

# ON $k$ -CONNECTED SETS IN MATROIDS

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ABSTRACT. In 1966, Tutte defined  $k$ -connected matroids for each  $k \geq 2$ . Robertson and Seymour built on Tutte's idea to define tangles as a way to identify highly connected parts of a matroid. Motivated by tangles, a set  $Z$  in a matroid  $M$  has been defined to be 3-connected if, for each  $j$  in  $\{1, 2\}$ , there is no  $j$ -separation  $(X, Y)$  of  $M$  such that each of  $|Z \cap X|$  and  $|Z \cap Y|$  is at least  $j$ . This paper solves a problem of Whittle for  $k = 3$  by proving that a minor-minimal matroid having  $Z$  as a 3-connected set has at most  $3|Z| - 4$  elements. We also identify the matroids for which this bound is attained.

## 1. INTRODUCTION

The matroid terminology used here will follow [8]. There has been extensive study of the properties of connected matroids, that is, of matroids that cannot be written as the direct sum of two non-empty matroids. In 1966, Tutte [12] defined  $k$ -connected matroids for all  $k \geq 2$ . In particular, a matroid is 2-connected if and only if it is connected. Tutte was motivated by a desire to extend vertex connectivity in graphs to matroids. His definition guaranteed that the dual of a  $k$ -connected matroid is also  $k$ -connected.

Robertson and Seymour [10] introduced tangles to identify highly connected parts of a graph or matroid. Geelen, Gerards, and Whittle have proved a number of important results for tangles in matroids (see, for example, [5, 6]). In work that uses tangles as the primary tool, Brettell et al. [2] defined a set  $Z$  in a matroid  $M$  to be  $k$ -connected for an integer  $k$  exceeding one provided that  $\lambda(A) \geq \min\{|Z \cap A|, |Z - A|, k - 1\}$  for all subsets  $A$  of  $E(M)$ .

Using tangles, the following theorem was proved in [2, Theorem 2]. It shows that the presence of a large  $k$ -connected set in a matroid  $M$  guarantees the presence of such a set in a minor  $N$  of  $M$  where  $N$  is

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*weakly 4-connected*. This means that  $N$  is 3-connected and  $\lambda(X) \geq 3$  for every subset  $X$  of  $E(N)$  with  $|X|$  and  $|E(N) - X|$  exceeding four.

**Theorem 1.1.** *Let  $Z$  be an  $n$ -element  $k$ -connected set in a matroid  $M$  where  $n \geq 3k - 5$  and  $k \geq 4$ . Then  $M$  has a weakly 4-connected minor that contains an  $n$ -element  $k$ -connected set.*

In this paper, we explore the properties of  $k$ -connected sets when  $k$  is 2 or 3. The first case can be completely described as follows by using a result of Tutte [12, 6.5].

**Proposition 1.2.** *The following are equivalent for a subset  $Z$  of the ground set of a matroid  $M$ .*

- (i)  $Z$  is a 2-connected set.
- (ii)  $Z$  is contained in a connected component of  $M$ .
- (iii)  $Z$  is the ground set of a 2-connected minor of  $M$ .

In a private communication, Geoff Whittle, after noting the preceding consequence of Tutte's work, posed the following problem. Find a sharp bound on the size of a matroid  $M$  that contains a  $k$ -connected set  $Z$  such that there is no element  $x$  of  $E(M) - Z$  for which  $Z$  is  $k$ -connected in  $M \setminus x$  or in  $M/x$ . He also sought a description of the extremal examples. The next theorem, which is our main result, solves these problems when  $k = 3$ .

The matroids that attain equality in the bound in the theorem are obtained by attaching copies of  $M(K_4)$  to a common line. More precisely, for  $k \geq 2$ , let  $(p_1, p_2, \dots, p_k)$  be a partition  $P$  of the integer  $k$ , that is,  $p_1 + p_2 + \dots + p_j = k$  for some  $j$  in  $[k]$  and  $p_1 \geq p_2 \geq \dots \geq p_j \geq 1$ , while  $p_{j+1} = p_{j+2} = \dots = p_k = 0$ . We construct the matroid  $N_P$  by beginning with a  $(k+2)$ -point line  $L$  on the set  $\{z_1, z_2, e_1, e_2, \dots, e_k\}$ . For each  $i$  in  $[k]$ , let  $M_i$  be a copy of  $M(K_4)$  that has  $\{z_1, z_2, a_i\}$  as a triangle and  $\{z_1, z_2, z_{i+2}\}$  as a triad.

Let  $p_0 = 0$ . For each  $t$  in  $[j]$  and each  $i$  with  $p_{t-1} + 1 \leq i \leq p_t$ , identify  $a_i$  with  $e_t$  and attach  $M_i$  to  $L$  via generalized parallel connection across the triangle  $\{z_1, z_2, e_t\}$ . Finally, delete  $\{e_1, e_2, \dots, e_k\}$  from the resulting matroid to obtain  $N_P$ . Then  $|E(N_P)| = 3k + 2$  and  $r(N_P) = k + 2$ . In particular,  $|E(N_P)| = 3|Z| - 4$  where  $Z = \{z_1, z_2, \dots, z_{k+2}\}$ . Let  $\mathcal{N}$  denote the class of all matroids that are isomorphic to  $N_P$  for some partition  $P$  of an integer  $k$  exceeding 1. We observe that, when  $P = (k, 0, 0, \dots, 0)$ , the matroid  $N_P$  is the cycle matroid of the graph that is obtained from  $K_{2,k}$  by adding a new vertex  $v$  that is adjacent to every other vertex. In this case,  $Z$  is the set of edges meeting  $v$ .

**Theorem 1.3.** *In a matroid  $M$ , let  $Z$  be a 3-connected subset of  $E(M)$ . Assume that  $M$  is minor-minimal having  $Z$  as a 3-connected set. If*

$|Z| \leq 3$ , then  $E(M) = Z$ . If  $|Z| \geq 4$ , then  $|E(M)| \leq 3|Z| - 4$ , and  $|E(M)| = 3|Z| - 4$  if and only if  $M$  or  $M^*$  is isomorphic to  $N_P$  for some partition  $P$  of an integer  $k$  exceeding one where  $Z = \{z_1, z_2, \dots, z_{k+2}\}$ .

## 2. PRELIMINARIES

This section presents some basic properties of  $k$ -connected matroids. Note that if  $Z$  is a  $k$ -connected set in a matroid  $M$ , then  $\text{cl}(Z)$  is not necessarily a  $k$ -connected set. For example, let  $Z$  be the set of spokes of  $M(\mathcal{W}_4)$ , and let  $M$  be obtained from  $M(\mathcal{W}_4)$  by adding an element in parallel to one of the elements of  $Z$ . Then  $Z$  is 3-connected in  $M$ , but  $\text{cl}(Z)$  is not 3-connected in  $M$ . Although adding elements to a  $k$ -connected set can destroy  $k$ -connectedness, removing elements cannot. In particular, for all  $k \geq 2$ , the empty set is  $k$ -connected in every matroid.

**Lemma 2.1.** *Every subset of a  $k$ -connected set in a matroid  $M$  is  $k$ -connected in  $M$ .*

*Proof.* Assume  $Z$  is  $k$ -connected in  $M$ . Suppose  $Z' \subseteq Z$ . We know that

$$\lambda(A) \geq \min\{|Z \cap A|, |Z - A|, k - 1\}$$

for all  $A \subseteq E(M)$ . But

$$\min\{|Z \cap A|, |Z - A|, k - 1\} \geq \min\{|Z' \cap A|, |Z' - A|, k - 1\}.$$

Thus  $Z'$  is  $k$ -connected.  $\square$

The following is an immediate consequence of the fact that  $\lambda_M(X) = \lambda_{M^*}(X)$  for all sets  $X$  in a matroid  $M$ .

**Lemma 2.2.** *A set  $X$  in a matroid  $M$  is  $k$ -connected if and only if  $X$  is  $k$ -connected in  $M^*$ .*

Cunningham and Edmonds [3, 4] described a decomposition of a connected matroid using repeated 2-sums. The details of this appear in [8, Section 8.3]. We now summarize these details. A *matroid-labeled tree*  $T$  is a tree with vertex set  $\{M_1, M_2, \dots, M_k\}$  for some positive integer  $k$  where each  $M_i$  is a matroid. Moreover,

- (i) if  $M_{j_1}$  and  $M_{j_2}$  are joined by an edge  $e_i$  of  $T$ , then  $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$ , and  $\{e_i\}$  is not a separator of  $M_{j_1}$  or  $M_{j_2}$ ; and
- (ii) if  $M_{j_1}$  and  $M_{j_2}$  are non-adjacent, then  $E(M_{j_1}) \cap E(M_{j_2})$  is empty.

Let  $e$  be an edge of a matroid-labeled tree  $T$ . Suppose  $e$  joins vertices labeled by  $N_1$  and  $N_2$ . Then  $T/e$  denotes the matroid-labeled tree in which the vertex that results by identifying the endpoints of  $e$  is labeled

by  $N_1 \oplus_2 N_2$  and all other vertex and edge labels are unchanged. A *tree decomposition* of a 2-connected matroid  $M$  is a matroid-labeled tree  $T$  such that if  $V(T) = \{M_1, M_2, \dots, M_k\}$  and  $E(T) = \{e_1, e_2, \dots, e_{k-1}\}$ , then

- (i)  $E(M) = (E(M_1) \cup E(M_2) \cup \dots \cup E(M_k)) - \{e_1, e_2, \dots, e_{k-1}\}$ ;
- (ii)  $|E(M_i)| \geq 3$  for all  $i$  unless  $|E(M)| < 3$ , in which case,  $k = 1$  and  $M_1 = M$ ; and
- (iii)  $M$  is the matroid that labels the only vertex of  $T/e_1, e_2, \dots, e_{k-1}$ .

The following result [4, 11] (see also [8, Proposition 8.3.9, Theorem 8.3.10]) notes that every 2-connected matroid can be written in terms of 2-sums of 3-connected matroids. We observe that these 3-connected matroids may be circuits or cocircuits of size less than four.

**Theorem 2.3.** *Let  $M$  be a 2-connected matroid. Then  $M$  has a tree decomposition  $T$  in which every vertex is labeled by a 3-connected matroid. Moreover, if no two adjacent vertices of  $T$  are labeled by circuits and no two adjacent vertices are labeled by cocircuits, then  $T$  is unique.*

The tree decomposition of  $M$  whose existence and uniqueness are asserted in the last theorem is called the *canonical tree decomposition* for  $M$ . If  $e$  is an edge in a tree decomposition  $T$  of a 2-connected matroid  $M$ , then  $T \setminus e$  consists of two subtrees  $T_1$  and  $T_2$ . Let  $X_i$  be the set of elements of  $M$  that are elements of some vertex label in  $T_i$ . Then  $M = N_1 \oplus_2 N_2$  where  $N_i$  is the matroid that labels the single-vertex tree that is obtained by contracting all of the edges of  $T_i$ . Observe that  $E(N_i) = X_i \cup e$  and that  $(X_1, X_2)$  is a 2-separation of  $M$ . We call this the 2-separation *corresponding* to the edge  $e$ .

Clearly, if  $k \geq 3$  and  $Z$  is a  $k$ -connected set in  $M$ , then  $Z$  is  $(k-1)$ -connected in  $M$ . The next two lemmas give some basic properties of  $k$ -connected sets.

**Lemma 2.4.** *Let  $M$  be a  $k$ -connected matroid. If  $Z \subseteq E(M)$ , then  $Z$  is  $k$ -connected in  $M$ .*

*Proof.* Suppose that  $Z$  is not  $k$ -connected. Then  $E(M)$  has a subset  $A$  for which  $\lambda(A) < \min\{|Z \cap A|, |Z - A|, k-1\}$ . As  $\lambda(A) = \lambda(E(M) - A)$ , we may assume that  $|Z \cap A| \leq |Z - A|$ . Suppose that  $|Z \cap A| \geq k-1$ . Then  $\lambda(A) < k-1$ . But  $|A|, |E(M) - A| \geq k-1$ , a contradiction to the fact that  $M$  is  $k$ -connected. Thus, for some  $s < k-1$ , we have  $\lambda(A) < s = |Z \cap A|$ . Then, as  $|A|, |E(M) - A| \geq |Z \cap A|$ , it follows that  $M$  has an  $s$ -separation, a contradiction.  $\square$

**Lemma 2.5.** *Let  $Z$  be the ground set of a  $k$ -connected minor  $M/C \setminus D$  of a matroid  $M$ . Then  $Z$  is  $k$ -connected in  $M$ .*

*Proof.* Suppose  $Z$  is not  $k$ -connected in  $M$ . Then there is a subset  $A$  of  $E(M)$  such that

$$\lambda(A) < \min\{|Z \cap A|, |Z - A|, k - 1\}.$$

Now, by, for example, [8, Corollary 8.2.5],

$$\lambda_{M/C \setminus D}(Z \cap A) \leq \lambda(A) < \min\{|Z \cap A|, |Z - A|, k - 1\}.$$

We may assume that  $|Z \cap A| \leq |Z - A|$ . Suppose  $|Z \cap A| \geq k - 1$ . Then, as  $M/C \setminus D$  is  $k$ -connected,

$$k - 1 \leq \lambda_{M/C \setminus D}(Z \cap A) \leq \lambda(A) < \min\{|Z \cap A|, |Z - A|, k - 1\} = k - 1.$$

This contradiction implies that  $|Z \cap A| = s$  for some  $s < k - 1$ . Then, since  $M/C \setminus D$  is  $k$ -connected,

$$s \leq \lambda_{M/C \setminus D}(Z \cap A) \leq \lambda(A) < \min\{|Z \cap A|, |Z - A|, k - 1\} = s.$$

This contradiction implies that  $Z$  is  $k$ -connected in  $M$ .  $\square$

The converse of the last result does not hold. For example, let  $M = M(\mathcal{W}_4)$  and  $Z$  be its set of spokes. Then, by Lemma 2.4,  $Z$  is 3-connected. But, as one can easily check,  $Z$  is not the ground set of a 3-connected minor of  $M$ .

By Proposition 1.2, we see that the converse of Lemma 2.4 also fails. For example, suppose  $k \geq 2$  and let  $Z$  be the ground set of a  $k$ -connected matroid  $N$ . Let  $M = N \oplus U_{1,1}$ . Then, by Lemma 2.5,  $Z$  is a  $k$ -connected set in  $M$ . Clearly  $M$  is not a  $k$ -connected matroid.

The behavior of  $k$ -connected sets when  $k = 2$  is described in Proposition 1.2. Although this is a straightforward consequence of the results of Tutte [12], we give the proof for completeness.

*Proof of Proposition 1.2.* Since (i)-(iii) hold when  $Z = \emptyset$ , we may assume that  $Z \neq \emptyset$ . By Lemma 2.5, (iii) implies (i). Now assume that (i) holds. Suppose that  $Z$  is not contained in a component of  $M$ . Let  $X$  be a component that meets  $Z$ . Then  $Z \cap (E - X) \neq \emptyset$ . As  $Z$  is 2-connected,

$$0 = \lambda(X) \geq \min\{|Z \cap X|, |Z - X|, 1\},$$

a contradiction. Thus  $Z$  is contained in a component of  $M$ , that is, (i) implies (ii).

Finally, we show that (ii) implies (iii). Assume that  $Z$  is contained in a component  $N$  of  $M$ . Then, by a theorem of Tutte [12, 6.5] (see also [8, Theorem 4.3.1]), there is a sequence  $N_1, N_2, \dots, N_s$  of 2-connected matroids such that  $N_1 = N$  and  $E(N_s) = Z$ , while, for all  $i$  in  $\{2, 3, \dots, s\}$ , the matroid  $N_i$  is a single-element deletion or contraction of  $N_{i-1}$ . Thus

$Z$  is the ground set of a 2-connected minor of  $M$ . Therefore (ii) implies (iii) and the theorem holds.  $\square$

For a positive integer  $j$ , let  $(X, Y)$  be a  $j$ -separation of a matroid  $M$ . We say that a set  $Z$  *exhibits* the  $j$ -separation  $(X, Y)$  if  $|Z \cap X| \geq j$  and  $|Z \cap Y| \geq j$ .

**Lemma 2.6.** *Suppose a set  $Z$  is  $(k-1)$ -connected in a matroid  $M$ . Then  $Z$  is not  $k$ -connected if and only if  $Z$  exhibits an exact  $(k-1)$ -separation of  $M$ .*

*Proof.* Suppose  $Z$  is not  $k$ -connected. Then there is a subset  $X$  of  $E(M)$  such that

$$\lambda(X) < \min\{|Z \cap X|, |Z - X|, k-1\}.$$

Since  $Z$  is  $(k-1)$ -connected,

$$\lambda(X) \geq \min\{|Z \cap X|, |Z - X|, k-2\}.$$

Therefore  $|Z \cap X| \geq k-1$  and  $|Z - X| \geq k-1$ . Thus  $\lambda(X) = k-2$ , so  $Z$  exhibits the  $(k-1)$ -separation  $(X, E-X)$  and this  $(k-1)$ -separation is exact.

Conversely, suppose  $Z$  exhibits an exact  $(k-1)$ -separation  $(X, Y)$  of  $M$ . Then  $\lambda(X) = k-2$  while  $|Z \cap X| \geq k-1$  and  $|Z - X| \geq k-1$ . Therefore

$$\lambda(X) < \min\{|Z \cap X|, |Z - X|, k-1\},$$

so  $Z$  is not  $k$ -connected.  $\square$

The next two results follow from Proposition 1.2 and Lemma 2.6.

**Corollary 2.7.** *In a matroid  $M$ , a set  $Z$  of size at least four is 3-connected if and only if  $Z$  is contained in a component of  $M$  and  $Z$  does not exhibit a 2-separation of  $M$ .*

**Corollary 2.8.** *Let  $Z$  be a 2-connected set in a matroid  $M$ , and suppose  $|Z| \in \{1, 2, 3\}$ . Then  $M$  has a 3-connected minor  $N$  with ground set  $Z$ .*

*Proof.* Since each 2-connected matroid with at most three elements is 3-connected, the result follows immediately by Proposition 1.2.  $\square$

### 3. MINOR-MINIMAL MATROIDS CONTAINING A 3-CONNECTED SET.

In this section, we prove Theorem 1.3. We begin with a preliminary result.

**Lemma 3.1.** *Suppose  $M$  is a minor-minimal matroid in which  $Z$  is a 3-connected set. If  $|Z| \geq 4$ , then  $M$  is 2-connected.*

*Proof.* Since  $Z$  is 3-connected,  $Z$  is 2-connected. Thus, by Proposition 1.2,  $Z$  is in a component  $N$  of  $M$ . We may assume that  $N \neq M$  otherwise the result holds. By the minimality of  $M$  and Proposition 1.2, in  $N$ , the set  $Z$  is not 3-connected, but  $Z$  is 2-connected. Thus  $N$  has a 2-separation  $(X, Y)$  in which  $|Z \cap X|$  and  $|Z \cap Y|$  are both at least 2. Since  $M = N \oplus (M \setminus E(N))$ , one easily checks that  $(X, Y \cup (E(M) - E(N)))$  is a 2-separation of  $M$ . Thus

$$\lambda_M(X) < \min\{|Z \cap X|, |Z - X|, 2\}.$$

Hence  $Z$  is not 3-connected in  $M$ , a contradiction.  $\square$

The following lemma will be used in the proof of the next theorem.

**Lemma 3.2.** *Let  $Z$  be a 3-connected set in a 2-connected matroid  $M$ . Suppose  $M = M_1 \oplus_2 M_2$  where  $p$  is the basepoint of the 2-sum.*

- (i) *If  $Z \subseteq E(M_1)$ , then  $Z$  is 3-connected in  $M_1$ .*
- (ii) *If  $Z \cap E(M_2) = \{z\}$  and  $M'_1$  is obtained from  $M_1$  by relabeling the basepoint  $p$  by  $z$ , then  $M'_1$  is a minor of  $M$  having  $Z$  as a 3-connected set.*

*Proof.* Since  $M$  is 2-connected,  $M_1$  is 2-connected. Assume that  $Z \subseteq E(M_1)$ . Then  $Z$  is 2-connected in  $M_1$ . Suppose that  $Z$  is not 3-connected in  $M_1$ . Then, by Lemma 2.6,  $Z$  exhibits an exact 2-separation  $(X, Y)$  of  $M_1$ . Without loss of generality, the basepoint  $p$  of the 2-sum  $M_1 \oplus_2 M_2$  lies in  $Y$ . Then  $(X, (Y \cup E(M_2)) - p)$  is an exact 2-separation of  $M$  that is exhibited by  $Z$ , a contradiction. Thus (i) holds.

Now suppose that  $Z \cap E(M_2) = \{z\}$ . Take a circuit  $C$  of  $M_2$  containing  $\{z, p\}$ . Then  $M_2 \setminus (E(M_2) - C) / (C - \{z, p\})$  is a 2-circuit with ground set  $\{z, p\}$ . It follows that  $M \setminus (E(M_2) - C) / (C - \{z, p\})$  is  $M'_1$  and that  $E(M'_1)$  contains  $Z$ . Then  $M'_1$  is 2-connected as  $M'_1 \cong M_1$ . Thus  $Z$  is 2-connected in  $M'_1$ . If  $Z$  exhibits an exact 2-separation  $(X, Y)$  of  $M'_1$  where  $z \in Y$ , then  $(X, Y \cup (E(M_2) - p))$  is an exact 2-separation of  $M$  that is exhibited by  $Z$ , a contradiction. Thus  $Z$  is 3-connected in  $M'_1$ .  $\square$

**Theorem 3.3.** *Let  $Z$  be a 3-connected set in a 2-connected matroid  $M$  and suppose that  $|Z| \geq 4$ . Then  $M$  has a 3-connected minor  $N$  such that  $Z \subseteq E(N)$ . Moreover, the canonical tree decomposition  $T$  of  $M$  has a vertex that is labeled by a 3-connected matroid  $M_0$  such that  $N$  can be obtained from  $M_0$  by relabeling each basepoint in  $E(M_0)$  by an element of  $N$ .*

*Proof.* We argue by induction on  $|V(T)|$ . If  $T$  has only one vertex, then  $M$  is 3-connected and we may take  $N = M$ . Assume the result holds

if  $|V(T)| < n$ , and let  $|V(T)| = n$ . Let  $p$  be an edge of  $T$ . Since there is a 2-separation  $(X_1, X_2)$  of  $M$  corresponding to  $p$ , we may assume that  $|X_2 \cap Z| \leq 1$  by Corollary 2.7. Then  $M = M_1 \oplus_2 M_2$  where  $E(M_i) = X_i \cup p$ . Let  $T_1$  be the canonical tree decomposition of  $M_1$ . Now, as  $Z$  is 3-connected in  $M$ , either  $Z \subseteq E(M_1)$  or  $Z \cap E(M_2) = \{z\}$ . Thus, by Lemma 3.2,  $Z$  is 3-connected in  $M_1$ , or  $Z$  is 3-connected in  $M'_1$  where  $M'_1$  is obtained from  $M_1$  by relabeling  $p$  by  $z$ . Since  $|V(T_1)| < |V(T)|$ , the result follows by induction.  $\square$

**Lemma 3.4.** *Let  $M$  be a matroid and suppose that  $Z \subseteq E(M)$ , that  $|Z| \geq 4$ , and that  $e \in E(M) - Z$ . Suppose  $M$  is minor-minimal having  $Z$  as a 3-connected set.*

- (i) *If  $\text{si}(M/e)$  is 3-connected, then  $e$  is in a triangle with two elements of  $Z$ .*
- (ii) *If  $\text{co}(M \setminus e)$  is 3-connected, then  $e$  is in a triad with two elements of  $Z$ .*
- (iii) *The element  $e$  is in a triangle or a triad with two elements of  $Z$ .*
- (iv) *The element  $e$  is not on a line of  $M$  or  $M^*$  having more than three elements.*

*Proof.* By Lemma 3.1 and Theorem 3.3,  $M$  is 3-connected. If  $\text{si}(M/e)$  is 3-connected, then the minimality of  $M$  implies that  $M$  has a triangle containing two elements of  $Z$ . Thus (i) holds, and (ii) follows by duality. Part (iii) follows by Bixby's Lemma.

Finally, suppose  $e$  is on a line  $L$  of  $M$  having at least four elements. If  $(X, Y)$  is a 2-separation of  $M \setminus e$ , then, without loss of generality,  $|X \cap L| \geq 2$ . Thus  $(X \cup e, Y)$  is a 2-separation of  $M$ . This contradiction and duality complete the proof of (iv).  $\square$

The next theorem proves the main result for  $|Z| = 4$ .

**Theorem 3.5.** *Suppose  $M$  is a minor-minimal matroid having  $Z$  as a 3-connected set where  $2 \leq |Z| \leq 4$ . Then  $|E(M) - Z| \leq 2|Z| - 4$ . Moreover, equality holds exactly when  $|Z| = 2$  and  $M$  is isomorphic to  $U_{1,2}$  and when  $|Z| = 4$  and  $M$  is isomorphic to  $M(\mathcal{W}_4)$  or  $\mathcal{W}^4$ .*

*Proof.* If  $|Z| \in \{2, 3\}$ , then, by Corollary 2.8,  $E(M) = Z$  and the result holds. Hence we may assume that  $|Z| = 4$ . Thus, by Theorem 3.3,  $M$  is 3-connected. Hence  $r(Z) \geq 2$ . If  $r(Z) = 2$ , then  $M|Z \cong U_{2,4}$  and the result holds since  $M = M|Z$ . Thus we may assume that  $r(Z) \geq 3$ . Now, as  $M$  is simple,  $M|Z \cong U_{2,3} \oplus U_{1,1}$ ,  $U_{3,4}$ , or  $U_{4,4}$ . Assume that  $M$  is a counterexample to the theorem and that  $Z = \{z_1, z_2, z_3, z_4\}$ .

**3.5.1.**  $E - Z$  has no elements  $a$  and  $b$  such that  $\{a, z_s, z_t\}$  is a triad and  $\{b, z_s, z_t\}$  is a triangle where  $\{s, t\} \subseteq \{1, 2, 3, 4\}$ .

Suppose that  $E - Z$  does have such elements  $a$  and  $b$  and that  $\{s, t\} = \{1, 2\}$ . Since  $M$  is 3-connected and  $r(M) \geq 3$ , the elements  $a$  and  $b$  are distinct. As  $\text{co}(M \setminus a)$  is not 3-connected, Bixby's Lemma implies that  $\text{si}(M/a)$  is 3-connected. Hence we may assume that  $\{a, z_1\}$  is in a triangle with an element  $z_3$  of  $Z - \{z_1, z_2\}$ . By duality,  $\{b, z_1\}$  or  $\{b, z_2\}$  is in a triad  $T^*$  with an element  $z_i$  of  $Z - \{z_1, z_2\}$ . Suppose  $i = 3$ . Then, for  $X = \{a, b, z_1, z_2, z_3\}$ , we have  $\lambda(X) = r(X) + r^*(X) - |X| \leq 3 + 3 - 5 = 1$ . As  $M$  is 3-connected,  $|E(M)| \leq 6$ , so  $|E - Z| < 2|Z| - 4$ , a contradiction. Thus  $i = 4$ . Since  $\{a, z_1, z_3\}$  is a triangle and  $\{b, z_j, z_4\}$  is a triad for some  $j$  in  $\{1, 2\}$ , we deduce, by orthogonality, that  $j = 2$ . Thus  $M$  has  $(z_3, a, z_1, z_2, b, z_4)$  as a fan where  $\{z_3, a, z_1\}$  is a triangle. Moreover,  $r(Z) \geq 4$ , otherwise  $M|(Z \cup \{a, b\})$  is isomorphic to  $M(K_4)$  or  $\mathcal{W}^3$ , in which case,  $|E - Z| = 2 < 2|Z| - 4$ , a contradiction.

Now, deleting the triads  $\{a, z_1, z_2\}$  and  $\{z_2, b, z_4\}$  from  $M$  gives a flat  $F$  containing  $z_3$ . Since  $|E - Z| \geq 4$ , it follows that  $|F - z_3| \geq 2$ . Suppose an element  $c$  of  $F - z_3$  is in a triangle  $T$  with two elements of  $Z$ . By orthogonality with the triads  $\{a, z_1, z_2\}$  and  $\{z_2, b, z_4\}$ , none of  $z_1, z_2$ , or  $z_4$  is in  $T$ , a contradiction. Thus, by Lemma 3.4 (iii),  $c$  is in a triad  $T^*$  with two elements of  $Z$ . The triangles  $\{z_1, a, z_3\}$  and  $\{z_1, z_2, b\}$  imply that  $z_1 \notin T^*$ . Moreover,  $z_2 \notin T^*$  otherwise, by orthogonality,  $b \in T^*$  and we obtain the contradiction that  $\{z_2, z_4\}$  is in a 4-point line in  $M^*$ . Thus  $\{z_3, z_4\} \subseteq T^*$ . Since  $c$  was arbitrarily chosen in  $F - z_3$ , it follows that  $M^*$  has a 4-point line containing  $\{z_3, z_4\}$ , a contradiction. We conclude that 3.5.1 holds.

**3.5.2.**  $E - Z$  has no distinct elements  $a$  and  $b$  such that  $\{a, z_s, z_t\}$  is a triad and  $\{b, z_u, z_v\}$  is a triangle where  $s, t, u, v \in \{1, 2, 3, 4\}$ .

Assume that such elements exist and take  $\{s, t\} = \{1, 2\}$ . By 3.5.1,  $\{u, v\} \neq \{1, 2\}$ . By orthogonality,  $\{u, v\} = \{3, 4\}$ . By Lemma 3.4 (iii), every element of  $E - Z$  is on the line of  $M$  spanned by  $\{z_3, z_4\}$  or the line of  $M^*$  spanned by  $\{z_1, z_2\}$ . Thus, by Lemma 3.4 (iv),  $|E - Z| \leq 2 < 2|Z| - 4$ , a contradiction. Hence 3.5.2 holds.

By 3.5.2 and duality, we may assume that every element of  $E - Z$  is in a triangle with two elements of  $Z$ .

**3.5.3.**  $r(Z) = 4$ .

Assume that  $r(Z) = 3$ . Then  $r(M) = 3$ . Suppose first that  $M|Z \cong U_{2,3} \oplus U_{1,1}$  where  $\{z_1, z_2, z_3\}$  is a triangle. By Lemma 3.4 (iv), we may assume that  $E - Z$  contains elements  $a$  and  $b$  such that  $\{a, z_4, z_1\}$  and  $\{b, z_4, z_2\}$  are triangles. Then  $M|(Z \cup \{a, b\})$  is 3-connected and so

equals  $M$ . This is a contradiction as  $|E - Z| = 2 < 2|Z| - 4$ . Thus  $M \not\cong U_{2,3} \oplus U_{1,1}$ .

We now know that  $M|Z \cong U_{3,4}$ . As  $|E - Z| \geq 4$ , there are distinct elements  $a, b$ , and  $c$  of  $E - Z$  such that  $\{a, z_p, z_q\}$ ,  $\{b, z_s, z_t\}$ , and  $\{c, z_u, z_v\}$  are triangles where  $\{p, q\}$ ,  $\{s, t\}$ , and  $\{u, v\}$  are distinct 2-element subsets of  $\{1, 2, 3, 4\}$ . Then  $M|(Z \cup \{a, b, c\})$  is 3-connected and we have a contradiction. Hence 3.5.3 holds.

By 3.5.3,  $M|Z \cong U_{4,4}$  and every element of  $M$  is in a triangle with two elements of  $Z$ . Construct an auxiliary graph  $G$  with  $V(G) = Z$  and  $E(G) = E - Z$  where an element  $e$  of  $E - Z$  joins vertices  $z_i$  and  $z_j$  if and only if  $\{e, z_i, z_j\}$  is a triangle. Then  $G$  is well defined and is simple. As  $M$  is 3-connected and minor-minimal having  $Z$  as a 3-connected set,  $G$  must be a 4-cycle. It follows that  $M$  is isomorphic to  $M(\mathcal{W}_4)$  or  $\mathcal{W}^4$ . This completes the proof of the theorem.  $\square$

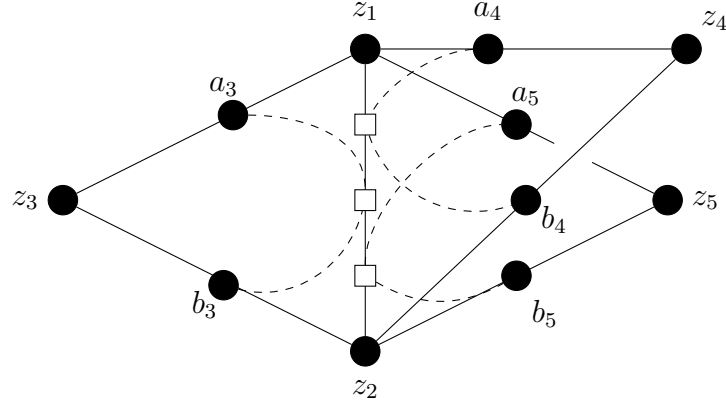
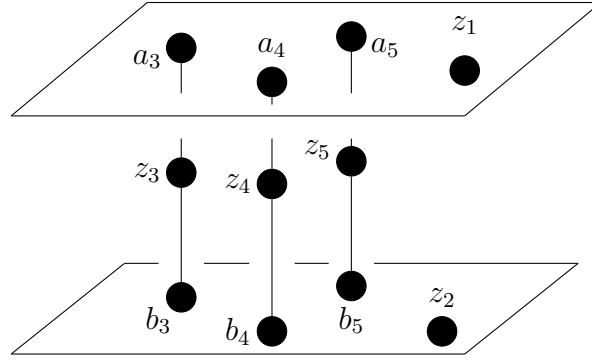
The proof of Theorem 1.3 will use the following results of Leo [7, Theorem 1.1] and Akkari and Oxley [1, Theorem 5.1]. Suppose  $\Delta$  is a triangle in both  $M(K_4)$  and in a matroid  $N$ . Then  $P_\Delta(M(K_4), N)$  denotes the generalized parallel connection of  $M(K_4)$  and  $N$  across the triangle  $\Delta$ .

**Theorem 3.6.** *Suppose that  $M$  is a 3-connected matroid with at least four elements. Let  $C$  be a circuit of  $M$  and  $f$  be an element of  $C$ . If  $M \setminus e$  is not 3-connected for all  $e$  in  $C - f$ , then  $C$  meets a triad of  $M$ .*

**Theorem 3.7.** *Let  $e$  and  $f$  be distinct elements of a 3-connected matroid  $M$ . Suppose that  $\{e, f, a\}$  is a triangle,  $\{e, f, b\}$  is a triad, and  $M$  is not isomorphic to  $U_{2,4}$ . Then  $M = P_\Delta(M(K_4), M') \setminus \{e', f'\}$  where  $\Delta = \{a, e', f'\}$ , the other triangles of  $M(K_4)$  are  $\{a, e, f\}$ ,  $\{b, e, e'\}$ , and  $\{b, f, f'\}$ , and  $M'$  is obtained from  $M/b$  by relabeling  $e$  and  $f$  as  $e'$  and  $f'$ , respectively.*

**Lemma 3.8.** *In a 3-connected matroid  $N$ , suppose  $(z_2, a_1, z_1, a_2, z_3)$  is a fan in which  $\{z_2, a_1, z_1\}$  is a triangle. Suppose  $N$  also has  $\{a_1, a_2, e\}$  and  $\{e, z_2, z_3\}$  as triangles. If  $|E(N)| > 6$ , then  $e$  is not in a triad of  $N$ , and  $N \setminus e$  is 3-connected.*

*Proof.* Let  $X = \{z_2, a_1, z_1, a_2, z_3, e\}$ . Suppose  $e$  is in a triad  $T^*$  of  $N$ . By orthogonality,  $T^* \subseteq X$ . Then  $\lambda(X) = r(X) + r^*(X) - |X| \leq 3 + 3 - 5 = 1$ . As  $N$  is 3-connected,  $|E(N) - X| \leq 1$ . Say  $E(N) - X = \{f\}$ . As  $\{a_1, z_1, a_2\}$  is a triad of  $N$ , it follows that  $\{z_2, e, f, z_3\}$  is a line of  $N$ . The triad  $T^*$  now gives a contradiction to orthogonality. Hence  $e$  is not in a triad of  $N$ . If the line containing  $\{e, z_2, z_3\}$  contains a fourth point, then  $N \setminus e$  is 3-connected. Therefore we may assume that  $r(N) \geq 4$  and

FIGURE 1. A depiction of the rank-5 matroid  $N_{(1,1,1)}$ .FIGURE 2. The rank-6 dual of  $N_{(1,1,1)}$  where  $\{a_3, a_4, a_5, z_1\}$  and  $\{b_3, b_4, b_5, z_2\}$  are skew circuits.

$\text{si}(N/e)$  is not 3-connected. Thus, as  $e$  is not in a triad, Bixby's Lemma implies that  $N \setminus e$  is 3-connected.  $\square$

For the following proof, recall the definition of the class  $\mathcal{N}$  given before the statement of Theorem 1.3. Depictions of the matroid  $N_{(1,1,1)}$  and its dual are given in Figures 1 and 2.

*Proof of Theorem 1.3.* When  $|Z| \leq 3$ , the result follows from Corollary 2.8. Moreover, the result holds when  $|Z| = 4$  by Theorem 3.5. Let  $M$  be a counterexample to the theorem that is minimal with respect to  $|Z|$ . Then  $|Z| \geq 5$ . Moreover, either

- (i)  $|E(M)| > 3|Z| - 4$ , or
- (ii)  $|E(M)| = 3|Z| - 4$ , but  $M \notin \mathcal{N} \cup \mathcal{N}^*$ .

The first part of our argument will show that (i) does not hold.

Suppose that  $Z$  spans both  $M$  and  $M^*$ . Then

$$|E(M)| = r(M) + r^*(M) = r(Z) + r^*(Z) \leq 2|Z| < 3|Z| - 4,$$

a contradiction. Hence, by duality, we may assume that

**3.8.1.**  *$Z$  does not span  $M^*$ .*

By 3.8.1,  $E - Z$  contains a circuit  $K$  of  $M$ . Here  $M \setminus e$  is not 3-connected for all  $e$  in  $K$ , otherwise  $M$  is not minor minimal having  $Z$  as a 3-connected set. Thus, by Theorem 3.6,  $M$  has a triad  $T^*$  that meets  $K$  in at least two elements. Suppose  $\{a_1, a_2\} \subseteq T^* \cap K$ . By Lemma 3.4 (iii) and orthogonality, each  $a_i$  is in a triangle  $T_{a_i}$  with two elements of  $Z$ . Since  $|T_{a_i} \cap T^*| \neq 1$ , it follows that  $T^* \cap Z$  contains a single element, say  $z_1$ . Moreover,  $z_1 \in T_{a_1} \cap T_{a_2}$ . Let  $T_{a_1} = \{a_1, z_1, z_2\}$  and  $T_{a_2} = \{a_2, z_1, z_3\}$  where  $\{z_2, z_3\} \subseteq Z$ .

As  $\text{co}(M \setminus a_2)$  is not 3-connected,  $\text{si}(M/a_2)$  is 3-connected. Now let  $M_1$  be  $M/a_2 \setminus z_1$  if  $\{a_1, a_2\}$  is not in a triangle of  $M$ , while  $M_1$  is  $M/a_2 \setminus z_1, a_1$  if  $M$  has  $\{a_1, a_2\}$  in a triangle. Then  $M_1$  is 3-connected since it is isomorphic to  $\text{si}(M/a_2)$ . Let  $M_2$  be a minor of  $M_1$  that is minimal having  $Z - z_1$  as a 3-connected set.

**3.8.2.**  *$M_2$  does not have a triangle containing  $\{z_2, z_3\}$ .*

Assume  $\{z_2, z_3, t\}$  is a triangle of  $M_2$ . Let  $N = P_\Delta(M(K_4), M_2)$  where  $\Delta = \{z_2, z_3, t\}$  and the other triangles of  $M(K_4)$  are  $T_{a_1}$ ,  $T_{a_2}$ , and  $\{a_1, a_2, t\}$ . Observe that, by Lemma 3.8,  $N \setminus t$  is 3-connected.

Suppose  $\{z_2, z_3, t\}$  is a triangle of  $M$ . Then  $N = M$ . Thus

$$3|Z| - 4 \leq |E(M)| = 3|E(M_2)| + 3 \leq 3|Z - z_1| - 4 + 3 = 3|Z| - 4,$$

so  $|E(M_2)| = 3|Z - z_1| - 4$ . Therefore  $M_2 \in \mathcal{N} \cup \mathcal{N}^*$ . Since no member of  $\mathcal{N} \cup \mathcal{N}^*$  has three collinear members of  $Z$ , it follows that  $t \notin Z$ . Then, by Lemma 3.8, we obtain the contradiction that  $M \setminus t$  is 3-connected.

We deduce that  $\{z_2, z_3, t\}$  is not a triangle of  $M$ . Thus  $M = N \setminus t$  and

$$|E(M)| = |E(M_2)| + 3 - 1 \leq 3|Z - z_1| - 4 + 2 \leq 3|Z| - 5.$$

This contradiction completes the proof of 3.8.2.

We now show that

**3.8.3.**  *$|E(M)| = 3|Z| - 4$  and  $M_2 = M \setminus a_1, z_1, a_2$ , so  $|E(M_2)| = 3|Z - z_1| - 4$ .*

Although  $M_2$  has no triangle containing  $\{z_2, z_3\}$ , there is a triangle  $\{z_2, z_3, t\}$  of  $M_1$ . If  $M_2 = M_1/C \setminus D$ , then  $t \in D$ , so  $t \notin Z$ . Moreover,  $t \notin E(M)$  otherwise, by Lemma 3.8,  $M \setminus t$  is 3-connected. It follows

that  $M$  has no triangle containing  $\{z_2, z_3\}$ . As  $M_1/C \setminus (D - t)$  has  $\{z_2, z_3, t\}$  as a triangle, we can obtain  $M$  from  $M_2$  by adjoining the elements  $\{a_1, z_1, a_2\}$ . Thus

$$3|Z| - 4 \leq |E(M)| = |E(M_2)| + 3 \leq 3|Z - z_1| - 4 + 3 = 3|Z| - 4,$$

that is, 3.8.3 holds.

By the choice of  $M$  and 3.8.3,  $M_2 \in \mathcal{N} \cup \mathcal{N}^*$ . To simplify notation, for some  $k \geq 2$ , let  $Z - z_1 = \{z', z'', x_1, x_2, \dots, x_k\}$  and let  $E(M_2) - (Z - z_1) = \{u_1, v_1, u_2, v_2, \dots, u_k, v_k\}$ . If  $M_2 \in \mathcal{N}$ , then  $Z - z_1$  spans  $M_2$  and, for all  $i$  in  $\{1, 2, \dots, k\}$ , we may assume that  $M_2$  has  $\{x_i, u_i, z'\}$  and  $\{x_i, v_i, z''\}$  as triangles and has  $\{u_i, x_i, v_i\}$  as a triad. When  $M_2 \in \mathcal{N}^*$ , we see that  $M_2$  has  $\{z', u_1, u_2, \dots, u_k\}$  and  $\{z'', v_1, v_2, \dots, v_k\}$  as skew circuits whose union spans  $M_2$ . Moreover,  $M_2$  has each  $\{u_i, x_i, v_i\}$  as a triangle.

We know that  $M$  is obtained from  $M_2$  by adding elements  $a_1, a_2$ , and  $z_1$  where  $M$  has  $\{a_1, a_2, z_1\}$  as a triad,  $z_1 \in Z$ , and  $a_1, a_2 \notin Z$ . Moreover, by 3.8.2,  $Z - z_1$  has no elements  $y_1$  and  $y_2$  such that  $\{z_1, a_1, y_1\}$  and  $\{z_1, a_2, y_2\}$  are triangles and  $\{y_1, y_2\}$  is in a triangle of  $M_2$ .

#### 3.8.4. $M_2 \notin \mathcal{N}$ .

Suppose  $M_2 \in \mathcal{N}$ . As  $M \notin \mathcal{N}$ , it follows that  $\{y_1, y_2\} \neq \{z', z''\}$ . By symmetry, we may assume that  $(y_1, y_2) = (x_1, x_2)$ . When  $|Z - z_1| = 4$ , this case is symmetric to the case when  $\{y_1, y_2\} = \{z', z''\}$ . Thus we may also assume that  $|Z - z_1| \geq 5$ . Now let  $X = \{z_1, a_1, x_1, u_1, z', u_2, x_2, a_2\}$ . Then, using the four triangles in  $M|X$ , it follows that this rank-4 matroid is isomorphic to  $M(\mathcal{W}_4)$  or  $\mathcal{W}^4$ . Moreover,  $M|Y \in \mathcal{N}$  where  $E(M) - Y = \{z_1, a_1, a_2, x_1, u_1, v_1\}$ . Since  $X \cap Y = \{z', u_2, x_2\}$ , it follows by a result of Oxley and Wu [9] (see also [8, Corollary 8.6.8]) that  $M|(X \cup Y)$  is 3-connected, that is,  $M \setminus v_1$  is 3-connected. This contradiction completes the proof of 3.8.4.

We now know that  $M_2 \in \mathcal{N}^*$ . By symmetry, we may assume that  $y_1 = z'$  or  $(y_1, y_2) = (x_1, x_2)$ . In the first case, since  $M \setminus u_i$  is not 3-connected for all  $i$  and  $\{u_1, u_2, \dots, u_k, z'\}$  is a circuit  $D_1$ , it follows by Theorem 3.6 that  $M$  has a triad  $T^*$  that meets  $D_1$ . Since  $|T^* \cap D_1| \geq 2$  and  $M$  has a disjoint set of triangles one containing each element of  $D_1$ , we obtain a contradiction to orthogonality.

It follows that  $(y_1, y_2) = (x_1, x_2)$ . Since  $M \setminus u_1$  and  $M \setminus v_1$  are not 3-connected, by Tutte's Triangle Lemma,  $M$  has a triad  $T^*$  that contains  $u_1$ . By orthogonality,  $T^*$  contains an element of  $D_1 - u_1$  and an element of  $\{v_1, x_1\}$ . By orthogonality again,  $T^*$  contains an element of  $\{v_1, v_2, \dots, v_k, z''\} - \{v_1\}$  or, as  $x_1 = y_1$ , an element of  $\{a_1, z_1\}$ . This contradiction completes the proof of the theorem.  $\square$

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