Suppose that  $|Z| = 3$ . Then  $Z = \{7, 8, 9\}$ . As  $Z$  is not a triangle of  $M_B$ , it follows that  $Z$  is independent in  $M_B$  and

a triangle in  $M$ . As  $\{2, 7, 8\}$  is a triangle in  $M_B \setminus 9 = M \setminus 9$ , we see that  $\{2, 7, 8, 9\}$  is a rank-2 flat of  $M$ . Thus  $M/2$  contains a parallel class of size three. But we concluded above that  $M/2 \setminus 7 \cong F_7$ , so we have a contradiction. Therefore  $|Z| = 4$ . There is no 4-element dependent set in  $M_B$  that contains  $\{7, 8, 9\}$ , so  $Z$  is a basis of  $M_B$ . Proposition 2.7 implies that  $Z$  is a 4-element circuit of  $M$ . Now  $\{2, 7, 8\}$  is a circuit of  $M_B$  and of  $M$ , and  $Z = \{7, 8, 9, x\}$  for some element  $x \in \{3, 4, 5, 6\}$ . By circuit elimination in  $M$ , there is a circuit  $C$  of  $M$  contained in  $\{2, 7, 9, x\}$ . Since this circuit does not contain 8, it is also a circuit of  $M_B$ . But there is no 3- or 4-element circuit in  $M_B$  containing  $\{2, 7, 9\}$ . The only 3-element circuits of  $M_B$  containing two of 2, 7, and 9 are  $\{2, 7, 8\}$  and  $\{1, 7, 9\}$ . But  $x \notin \{1, 8\}$ , so we have a contradiction.

Now we suppose that case (iii) holds. We note that  $\{1, 4, 6, 7\}$  is a basis of  $M_B$  and hence of  $M$ , and the fundamental circuits of  $M$  and  $M_B$  with respect to this basis are the same since no such circuit can contain  $\{8, 9\}$ . Thus the matrix in Figure 6 is a representation for  $M_B$  and a partial representation for  $M$ .

$$\begin{array}{cccccccc} 1 & 6 & 7 & 4 & 5 & 2 & 3 & 8 & 9 \\ \left[ \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \end{array}$$

FIGURE 6. A partial representation for  $M$ .

Since  $M_B/7$  has  $\{2, 8\}$  as a circuit, so does  $M/7$ . Thus

$$M/7 \setminus 2 \cong M/7 \setminus 8 = M_B/7 \setminus 8 \cong F_7.$$

Hence  $M/7 \setminus 2$  is non-ternary. Therefore  $M/7$  and  $M \setminus 2$  are binary, so  $M/7 = M_B/7$  and  $M \setminus 2 = M_B \setminus 2$ . It follows that  $2 \in Z$  and that  $Z \subseteq \{2, 3, 4, 5, 6, 8, 9\}$ .

Suppose that  $|Z| = 3$ , so that  $Z = \{2, 8, 9\}$ . As  $\{2, 8, 9\}$  is independent in  $M_B$ , we see that  $Z$  is a triangle in  $M$ . As  $\{2, 7, 8\}$  is also a triangle of  $M$ , it follows that  $M/7 \setminus 2$  cannot be isomorphic to  $F_7$ , a contradiction.

We know now that  $|Z| = \{2, 8, 9, x\}$  for some  $x \in \{3, 4, 5, 6\}$ . By circuit exchange in  $M$  between  $Z$  and  $\{2, 7, 8\}$ , we conclude that  $\{2, 7, 9, x\}$  contains a circuit of  $M \setminus 8 = M_B \setminus 8$ . But the only circuits of  $M_B$  that meet  $\{2, 7, 9\}$  in more than one element are  $\{1, 2, 9\}$  and  $\{2, 7, 8\}$ . As  $x \notin \{1, 8\}$ , we have arrived at a contradiction that completes the proof.  $\square$

In the light of Lemmas 3.2, 3.3, 4.1, and 4.6, we need now only characterize the excluded minors for  $\mathcal{M}$  that are 3-connected with rank and corank at least four, and which have a ground set containing at least ten elements. In the next section we begin to move towards this goal.

## 5. A STRUCTURE THEOREM FOR EXCLUDED MINORS

The following theorem is the main result of this section.

**Theorem 5.1.** *Let  $M$  be a 3-connected excluded minor for  $\mathcal{M}$  such that  $|E(M)| \geq 10$  and both the rank and corank of  $M$  exceed three. Then there is a 3-connected binary matroid  $M_B$  such that  $E(M_B) = E(M)$  and:*

- (i) *There are disjoint circuit-hyperplanes  $J$  and  $K$  in  $M_B$  such that  $E(M_B) = J \cup K$ ;*
- (ii)  *$M$  is obtained from  $M_B$  by relaxing  $J$ ; and*
- (iii) *The matroid  $M_T$  that is obtained from  $M_B$  by relaxing  $J$  and  $K$  is ternary.*

Before we prove Theorem 5.1, we discuss some preliminary facts. The binary matroid  $P_9$  is a 3-connected extension of  $S_8$ , and is represented over  $\text{GF}(2)$  by the matrix in Figure 7.

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \left[ \begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \end{array}$$

FIGURE 7. A representation of  $P_9$ .

The following fact follows from [17, Lemma (2.6)].

**Proposition 5.2.** *Every binary 3-connected single-element extension of  $S_8$  is either isomorphic to  $P_9$  or has an  $\text{AG}(3, 2)$ -minor.*

**Proposition 5.3.** *Suppose that  $M$  is a 3-connected excluded minor for  $\mathcal{M}$  and that  $|E(M)| \geq 10$ . Then  $M$  has either  $P_9$  or  $P_9^*$  as a minor.*

*Proof.* Proposition 4.5 implies that  $M$  has a minor  $M_1$  isomorphic to  $S_8$ . Now the Splitter Theorem implies that  $M$  has a minor  $M_2$  that is a 3-connected extension or coextension of  $S_8$ . If  $M_2$  is non-binary, then  $M_2$  is both non-binary and non-ternary, so  $M_2 = M$  and hence  $|E(M)| = 9$ . This is a contradiction, so  $M_2$  is binary. Thus, by Propositions 5.2 and 4.3 and duality, we see that  $M_2$  is isomorphic to either  $P_9$  or  $P_9^*$ .  $\square$

*Proof of Theorem 5.1.* Let  $M$  be a 3-connected excluded minor for  $\mathcal{M}$  such that  $r(M), r(M^*) \geq 4$  and  $|E(M)| \geq 10$ . By duality, we may assume that  $r(M) \leq r^*(M)$ . By Proposition 5.3,  $M$  has a minor  $N$  that is isomorphic to  $P_9$  or  $P_9^*$ . Suppose that  $N = M \setminus X/Y$ , where we may assume that  $Y$  is independent and that  $X$  is coindependent in  $M$ . As  $|E(M)| \geq 10$ , it follows that  $X \cup Y$  is non-empty. We assume that  $N$  has ground set  $\{1, 2, \dots, 9\}$  and that if  $N$  is  $P_9$ , then  $N$  is represented over  $\text{GF}(2)$  by the matrix in Figure 7, while if  $N$  is  $P_9^*$ , then  $N$  is represented over  $\text{GF}(2)$  by the matrix

in Figure 8. We note, for future reference, that  $P_9/1\setminus 7$ ,  $P_9/1\setminus 9$ ,  $P_9/2\setminus 7$ , and  $P_9/2\setminus 8$  are all isomorphic to  $F_7$ .

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \left[ \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \end{array}$$

FIGURE 8. A representation of  $P_9^*$ .

We wish to fix a basis  $B$  and a cobasis  $B'$  of  $N$ . If  $N = P_9$ , we choose  $B = \{1, 2, 3, 4\}$ , and  $B' = \{5, 6, 7, 8, 9\}$ , while if  $N = P_9^*$ , we choose  $B = \{5, 6, 7, 8, 9\}$ , and  $B' = \{1, 2, 3, 4\}$ . Now  $Y \cup B$  is a basis of  $M$ . Suppose that  $[I_r|A]$  is a partial representation of  $M$  with respect to the basis  $B$ . Let  $M_B$  be the binary matroid represented over  $\text{GF}(2)$  by  $[I_r|A]$ . The rest of the proof involves showing that  $M_B$  has the properties specified in the theorem. Note that there must be at least one subset of  $E(M)$  that is independent in one of  $M$  and  $M_B$ , and dependent in the other. Recall that we call any such set a distinguishing set.

**Lemma 5.4.** *If  $x \in X \cup B'$  and  $M \setminus x$  is binary, then  $M \setminus x = M_B \setminus x$ . If  $y \in Y \cup B$  and  $M/y$  is binary, then  $M/y = M_B/y$ .*

*Proof.* Suppose  $x \in X \cup B'$  and that  $M \setminus x$  is binary. Then, by deleting the column of  $[I_r|A]$  labeled by  $x$ , we obtain a partial representation for  $M \setminus x$ . Since  $M \setminus x$  is binary, this matrix in fact represents  $M \setminus x$  over  $\text{GF}(2)$ . It also represents  $M_B \setminus x$  over  $\text{GF}(2)$ , so  $M \setminus x = M_B \setminus x$ . The second statement follows by a similar argument.  $\square$

**Lemma 5.5.** *There are subsets  $X' \subseteq B'$  and  $Y' \subseteq B$  of  $E(M)$  with the following properties:*

- (i)  $X \subseteq X' \subseteq E(M) - Y' \subseteq E(M) - Y$ ;
- (ii)  $E(M) - (X' \cup Y') = \{3, 4, 5, 6\}$ ;
- (iii) if  $x \in X'$ , then  $M \setminus x$  is non-ternary and  $M \setminus x = M_B \setminus x$ ;
- (iv) if  $y \in Y'$ , then  $M/y$  is non-ternary and  $M/y = M_B/y$ ;
- (v)  $|X'| = r^*(M) - 2$ ;
- (vi)  $|Y'| = r(M) - 2$ ;
- (vii) if  $e \in E(M) - (X' \cup Y')$ , then  $X' \cup Y' \cup \{e\}$  spans both  $M$  and  $M_B$ ; and
- (viii) if  $Z$  is a distinguishing set for  $M$  and  $M_B$  then  $X' \subseteq Z \subseteq E(M) - Y'$ .

*Proof.* We first consider the case that  $N \cong P_9$ . Let  $X' = X \cup \{7, 8, 9\}$ , and let  $Y' = Y \cup \{1, 2\}$ . Then (i) and (ii) are certainly true. Suppose that  $x \in X$ . Then  $M \setminus x$  has a  $P_9$ -minor, and as  $P_9$  has an  $F_7$ -minor, it follows that  $M \setminus x$  is non-ternary, and therefore binary. The fact that  $M \setminus x = M_B \setminus x$

follows from Lemma 5.4. Moreover, if  $x \in \{7, 8, 9\}$ , then  $N \setminus x$  has an  $F_7$ -minor, so  $M \setminus x$  is non-ternary, and hence binary. Therefore (iii) holds. A similar argument shows that (iv) holds. Statement (viii) follows immediately from (iii) and (iv). As  $N$  has rank four and corank five, it follows that  $|X| = r^*(M) - 5$  and  $|Y| = r(M) - 4$ . Thus (v) and (vi) are immediate. To see that (vii) is true, note that  $M \setminus X/Y = M_B \setminus X/Y = P_9$ . Since  $\{3, 4, 5, 6\}$  is a cocircuit of  $P_9$ , it follows that if  $e \in \{3, 4, 5, 6\}$ , then  $\{1, 2, 7, 8, 9, e\}$  contains a basis of  $N$ . Thus  $\{1, 2, 7, 8, 9, e\} \cup Y$  contains a basis  $B_0$  of  $M$ . Suppose that  $B_0$  is not a basis of  $M_B$ . Then there is a minimal distinguishing set  $Z$  for  $M$  and  $M_B$  such that  $Z \subseteq B_0$  and  $Z$  is independent in  $M$  and dependent in  $M_B$ . Part (viii) shows that  $\{7, 8, 9\} \subseteq Z$ , but  $Z$  does not contain any element in  $\{1, 2\} \cup Y$ . It follows that there is a circuit of  $M_B \setminus X/Y = N$  that is contained in  $\{7, 8, 9, e\}$ . But no such circuit exists, so  $B_0$  is a basis of both  $M$  and  $M_B$ . Therefore (vii) holds.

In the case that  $N \cong P_9^*$ , we set  $X'$  to be  $X \cup \{1, 2\}$  and  $Y'$  to be  $Y \cup \{7, 8, 9\}$ . Then similar arguments show that the lemma holds.  $\square$

For the rest of the proof,  $X'$  and  $Y'$  refer to the sets described in Lemma 5.5. We will make frequent use of the following fact.

**Proposition 5.6.** *Suppose that  $\{M_1, M_2\} = \{M, M_B\}$  and that  $C$  is a circuit of  $M_1$ . If  $C$  does not contain  $X'$ , then it is also a circuit of  $M_2$ .*

*Proof.* Suppose that  $C$  does not contain  $X'$ . Then  $C$  cannot be a distinguishing set by Lemma 5.5 (viii). Since  $C$  is dependent in  $M_1$ , it must therefore be dependent in  $M_2$ . If  $C$  is not a circuit of  $M_2$ , it properly contains a circuit  $C'$  of  $M_2$ . Now  $C'$  is independent in  $M_1$ , so it is a distinguishing set of  $M_1$  and  $M_2$ . However,  $C'$  does not contain  $X'$ , so we have a contradiction to Lemma 5.5 (viii). Therefore  $C$  is a circuit of  $M_2$ .  $\square$

**Lemma 5.7.** *Suppose that  $Z$  is a distinguishing set for  $M$  and  $M_B$ . Then  $|Z| = r(M) = r(M_B)$ .*

*Proof.* Let  $r$  be the common rank of  $M$  and  $M_B$ , and let  $r^* = r^*(M) = r^*(M_B)$ . Suppose that  $Z$  is a distinguishing set for  $M$  and  $M_B$  and that  $|Z| \neq r$ . Obviously  $|Z| \leq r$ , so  $|Z| \leq r - 1$ . Let  $\{M_1, M_2\} = \{M, M_B\}$ , where we assume that  $Z$  is dependent in  $M_1$  and independent in  $M_2$ . We can assume that  $Z$  is a minimal distinguishing set, so  $Z$  is in fact a circuit of  $M_1$ .

Lemma 5.5 (v) and (vii) imply that  $X' \subseteq Z$ , and  $|X'| = r^* - 2$ . Therefore  $r^* - 2 \leq r - 1$ . But we have assumed that  $r \leq r^*$ , so  $r^* \in \{r, r + 1\}$ . Hence  $|X'| \in \{r - 2, r - 1\}$ .

Note that  $Z \cap Y' = \emptyset$  by Lemma 5.5 (viii). Suppose that  $y \in Y'$ . Then  $M_1/y = M_2/y$  by Lemma 5.5 (iv). As  $Z$  is dependent in  $M_1/y$ , it follows that  $Z$  is dependent in  $M_2/y$ , so  $Z \cup y$  is dependent in  $M_2$ . As  $Z$  is independent in  $M_2$ , this means that  $y \in \text{cl}_{M_2}(Z)$ . Therefore  $Y' \subseteq \text{cl}_{M_2}(Z)$ .

Suppose that  $Z \neq X'$ . Then  $|X'| = r - 2$  and  $|Z| = r - 1$ , so  $Z - X'$  contains a unique element  $z$ . Moreover,  $z \notin X' \cup Y'$  by Lemma 5.5 (vii). We

have already shown that  $Y' \subseteq \text{cl}_{M_2}(Z)$ , so Lemma 5.5 (vii) implies that  $Z$  is spanning in  $M_2$ . This is a contradiction since  $|Z| < r(M_2)$ . We conclude that  $Z = X'$ .

Now suppose that  $y \in Y'$  and that  $y \in \text{cl}_{M_1}(Z)$ . Then there is a circuit  $C \subseteq Z \cup y$  such that  $y \in C$ . Note that  $C$  does not contain  $X' = Z$ , as  $Z$  is a circuit of  $M_1$ . Proposition 5.6 implies that  $C$  is a circuit of  $M_2$ . The fact that  $M$  and  $M_B$  are loopless means that  $C \neq \{y\}$ , so there is an element  $e \in X' \cap C$ .

By circuit elimination between  $Z$  and  $C$  in  $M_1$ , there is a circuit  $C'$  of  $M_1$  such that  $y \in C'$  and  $C' \subseteq (Z - e) \cup y$ . As  $C'$  does not contain  $e$ , Proposition 5.6 implies that  $C'$  is a circuit of  $M_2$ . Now  $C$  and  $C'$  are circuits of  $M_2$  contained in  $Z \cup y$ , and  $C \neq C'$  as  $e \notin C'$ . But  $Z$  is independent in  $M_2$ , so this leads to a contradiction. This shows that  $\text{cl}_{M_1}(Z) \cap Y' = \emptyset$ .

We have shown that if  $y \in Y'$  then  $y \in \text{cl}_{M_2}(Z)$ . In fact, we can prove something stronger: that  $Z \cup y$  is a circuit of  $M_2$ . Suppose that this is not the case. Then there is a circuit  $C$  that is properly contained in  $Z \cup y$ , such that  $y \in C$ . Certainly  $C$  does not contain  $X' = Z$ , so  $C$  is a circuit of  $M_1$ . Therefore  $y \in \text{cl}_{M_1}(Z)$ , contrary to our earlier conclusion. Thus  $Z \cup y$  is indeed a circuit of  $M_2$ .

We know that  $|Y'| \geq 2$ , so let  $y$  and  $y'$  be distinct members of  $Y'$ . Then  $Z \cup y$  and  $Z \cup y'$  are circuits of  $M_2$ . If  $M_2$  is binary, then  $\{y, y'\}$  contains a circuit of  $M_2$ . But  $Y'$  is contained in the common basis  $B$  so this leads to a contradiction. Therefore  $M_2 \neq M_B$ , so  $M_1 = M_B$  and  $M_2 = M$ .

Suppose that  $r^* = r$ . Then  $|Z| = |X'| = r^* - 2 = r - 2$ . As  $Z$  is independent in  $M_2$ , it follows that  $r_{M_2}(Z) = r - 2$ . Moreover  $r - 2 = r_{M_2}(Z \cup Y') = r_{M_2}(X' \cup Y')$ , from our earlier conclusion that  $Y' \subseteq \text{cl}_{M_2}(Z)$ . However, it follows from Lemma 5.2 (vii) that  $r_{M_2}(X' \cup Y') \geq r - 1$ . Therefore we have a contradiction, and we conclude that  $r^* = r + 1$ , so  $|Z| = r - 1$ . Thus  $r_{M_2}(Z) = r - 1$ .

Let  $W = E(M) - (X' \cup Y') = \{3, 4, 5, 6\}$ . We already know that  $\text{cl}_{M_1}(Z)$  does not meet  $Y'$ . Suppose that there is some element  $w \in W$  such that  $w \in \text{cl}_{M_1}(Z)$ . Then there is a circuit  $C$  of  $M_1$  such that  $C \subseteq Z \cup w$  and  $w \in C$ . As  $Z$  is a circuit of  $M_1$ , it follows that there is an element  $z \in Z$  such that  $z \notin C$ . Therefore  $C$  does not contain  $X' = Z$ , so  $C$  is a circuit of  $M_2$ . Thus  $\text{cl}_{M_2}(Z)$  contains  $w$  and  $Y'$ , and therefore  $Z$  is spanning in  $M_2$ . But this is a contradiction as  $r_{M_2}(Z) = r - 1$ . We conclude that  $Z$  is a flat of  $M_1$ .

Recall that  $M_1 = M_B$  and that  $Z$  is a circuit and a flat of  $M_1$  with cardinality  $r - 1$ . Consider  $M_1/Z$ . This is a loopless rank-2 binary matroid on the ground set  $W \cup Y'$ . Obviously  $M_1/Z$  contains no more than three parallel classes. As  $|W| = 4$ , we deduce that some parallel class of  $M_1/Z$  contains  $w$  and  $w'$ , two distinct elements of  $W$ . Therefore there is a circuit  $C$  of  $M_1$  such that  $C \subseteq Z \cup \{w, w'\}$  and  $w, w' \in C$ . Note that  $C$  must meet  $Z$ , for  $w$  and  $w'$  are not parallel in  $M_B \setminus X/Y = N$ , so they are not parallel in  $M_1$ .

Let  $C' = (Z - C) \cup \{w, w'\}$ . Since  $M_1$  is binary,  $C'$ , which is the symmetric difference of  $C$  and  $Z$ , is a disjoint union of circuits of  $M_1$ . Any circuit in  $C'$  that contains  $w$  must also contain  $w'$ , for  $w \notin \text{cl}_{M_1}(Z)$ . Note that  $C' \cap Z$  is a proper subset of  $Z$ , as  $C \cap Z$  is non-empty. These observations imply that  $C'$  must in fact be a circuit of  $M_1$ . Moreover,  $C' \cap Z$  is non-empty, as  $C$  cannot contain the circuit  $Z$ .

Both  $C$  and  $C'$  are circuits of  $M_2$  since neither contains  $Z$ . Thus  $M_2$  has a circuit contained in  $(C \cup C') - w'$ . This circuit must contain  $w$ , so  $w \in \text{cl}_{M_2}(Z)$ . Hence, by Lemma 5.5 (vii),  $Z$  is spanning in  $M_2$ ; a contradiction.  $\square$

**Corollary 5.8.** *Both  $M$  and  $M_B$  are simple.*

*Proof.* Certainly  $M$  is simple as it is 3-connected and  $|E(M)| \geq 10$ . If  $M_B$  contains a circuit of at most two elements, then that set contains a distinguishing set. But Lemma 5.7 implies that any distinguishing set has cardinality at least four.  $\square$

**Corollary 5.9.** *Suppose that  $Z$  is a distinguishing set of  $M$  and  $M_B$ . Then  $Z$  is a circuit in one of  $M$  and  $M_B$ , and a basis in the other. Moreover,  $r^*(M) \in \{r(M), r(M) + 1, r(M) + 2\}$ .*

*Proof.* It follows from Lemma 5.7 that any distinguishing set of  $M$  and  $M_B$  is in fact a minimal distinguishing set. The fact that  $Z$  is a circuit in one of  $M$  and  $M_B$  and a basis in the other now follows easily.

Lemma 5.5 implies that  $X' \subseteq Z$  and that  $|X'| = r^*(M) - 2$ . Thus  $r^*(M) - 2 \leq |Z| = r(M)$  by Lemma 5.7. The corollary follows from our assumption that  $r(M) \leq r^*(M)$ .  $\square$

We now set to the task of showing that  $M$  and  $M_B$  have a unique distinguishing set.

**Lemma 5.10.** *Let  $\{M_1, M_2\} = \{M, M_B\}$  and suppose that the distinguishing set  $Z$  is a circuit in  $M_1$  and a basis in  $M_2$ . Then*

- (i) *if  $M_1$  is binary, then  $Z$  is a hyperplane of  $M_1$ ; and*
- (ii) *if  $Z$  is not a hyperplane of  $M_1$ , then  $r(M) = r^*(M)$ .*

*Moreover  $|\text{cl}_{M_1}(Z)| \leq |Z| + 1$ .*

*Proof.* Let  $r = r(M)$  and let  $r^* = r^*(M)$ . We note that  $X' \subseteq Z$  and that  $|X'| = r^* - 2$  by Lemma 5.5. Corollary 5.9 states that  $r^* \in \{r, r + 1, r + 2\}$ . Note that  $|Z - X'| = r - r^* + 2$ , so  $r^* = r$  if and only if  $Z - X'$  contains exactly two elements. We prove the following claim:

**5.10.1.** *Suppose that  $y$  is in  $\text{cl}_{M_1}(Z) - Z$ . Then  $(Z - X') \cup y$  is a circuit of both  $M_1$  and  $M_2$ .*

*Proof.* There is a circuit  $C$  of  $M_1$  such that  $y \in C$  and  $C \subseteq Z \cup y$ . Assume that  $C$  is not a circuit of  $M_2$ . If  $C$  is a distinguishing set, then  $X' \subseteq C$ . On the other hand, if  $C$  is not a distinguishing set, then  $C$  is dependent in  $M_2$

and  $C$  must properly contain a circuit  $C'$  of  $M_2$ . Since  $C'$  is a proper subset of the circuit  $C$  of  $M_1$ , it follows that  $C'$  is a distinguishing set of  $M_1$  and  $M_2$ , and therefore  $X' \subseteq C'$ . Hence  $X' \subseteq C$  in either case.

Choose  $e$  in  $X'$ . Then  $e \in Z \cap C$ . By circuit elimination in  $M_1$ , there is a circuit  $C' \subseteq (Z - e) \cup y$  such that  $y \in C'$ . Note that  $C'$  does not contain  $X'$ , so  $C'$  is a circuit of  $M_2$  by Proposition 5.6. Therefore we can relabel  $C'$  with  $C$ , and assume that  $C$  is a circuit of both  $M_1$  and  $M_2$  such that  $C \subseteq Z \cup y$  and  $y \in C$ .

If  $C$  does not avoid  $X'$ , then  $C \cap X'$  contains an element  $e$  and, by circuit elimination in  $M_1$ , there is a circuit  $C'$  of  $M_1$  such that  $y \in C'$  and  $C' \subseteq (Z - e) \cup y$ . Since  $C'$  does not contain  $X'$ , Proposition 5.6 implies that  $C'$  is a circuit of  $M_2$ . Thus  $Z$  is independent in  $M_2$ , but  $Z \cup y$  contains two distinct circuits of  $M_2$ , namely  $C$  and  $C'$ . This contradiction means that  $C$  avoids  $X'$ , so  $C \subseteq (Z - X') \cup y$ . But  $|Z - X'| \leq 2$  and  $M_2$  is simple by Corollary 5.8. Thus  $C = (Z - X') \cup y$  is a circuit of both  $M_1$  and  $M_2$ .  $\square$

Suppose that  $Z$  is not a hyperplane of  $M_1$ . As  $r_{M_1}(Z) = |Z| - 1 = r - 1$ , there must be some element  $y$  in  $\text{cl}_{M_1}(Z) - Z$ . Then (5.10.1) implies that  $(Z - X') \cup y$  is a circuit of  $M_1$ . As  $|Z - X'| \leq 2$  and  $M_1$  is simple, we conclude that  $|Z - X'| = 2$ , and that therefore  $r = r^*$  by our earlier observation. We have shown that statement (ii) of the lemma holds.

Suppose that  $M_1 = M_B$ . Then  $M_1$  is binary and  $((Z - X') \cup y) \triangle Z$  is a disjoint union of circuits of  $M_1$ . Thus  $X' \cup y$  contains a circuit  $C'$  of  $M_1$  that contains  $y$ . Clearly  $|C'| \leq |X'| + 1 = r^* - 1 = r - 1$ . Therefore  $C'$  cannot be a distinguishing set by Lemma 5.7. Hence  $C'$  is dependent in  $M_2$ . If  $C'$  is not a circuit of  $M_2$ , then it properly contains a circuit of  $M_2$ , and this circuit is a distinguishing set with cardinality less than  $r$ , a contradiction. Therefore  $C'$  is a circuit of  $M_2$ . Note that  $C' \neq (Z - X') \cup y$ , so  $Z \cup y$  contains two distinct circuits of  $M_2$ . This is a contradiction. Therefore  $|\text{cl}_{M_1}(Z)| > |Z|$  implies that  $M_1$  is not binary. It follows that if  $M_1$  is binary, then  $Z$  is a hyperplane of  $M_1$ . This completes the proof of statement (i).

We may now assume that  $M_1$  is non-binary, so that  $M_2$  is binary. Suppose that  $y_1$  and  $y_2$  are distinct elements of  $\text{cl}_{M_1}(Z) - Z$ . Then  $(Z - X') \cup y_1$  and  $(Z - X') \cup y_2$  are both circuits of  $M_2$  by (5.10.1). By taking the symmetric difference of these circuits, we deduce that  $\{y_1, y_2\}$  is a disjoint union of circuits of  $M_2$ , and this is a contradiction. It follows that  $\text{cl}_{M_1}(Z)$  can contain at most one element not in  $Z$ . This completes the proof.  $\square$

**Lemma 5.11.** *Suppose that  $Z$  is a distinguishing set for  $M$  and  $M_B$ . Then  $Z$  is a circuit-hyperplane in  $M_B$  and a basis in  $M$ .*

*Proof.* Let  $\{M_1, M_2\} = \{M, M_B\}$ , and assume that  $Z$  is a circuit in  $M_1$  and a basis in  $M_2$ . If  $M_1$  is binary, then  $Z$  is a hyperplane of  $M_1$  by Lemma 5.10 and there is nothing left to prove, so we assume that  $M_1 = M$  and  $M_2 = M_B$ .



Note that  $Z$  does not meet  $Y'$  by Lemma 5.5 (viii). Suppose that  $Z$  is a hyperplane of  $M_1$ . Then

$$|Y' - \text{cl}_{M_1}(Z)| = |Y'| = r(M) - 2 \geq 2.$$

On the other hand, if  $Z$  is not a hyperplane of  $M_1$ , then  $r(M) = r^*(M)$  and  $|\text{cl}_{M_1}(Z)| \leq |Z| + 1$  by Lemma 5.10. In this case,  $r(M) = |E(M)|/2 \geq 5$ . Hence

$$|Y' - \text{cl}_{M_1}(Z)| \geq |Y'| - 1 = r(M) - 3 \geq 2.$$

In either case,  $Y' - \text{cl}_{M_1}(Z)$  contains distinct elements  $y_1$  and  $y_2$ .

Clearly  $Z$  is a circuit of  $M_1/y_i$  for  $i = 1, 2$ . Lemma 5.5 (iv) implies that  $M_1/y_i = M_2/y_i$ . Therefore  $Z$  is a circuit of  $M_2/y_i$  and, as  $Z$  is independent in  $M_2$ , this means that  $Z \cup y_i$  is a circuit of  $M_2$ . Therefore  $Z \cup y_1$  and  $Z \cup y_2$  are distinct circuits of the binary matroid  $M_2$ , so  $\{y_1, y_2\}$  is a union of circuits in  $M_2$ . This contradiction completes the proof.  $\square$

**Lemma 5.12.** *Suppose that  $Z_1$  and  $Z_2$  are distinct distinguishing sets for  $M$  and  $M_B$ . Then*

- (i)  $|Z_1| = |Z_2| = r(M) = r^*(M)$ ;
- (ii)  $Z_1 - X'$  and  $Z_2 - X'$  are disjoint sets;
- (iii)  $|Z_1 - X'| = |Z_2 - X'| = 2$ ;
- (iv)  $Z_1 \triangle Z_2 = \{3, 4, 5, 6\}$ ; and
- (v)  $Z_1 \triangle Z_2$  is a circuit of  $M$ .

*Proof.* Let  $r = r(M)$  and  $r^* = r^*(M)$ . From Lemma 5.11, we see that  $Z_1$  and  $Z_2$  are circuit-hyperplanes of  $M_B$ , so  $|Z_1| = |Z_2| = r$ . Moreover,  $X' \subseteq Z_i \subseteq E(M) - Y'$  by Lemma 5.5. Note that  $r^* - 2 = |X'| \leq |Z_i| = r$  for  $i = 1, 2$ . As  $r \leq r^*$ , this means that  $|Z_i - X'| \leq 2$ . Since

$$(5.1) \quad Z_1 \triangle Z_2 \subseteq (Z_1 - X') \cup (Z_2 - X'),$$

it follows that  $|Z_1 \triangle Z_2| \leq 4$ . Moreover  $|Z_1 \triangle Z_2|$  is even, as  $|Z_1| = |Z_2|$ .

Now  $Z_1 \triangle Z_2$  is a disjoint union of circuits in the simple matroid  $M_B$ . It follows that  $Z_1 \triangle Z_2$  is a circuit, and that  $|Z_1 \triangle Z_2| = 4$ . Equation (5.1) implies that  $Z_1 - X'$  and  $Z_2 - X'$  must be disjoint sets, each of cardinality two. Since

$$Z_i - X' \subseteq E(M) - (X' \cup Y') = \{3, 4, 5, 6\}$$

for  $i = 1, 2$ , we have that  $Z_1 \triangle Z_2 = \{3, 4, 5, 6\}$  is a circuit. As  $|Z_1 - X'| = |Z_2 - X'| = 2$ , it follows that  $|Z_1| = |Z_2| = |X'| + 2 = r^*$ , so we are done.  $\square$

**Proposition 5.13.** *Suppose that  $Z$  is a distinguishing set for  $M$  and  $M_B$ . If  $e \in E(M) - Z$ , then  $Z \cup e$  is a circuit of  $M$ .*

*Proof.* As  $Z$  is a basis of  $M$ , there is a circuit  $C$  of  $M$  such that  $e \in C$  and  $C \subseteq Z \cup e$ . If  $C$  is a distinguishing set for  $M$  and  $M_B$ , then it is a circuit of  $M_B$  by Lemma 5.11, implying that  $e \in \text{cl}_{M_1}(Z)$ . But  $Z$  is a hyperplane of  $M_1$ , so this is a contradiction. Hence  $C$  is not a distinguishing set, and is therefore dependent in  $M_B$ . But there is only one circuit of  $M_B$  that is

contained in  $Z \cup e$ , namely  $Z$  itself. Therefore  $C$  contains  $Z$ , so  $C = Z \cup e$  is a circuit of  $M$ , as desired.  $\square$

For the next step we will need a result due to Kahn and Seymour [11] (see [19, Lemma 10.3.7]).

**Lemma 5.14.** *Let  $N_1$  and  $N_2$  be matroids having distinct elements  $a$  and  $b$  such that the following conditions hold:*

- (i)  $N_1$  and  $N_2$  are distinct connected matroids having a common ground set;
- (ii)  $N_1 \setminus a = N_2 \setminus a$  and  $N_1 \setminus b = N_2 \setminus b$ ;
- (iii)  $N_1 \setminus a \setminus b = N_2 \setminus a \setminus b$  and this matroid is connected; and
- (iv)  $\{a, b\}$  is not a cocircuit of  $N_1$  or of  $N_2$ .

*Then at most one of  $N_1$  and  $N_2$  is ternary.*

**Lemma 5.15.** *There is a unique distinguishing set for  $M$  and  $M_B$ .*

*Proof.* Lemma 5.12 says that if  $Z_1$  and  $Z_2$  are distinct distinguishing sets, then  $Z_1 - X'$  and  $Z_2 - X'$  are disjoint sets of cardinality two. As both  $Z_1 - X'$  and  $Z_2 - X'$  are contained in  $\{3, 4, 5, 6\}$ , it follows that there can be no more than two distinguishing sets for  $M$  and  $M_B$ . Suppose that there are precisely two,  $Z_1$  and  $Z_2$ . We deduce from Lemma 5.12 that  $r(M) = r^*(M)$  and that  $Z_1 - X' = \{a, b\}$  while  $Z_2 - X' = \{c, d\}$ , where  $\{a, b, c, d\} = \{3, 4, 5, 6\}$ .

**5.15.1.** *If  $e \in E(M) - X'$  then  $M \setminus e$  is non-binary.*

*Proof.* Since  $X' = Z_1 \cap Z_2$ , we can assume without loss of generality that  $e \notin Z_1$ . Let  $a$  and  $b$  be distinct elements in  $E(M) - (Z_1 \cup e)$ . Proposition 5.13 implies that  $Z_1 \cup a$  and  $Z_1 \cup b$  are circuits of  $M \setminus e$ . If  $M \setminus e$  is binary, then this would imply that  $\{a, b\}$  is a union of circuits in  $M$ ; a contradiction. Therefore  $M \setminus e$  is non-binary.  $\square$

Suppose that  $e \in E(M) - Z_1$ . Then  $Z_1 \cup e$  is a circuit of  $M$  by Proposition 5.13. This observation means that if  $A$  is a proper subset of  $E(M) - Z_1$ , then  $M \setminus A$  is connected. Similarly, if  $A$  is a proper subset of  $E(M) - Z_2$ , then  $M \setminus A$  is connected.

We have shown in (5.15.1) that  $M \setminus a$  and  $M \setminus b$  are non-binary, and therefore ternary. Obviously  $M \setminus a \setminus b$  is ternary. Suppose that  $M \setminus a \setminus b$  is represented over  $\text{GF}(3)$  by the matrix  $[I_r | A]$ . It is known [6] that ternary matroids are uniquely representable over  $\text{GF}(3)$ . One consequence of this is that there are column vectors  $\mathbf{a}$  and  $\mathbf{b}$  over  $\text{GF}(3)$  such that  $[I_r | A | \mathbf{a}]$  and  $[I_r | A | \mathbf{b}]$  represent  $M \setminus b$  and  $M \setminus a$  respectively over  $\text{GF}(3)$ . Let  $M_T$  be the ternary matroid that is represented over  $\text{GF}(3)$  by the matrix  $[I_r | A | \mathbf{a} | \mathbf{b}]$ . Thus  $M_T \setminus a = M \setminus b$  and  $M_T \setminus b = M \setminus a$ .

Let  $e$  be an arbitrary element in  $Y'$ . We wish to show that  $M \setminus e = M_T \setminus e$ . We know that  $M \setminus e$  is non-binary, and hence ternary, by (5.15.1). Certainly  $M_T \setminus e \setminus a = M \setminus e \setminus a$  and  $M_T \setminus e \setminus b = M \setminus e \setminus b$ . Moreover, our earlier observation implies that  $M \setminus e$  and  $M \setminus e \setminus a$  are connected. If  $M_T \setminus e$  is not connected, then

$e$  must be a loop or a coloop in  $M_T \setminus a$ . This means that  $e$  is a loop of  $M_T$ , or  $\{a, e\}$  is a series pair in  $M_T$ . But  $M_T$  contains no loops as  $M$  contains no loops; and  $\{a, e\}$  is not a series pair of  $M_T$ , as  $M_T \setminus a = M \setminus a$  is connected. Therefore  $M_T$  is connected. The set  $\{a, b\}$  is not a cocircuit of either  $M_T \setminus e$ , or  $M \setminus e$ , for  $M \setminus e \setminus a = M_T \setminus e \setminus a$  and  $M \setminus e \setminus b = M_T \setminus e \setminus b$ , and both these matroids are connected. Finally,  $M \setminus e \setminus a \setminus b = M_T \setminus e \setminus a \setminus b$  and this matroid is connected, since  $Z_2$  avoids all of  $e, a$ , and  $b$ . We have shown that the hypotheses of Lemma 5.14 apply to  $M \setminus e$  and  $M_T \setminus e$ . Since  $M \setminus e$  and  $M_T \setminus e$  are both ternary, Lemma 5.14 implies that  $M \setminus e$  and  $M_T \setminus e$  are not distinct. Therefore  $M \setminus e = M_T \setminus e$ .

The matroids  $M$  and  $M_T$  are distinct as  $M$  is not ternary. Let  $Z$  be a minimal distinguishing set for  $M$  and  $M_T$ . We have deduced that  $M \setminus x = M_T \setminus x$  for every  $x \in Y' \cup \{a, b\} = E(M) - Z_2$ . Thus  $Y' \cup \{a, b\} \subseteq Z$ . But  $|Y'| = r^*(M) - 2 = r(M) - 2$ , so  $|Z| \geq r(M)$ . However,  $|Z| \leq r(M)$ , so  $|Z| = r(M)$ , and  $Z = Y' \cup \{a, b\}$ . Therefore there is a unique distinguishing set for  $M$  and  $M_T$ , and Proposition 2.8 implies that  $Z$  is a circuit-hyperplane in one of these matroids, and a basis in the other.

Suppose that  $Z$  is a basis of  $M$ . Then Proposition 2.8 states that  $Z \cup e$  is a circuit of  $M$  for all  $e$  in  $E(M) - Z$ . Since  $Z \cup e$  contains neither  $Z_1$  nor  $Z_2$ , we deduce that  $Z \cup e$  is a circuit of  $M_B$  for all  $e \in E(M) - Z$ . If  $e$  and  $e'$  are distinct elements in  $E(M) - Z$ , then  $\{e, e'\}$  is a union of circuits in  $M_B$ , a contradiction. Therefore, from Proposition 2.8, we conclude that  $Z$  is a circuit-hyperplane of  $M$ , and  $M_T$  is obtained from  $M$  by relaxing  $Z$ .

We know from (5.15.1) that  $M \setminus c$  is non-binary, and hence ternary. We have already noted that  $M \setminus c$  is connected. Moreover, Proposition 2.8 implies that  $Z \cup e$  is a circuit of  $M_T$  for every  $e \in E(M) - Z$ . Thus  $M_T \setminus c$  is connected. We have shown that if  $y_1, y_2 \in Y'$ , then  $M \setminus y_i = M_T \setminus y_i$  for  $i \in \{1, 2\}$ . Therefore  $M \setminus c \setminus y_i = M_T \setminus c \setminus y_i$ . Also  $M \setminus c \setminus y_1 \setminus y_2 = M_T \setminus c \setminus y_1 \setminus y_2$  and this last matroid is connected, since  $Z_1$  avoids  $c, y_1$ , and  $y_2$ . Finally,  $\{y_1, y_2\}$  is not a cocircuit of  $M \setminus c$  or of  $M_T \setminus c$ , for  $M \setminus c \setminus y_i = M_T \setminus c \setminus y_i$  for  $i = 1, 2$ , and these matroids are connected. We apply Lemma 5.14. Since both  $M \setminus c$  and  $M_T \setminus c$  are ternary, we conclude that  $M \setminus c = M_T \setminus c$ . But  $Z$  is a circuit of  $M \setminus c$ , and a basis of  $M_T \setminus c$ . This contradiction completes the proof.  $\square$

**Lemma 5.16.** *Suppose that  $Z$  is a distinguishing set for  $M$  and  $M_B$ . Then  $E(M) - Z$  is a circuit-hyperplane of  $M$ , and the matroid obtained from  $M$  by relaxing  $E(M) - Z$  is ternary.*

*Proof.* Much of the argument in this lemma is similar to that in Lemma 5.15. Note that  $Z$  is a circuit-hyperplane of  $M_B$  by Lemma 5.11. Since  $Z$  is a unique distinguishing set by Lemma 5.15, we see from Proposition 2.8 that  $M$  is obtained from  $M_B$  by relaxing  $Z$ .

Suppose that  $e \in E(M) - Z$ . Let  $a$  and  $b$  be distinct elements in  $E(M) - (Z \cup e)$ . Then  $Z \cup a$  and  $Z \cup b$  are circuits of  $M \setminus e$  by Proposition 5.13.

If  $M \setminus e$  were binary, then  $\{a, b\}$  would be a union of circuits in  $M$ . This contradiction implies that  $M \setminus e$  is ternary for every element  $e \in E(M) - Z$ .

The fact that  $Z \cup e$  is a circuit of  $M$  for every  $e \in E(M) - Z$  means that  $M \setminus A$  is connected for every proper subset of  $E(M) - Z$ .

Choose elements  $a, b \in E(M) - Z$ . Then  $M \setminus a$ ,  $M \setminus b$ , and  $M \setminus a \setminus b$  are all ternary. Suppose that these three matroids are represented over  $\text{GF}(3)$  by the matrices  $[I_r | A | \mathbf{b}]$ ,  $[I_r | A | \mathbf{a}]$ , and  $[I_r | A]$  respectively. Let  $M_T$  be the ternary matroid represented over  $\text{GF}(3)$  by  $[I_r | A | \mathbf{a} | \mathbf{b}]$ , so that  $M_T \setminus a = M \setminus a$  and  $M_T \setminus b = M \setminus b$ .

Suppose that  $e \in E(M) - Z$ . Then  $M \setminus e$  is ternary. Furthermore,  $M \setminus e \setminus a = M_T \setminus e \setminus a$  and  $M \setminus e \setminus b = M_T \setminus e \setminus b$ , and these matroids are both connected. We have already observed that  $M \setminus e$  is connected. If  $M_T \setminus e$  is not connected, then  $e$  is a loop or a coloop in  $M_T \setminus a$ . But  $M_T$  has no loops, and  $\{a, e\}$  is not a series pair of  $M_T$  as  $M_T \setminus a = M \setminus a$ , and this matroid is connected. We also note that  $M \setminus e \setminus a \setminus b = M_T \setminus e \setminus a \setminus b$ , and this last matroid is connected. Finally,  $\{a, b\}$  is not a series pair of  $M \setminus e$  or  $M_T \setminus e$  as  $M \setminus e \setminus a = M_T \setminus e \setminus b$  and  $M \setminus e \setminus b = M_T \setminus e \setminus a$  are connected.

We have shown that Lemma 5.14 applies to  $M \setminus e$  and  $M_T \setminus e$ . Since both these matroids are ternary, we deduce that  $M \setminus e = M_T \setminus e$ .

Let  $Z'$  be a distinguishing set for  $M$  and  $M_T$ . Then  $E(M) - Z \subseteq Z'$ , so  $r^*(M) = |E(M) - Z| \leq |Z'|$ . But  $|Z'| \leq r(M) \leq r^*(M)$ , so  $Z' = E(M) - Z$ . Thus  $E(M) - Z$  is the unique distinguishing set for  $M$  and  $M_T$ , and one of these matroids is obtained from the other by relaxing  $E(M) - Z$ .

Suppose that  $M$  is obtained from  $M_T$  by relaxing the circuit-hyperplane  $E(M) - Z$ . Then  $(E(M) - Z) \cup e$  is a circuit of  $M$  for all  $e$  in  $Z$ . Thus  $(E(M) - Z) \cup e$  is a circuit of  $M_B$  for all  $e \in Z$ . It follows that  $M_B$  has a circuit of size at most two. This contradiction shows that  $M_T$  is obtained from  $M$  by relaxing the circuit-hyperplane  $E(M) - Z$ , and this completes the proof.  $\square$

To complete the proof of Theorem 5.1, we suppose that  $Z$  is a distinguishing set for  $M$  and  $M_B$ . Then  $Z$  is a circuit-hyperplane of  $M_B$  by Lemma 5.11. Lemma 5.15 says that  $Z$  is unique, so  $M$  is obtained from  $M_B$  by relaxing the circuit-hyperplane  $Z$  (Proposition 2.8). Also,  $E(M) - Z$  is a circuit-hyperplane of  $M$ , and the matroid  $M_T$  produced by relaxing  $E(M) - Z$  in  $M$  is ternary by Lemma 5.16. It is an easy exercise to see that  $E(M) - Z$  is a circuit-hyperplane of  $M_B$ , and that relaxing both  $Z$  and  $E(M) - Z$  in  $M_B$  produces  $M_T$ . Thus, if we can show that  $M_B$  is 3-connected, the result follows by renaming  $Z$  with  $J$  and  $E(M) - Z$  with  $K$ .

Suppose that  $(X_1, X_2)$  is a  $k$ -separation of  $M_B$  for some  $k < 3$ . As  $M$  is 3-connected,  $(X_1, X_2)$  is not a  $k$ -separation of  $M$ . Thus  $r_M(X_i) > r_{M_B}(X_i)$ , where  $\{i, j\} = \{1, 2\}$ . It is easy to see that this means  $X_i = Z$ . Thus  $r_{M_B}(X_i) = r(M_B) - 1$ . Therefore  $r_{M_B}(X_j) \leq 2$ . If  $r_{M_B}(X_j) = 1$ , then  $M_B$  is not simple, a contradiction. Therefore  $r_{M_B}(X_j) = 2$ . Suppose that  $X_j$  contains three distinct elements,  $a, b$ , and  $c$ . Then  $\{a, b, c\}$  is a

triangle of  $M_B$ . Moreover,  $M_B$  has a circuit  $C$  contained in  $X_i \cup \{a, b\}$  such that  $a, b \in C$ . Taking the symmetric difference of  $C$  and  $X_i$  shows that  $c \in \text{cl}_{M_B}(X_i) = \text{cl}_{M_B}(Z) = Z$ . Hence  $|X_j| = 2$ . Thus  $M$  has corank two, and  $Z$  is a series class of  $M$ . It follows that  $M$  is ternary, a contradiction.  $\square$

We close this section with some simple consequences of Theorem 5.1.

**Corollary 5.17.** *Let  $M$  be a 3-connected excluded minor for  $\mathcal{M}$  such that  $|E(M)| \geq 10$  and both the rank and corank of  $M$  exceed three. Let  $M_B$  be the binary matroid supplied by Theorem 5.1, and let  $J$  and  $K$  be the circuit-hyperplanes that partition  $E(M_B)$ . Then*

- (i)  $r(M) = r^*(M)$ ;
- (ii)  $|E(M)|$  is divisible by 4;
- (iii) every non-spanning circuit of  $M$  has even cardinality;
- (iv) every non-cospanning cocircuit of  $M$  has even cardinality;
- (v)  $M_B$  contains no triangles and no triads; and
- (vi) the matroid obtained from  $M_B$  by relaxing  $K$  is an excluded minor for  $\mathcal{M}$ .

*Proof.* Statement (i) is clear. Observe that  $J$  and  $K$  are both circuits and cocircuits of  $M_B$ . As  $M_B$  is binary, this means that  $|J| = |K|$  is even. Therefore  $|E(M)| = |J| + |K|$  is a multiple of 4.

Any non-spanning circuit of  $M$  is also a circuit in  $M_B$ , and must therefore meet both  $J$  and  $K$  in an even number of elements. If  $H$  is the complement of a non-cospanning cocircuit in  $M$ , then  $H$  is dependent in  $M$ , and hence in  $M_B$ . It follows easily that  $E(M) - H$  is a cocircuit of  $M_B$ , and therefore meets  $J$  and  $K$  in an even number of elements.

If  $T$  is a triangle of  $M_B$ , then it is also a triangle of  $M$ , which contradicts statement (iii). If  $T^*$  is a triad of  $M_B$ , then it cannot be a triad of  $M$ , by statement (iv). Since  $M$  is obtained from  $M_B$  by relaxing the circuit-hyperplane  $J$ , it follows that  $T^*$  must be  $E(M) - J = K$ . As  $|E(M)| = |J| + |K| = 2|K|$ , this means that  $|E(M)| = 6$ , a contradiction.

Finally, let  $M'$  be the matroid obtained from  $M_B$  by relaxing  $K$ . Suppose that  $M'$  is binary. Then, for any two elements  $j, j' \in J$ , both  $K \cup j$  and  $K \cup j'$  are circuits of  $M'$ , and hence  $\{j, j'\}$  is a circuit of  $M'$ . This implies that  $M_B$  contains a parallel pair, and this contradicts the fact that  $M_B$  is 3-connected. Suppose that  $M'$  is ternary. Then the matroid,  $M_T$ , obtained from  $M'$  by relaxing  $J$  is also ternary, and Lemma 2.9 implies that  $M'$  is binary, a contradiction. Therefore  $M'$  does not belong to  $\mathcal{M}$ . By applying Proposition 2.6 we see that a single-element deletion or contraction of  $M'$  is equal to a single-element deletion or contraction of either the binary matroid  $M_B$  or the ternary matroid  $M_T$ . The result follows.  $\square$

## 6. ALMOST-REGULAR MATROIDS

In this section we establish a connection between the excluded minors for  $\mathcal{M}$ , and Truemper's class of almost-regular matroids, defined in Section 2.6.

**Theorem 6.1.** *Let  $M$  be an excluded minor for  $\mathcal{M}$  with  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ . Let  $M_B$  be the binary matroid supplied by Theorem 5.1, so that  $E(M)$  is partitioned into two circuit-hyperplanes,  $J$  and  $K$ , of  $M_B$ . Then  $M_B \setminus e$  and  $M_B/e$  are almost-regular, for every element  $e \in E(M)$ . In particular, if  $e \in J$ , then  $M_B \setminus e$  is almost-regular, with  $\text{con} = J - e$  and  $\text{del} = K$ , and  $M_B/e$  is almost-regular with  $\text{con} = K$  and  $\text{del} = J - e$ . If  $e \in K$ , then  $M_B \setminus e$  is almost-regular, with  $\text{con} = K - e$  and  $\text{del} = J$ , and  $M_B/e$  is almost-regular with  $\text{con} = J$  and  $\text{del} = K - e$ .*

*Proof.* Theorem 5.1 states that relaxing both  $J$  and  $K$  in  $M_B$  produces a ternary matroid  $M_T$ . Let  $e$  be an element in  $J$ . Let  $\text{con} = J - e$  and let  $\text{del} = K$ . It follows from Proposition 2.6 that if  $f$  is in  $K$ , then  $M_B \setminus e \setminus f = M_T \setminus e \setminus f$ . Hence  $M_B \setminus e \setminus f$  is both binary and ternary, and is therefore regular. On the other hand, if  $f \in J - e$ , then  $M_B \setminus e / f = M_T \setminus e / f$ , so  $M_B \setminus e / f$  is regular.

Next we show that  $M_B \setminus e$  itself is not regular. Suppose that it is. Then, in particular,  $M_B \setminus e$  is ternary. Note that  $K$  is a circuit-hyperplane of  $M_B \setminus e$ , and that relaxing this circuit-hyperplane in  $M_B \setminus e$  produces  $M_T \setminus e$ , by Proposition 2.6. Therefore  $M_B \setminus e$  and  $M_T \setminus e$  are both ternary matroids, and the second is produced from the first by relaxing  $K$ . Lemma 2.9 asserts that there is an enlarged wheel  $G$  such that  $K$  is the rim of  $G$  and  $M_B \setminus e = M(G)$ . Now  $M_B$  is simple, so  $G$  contains no parallel edges. Since  $J - e$  makes up the spoke edges of  $G$ , and  $|K| = |J - e| + 1$ , it follows that the rim of  $G$  contains precisely one series pair. But  $M_B$  contains no series pair, as it is 3-connected. Therefore  $M_B$  contains at least one triad, contradicting Corollary 5.17. Hence  $M_B \setminus e$  is not regular.

We note that  $J - e$  is a cocircuit of  $M_B \setminus e$ , so any circuit of this matroid meets  $J - e$  in an even number of elements. Similarly,  $K$  is a circuit of  $M_B \setminus e$ , so any cocircuit of  $M_B \setminus e$  meets  $K$  in a set of even cardinality. We conclude that  $M_B \setminus e$  is almost-regular.

Next we consider  $M_B/e$ . Let  $\text{con} = K$  and let  $\text{del} = J - e$ . If  $f \in K$ , then  $M_B/e / f = M_T/e / f$ , and if  $f \in J - e$ , then  $M_B/e \setminus f = M_T/e \setminus f$ , so both these matroids are regular. Suppose that  $M_B/e$  is regular. Now  $J - e$  is a circuit-hyperplane of  $M_B/e$ , and the matroid produced from  $M_B/e$  by relaxing  $J - e$  is  $M_T/e$ . Therefore  $M_B/e$  is the cycle matroid of an enlarged wheel  $G$ , and  $J - e$  is the rim of  $G$ . Since  $M_B$  is 3-connected, it follows that  $M_B/e$  has no series pairs. As the rim of  $G$  has cardinality  $r(M) - 1$  and the complement of the rim contains  $r(M)$  elements, this means that  $G$  must contain a parallel pair. Therefore  $M_B$  contains a triangle, so we have a contradiction to Corollary 5.17. Finally, we observe that  $K$  is a cocircuit of  $M_B/e$ , so any circuit of this matroid meets  $K$  in an even number of elements, and  $J - e$  is a circuit of  $M_B/e$ , so any cocircuit meets  $J - e$  in a set with even cardinality. It follows that  $M_B/e$  is almost-regular.

An identical argument works in the case that  $e \in K$ . □

**Proposition 6.2.** *Let  $M$  be an excluded minor for  $\mathcal{M}$  with  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ . Let  $M_B$  be the binary matroid supplied by Theorem 5.1. Then  $M_B \setminus e$  and  $M_B/e$  are internally 4-connected, for every  $e$  in  $E(M)$ .*

*Proof.* By duality it suffices to prove that  $M_B \setminus e$  is internally 4-connected. Since  $M_B$  is 3-connected,  $M_B \setminus e$  is certainly 2-connected. Suppose that  $M_B \setminus e$  is not 3-connected. Since  $M_B \setminus e$  is almost-regular, Theorem 22.1 of [27] implies that  $M_B \setminus e$  must contain a series pair. But this implies that  $M_B$  contains a triad, a contradiction to Corollary 5.17.

We now assume that  $M_B \setminus e$  is 3-connected. If  $M_B \setminus e$  is not internally 4-connected, then [27, Theorem 22.1] implies that  $M_B \setminus e$  contains both a triangle and a triad. Thus  $M_B$  contains a triangle, and again we have a contradiction to Corollary 5.17.  $\square$

## 7. REDUCTION TO A FINITE LIST OF EXCLUDED MINORS

We are now ready to proceed with the proof of Theorem 1.1. In what follows,  $M$  will be an excluded minor for  $\mathcal{M}$  such that  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ . Theorem 5.1 supplies us with the matroid  $M_B$ . We consider three cases. In the first,  $M_B$  has an  $R_{10}$ -minor; in the second,  $M_B$  has an  $R_{12}$ -minor; and, in the last case,  $M_B$  has no  $R_{10}$ -minor and no  $R_{12}$ -minor. In each case, we bound the size of  $|E(M)|$ , and thereby reduce the remainder of the proof to a finite case check.

**7.1. The  $R_{10}$  case.** In this section we consider the easiest case, namely when  $M_B$  has an  $R_{10}$ -minor. The arguments of this section closely follow those of Truemper in Section 26 of [27].

The matroid  $N_{11}$  plays an important role in Truemper's characterization of the almost-regular matroids. It is the rank-5 binary matroid with eleven elements and a distinguished element  $z$  such that  $N_{11} \setminus z \cong R_{10}$ , where  $z$  is in a triangle with two elements of  $E(N_{11}) - z$ . Since the automorphism group of  $R_{10}$  is transitive on pairs of elements ([25, p. 328]), this matroid is well-defined up to isomorphism. As  $R_{10}$  contains no triangles, it follows that  $z$  is in no parallel pair of  $N_{11}$ . Therefore  $N_{11}$  is 3-connected. Since  $R_{10}$  is a splitter for the class of regular matroids (Proposition 2.12), it follows that  $N_{11}$  is not regular. However, it is not difficult to see that  $N_{11}$  is almost-regular. The following matrix is a reduced representation of  $N_{11}$  over  $\text{GF}(2)$ .

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Deleting the last column of this matrix produces a reduced representation of  $R_{10}$ .

**Proposition 7.1.** *The matroid  $N_{11}/z$  is not regular.*

*Proof.* Let  $A$  be the matrix displayed above, so that  $[I_5|A]$  represents  $N_{11}$ . Suppose that the columns of  $[I_5|A]$  are labeled with the integers  $1, \dots, 11$ , so that  $z$  corresponds to the column labeled by 11. By pivoting on the first entry in column 11 and then deleting the first row, and columns 1, 6, and 7, we see that  $N_{11}/z$  has an  $F_7^*$ -minor, and is therefore not regular.  $\square$

**Proposition 7.2.** *Suppose that  $N$  is a 3-connected almost-regular matroid such that  $|E(N)| = 11$  and  $N$  has an  $R_{10}$ -minor. Then  $N$  is isomorphic to either  $N_{11}$  or  $N_{11}^*$ .*

*Proof.* Since  $R_{10}$  is self-dual, we can assume that  $N$  is an extension of  $R_{10}$  by the element  $z$ . We will be done if we can show that  $z$  is contained in a triangle of  $N$ .

Consider the partition  $(del, con)$  of  $E(N)$ . The set  $con$  is non-empty, by definition. Suppose that  $con$  contains only a single element. This element is contained in a circuit, as  $N$  is connected. But this circuit meets  $con$  in precisely one element, which contradicts the definition of almost-regular matroids. Thus we can choose an element  $e \in con$  such that  $e \neq z$ .

Suppose that  $N/e$  is 3-connected. It is regular as  $e \in con$ . Since  $N/e$  has rank four, and contains ten elements, it has neither an  $R_{10}$ - nor an  $R_{12}$ -minor, and is therefore either graphic or cographic by Lemma 2.13. Every single-element contraction of  $R_{10}$  is isomorphic to  $M^*(K_{3,3})$ , so  $N/e$  is a 3-connected cographic extension of  $M^*(K_{3,3})$  by the element  $z$ . But it is easy to see that no such cographic matroid exists, so we have a contradiction.

We now know that  $N/e$  is not 3-connected. As  $N/e$  is a single-element extension of  $M^*(K_{3,3})$ , a 3-connected matroid, it follows that  $z$  is in a parallel pair in  $N/e$ . Therefore  $z$  is in a triangle in  $N$ . Thus we are done.  $\square$

**Proposition 7.3.** *Let  $e$  be an element of  $E(N_{11})$  such that no triangle of  $N_{11}$  contains  $\{e, z\}$ . Let  $M$  be the binary matroid obtained by adding the element  $f$  to  $N_{11}$  so that  $\{e, f, z\}$  is a triangle. Then  $M \setminus e \setminus z$  is not regular.*

*Proof.* Note that  $z$  is contained in at least one triangle in  $M \setminus f$ . Let  $\{a, b, z\}$  be such a triangle. We start by showing that  $M/a \setminus z$  is simple. Since  $M$  is simple by construction, if  $M/a \setminus z$  is not simple, there is a triangle  $T$  of  $M$  such that  $a \in T$ , but  $T$  avoids  $z$ . Note that  $M/a \setminus z \setminus f$  is isomorphic to a single-element contraction of  $R_{10}$ , and is therefore simple. Thus  $f \in T$ . Let  $x$  be the single element in  $T - \{a, f\}$ . Then  $x$  is not equal to  $b$ , for that would imply that  $f$  and  $z$  are parallel in  $M$ , and  $x \neq e$ , as that would imply that  $a$  and  $z$  are parallel in  $M$ .

Now  $a, b, e, f, x, z$  are distinct elements of  $M$ , and  $\{a, b, z\}$ ,  $\{e, f, z\}$ , and  $\{a, f, x\}$  are triangles of  $M$ . The symmetric difference of these sets is  $\{b, e, x\}$ , which is therefore a triangle of  $M \setminus z \setminus f \cong R_{10}$ . As  $R_{10}$  has no triangles, we conclude that  $M/a \setminus z$  is simple.

Next we observe that  $M/a \setminus z \setminus f$  is a single-element contraction of  $R_{10}$ , and is therefore isomorphic to  $M^*(K_{3,3})$ . Moreover,  $\{b, e, f\}$  is a triangle of  $M/a \setminus z$ . Thus  $M/a \setminus z$  is isomorphic to the matroid obtained from  $M^*(K_{3,3})$



by adding the element  $f$  so that it forms a triangle with  $b$  and  $e$ , where no triangle of  $M^*(K_{3,3})$  contains  $\{b, e\}$ .

There is a unique four-element circuit  $C$  of  $M^*(K_{3,3})$  that contains  $b$  and  $e$ . It follows easily that  $f$  is contained in precisely two triangles in  $M/a \setminus z$ , namely  $\{b, e, f\}$ , and  $(C - \{b, e\}) \cup f$ . Thus  $f$  is in exactly one triangle in  $M/a \setminus z \setminus e$ , so  $M/a/f \setminus z \setminus e$  has exactly one parallel pair. Now  $\text{si}(M/a/f \setminus z \setminus e)$  has rank three, and contains seven elements. It is therefore isomorphic to  $F_7$ . Hence  $M \setminus e \setminus z$  is non-regular, as desired.  $\square$

The following result is the key step in this part of the case analysis (see also [27, Theorem 26.1]).

**Lemma 7.4.** *Let  $N$  be an internally 4-connected almost-regular matroid having an  $R_{10}$ -minor. Then  $N$  is isomorphic to either  $N_{11}$  or  $N_{11}^*$ .*

*Proof.* Since  $N$  is not regular, it cannot be isomorphic to  $R_{10}$ . By the Splitter Theorem (Theorem 2.10), there is a 3-connected minor  $N_0$  of  $N$  such that  $N_0$  is a single-element extension or coextension of  $R_{10}$ . Proposition 7.2 implies that  $N_0$  is isomorphic to either  $N_{11}$  or  $N_{11}^*$ . By exploiting duality, we can assume the former. Let  $z$  be the distinguished element of  $E(N_0)$  such that  $N_0 \setminus z \cong R_{10}$  and  $z$  is contained in a triangle of  $N_0$ .

If  $N$  is equal to  $N_0$ , we are done, so assume that  $N_0$  is a proper minor of  $N$ . Since  $N_0/z$  is non-regular by Proposition 7.1, it follows that  $N/z$  is non-regular. Thus  $N \setminus z$  is regular and has a proper  $R_{10}$ -minor. But  $R_{10}$  is a splitter for the class of regular matroids, so  $N \setminus z$  is not 3-connected. As  $N$  is 3-connected, we see that  $N \setminus z$  is certainly 2-connected.

Suppose that  $(X_1, X_2)$  is a 2-separation of  $N \setminus z$ , and that  $|X_1|, |X_2| \geq 3$ . Then both  $(X_1 \cup z, X_2)$  and  $(X_1, X_2 \cup z)$  are 3-separations of  $N$ , and we have a contradiction to the fact that  $N$  is internally 4-connected. We deduce from this that if  $(X_1, X_2)$  is a 2-separation of  $N \setminus z$ , then either  $X_1$  or  $X_2$  is a series pair of  $N \setminus z$ . This implies that  $\text{co}(N \setminus z)$  is 3-connected. As  $\text{co}(N \setminus z)$  is regular with an  $R_{10}$ -minor,  $\text{co}(N \setminus z)$  must in fact be isomorphic to  $R_{10}$ .

Consider a series pair  $P$  of  $N \setminus z$ , and suppose that  $P \subseteq E(N_0)$ . Then  $N_0 \setminus z$  must contain a cocircuit of size at most two, and this is a contradiction, as  $N_0 \setminus z \cong R_{10}$ . Therefore we can find a set  $S$  containing exactly one element from each series pair of  $N \setminus z$ , such that  $S$  does not meet  $E(N_0)$ . Note that  $N \setminus z/S \cong \text{co}(N \setminus z)$ . Thus  $|E(N \setminus z/S)| = 10$ . But  $E(N \setminus z/S)$  contains  $E(N_0 \setminus z)$ , and this set also has cardinality ten. Thus every element of  $E(N)$  not in  $S$  is an element of  $N_0$ .

Let  $P$  be a series pair of  $N \setminus z$ . Then  $P \cup z$  is a triad of  $N$ . Let  $s$  be the unique element in  $P \cap S$ . Suppose that  $N_0$  is a minor of  $N \setminus s$ . Then  $N_0$  contains  $(P - s) \cup z$ , and this set is a series pair of  $N \setminus s$ . Thus  $N_0$  contains a cocircuit of size at most two, a contradiction. Therefore  $N_0$  is not a minor of  $N \setminus s$ , for any element  $s \in S$ . It follows that  $N_0 = N/S$ .

Next we suppose that  $P$  is a series pair of  $N \setminus z$ , that  $P = \{e, s\}$  where  $s \in S$ , and that there is no triangle of  $N_0$  that contains both  $\{e, z\}$ . Consider the matroid  $N/(S - s)$ . This matroid cannot be regular, since it has  $N_0$

as a minor. Hence it is almost-regular, by Proposition 2.16. Note that  $N/(S-s)/s$  is not regular, so  $N/(S-s)\setminus s$  must be regular. However,  $P \cup z$  is a triad of  $N/(S-s)$ . Let  $M$  be the binary matroid obtained from  $N_0$  by adding an element so that it forms a triangle with  $z$  and  $e$ . Then  $N/(S-s)\setminus s/e \cong M\setminus e\setminus z$ . The last matroid is not regular by Proposition 7.3. Thus we have a contradiction, and conclude that if  $P$  is a series pair of  $N\setminus z$ , then the single element in  $P-S$  is contained in a triangle of  $N_0$  with  $z$ .

Suppose that there are distinct triangles  $T_1$  and  $T_2$  of  $N_0$  such that  $z \in T_1 \cap T_2$ , and there are elements  $e_1 \in T_1 - z$  and  $e_2 \in T_2 - z$  such that  $e_i$  is contained in the series pair  $\{e_i, s_i\}$  of  $N\setminus z$  for  $i = 1, 2$ . Let  $N'$  be  $N/(S - \{s_1, s_2\})$ . Then  $N'$  is not regular, since it has  $N_0$  as a minor. Thus  $N'$  is almost-regular.

Note that  $N'/s_1/s_2/z = N_0/z$  is non-regular by Proposition 7.1. But  $e_1$  is in a parallel pair of  $N_0/z$ , so  $N'/s_1/s_2/z\setminus e_1$ , and hence  $N'\setminus e_1$  is non-regular. It follows that  $N'/e_1$  is regular.

We observe that  $N_0\setminus z$ , and hence  $N'\setminus z$ , has an  $R_{10}$ -minor. But  $\{e_1, s_1\}$  is a series pair of  $N'\setminus z$ , so  $N'\setminus z/e_1$ , and hence  $N'/e_1$  has an  $R_{10}$ -minor. Thus  $N'/e_1$  is regular with a proper  $R_{10}$ -minor. We will obtain a contradiction by showing that  $N'/e_1$  is 3-connected.

First we show that  $N'$  is 3-connected. The matroid  $N'/s_1/s_2$  is 3-connected, as it is isomorphic to  $N_{11}$ . Neither  $s_1$  nor  $s_2$  is a loop of  $N'$ , so if  $N'$  is not 3-connected, it contains a cocircuit of size at most two. Hence so does  $N$ , a contradiction. Thus  $N'$  is 3-connected.

Suppose that  $N'/e_1$  is not simple. Then there is a triangle  $T$  of  $N'$  that contains  $e_1$ . The triad  $\{e_1, s_1, z\}$  must meet  $T$  in two elements. If  $s_1$  were in  $T$ , then  $N'/s_1/s_2 = N_0$  would contain a circuit of size at most two, a contradiction. Therefore  $z \in T$ . The triad  $\{e_2, s_2, z\}$  must meet  $T$  in two elements, and  $s_2$  is not in  $T$ , by the previous argument, so  $T = \{e_1, e_2, z\}$ . Now  $T_1, T_2$ , and  $\{e_1, e_2, z\}$  are triangles of  $N_0$ , and this implies the existence of a parallel pair in  $N_0$ . This contradiction means that  $N'/e_1$  is simple.

Suppose that  $(X_1, X_2)$  is a 2-separation of  $N'/e_1$ . It follows easily from the fact that  $N'/e_1$  contains no series or parallel pairs that  $|X_1|, |X_2| \geq 4$ . Note that  $N'/e_1/s_1/s_2 \cong N_0/e_1$  and the last matroid is obtained from  $M^*(K_{3,3})$  by adding a single parallel element. Thus if  $(Y_1, Y_2)$  is a 2-separation of  $N'/e_1/s_1/s_2$ , then either  $Y_1$  or  $Y_2$  is a parallel pair. Now Proposition 2.3 implies that  $s_1$  and  $s_2$  must be contained in either  $X_1$  or  $X_2$ . Without loss of generality, we assume the former. It follows that  $X_1 - \{s_1, s_2\}$  is the unique parallel pair of  $N'/e_1/s_1/s_2$ . Thus  $|X_1| = 4$ .

As  $N'/e_1$  is simple,  $r_{N'/e_1}(X_1) \geq 3$ . Thus  $r_{N'/e_1}(X_2) \leq r(N'/e_1) - 2$ , so  $X_1$  contains at least two cocircuits of  $N'/e_1$ . This implies the existence of a cocircuit of size at most two in  $N'$ , and we have a contradiction.

This argument shows that there is a triangle  $T$  of  $N_0$ , such that if  $P$  is a series pair of  $N\setminus z$ , then the unique element in  $P-S$  is contained in  $T$ . There is a circuit  $C \subseteq T \cup S$  such that  $C$  contains  $T$ . But  $C$  must be equal to  $T$ , for otherwise  $C$  meets a triad of  $N$  in three elements. Thus  $T$

is a triangle of  $N$  which meets at least one triad. This is impossible in an internally 4-connected matroid, so we have arrived at a contradiction that completes the proof of the lemma.  $\square$

Now we can state the conclusion of this analysis.

**Lemma 7.5.** *Suppose that  $M$  is an excluded minor for the class  $\mathcal{M}$  such that  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ . Let  $M_B$  be the binary matroid supplied by Theorem 5.1. If  $M_B$  has an  $R_{10}$ -minor, then  $M$  is a single-element extension of  $N_{11}$  or  $N_{11}^*$ , and hence  $|E(M)| = 12$ .*

*Proof.* By duality we can assume that there is an element  $e \in E(M)$  such that  $M_B \setminus e$  has an  $R_{10}$ -minor. Then  $M_B \setminus e$  is almost-regular and internally 4-connected by Theorem 6.1 and Proposition 6.2. The result follows from Lemma 7.4.  $\square$

**7.2. The  $R_{12}$  case.** In this section we assume that  $M$  is an excluded minor for  $\mathcal{M}$  with  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ , and that  $M_B$ , the matroid supplied by Theorem 5.1, has an  $R_{12}$ -minor.

Recall that Truemper graphs were defined in Section 2.8. We use these graphs repeatedly in this section and the next.

**Proposition 7.6.** *Let  $G = (R, S)$  be a simple Truemper graph. Assume that both  $R$  and  $S$  contain at least two edges, and that every vertex is incident with at least one cross edge. Then either:*

- (i)  $G$  contains a triangle;
- (ii) an internal vertex of  $G$  has degree three; or
- (iii)  $G$  has an XX-minor.

*Proof.* Let  $r_1, \dots, r_m$  and  $s_1, \dots, s_n$  be the vertices of  $R$  and  $S$  respectively. Thus  $m, n \geq 3$ . Assume that the result fails, and that  $G$  is a counterexample, but that the result holds for graphs with fewer edges than  $G$ .

We first suppose that  $m = 3$ . Consider  $s_2$  and  $s_{n-1}$ . Because  $G$  is a counterexample, both these vertices meet at least two cross edges. Neither can be adjacent to  $r_2$ , for that implies that  $G$  contains a triangle. Thus  $s_2$  and  $s_{n-1}$  are adjacent to  $r_1$  and  $r_3$ . If  $s_1$  were adjacent to  $r_1$  or  $r_3$ , then  $G$  would contain a triangle. Thus  $s_1$  is adjacent to precisely one vertex in  $R$ , namely  $r_2$ . Similarly,  $s_n$  is adjacent to  $r_2$ , and no other vertex in  $R$ . But now the edges  $r_1s_2, r_2s_1, r_2s_n, r_3s_2$  form an XX-minor. This contradiction means that  $m > 3$  and, by symmetry,  $n > 3$ .

Proposition 2.20 implies that there is an edge joining two terminal vertices. By relabeling if necessary, we assume that there is an edge  $e$  joining  $r_1$  and  $s_1$ . Suppose that both  $r_1$  and  $s_1$  meet at least two cross edges in  $G$ . Then the hypotheses of the proposition apply to  $G \setminus e$ , so our minimality assumption implies that  $G \setminus e$  contains either a triangle, an XX-minor, or an internal vertex with degree three. However, in any of these cases, the result also holds for  $G$ , and we have a contradiction. Hence either  $r_1$  or  $s_1$  has degree exactly two. By symmetry, we assume that  $r_1$  has degree two.

Let  $f$  be the edge  $r_1r_2$ . Assume that  $s_1$  has degree greater than two. Since  $m, n > 3$ , the hypotheses of the lemma apply to  $G \setminus e/f$ . Therefore  $G \setminus e/f$  contains (a) an XX-minor, (b) an internal vertex with degree three, or (c) a triangle. If  $G \setminus e/f$  has an XX-minor, then so does  $G$ , and we have a contradiction. The internal vertices of  $G \setminus e/f$  are internal vertices of  $G$ , and the degree of such a vertex in  $G \setminus e/f$  is the same as its degree in  $G$ . Therefore (b) cannot occur. Finally, we suppose that (c) occurs. Then  $G \setminus e/f$  has a triangle, but  $G$  does not. Thus  $f$  is contained in a cycle of length four in  $G \setminus e$ . But  $f$  is a pendant edge in this graph, and we have a contradiction.

We may now assume that the degree of  $s_1$  is two. Let  $g$  be the edge  $s_1s_2$ . The result holds for  $G \setminus e/f/g$ , so  $G \setminus e/f/g$  has an XX-minor, an internal vertex with degree three, or a triangle. The first two cases quickly lead to contradictions. Thus  $G \setminus e/f/g$  has a triangle, but  $G$  does not. Therefore there is a cycle of  $G \setminus e$  that contains either  $f$  or  $g$ . As these are pendant edges in  $G \setminus e$ , we have a contradiction.  $\square$

Truemper introduced a particular almost-regular matroid,  $V_{13}$ . There is a distinguished element  $z$  in  $V_{13}$  such that  $V_{13} \setminus z$  is isomorphic to  $R_{12}$ . The dual matroid,  $V_{13}^*$ , has the reduced representation shown in Figure 9.

	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$
$a_1$	1	0	1	1	0	0
$a_2$	0	1	1	1	0	0
$a_3$	1	0	1	0	1	1
$a_4$	0	1	0	1	1	1
$a_5$	1	0	1	0	1	0
$a_6$	0	1	0	1	0	1
$z$	0	0	0	1	0	1

FIGURE 9. A reduced representation of  $V_{13}^*$ .

Let  $A_0$  be the matrix in Figure 9. We assume that the columns of  $[I_7|A_0]$  are labeled  $a_1, \dots, a_6, z, b_1, \dots, b_6$ . Thus the rows of  $A_0$  correspond in a natural way with the columns of the identity matrix, as reflected by the labels in Figure 9.

The next result follows from Theorem 25.9 of [27].

**Lemma 7.7.** *Let  $N$  be a 3-connected almost-regular matroid having an  $R_{12}$ -minor. Then  $N$  has a minor isomorphic to either  $V_{13}$  or  $V_{13}^*$ .*

**Proposition 7.8.** *The matroid  $V_{13}^* \setminus z$  is non-regular.*

*Proof.* Consider the matrix obtained from that in Figure 9 by pivoting on the entry labeled by  $z$  and  $b_6$ , on the entry labeled by  $a_1$  and  $b_1$ , and on the entry labeled by  $a_2$  and  $b_2$ . If we then delete the second and sixth columns, and the second, third, sixth, and seventh rows, we obtain a reduced representation of  $F_7$ . Thus deleting  $z$  from  $V_{13}^*$  produces a non-regular matroid.  $\square$

Suppose that  $A$  is a matrix, and that  $X$  (respectively  $Y$ ) is a set of rows (columns) of  $A$ . We use  $A[X, Y]$  to denote the submatrix of  $A$  induced by  $X$  and  $Y$ .

**Lemma 7.9.** *Suppose that  $N$  is an almost-regular matroid with a minor  $N_0$  such that  $N_0 \cong V_{13}^*$ . Let  $E(N_0) = \{a_1, \dots, a_6, b_1, \dots, b_6, z\}$ , and assume that  $A_0$  is a reduced representation of  $N_0$  over  $\text{GF}(2)$ , where  $A_0$  is the matrix in Figure 9. Let  $A$  be a reduced representation of  $N$  over  $\text{GF}(2)$ , and assume that  $\{a_1, \dots, a_6, z\}$  label rows of  $A$ , while  $\{b_1, \dots, b_6\}$  label columns. Then, up to row and column permutations,  $A$  has the form shown in Figure 10, and the following conditions hold:*

- (i)  $A[A_1, B_2]$  is the zero matrix; and
- (ii)  $A[A_2, B_1]$  has rank three, while  $A[A_2 - z, B_1]$  has rank two.

		$B_1$				$B_2$	
		$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$
$A_1$	$a_1$	1	0	1	1	0	0
$A_1$	$a_2$	0	1	1	1	0	0
$A_2$	$a_3$	1	0	1	0	1	1
$A_2$	$a_4$	0	1	0	1	1	1
$A_2$	$a_5$	1	0	1	0	1	0
$A_2$	$a_6$	0	1	0	1	0	1
$A_2$	$z$	0	0	0	1	0	1

FIGURE 10

*Proof.* Proposition 7.8 implies that  $N \setminus z$  is non-regular, so  $N/z$  is regular. Recall that  $V_{13}^*/z \cong R_{12}$ . Thus  $V_{13}^*/z$  has a 3-separation  $(X_1, X_2)$  such that  $|X_1| = |X_2| = 6$ . In particular,  $X_1 = \{a_1, a_2, b_1, b_2, b_3, b_4\}$  and  $X_2 = \{a_3, a_4, a_5, a_6, b_5, b_6\}$ , so  $(X_1, X_2)$  is the 3-separation of  $V_{13}^*/z$  indicated by the division of the matrix in Figure 9.

Now  $N/z$  is a regular matroid with an  $R_{12}$ -minor, and therefore  $N/z$  has a 3-separation  $(Y_1, Y_2)$  such that  $X_i \subseteq Y_i$  for  $i = 1, 2$  (see [25, (9.2)]). From this fact, Truemper deduces that any reduced representation of  $N$  must be as is illustrated in Figure 10. (Note that the figure (25.12) of [27] contains an error. The upper right submatrix should consist of zeroes.) He concludes, moreover, that  $A[A_1, B_2]$  is the zero matrix,  $A[A_2, B_1]$  has rank three, and  $A[A_2 - z, B_1]$  has rank two (see [27, p. 294]).  $\square$

Suppose that  $A$  is any matrix of the form in Figure 10, and that  $A$  is a reduced representation of an almost-regular matroid  $N$ . We let  $A_{11} =$

$A_1 - \{a_1, a_2\}$  and  $B_{11} = B_1 - \{b_1, b_2, b_3, b_4\}$ . Similarly, we let  $A_{22} = A_2 - \{a_3, a_4, a_5, a_6, z\}$  and  $B_{22} = B_2 - \{b_5, b_6\}$ . If the column  $b \notin \{b_1, \dots, b_6\}$  has zero entries for all rows in  $A_{11}$ , then we shall say that  $b$  is a *righthand* column. Otherwise, we shall say that  $b$  is a *lefthand* column. Similarly, if  $a$  is a row of  $A - \{a_1, \dots, a_6, z\}$ , and the row vector  $A[\{a\}, B_1]$  is in the row space of  $A[\{a_3, a_4\}, B_1]$ , then we shall say that  $a$  is a *lower* row. Otherwise we say that  $a$  is an *upper* row. Note that the rank conditions upon the matrix  $A[A_2, B_1]$  mean that if  $b$  is a lefthand column, then the entry in column  $b$  and row  $a$ , where  $a \in A_2 - z$ , is completely determined by the entries of  $b$  in rows  $a_3$  and  $a_4$ .

Truemper studies the matroid  $N/A_{11} \setminus B_{11}$ , that is, the matroid with the reduced representation  $A[A_2 \cup \{a_1, a_2\}, B_2 \cup \{b_1, b_2, b_3, b_4\}]$ . He starts by considering the rows of the matrix  $A[A_{22}, \{b_1, \dots, b_6\}]$ . Any such row must be one of the following vectors (see [27, (25.15)]).

I	[1 0 1 0 0 0]	II	[0 0 0 0 1 0]
III	[0 1 0 1 0 1]	IV	[1 1 1 1 0 0]
V	[1 0 1 0 1 1]	VI	[0 0 0 0 0 1]

If  $a$  is an element of  $A_{22}$  that corresponds to a row of type I, then we shall say that  $a$  is *type I element*, and so on.

Consider the family of graphs illustrated in Figure 11. In this diagram, the dashed edges represent (possibly empty) paths. We let  $G_0$  stand for any graph in this family. We let  $R$  (respectively  $S$ ) be the path consisting of the horizontal edges joining vertices 1 and 7 (respectively 8 and 14).

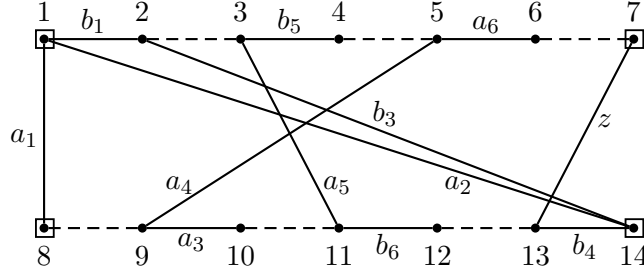


FIGURE 11. The graph  $G_0$ .

**Lemma 7.10.** *Suppose that  $N$  is an almost-regular matroid with a reduced representation  $A$ , where  $A$  is as shown in Figure 10. Then  $N/A_{11} \setminus B_{11}$  is equal to a graft of the form  $M(G, D)$ , where  $G$  is obtained from  $G_0$  by adding edges between  $R$  and  $S$ , and  $D = \{1, 7, 8, 14\}$ . Here the graft element is  $b_2$ . In the graph  $G$ :*

- (i) *The subpath of  $R$  between 2 and 3 consists of type I elements.*
- (ii) *The subpath of  $R$  between 4 and 5 consists of type II elements.*
- (iii) *The subpath of  $R$  between 6 and 7 consists of type III elements.*
- (iv) *The subpath of  $S$  between 8 and 9 consists of type IV elements.*

- (v) The subpath of  $S$  between 10 and 11 consists of type  $V$  elements.
- (vi) The subpath of  $S$  between 12 and 13 consists of type  $VI$  elements.

*Proof.* This follows immediately from Lemma 25.20 of [27].  $\square$

Note that the graph  $G$  in Lemma 7.10 is a Truemper graph, as defined in Section 2.8. We remark that the cross edges added to  $G_0$  to obtain  $G$  are precisely the members of  $B_{22}$ . Similarly, every element in  $A_{22}$  is an edge that appears in one of the paths represented by dashed edges.

**Proposition 7.11.** *Suppose that  $N$  is an almost-regular matroid and that  $A$  is a reduced representation of  $M$ , where  $A$  is a matrix of the type in Figure 10. Let  $(G, D)$  be the graft supplied by Lemma 7.10, so that  $M(G, D) = N/A_{11} \setminus B_{11}$ . Let  $v$  be an internal vertex of  $G$  other than 2 or 13, and let  $C^*$  be the set of edges incident to  $v$  in  $G$ . Then  $C^*$  is a cocircuit of  $N$ .*

*Proof.* This follows by examining the matrix in Figure 10. (See [27, p. 298].)  $\square$

Now we are ready to prove the concluding result in this case.

**Lemma 7.12.** *Suppose that  $M$  is an excluded minor for the class  $\mathcal{M}$  such that  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ . Let  $M_B$  be the binary matroid supplied by Theorem 5.1. Then  $M_B$  has no  $R_{12}$ -minor.*

*Proof.* Let us assume that lemma fails, and that  $M_B$  does have a minor isomorphic to  $R_{12}$ . Corollary 5.17 implies the following:

**7.12.1.**  $M_B$  has no triangles and no triads.

**7.12.2.** *By exploiting duality, we can assume that there is an element  $e$  of  $E(M_B)$  such that  $M_B \setminus e$  is internally 4-connected, almost-regular, and has a  $V_{13}^*$ -minor.*

*Proof.* Since  $M_B$  is not regular, it follows that  $M_B$  has a proper  $R_{12}$ -minor. Theorem 2.10 implies that there is an element  $e \in E(M_B)$  such that either  $M_B \setminus e$  or  $M_B/e$  is 3-connected with an  $R_{12}$ -minor.

Suppose that  $M_B \setminus e$  is 3-connected with an  $R_{12}$ -minor. Theorem 6.1 says that  $M_B \setminus e$  is almost-regular, so  $M_B \setminus e$  has either a  $V_{13}$ - or a  $V_{13}^*$ -minor, by Lemma 7.7. If  $M_B \setminus e$  has a  $V_{13}^*$ -minor, then we are done, since  $M_B \setminus e$  is internally 4-connected by Proposition 6.2, so we assume that  $M_B \setminus e$  has a  $V_{13}$ -minor. Thus  $M_B/e$  has a  $V_{13}^*$ -minor.

Next we assume that  $M_B/e$  is 3-connected with an  $R_{12}$ -minor. Now  $M_B/e$  is internally 4-connected and almost-regular, so it has either a  $V_{13}$ - or a  $V_{13}^*$ -minor. If it has a  $V_{13}$ -minor, then we can switch to the dual, and we are done. Thus we assume that  $M_B/e$  has a  $V_{13}^*$ -minor.

In either case, we have assumed that  $M_B/e$  has a  $V_{13}^*$ -minor. It follows from Lemma 7.9 that we can assume  $M_B/e$  has a reduced representation  $A$  over  $\text{GF}(2)$ , where  $A$  is as shown in Figure 10. If  $B_{11} \cup B_{22}$  is non-empty,

then there is an element  $f \in B_{11} \cup B_{22}$  such that  $M_B/e \setminus f$ , and hence  $M_B \setminus f$ , has a  $V_{13}^*$ -minor. As  $M_B \setminus f$  is internally 4-connected and almost-regular, we are done. Therefore we assume that  $B_{11} \cup B_{22} = \emptyset$ .

Lemma 7.10 implies that  $M_B/e/A_{11}$  is equal to a graft  $M(G, D)$ . As  $B_{22}$  is empty, no cross edges are added to  $G_0$  to obtain  $G$ . But this means that the set of edges incident with vertex 5 in  $G$  is a triad of  $M(G, D) = M_B/e/A_{11}$ . Thus  $M_B$  contains a triad. This contradicts 7.12.1.  $\square$

In what follows, we will utilize 7.12.2, and assume that  $e$  is an element of  $M_B$  such that  $M_B \setminus e$  is an internally 4-connected almost-regular matroid with a  $V_{13}^*$ -minor. Thus we can assume, by Lemma 7.9, that  $M_B \setminus e$  has a reduced representation,  $A$ , over  $\text{GF}(2)$ , of the type shown in Figure 10. There is a column which we can add to  $A$  so that the resulting matrix is a reduced representation of  $M_B$  over  $\text{GF}(2)$ . We will abuse notation, and refer to this column as  $e$ .

**7.12.3.** *The set  $A_{11}$  is non-empty.*

*Proof.* By considering the six possibilities for rows of  $A[A_{22}, \{b_1, \dots, b_6\}]$ , we see that the columns of  $A$  labeled by  $b_1$  and  $b_3$  are identical in all rows except  $a_2$ , and possibly rows in  $A_{11}$ . Thus, if  $A_{11}$  is empty, then  $\{a_2, b_1, b_3\}$  is a triangle of  $M_B \setminus e$ , and hence of  $M_B$ . This contradiction completes the proof of the claim.  $\square$

**7.12.4.** *The set  $B_{11} \cup B_{22}$  is non-empty.*

*Proof.* Suppose that  $B_{11} \cup B_{22}$  is empty. By 7.12.3, there is an element  $a$  in  $A_{11}$ . Let  $A_a$  be the matrix obtained from  $A$  by adding the column  $e$ , and then deleting the row  $a$ . Now  $M_B/a$  is almost-regular by Theorem 6.1, and  $A_a$  is a reduced representation of  $M_B/a$ . Lemma 7.9 implies that  $A_a$  must have the form illustrated in Figure 10. This means that either:

- (i) the column  $e$  has zero entries in all rows labeled by  $A_{11} - a$ ; or
- (ii) the entries of  $e$  in  $A_2 - z$  are completely determined by the entries of  $e$  in  $a_3$  and  $a_4$ .

In the first case, we call  $e$  a righthand column of  $A_a$ , and in the second we call it a lefthand column. If  $e$  is a righthand column of  $A_a$ , then we let  $B'_{11}$  be  $B_{11} = \emptyset$ , and if  $e$  is a lefthand column, we let  $B'_{11}$  be  $\{e\}$ .

In either case,  $M_B/A_{11} \setminus B'_{11}$  is equal to a graft  $M(G, D)$ , as described in Lemma 7.10. But  $G$  is obtained from a graph  $G_0$ , either by adding a single edge (if  $e$  is a righthand column), or by adding no edges at all (if  $e$  is a lefthand column). If the second case applies, then the set of edges incident with the vertex 5 is a triad of  $M(G, D)$ , and of  $M_B/a$ , by Proposition 7.11. Thus  $M_B$  contains a triad, a contradiction.

We may now assume that  $e$  is a righthand column, and that we obtain  $G$  by adding the edge  $e$  to the graph  $G_0$ . It is easy to check that all the dashed edges in Figure 11 must represent empty paths, for otherwise  $G_0$  has at least three internal vertices (other than 2 and 13) of degree two or three. This



means that  $G$  contains an internal vertex of degree at most three, so  $M_B/a$  has a cocircuit of size at most three. This argument shows that  $A_a$  has no lower rows. A lower row of  $A$  is also a lower row of  $A_a$ , so this argument shows that  $A$  has no lower row, and that therefore  $A_{22} = \emptyset$ .

We now know that  $G_0$  has exactly three internal vertices with degree three: those in Figure 11 labeled by 5, 11, and 13. Proposition 7.11 implies that in the edge  $e$  must join 5 and 11 in  $G$ . Now  $\{a_5, b_5, e\}$  is a triangle of  $M(G, D) = M_B/A_{11}$ . By considering the matrix in Figure 10, we see that the column  $e$  has non-zero entries in rows  $a_3$  and  $a_4$ , and that if  $s$  is any other row in  $A_2$ , then  $e$  has a zero in row  $s$ .

Suppose now that  $e$  has a zero entry in row  $a$ . Then  $e$  contains precisely two non-zero entries: in rows  $a_3$  and  $a_4$ . This means that  $\{a_5, b_5, e\}$  is a triangle of  $M_B$ , and we have a contradiction. Therefore  $e$  contains precisely three non-zero entries: in rows  $a, a_3$ , and  $a_4$ .

Now we suppose that  $A_{11} - a$  is non-empty, and that  $a'$  is an element in this set. We let  $A_{a'}$  be the matrix obtained from  $A$  by adding the column  $e$ , and deleting the row  $a'$ . As before,  $A_{a'}$  is a reduced representation of the almost-regular matroid  $M_B/a'$ , and  $A_{a'}$  must have the form described in Lemma 7.9. But  $e$  has a non-zero entry in  $A_{11} - a'$ , so  $e$  cannot be a righthand column of  $A_{a'}$ . Thus  $e$  is a lefthand column of  $A_{a'}$ , and  $M_B/A_{11}$  is equal to a graft  $M(G', D')$ . In this case,  $G'$  is obtained from a graph  $G_0$  by adding no edges. Thus  $M(G', D')$  contains a triad at the internal vertex 5, and hence  $M_B/a'$  contains a triad by Proposition 7.11. This contradiction means that  $A_1 - a$  is empty.

We have shown that  $A_{22} = \emptyset$ , and that  $|A_{11}| = 1$ . Since  $B_{11} \cup B_{22} = \emptyset$ , we conclude that  $|E(M_B \setminus e)| = 14$ . Thus  $|E(M_B)| < 16$ , and Corollary 5.17 implies that  $|E(M_B)| \leq 12$ . This is a contradiction as  $M_B$  has a proper  $R_{12}$ -minor.  $\square$

By virtue of 7.12.4, there is a column  $b \in B_{11} \cup B_{22}$ . Consider the matrix  $A_b$  produced by adding the column  $e$  to  $A$  and then deleting  $b$ . Then  $A_b$  is a reduced representation  $M_B \setminus b$ , an almost-regular matroid with a  $V_{13}^*$ -minor. Thus  $A_b$  is of the form described in Lemma 7.9. Thus  $e$  is either a righthand or a lefthand column of  $A_b$ . We say that  $e$  is a right or lefthand element, according to which of these cases is true. Clearly this definition is independent of our choice of  $b$ .

By 7.12.3, there is a row  $a$  in  $A_{11}$ . Let  $A_a$  be the matrix obtained from  $A$  by adding the column  $e$ , and deleting  $a$ . Thus  $A_a$  is a reduced representation of the almost-regular matroid  $M_B/a$ . If  $e$  is a lefthand element, then let  $B'_{11} = B_{11} \cup e$ , and otherwise let  $B'_{11} = B_{11}$ . Now consider  $M_B/A_{11} \setminus B'_{11}$ . Lemma 7.10 says that this matroid is equal to a graft  $M(G, D)$ , where  $G$  is a member of the family illustrated in Figure 11.

**7.12.5.** *The vertices 1 and 14 have degree three in  $G$ .*

*Proof.* Let  $X$  be the set of edges that are incident with 1 in  $G$ . Assume that  $X - \{a_1, a_2, b_1\}$  is non-empty, and let  $b$  be an element of this set, so

that either  $b \in B_{22}$  or  $b = e$  (in which case  $e$  is a righthand element). Then  $X \cup b_2$  is a cocircuit of  $M(G, D) = M_B/A_{11} \setminus B'_{11}$ . By examining the matrix in Figure 10, we see that this means that the column of  $A$  labeled by  $b$  has a non-zero entry in the one of the rows labeled by  $a_1$  or  $a_2$ . This means that  $b$  cannot be equal to  $e$ , for if it were  $e$  would not be a righthand element. Thus  $b \in B_{22}$ , and this contradicts the fact that  $A[A_1, B_2]$  is the zero matrix.

Now let  $X$  be the set of edges incident with 14. If  $X - \{a_2, b_3, b_4\}$  is non-empty, we can deduce, using the same type of argument, that either  $e$  is a righthand column, and has a non-zero entry in row  $a_2$ , or that some member of  $B_{22}$  has a non-zero entry in row  $a_2$ . In either case, we have a contradiction that completes the proof.  $\square$

Let  $G'$  be the graph obtained from  $G$  by deleting  $a_1, a_2$ , and  $b_3$ . We obtain  $G''$  from  $G'$  by contracting  $b_1$  and  $b_4$ , and possibly two other edges: if vertex 2 has degree two in  $G'$ , then we contract both of its incident edges, and if 8 has degree one in  $G'$ , then we contract its incident edge. Every vertex in  $G''$  must be incident with at least one cross edge, for otherwise  $G$  contains an internal vertex with degree two and, in this case, Proposition 7.11 would imply that  $M_B/a$ , and hence  $M_B$ , contains a series pair. Certainly the two vertex-disjoint paths in  $G''$  contain at least two edges each, so we can apply Proposition 7.6 to  $G''$ .

If  $G''$  has an XX-minor, then  $M/A_{11} \setminus B'_{11}$  has a minor isomorphic to  $AG(3, 2)$ , and is therefore neither regular nor almost-regular. This contradicts Proposition 2.16. The internal vertices of  $G''$  are internal vertices of  $G$ , and 2 and 13 are not internal vertices of  $G''$ . The degree of an internal vertex in  $G''$  equals its degree in  $G$ . Therefore no internal vertex of  $G''$  has degree three, by Proposition 7.11 and 7.12.1. We conclude from Proposition 7.6 that  $G''$  contains a triangle  $T$ . Now  $G''$  can be obtained from  $G' = G \setminus a_1 \setminus a_2 \setminus b_3$  by contracting pendant edges, so  $T$  is also a triangle of  $G'$ , and hence of  $G$ .

Clearly  $T$  must contain at least one element corresponding to a column of  $A_a$ . Since  $T$  is a triangle of  $G''$ , it does not contain  $b_1, b_2, b_3$ , or  $b_4$ . Thus any column contained in  $T$  is either a member of  $B_2$ , or is equal to  $e$  (in which case  $e$  is a righthand element). This implies that any column in  $T$  has zero entries in any row in  $A_{11}$ . It follows that  $T$  is a triangle of  $M[A] = M_B$ . This contradiction completes the proof of the lemma.  $\square$

**7.3. The no  $R_{10}$  and no  $R_{12}$  case.** The two previous sections mean that we now need only consider the case that the binary matroid  $M_B$  has no minor isomorphic to  $R_{10}$  or  $R_{12}$ .

**Lemma 7.13.** *Suppose that  $N$  is an internally 4-connected almost-regular matroid and assume that  $N$  has no  $R_{10}$ - or  $R_{12}$ -minor. Suppose also that  $N = M(G, D)$  for some graft  $(G, D)$ . If  $D$  is minimal under switching, then  $|D| = 4$  and  $G = (R, S)$  is a Truemper graph. Moreover*

- (i) *the set  $\text{del}$  consists of all path edges, along with the graft element;*

- (ii) the set  $\text{con}$  consists of all cross edges; and
- (iii) the vertices in  $D$  are precisely the terminal vertices of  $G$ .

*Proof.* Theorem 23.41 of [27] proves this result in the case that  $N$  is an irreducible almost-regular matroid. A close examination of [27, Section 23] up to the proof of Theorem 23.41 reveals that the hypothesis of  $N$  being irreducible is not needed. Truemper shows that an irreducible almost-regular matroid is necessarily internally 4-connected [27, Theorem 22.1], and the proof of Theorem 23.41 holds under the weaker hypothesis that  $N$  is internally 4-connected.  $\square$

**Lemma 7.14.** *Let  $G = (R, S)$  be a Truemper graph with no XX-minor. Assume that the cross edges of  $G$  form a spanning path  $P$  and that the end-vertices of  $P$  are terminal vertices of  $G$ . If both  $R$  and  $S$  contain at least four vertices, then  $G$  contains distinct triangles  $T_1, T_2$ , and  $T_3$ , two of which are edge-disjoint.*

*Proof.* Assume that  $G$  is a minimal counterexample to the proposition. Thus  $|V(R)| \geq 4$  and  $|V(S)| \geq 4$ . Suppose the terminal vertices of  $G$  are  $\{v_1, v_2, v_3, v_4\}$  and that the end-vertices of  $P$  are  $v_1$  and  $v_4$ . Let  $e_1$  and  $e_4$  respectively be the cross edges incident with  $v_1$  and  $v_4$ . Now  $v_2$  and  $v_3$  are incident with exactly two cross edges each. It follows that we can find distinct cross edges  $e_2$  and  $e_3$  such that  $e_2$  is incident with  $v_2$  and  $e_3$  is incident with  $v_3$ , and neither  $e_2$  nor  $e_3$  joins  $v_2$  to  $v_3$ . Since no cross edge joins  $v_1$  to  $v_4$ , we conclude, by applying Proposition 2.20 to  $\{e_1, e_2, e_3, e_4\}$ , that one of  $v_1$  or  $v_4$  is adjacent to one of  $v_2$  or  $v_3$ . We will assume without loss of generality that  $v_1$  is adjacent to  $v_2$ .

Suppose that  $\max\{|V(R)|, |V(S)|\} > 4$ . If  $|V(R)| = |V(S)|$ , then  $R$  and  $S$  each contain one of the vertices  $v_1$  and  $v_4$ . In this case, we will assume by relabelling if necessary that  $v_1$  is in  $R$ . If  $|V(R)| \neq |V(S)|$ , then let us assume, by relabelling if necessary, that  $|V(R)| > |V(S)|$ . In this case, both  $v_1$  and  $v_4$  are contained in  $R$ . Thus  $v_1$  is in  $R$  and  $|V(R)| > 4$  in either case, so  $R - v_1$  contains at least four vertices. Moreover,  $P - v_1$  is a spanning path of  $G - v_1$  and the end-vertices of  $P - v_1$  are  $v_2$  and  $v_4$ , which are terminal vertices of the Truemper graph  $G - v_1 = (R - v_1, S)$ . By our assumption on the minimality of  $G$ , it follows that  $G - v_1$  contains distinct triangles  $T_1, T_2$ , and  $T_3$ , two of which are edge-disjoint. This implies that  $G$  is not a counterexample to the proposition, so we must assume that  $|V(R)| = |V(S)| = 4$ .

It remains only to show that the result holds when both  $R$  and  $S$  have exactly four vertices each. This is easily done: We simply construct all relevant Truemper graphs  $G = (R, S)$  where  $R$  and  $S$  have vertices  $r_1, r_2, r_3, r_4$  and  $s_1, s_2, s_3, s_4$  respectively. We identify  $(v_1, v_2, v_4)$  with  $(r_1, s_1, s_4)$ . Thus  $r_1$  is adjacent to  $s_1$  and the cross edges form a spanning path with end-vertices  $r_1$  and  $s_4$ . Ignoring automorphisms, there are exactly twelve such graphs. These are obtained from the graphs in Figure 13 on page 56 by deleting the extra edge joining  $r_1$  and  $s_4$ . Four of the twelve graphs have XX-minors,

marked by heavy edges. The remaining eight graphs each contain three triangles, two of which are edge-disjoint. Therefore the proposition holds in the case that  $|V(R)| = |V(S)| = 4$ , and hence holds in general.  $\square$

**Lemma 7.15.** *Let  $G = (R, S)$  be a Truemper graph and assume that the cross edges of  $G$  form a spanning cycle. Let the vertices of  $R$  and  $S$  be  $r_1, \dots, r_n$  and  $s_1, \dots, s_n$  respectively where  $n \geq 3$ . Assume that  $r_1$  is adjacent to both  $s_1$  and  $s_n$  and that  $s_n$  is not adjacent to  $r_2$ . Suppose that  $f$  is the edge  $r_1r_2$  and that  $g$  is the edge  $s_1s_2$ . If  $s_1$  is not adjacent to  $r_2$ , then let  $G' = G/f$ . Otherwise let  $G' = G/f/g$ . In either case,  $G'$  is a 3-connected graph. Moreover, if  $T$  is the edge set of a triangle of  $G$  and  $T$  is also a triangle in  $G'$ , then  $G'/T$  is 2-connected.*

*Proof.* We start by proving the following claim.

**7.15.1.** *Suppose that  $u$  and  $v$  are distinct vertices of  $G$  and that  $\{r_1, s_1\} \cap \{u, v\} = \emptyset$ . There are three paths  $P_1, P_2$ , and  $P_3$ , such that  $u$  and  $v$  are the end-vertices of  $P_1, P_2$ , and  $P_3$ , and:*

- (i)  $P_1, P_2$ , and  $P_3$  are internally disjoint;
- (ii) at most one of  $P_1 - \{u, v\}, P_2 - \{u, v\}$ , and  $P_3 - \{u, v\}$  meets  $\{r_1, r_2\}$ ;
- (iii) if  $s_1$  is adjacent to  $r_2$ , then at most one of  $P_1 - \{u, v\}, P_2 - \{u, v\}$ , and  $P_3 - \{u, v\}$  meets  $\{s_1, s_2\}$ ; and
- (iv) if  $T$  is a triangle of  $G$ , then at most two of  $P_1 - \{u, v\}, P_2 - \{u, v\}$ , and  $P_3 - \{u, v\}$  meet the vertices of  $T$ .

*Proof.* The proof of the claim is divided into several cases and subcases.

**Case 1.**  $u = s_i$  and  $v = s_j$  where  $1 < i < j \leq n$ .

We let  $P_1$  be the path  $s_i, \dots, s_j$  and let  $P_2$  be the path with vertex sequence  $s_i, \dots, s_1, r_1, s_n, \dots, s_j$ . Since every vertex of  $G$  is incident with two cross edges, there are vertices  $r_{i_1}$  and  $r_{j_1}$  such that  $s_i r_{i_1}$  and  $s_j r_{j_1}$  are edges. Since  $1 < i < n$ , it follows that  $r_1$  is not adjacent to  $s_i$ . Thus we can choose  $i_1$  so that  $2 < i_1$ . Similarly, by using the assumption that  $s_n$  is not adjacent to  $r_2$ , we can assume that  $2 < j_1$ . We let  $P_3$  be the path formed by  $s_i r_{i_1}$  and  $s_j r_{j_1}$  and the segment of  $R$  between  $r_{i_1}$  and  $r_{j_1}$ .

It is easy to see that condition (i) is satisfied. Since  $2 < i_1, j_1$ , it also follows that (ii) is satisfied, and it is clear that (iii) holds. To see that condition (iv) is satisfied, we note that the vertex set of any triangle in  $G$  contains either two adjacent vertices in  $R$  or two adjacent vertices in  $S$ . Since  $2 < i_1, j_1$ , it follows that no triangle of  $G$  can meet all three of the sets  $P_1 - \{u, v\}, P_2 - \{u, v\}$ , and  $P_3 - \{u, v\}$ .

**Case 2.**  $u = r_i$  and  $v = r_j$ , where  $1 < i < j \leq n$ .

We let  $P_1$  be the path  $r_i, \dots, r_j$ . Assume that  $u$  is adjacent to  $s_{i_1}$  and  $s_{i_2}$  and that  $v$  is adjacent to  $s_{j_1}$  and  $s_{j_2}$  where  $1 \leq i_1 < i_2 \leq n$  and  $1 \leq j_1 < j_2 \leq n$ .

**Case 2.1.**  $j_1 \leq i_1$ .

In this case  $j_1 < i_2$ . We let  $P_2$  be the path  $r_i, \dots, r_1, s_1, \dots, s_{j_1}, r_j$  and we let  $P_3$  be the path formed from  $r_i s_{i_2}$ ,  $r_j s_{j_2}$ , and the segment of  $S$  between  $s_{i_2}$  and  $s_{j_2}$ . It is clear that conditions (i) and (ii) are satisfied. If more than one of these three sets has a non-empty intersection with  $\{s_1, s_2\}$ , then  $j_1 = 1$  and either  $i_2 = 2$  or  $j_2 = 2$ . As  $j_1 = 1$ , we have  $s_1$  adjacent to both  $r_1$  and  $r_j$ , and therefore  $s_1$  is not adjacent to  $r_2$ . Thus (iii) is satisfied.

If condition (iv) is violated, then  $j_1 + 1 \in \{i_2, \dots, j_2\}$ , and some triangle contains  $s_{j_1}, s_{j_1+1}$ , and a vertex  $w$  in  $\{r_{i+1}, \dots, r_{j-1}\}$ . Thus either  $i_2 = j_1 + 1$  or  $j_2 = j_1 + 1$ . In the first case,  $i_1 = j_1$ , so the only vertices in  $R$  that  $s_{j_1}$  is adjacent to are  $r_i$  and  $r_j$ . Thus the triangle cannot exist. In the second case, the cross edges contain the cycle  $\{ws_{j_1}, s_{j_1}r_j, r_j s_{j_2}, s_{j_2}w\}$ . This is a contradiction as  $n \geq 3$  and the cross edges form a spanning cycle.

**Case 2.2.**  $i_2 \leq j_2$  and  $i_1 < j_1$ .

In this case,  $i_1 < j_2$ . We let  $P_2$  be the path  $r_i, \dots, r_1, s_n, \dots, s_{j_2}, r_j$  and we let  $P_3$  be the path formed from  $r_i s_{i_1}$  and  $r_j s_{j_1}$  and the segment of  $S$  between  $s_{i_1}$  and  $s_{j_1}$ . As in the previous case, it is easy to check that conditions (i) and (ii) hold. Moreover, (iii) holds as  $i_1 < j_1 < j_2$ .

If (iv) is violated, then  $j_2 - 1 \in \{s_{i_1}, \dots, s_{j_1}\}$  and there is a triangle with vertices  $s_{j_2}, s_{j_2-1}$  and  $w \in \{r_{i+1}, \dots, r_{j-1}\}$ . Thus  $j_1 = j_2 - 1$  and the cross edges contain the cycle  $\{ws_{j_1}, s_{j_1}r_j, r_j s_{j_2}, s_{j_2}w\}$ ; a contradiction.

**Case 2.3.**  $i_1 < j_1$  and  $j_2 < i_2$ .

We let  $P_2$  be  $r_i, s_{i_1}, \dots, s_{j_1}, r_j$  and we let  $P_3$  be  $r_i, s_{i_2}, \dots, s_{j_2}, r_j$ . Because  $j_1 < j_2$ , it follows that condition (i) holds, and it is obvious that (ii) and (iii) hold. The only way in which (iv) can fail is if  $j_2 = j_1 + 1$  and there is a triangle with vertices  $s_{j_1}, s_{j_2}$  and  $w \in \{r_{i+1}, \dots, r_{j-1}\}$ . In this case, the cross edges contain the cycle  $\{ws_{j_1}, s_{j_1}r_j, r_j s_{j_2}, s_{j_2}w\}$ .

**Case 3.**  $u = r_i$  and  $v = s_j$  where  $1 < i, j \leq n$ .

Suppose that  $u$  is adjacent to  $s_{i_1}$  and  $s_{i_2}$  where  $1 \leq i_1 < i_2 \leq n$  and that  $v$  is adjacent to  $r_{j_1}$  and  $r_{j_2}$ , where  $1 \leq j_1 < j_2 \leq n$ .

**Case 3.1.**  $i \leq j_2$ .

**Case 3.1.1.**  $j \leq i_2$

We let  $P_1$  be the path  $r_i, s_{i_2}, \dots, s_j$  and let  $P_2$  be the path  $r_i, \dots, r_{j_2}, s_j$ . We also let  $P_3$  be the path  $r_i, \dots, r_1, s_1, \dots, s_j$ . It is easy to see that conditions (i), (ii), (iii), and (iv) are satisfied.

**Case 3.1.2.**  $i_2 < j$ .

We let  $P_1$  be the path  $r_i, s_{i_2}, \dots, s_j$ , we let  $P_2$  be the path  $r_i, \dots, r_{j_2}, s_j$ , and we let  $P_3$  be the path  $r_i, \dots, r_1, s_n, \dots, s_j$ . In this case, the result holds.

**Case 3.2.**  $j_2 < i$ .

**Case 3.2.1.**  $i_1 \leq j$  and  $j \leq i_2$ .

We let  $P_1$  be the path  $r_i, s_{i_1}, \dots, s_j$ , we let  $P_2$  be the path  $r_i, s_{i_2}, \dots, s_j$ , and we let  $P_3$  be the path  $r_i, \dots, r_{j_2}, s_j$ . It is easy to see that conditions (i), (ii), (iii), and (iv) hold.

**Case 3.2.2.**  $j < i_1$ .

Let  $P_1$  be the path  $r_i, s_{i_1}, \dots, s_j$ , let  $P_2$  be the path  $r_i, \dots, r_{j_2}, s_j$ , and let  $P_3$  be the path  $r_i, s_{i_2}, \dots, s_n, r_1, \dots, r_{j_1}, s_j$ . Statement (i) holds. If (ii) fails, then  $j_2 = 2$ , so  $j_1 = 1$ . Since the only vertices in  $S$  adjacent to  $r_1$  are  $s_1$  and  $s_n$ , it follows that  $j = n$ . But then  $j_2 < i_1 \leq n$ , so we have a contradiction. Clearly (iii) is satisfied.

If condition (iv) fails, then either:  $j_2 = j_1 + 1$  and some triangle contains  $r_{j_1}, r_{j_2}$  and some vertex in  $\{s_{j_1+1}, \dots, s_{i_1}\}$ ; or  $i_2 = i_1 + 1$ , and some triangle contains  $s_{i_1}, s_{i_2}$ , and some vertex in  $\{r_{j_2}, \dots, r_{i_1-1}\}$ . In either of these cases, the set of cross edges contains a cycle of length four, which is a contradiction as we have assumed  $n \geq 3$ .

**Case 3.2.3.**  $i_2 < j$ .

Let  $P_1$  be the path  $r_i, s_{i_2}, \dots, s_j$ , let  $P_2$  be the path  $r_i, \dots, r_{j_2}, s_j$ , and let  $P_3$  be the path  $r_i, s_{i_1}, \dots, s_1, r_1, \dots, r_{j_1}, s_j$ . Clearly (i) is true. If (ii) is not true, then  $j_1 = 1$  and  $j_2 = 2$ . This implies that  $j = n$  and that  $s_n$  is adjacent to  $r_2$ , a contradiction. For (iii) to be false, we must have  $i_1 = 1$  and  $i_2 = 2$ , and  $s_1$  is adjacent to  $r_2$ . Thus  $r_i$  is adjacent to  $s_1$ . However,  $j_2 < i$ , so  $2 < i$ . Thus  $s_1$  is adjacent to three vertices in  $R$ :  $r_1, r_2$ , and  $r_i$ . This is a contradiction.

We again see that if (iv) fails then the cross edges of  $G$  contain a cycle of length four, a contradiction.

We have now exhausted all possible cases, so the claim must hold.  $\square$

We continue with the proof of the lemma. First suppose that  $s_1$  is adjacent to  $r_2$ . Then  $G' = G/f/g$ . Let  $T$  be an arbitrary triangle of  $G$  that is also a triangle in  $G'$ . Suppose that  $u'$  and  $v'$  are distinct vertices of  $G'$ . Let  $u$  and  $v$  be vertices of  $G$  that correspond to  $u'$  and  $v'$  respectively. Since  $r_1$  is identified with  $r_2$  and  $s_1$  is identified with  $s_2$  in  $G'$ , we may assume that  $\{u, v\} \cap \{r_1, s_1\} = \emptyset$ . Claim 7.15.1 says that there are three internally disjoint paths in  $G$  joining  $u$  to  $v$ , and conditions (ii) and (iii) imply that these paths lead to three internally disjoint paths in  $G'$  joining  $u'$  to  $v'$ . Since  $u'$  and  $v'$  were arbitrary distinct vertices in  $G'$ , this means that  $G'$  is 3-connected. Moreover, condition (iv) implies the existence of two internally disjoint paths in  $G'/T$  joining  $u'$  to  $v'$ . Thus  $G'/T$  is 2-connected.

Next we suppose that  $s_1$  is not adjacent to  $r_2$ . In this case,  $G' = G/f$ . Suppose that  $G'$  is not 3-connected. Then there are subsets  $X, Y \subseteq V(G')$  such that (i)  $X \cup Y = V(G')$ ; (ii)  $|X \cap Y| \leq 2$ ; (iii) neither  $X - Y$  nor  $Y - X$  is empty; and (iv) no edge of  $G'$  joins a vertex in  $X - Y$  to a vertex in  $Y - X$ .

Let  $u'$  and  $v'$  be vertices in  $X - Y$  and  $Y - X$  respectively, and let  $u$  and  $v$  be vertices of  $G$  which correspond to  $u'$  and  $v'$ . Since  $r_1$  is identified with  $r_2$  in  $G'$ , we may assume that neither  $u$  nor  $v$  is equal to  $r_1$ . If neither  $u$  nor

$v$  is equal to  $s_1$ , then Claim 7.15.1 implies there are three internally disjoint paths joining  $u$  to  $v$  in  $G$ , and that furthermore these paths lead to three internally disjoint paths from  $u'$  to  $v'$  in  $G'$ . This is a contradiction as any path from  $u'$  to  $v'$  contains a vertex in  $X \cap Y$ . Thus we assume that  $u = s_1$ . Since  $s_1$  is not incident with  $f$ , this means  $u' = s_1$ . As  $u'$  was an arbitrary vertex in  $X - Y$ , it follows that  $X - Y = \{u\}$ . Now any vertex that is adjacent with  $u$  in  $G'$  must be in  $X \cap Y$ . However,  $u$  is adjacent to distinct vertices  $s_2$ ,  $r_1$ , and  $r_i$  in  $G$ , where  $2 < i \leq n$ , and these three vertices are distinct in  $G'$ . Thus  $|X \cap Y| > 2$ , a contradiction.

Next we suppose that  $T$  is an arbitrary triangle of  $G$  and that  $T$  is a triangle in  $G'$ . Suppose that  $G'/T$  is not 2-connected. Then there are subsets  $X, Y \subseteq V(G'/T)$  such that: (i)  $X \cup Y = V(G'/T)$ ; (ii)  $|X \cap Y| \leq 1$ ; (iii) neither  $X - Y$  nor  $Y - X$  is empty; and (iv) no edge of  $G'/T$  joins a vertex in  $X - Y$  to a vertex in  $Y - X$ .

Assume that  $u'$  and  $v'$  are vertices in  $X - Y$  and  $Y - X$  respectively, and let  $u$  and  $v$  be corresponding vertices of  $G$ . We may assume that neither  $u$  nor  $v$  is  $r_1$ . If neither  $u$  nor  $v$  is  $s_1$ , then there are three internally disjoint paths between  $u$  and  $v$ , and these paths lead to two internally disjoint paths in  $G'/T$ , a contradiction. Thus  $u = s_1$ , without loss of generality, and if we assume that  $u$  is also a vertex of  $G'/T$ , then  $X - Y = \{u\}$ . Now every vertex adjacent to  $u$  must be in  $X \cap Y$ . Since  $|X \cap Y| \leq 1$ , this means that all vertices of  $G$  that are adjacent to  $u$  must be identified in  $G'/T$ . Thus the vertices of  $T$  are  $s_2$ ,  $r_1$ , and  $r_i$ . But  $n \geq 3$ , so  $r_1$  is not adjacent to  $s_2$ , and we have a contradiction. This completes the proof of the lemma.  $\square$

**Definition 7.16.** Suppose that  $M$  is a connected matroid. A triangle  $T$  of  $M$  is a *separating triangle* if  $M/T$  is not connected.

**Lemma 7.17.** Let  $G'$  be a 2-connected graph and let  $T_1$ ,  $T_2$ , and  $T_3$  be distinct non-separating triangles of  $M(G')$ . If  $M'$  is a single-element coextension of  $M(G')$ , and none of  $T_1$ ,  $T_2$ , or  $T_3$  is a triangle in  $M'$ , then  $M'$  is not cographic.

*Proof.* Assume that  $M'$  is a coextension of  $M(G')$  by the element  $e$ . Suppose that  $M'$  is cographic, so that  $M' = M^*(H)$  for some graph  $H$ . Now  $T_1$ ,  $T_2$ , and  $T_3$  are triads in

$$M^*(G') = (M'/e)^* = M(H \setminus e).$$

Thus  $T_1$ ,  $T_2$ , and  $T_3$  are minimal edge cut-sets in  $H \setminus e$ .

Let the two components of  $H \setminus e \setminus T_1$  be  $H_1$  and  $H_2$ . If both  $H_1$  and  $H_2$  contain at least one edge, then  $M(H \setminus e \setminus T_1)$ , and hence  $M^*(H \setminus e \setminus T_1)$ , is not connected. But  $M^*(H \setminus e \setminus T_1) = M(G')/T_1$ , so this contradicts the fact that  $T_1$  is not a separating triangle of  $M(G')$ . Thus we assume that  $H_1$  contains no edges. Every vertex in  $H \setminus e$  has degree at least two. It follows that  $H_1$  must contain a single vertex, so  $T_1$  is the set of edges incident with a vertex  $v_1$  in  $H \setminus e$ . The same argument implies that  $T_2$  and  $T_3$  are the sets of edges incident with vertices  $v_2$  and  $v_3$  in  $H \setminus e$ .

None of  $T_1$ ,  $T_2$ , or  $T_3$  is a triad in  $H$ , so  $e$  must be incident in  $H$  with the distinct vertices  $v_1$ ,  $v_2$ , and  $v_3$ , an impossibility.  $\square$

**Proposition 7.18.** *Suppose that  $M$  is an excluded minor for the class  $\mathcal{M}$  such that  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ , and that  $M_B$  is the binary matroid supplied by Theorem 5.1. Assume that  $M_B$  has no  $R_{10}$ -minor and no  $R_{12}$ -minor. Then there are distinct elements  $e$  and  $d$  in  $E(M_B)$  such that either  $M_B/e \setminus d$  or  $M_B^*/e \setminus d$  is graphic.*

*Proof.* Let  $e$  be an arbitrary element of  $E(M_B)$ . Then  $M_B/e$  is almost-regular by Theorem 6.1, so  $E(M_B/e)$  can be partitioned into non-empty *del* and *con* sets. Let  $d$  be an element in *del*. Then  $M_B/e \setminus d$  is regular.

Proposition 6.2 implies that  $M_B/e$  is internally 4-connected. If  $M_B/e \setminus d$  is not 3-connected, then  $M_B/e$  must contain a triad, which contradicts Corollary 5.17. As  $M_B$  has no  $R_{10}$ - or  $R_{12}$ -minor, Lemma 2.13 implies that  $M_B/e \setminus d$  is either graphic or cographic. If  $M_B/e \setminus d$  is graphic, then we are done. Therefore we assume that  $M_B/e \setminus d$  is cographic. In this case,

$$(M_B/e \setminus d)^* = M_B^*/d \setminus e$$

is graphic and the result follows by swapping the labels on  $e$  and  $d$ .  $\square$

We can now prove the main result in this part of the case analysis.

**Lemma 7.19.** *Let  $M$  be an excluded minor for the class  $\mathcal{M}$  with  $|E(M)| \geq 10$  and  $r(M), r^*(M) \geq 4$ . Let  $M_B$  be the binary matroid supplied by Theorem 5.1. If  $M_B$  has no  $R_{10}$ -minor and no  $R_{12}$ -minor, then  $|E(M)| \leq 16$ .*

*Proof.* Let us assume that  $|E(M)| > 16$ .

Corollary 5.17 implies the following fact:

**7.19.1.**  *$M_B$  has no triangles and no triads.*

By virtue of Proposition 7.18, and by switching to the dual if necessary, we will henceforth assume that  $e$  and  $d$  are distinct elements of  $E(M_B)$  such that  $M_B/e \setminus d$  is graphic. Thus  $M_B/e$  is almost-regular and a graft, where  $d$  is the graft element. By Proposition 6.2 and Lemma 7.13, we can assume that  $M_B/e = M(G, D)$ , where  $G = (R, S)$  is a Truemper graph and  $D$  is exactly the set of terminal vertices of  $G$ . Proposition 2.19 implies that:

**7.19.2.**  *$G$  has no  $XX$ -minor.*

By virtue of Corollary 5.17 (vi), we can relabel  $J$  and  $K$ , so we assume that  $e \in J$ . It follows from Theorem 6.1 that  $M_B/e$  is almost-regular with *del* =  $J - e$  and *con* =  $K$ . By Lemma 7.13, *del* =  $J - e$  consists of the path edges of  $G$  along with  $d$ ; *con* =  $K$  consists of the cross edges. Thus  $d \in J$ . But  $K$  is a spanning circuit of  $M_B/e$ . From this, we deduce the following.

**7.19.3.** *The paths  $R$  and  $S$  have the same length, and the cross edges form a spanning cycle of  $G$ .*



Suppose that the vertices of  $R$  and  $S$  are  $r_1, \dots, r_n$  and  $s_1, \dots, s_n$  respectively. Since we are assuming  $|E(M)| > 16$ , it follows that  $n \geq 5$ . By Corollary 2.21, we may assume the following without loss of generality.

**7.19.4.**  $r_1$  is adjacent to both  $s_1$  and  $s_n$ .

Both  $s_1$  and  $s_n$  cannot be adjacent to  $r_2$ , otherwise the cross edges contain the cycle  $\{r_1s_1, s_1r_2, r_2s_n, s_nr_1\}$ . Thus we will assume the following.

**7.19.5.**  $s_n$  is not adjacent to  $r_2$ .

Let  $f$  and  $g$  be the edges  $r_1r_2$  and  $s_1s_2$  respectively. First let us suppose that  $s_1$  is adjacent to  $r_2$ . Let  $G'$  be  $G/f/g$  and let  $M'$  be  $M_B \setminus d/f/g$ . Thus  $M'$  is a coextension of  $M(G')$  by the element  $e$ .

Note that  $G - \{r_1, s_1\}$  is a subgraph of  $G'$ . Furthermore,  $G - \{r_1, s_1\} = (R - r_1, S - s_1)$  is a Truemper graph and both  $R - r_1$  and  $S - s_1$  contain at least four vertices. The cross edges of  $G - \{r_1, s_1\}$  form a spanning path joining the terminal vertex  $r_2$  to the terminal vertex  $s_n$ . From Lemma 7.14, we conclude that  $G - \{r_1, s_1\}$ , and hence  $G'$ , contains distinct triangles,  $T_1$ ,  $T_2$ , and  $T_3$ , such that at least two of these triangles are edge-disjoint.

Since  $T_1$ ,  $T_2$ , and  $T_3$  are triangles of  $M_B \setminus e \setminus d$ , but  $M_B$  has no triangles,  $T_1 \cup e$ ,  $T_2 \cup e$ , and  $T_3 \cup e$  are circuits of  $M_B$ . Hence each  $T_i \cup e$  is a union of non-loop circuits of  $M_B \setminus d/f/g = M'$ . As  $T_i$  is a circuit of  $M'/e$ , it follows that  $T_i \cup e$  is a circuit of  $M'$  for each  $i$  in  $\{1, 2, 3\}$ .

Since  $T_1$ ,  $T_2$ , and  $T_3$  are triangles of  $G - \{r_1, s_1\}$ , and hence of  $G$ , Lemma 7.15 implies that  $G'/T_i$  is 2-connected for all  $i \in \{1, 2, 3\}$ . Thus  $T_1$ ,  $T_2$ , and  $T_3$  are non-separating triangles of  $M(G')$ .

As  $d$  is a member of  $J$ , Theorem 6.1 states that  $M_B \setminus d$  is almost-regular with  $del = K$  and  $con = J - d$ . Both  $f$  and  $g$  are path edges of  $G$ , and are therefore in  $J - e$ , so  $f, g \in con$ . Thus  $M' = M_B \setminus d/f/g$  is regular. Furthermore,  $G'$  is a 3-connected graph by Lemma 7.15, and  $M'$  is a single-element coextension of  $M(G')$ . It is not difficult to check that  $M'$  can be obtained from a 3-connected matroid  $M''$  by a sequence of parallel or series extensions. Since  $M'$  has no  $R_{10}$ - or  $R_{12}$ -minor, Lemma 2.13 tells us that  $M''$  is either graphic or cographic. Therefore  $M'$  is either graphic or cographic.

Lemma 7.17 implies that  $M'$  is not cographic. So we may assume that  $M'$  is graphic. Thus  $M' = M(H)$  for some graph  $H$  so that  $e$  is an edge of  $H$  and  $M(H/e) = M(G')$ . Since  $G'$  is 3-connected by Lemma 7.15, it follows from Whitney's 2-isomorphism theorem that  $H/e = G'$ .

Suppose that  $e$  is incident with vertices  $v_0$  and  $v_1$  in  $H$ , and let  $v$  be the vertex of  $H/e = G'$  that results from identifying  $v_0$  and  $v_1$ .

We will suppose that  $v$  has degree at most four. Since  $M_B$  is 3-connected having no triads and  $M(G') = M_B \setminus d/e/f/g$ , both  $v_0$  and  $v_1$  have degree three in  $H$ . Thus if  $T$  is the set of edges incident with  $v_0$  in  $H$ , then  $T \cup d$  is a cocircuit of  $M_B$ . Moreover, as both  $e$  and  $d$  are contained in  $J$ , and both  $J$  and  $K$  are circuits of  $M_B$ , it follows that either  $T - e \subseteq J$  or  $T - e \subseteq K$ . If  $T - e \subseteq J$ , then  $J$  contains the cocircuit  $T \cup d$  and, as  $J$  is a cocircuit

of  $M_B$ , this means that  $J = T \cup d$ . This implies that  $|E(M_B)| = 2|J| = 8$ , a contradiction. Therefore  $T - e \subseteq K$ , so the two edges other than  $e$  that are incident with  $v_0$  in  $H$  are members of  $K$ , implying that they are cross edges of  $G$ . The same argument shows that two edges other than  $e$  that are incident with  $v_1$  in  $H$  are cross edges of  $G$ . Thus  $v$  is incident with precisely four edges in  $G'$ , and they are all cross edges of  $G$ . But no such vertex of  $G'$  exists, so we conclude that  $v$  has degree at least five in  $G'$ .

We may assume that  $r_2$  and  $s_2$  are vertices of  $G'$ . Then they are the only two vertices of degree at least five. Thus  $v = r_2$  or  $v = s_2$ . Since  $T_1 \cup e$ ,  $T_2 \cup e$ , and  $T_3 \cup e$  are circuits in  $M' = M(H)$ , it follows that all of  $T_1$ ,  $T_2$ , and  $T_3$  are incident with  $v$  in  $H/e = G'$ , and hence in  $G - \{r_1, s_1\}$ . But  $r_2$  and  $s_2$  have degree at most three in  $G - \{r_1, s_1\}$ , so no pair of triangles in  $\{T_1, T_2, T_3\}$  can be edge-disjoint, a contradiction.

This completes the argument in the case that  $s_1$  is adjacent to  $r_2$ . The argument when  $s_1$  is not adjacent to  $r_2$  is very similar: Let  $G'$  be  $G/f$  and let  $M'$  be  $M_B \setminus d/f$ . Both  $R - r_1$  and  $S$  contain at least four vertices, and  $G - r_1 = (R - r_1, S)$  is a Truemper graph in which the cross edges form a spanning path joining two terminal vertices. Thus  $G - r_1$ , and hence  $G'$ , contains distinct triangles  $T_1$ ,  $T_2$ , and  $T_3$ , two of which are edge-disjoint. None of these triangles are triangles in  $M'$ .

We observe that  $M_B \setminus d$  is almost-regular with  $del = K$  and  $con = J - d$ . Since  $f \in J - d$ , it follows that  $M'$  is regular. Hence  $M'$  is graphic or cographic. Since  $T_1$ ,  $T_2$ , and  $T_3$  are non-separating triangles of  $M(G')$ , it follows that  $M'$  is not cographic.

We now know that  $M' = M(H)$  for some graph  $H$ , where  $H/e = G'$ . If  $v$  is the vertex of  $H/e$  formed by identifying the two end-vertices of  $e$ , then  $v$  must have degree at least five, so  $v = r_2$ . Thus  $T_1$ ,  $T_2$ , and  $T_3$  are incident with  $r_2$  in  $G - r_1$ . However,  $r_2$  has degree three in  $G - r_1$ , so no two of  $T_1$ ,  $T_2$ , and  $T_3$  are edge-disjoint, a contradiction.

This completes the proof of the lemma.  $\square$

## 8. CASE-CHECKING

The results in Section 7 mean that the proof of our main theorem is reduced to a finite case check. In this section we develop the tools required for such a check, and we prove our principal result. We start by deducing some information about representations of the binary matroid  $M_B$ .

**Lemma 8.1.** *Suppose that  $M$  is an excluded minor for  $\mathcal{M}$  such that  $|E(M)| \geq 10$ , while  $r(M), r^*(M) \geq 4$ . Let  $r = r(M)$ , and let  $M_B$  be the rank- $r$  binary matroid supplied by Theorem 5.1, so that  $M_B$  contains two disjoint circuit-hyperplanes,  $J$  and  $K$ . For all  $j$  in  $J$  and all  $k$  in  $K$ , there is a matrix  $A(j, k)$  such that  $M_B$  is represented over  $\text{GF}(2)$  by the following matrix.*

$$\begin{array}{c} j \quad K-k \quad k \quad J-j \\ \left[ \begin{array}{c|c|c|c} 1 & \mathbf{0}^T & 0 & \mathbf{1}^T \\ \hline \mathbf{0} & I_{r-1} & \mathbf{1} & A(j, k) \end{array} \right] \end{array}$$

*Proof.* It is clear that  $(K-k) \cup j$  is a basis of  $M_B$ . Moreover  $(K-k) \cup k = K$  is a circuit, and no element of  $J-j$  is spanned by  $K-k$ . The result follows.  $\square$

Before proving the next result, we give an alternative reduced representation of  $T'_{12}$ .

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

This representation is obtained from the one in the introduction by pivoting so that, if the columns in the original representation are ordered  $1, 2, \dots, 12$ , those in the new representation are ordered  $5, 2, 10, 4, 6, 8, 12, 11, 3, 9, 1, 7$ .

**Lemma 8.2.**  $T'_{12}$  is the unique 12-element excluded minor for  $\mathcal{M}$ .

*Proof.* Let  $M$  be a 12-element excluded minor for  $\mathcal{M}$ . Then  $M$  is 3-connected, and  $r(M), r^*(M) \geq 4$ , by Lemmas 3.2 and 3.3. Theorem 5.1 implies that there is a binary matroid  $M_B$  having two complementary circuit-hyperplanes,  $J$  and  $K$ , such that  $M$  is obtained from  $M_B$  by relaxing  $J$ . Corollary 5.17 implies that  $r(M_B) = r^*(M_B) = 6$ .

We start by proving that  $M_B$  has no  $R_{10}$ -minor. Assume otherwise. We can assume that there is an element  $e \in E(M_B)$  such that  $M_B \setminus e$  has an  $R_{10}$ -minor. Theorem 6.1 and Proposition 6.2 imply that  $M_B \setminus e$  is an internally 4-connected almost-regular matroid. As  $|E(M_B \setminus e)| = 11$  and  $r(M_B \setminus e) = 6$ , Lemma 7.4 implies that  $M_B \setminus e \cong N_{11}^*$ . Therefore  $M_B \setminus e$  is represented by  $[I_6|A]$ , where  $A$  is the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Assume the columns of  $[I_6|A]$  are labeled  $1, \dots, 11$ . It is routine to check that  $M_B \setminus e$  has a unique circuit-hyperplane, namely  $\{1, 2, 7, 8, 9, 11\}$ . Thus  $\{3, 4, 5, 6, 10, e\}$  is a circuit of  $M_B$ . But  $\{3, 4, 5, 10\}$  is a circuit of  $M_B$ , so  $\{6, e\}$  must also be a circuit of  $M_B$ , and this contradicts the fact that  $M_B$  is 3-connected. Therefore  $M_B$  has no  $R_{10}$ -minor, as desired.

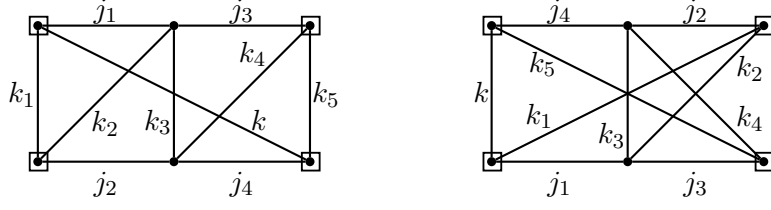


FIGURE 12. Two Truemper graphs.

Lemma 7.12 says that  $M_B$  has no  $R_{12}$ -minor. By using Proposition 7.18 and duality, we can assume there are distinct elements  $e, d \in E(M_B)$  such that  $M_B/e \setminus d$  is graphic. By Corollary 5.17 (vi), we assume that  $e \in J$ . As  $M_B/e$  is almost-regular and internally 4-connected, Lemma 7.13 says that it is isomorphic to a graft  $M(G, D)$ , where  $G = (R, S)$  is a Truemper graph. As  $(\text{del}, \text{con}) = (J - e, K)$  by Theorem 6.1, the cross edges of  $G$  comprise  $K$ , and therefore form a spanning cycle of  $G$ . Thus  $R$  and  $S$  both contain exactly three vertices. Since  $G$  has no  $XX$ -minor, we can assume by Corollary 2.21 that  $r_1$  is adjacent to both  $s_1$  and  $s_3$ . We enumerate the Truemper graphs having these properties, and we see that  $G$  must be one of the two (isomorphic) graphs in Figure 12. In either case, we let  $j = e$ , and we let  $k$  be the edge labeled as such in Figure 12. If the elements of  $K - k$  and  $J - j$  are ordered  $k_1, \dots, k_5$  and  $j_1, \dots, j_5$  respectively (where  $j_5$  is the graft element  $d$ ), then  $A(j, k)$  is the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus  $M_B$  is isomorphic to  $T_{12}$ , so  $M$  is isomorphic to  $T'_{12}$ , as desired.  $\square$

**Lemma 8.3.** *There is no 16-element excluded minor for  $\mathcal{M}$ .*

*Proof.* Suppose that  $M$  is a 16-element excluded minor for  $\mathcal{M}$ , and that  $M_B$  is the binary matroid appearing in Theorem 5.1. Recall that  $\text{AG}(3, 2)$  has the following reduced representation.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

We will deduce that  $M_B$  has a minor isomorphic to  $\text{AG}(3, 2)$ . Since every proper minor of  $M_B$  is either regular or almost-regular (Proposition 2.16 and Theorem 6.1), and  $\text{AG}(3, 2)$  is neither, this will yield a contradiction.

Let  $J$  and  $K$  be the complementary circuit-hyperplanes of  $M_B$ . Now  $M_B$  has no  $R_{10}$ - or  $R_{12}$ -minor, by Lemmas 7.5 and 7.12. As in the proof of

Lemma 8.2, we deduce that, up to duality, there are elements  $e$  and  $d$  in  $M_B$  such that  $M_B/e \setminus d$  is graphic, and  $M_B/e \cong M(G, D)$ . Here  $G = (R, S)$  is a Truemper graph, the cross edges of  $G$  form a spanning path, and both  $R$  and  $S$  have exactly four vertices. We assume that  $e \in J$ . We also assume that  $r_1$  is adjacent to both  $s_1$  and  $s_4$ . The twelve Truemper graphs satisfying these constraints are enumerated in Figure 13 (we ignore symmetries).

Four of these Truemper graphs have XX-minors, and so can be disregarded. In the remaining cases, we assume that  $j = e$ . One of the edges in  $G$  is labeled by  $k$ . We also assume that the elements of  $J - j$  and  $K - k$  are ordered  $j_1, \dots, j_7$  and  $k_1, \dots, k_7$  respectively (where  $j_7$  is the graft element  $d$ ). Now it is easy to see that  $A(j, k)$  is one of the following three matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

In each of these three cases, we demonstrate that  $M_B$  has an AG(3, 2)-minor.

Recall that  $M_B$  has the following reduced representation.

$$A = \left[ \begin{array}{c|c} 0 & \mathbf{1}^T \\ \hline \mathbf{1} & A(j, k) \end{array} \right].$$

Suppose that  $A(j, k)$  is equal to the first of the three matrices above. We pivot on the entry of  $A$  in the second row and first column, and the entry in the first row and second column. By deleting all rows from  $A$  except the first, fourth, fifth, and seventh, and all columns except the first, third, fifth, and sixth, we see that  $M_B$  has an AG(3, 2)-minor.

Now suppose  $A(j, k)$  is the second displayed matrix. We pivot on the entry in the first row and second column of  $A$ , and then delete all rows except the first, second, sixth, and eighth, and all columns except the first, fourth, seventh, and eighth. This reveals an AG(3, 2)-minor. Similarly, if  $A(j, k)$  is the third displayed matrix, we pivot on the entry of  $A$  in the first row and second column, and then delete all rows except the first, fourth, sixth, and seventh, and all columns except the first, fourth, fifth, and seventh.  $\square$

We are now ready to prove our main theorem, which we restate here.

**Theorem 8.4.** *The excluded minors for the class of matroids that are binary or ternary are  $U_{2,5}$ ,  $U_{3,5}$ ,  $U_{2,4} \oplus F_7$ ,  $U_{2,4} \oplus F_7^*$ ,  $U_{2,4} \oplus_2 F_7$ ,  $U_{2,4} \oplus_2 F_7^*$ , AG(3, 2)', and  $T'_{12}$ .*

*Proof.* Let  $M$  be an excluded minor for  $\mathcal{M}$ . If  $M$  is not 3-connected, or if the rank or corank of  $M$  is less than four, then  $M$  is isomorphic to one of  $U_{2,5}$ ,  $U_{3,5}$ ,  $U_{2,4} \oplus F_7$ ,  $U_{2,4} \oplus F_7^*$ ,  $U_{2,4} \oplus_2 F_7$ , or  $U_{2,4} \oplus_2 F_7^*$ , by Lemmas 3.2 and 3.3. Thus we assume that  $M$  is 3-connected, and that  $r(M)$ ,  $r^*(M) \geq 4$ . Hence

$|E(M)| \geq 8$ . If  $|E(M)| = 8$  then  $M \cong \text{AG}(3, 2)'$ , by Lemma 4.1. Thus we assume that  $|E(M)| \geq 9$ . This implies that  $|E(M)| \geq 10$ , by Lemma 4.6.

Now we apply Theorem 5.1 to deduce the existence of a binary matroid  $M_B$  such that  $M$  is obtained from  $M_B$  by relaxing a circuit-hyperplane. Lemma 7.12 says that  $M_B$  has no  $R_{12}$ -minor. If  $M_B$  has an  $R_{10}$ -minor, then  $|E(M)| = 12$ , by Lemma 7.5. On the other hand, if  $M_B$  has no  $R_{10}$ -minor, then  $|E(M)| \leq 16$ , by Lemma 7.19. Therefore we have established that  $|E(M)| \leq 16$ . Corollary 5.17 implies that we need only consider the case that  $|E(M)| = 12$  or 16. If  $|E(M)| = 12$  then  $M \cong T'_{12}$ , by Lemma 8.2, and Lemma 8.3 implies that  $|E(M)| \neq 16$ . Therefore the proof is complete.  $\square$

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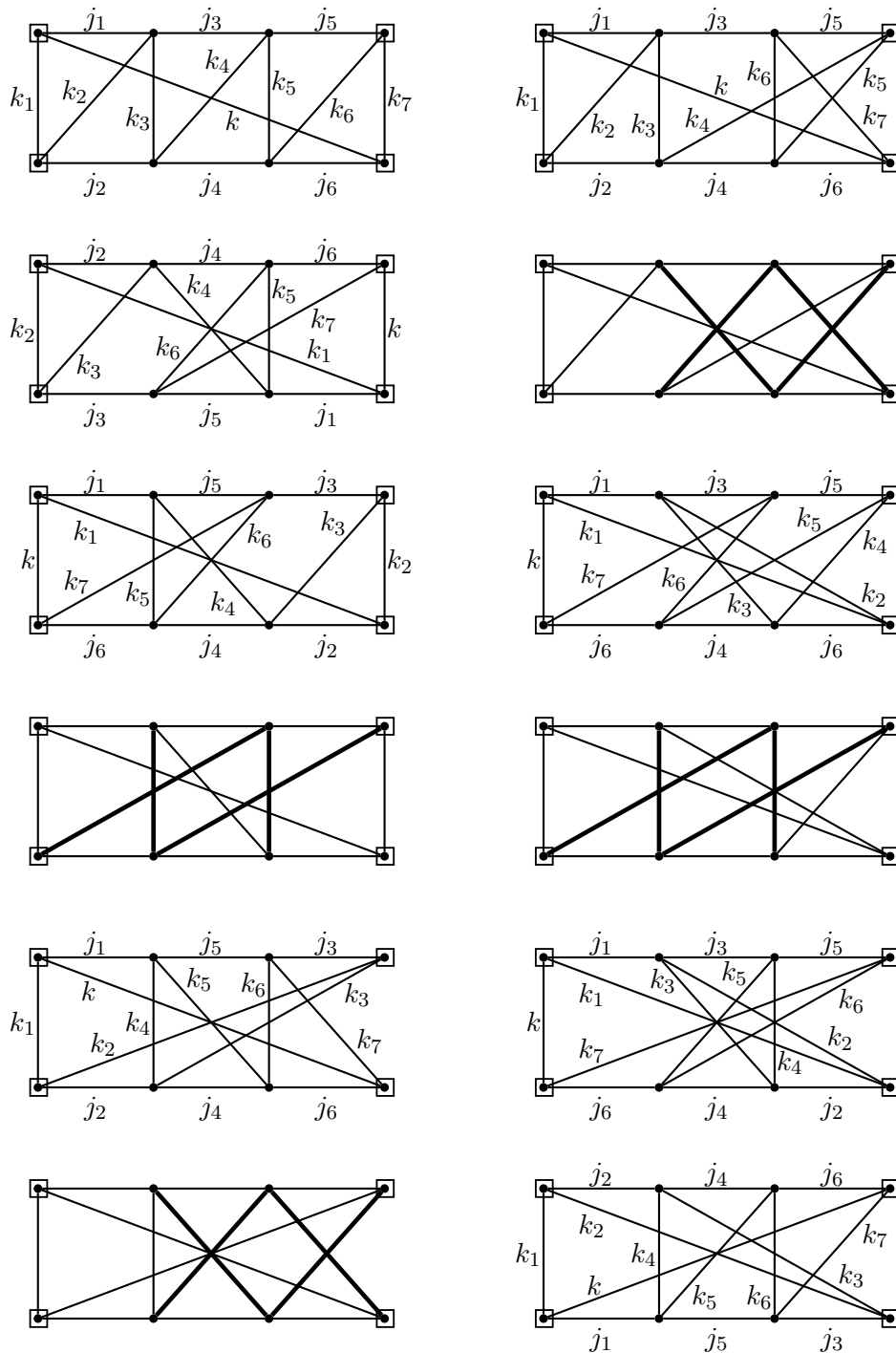


FIGURE 13. Twelve Truemper graphs.