CHORDAL MATROIDS ARISING FROM GENERALIZED PARALLEL CONNECTIONS II

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Abstract. In 1961, Dirac showed that chordal graphs are exactly the graphs that can be constructed from complete graphs by a sequence of clique-sums. In an earlier paper, by analogy with Dirac's result, we introduced the class of $GF(q)$-chordal matroids as those matroids that can be constructed from projective geometries over $GF(q)$ by a sequence of generalized parallel connections across projective geometries over $GF(q)$. Our main result showed that when $q = 2$, such matroids have no induced minor in $\{M(C_4), M(K_4)\}$. In this paper, we show that the class of $GF(2)$-chordal matroids coincides with the class of binary matroids that have none of $M(K_4)$, $M^*(K_{3,3})$, or $M(C_n)$ for $n \geq 4$ as a flat. We also show that $GF(q)$-chordal matroids can be characterized by an analogous result to Rose's 1970 characterization of chordal graphs as those that have a perfect elimination ordering of vertices.

1. Introduction

The notation and terminology in this paper will follow [5] for graphs and [10] for matroids. Unless stated otherwise, all graphs and matroids considered here are simple. Thus every contraction of a set from a matroid is immediately followed by the simplification of the resulting matroid. We will also assume all matroids are binary unless otherwise specified. Following Cordovil, Forge, and Klein [4], we define a simple or non-simple matroid M to be *chordal* if, for each circuit D that has at least four elements, there are circuits D_1 and D_2 and an element e such that $D_1 \cap D_2 = \{e\}$ and $D = (D_1 \cup D_2) - e$. Therefore, a simple binary matroid is chordal precisely when it has no member of $\{M(C_n) : n \geq 4\}$ as a flat, where C_n is the n-edge cycle. In Section 2, we prove an assertion made in [7] that the class of such matroids coincides with the class of binary matroids with no $M(C_4)$ as an induced minor, where an *induced minor* of a matroid M is any matroid that can be obtained from M by a sequence of contractions and restrictions to flats. This

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matroid notion is analogous to a graph notion, an induced minor of a graph G being a graph that can be obtained from G by a sequence of vertex deletions and edge contractions.

For a prime power q, we denote the projective geometry $PG(r-1,q)$ by P_r when context makes the field clear. A matroid is *$GF(q)$-chordal* if it can be obtained by repeated generalized parallel connections across projective geometries over $GF(q)$ starting with projective geometries over $GF(q)$. In Section 3, we prove the next theorem, which is the main result of the paper, and we give an analogous result for $q > 2$. The equivalence of (i) and (ii) was shown in [7].

Theorem 1.1. *The following are equivalent for a binary matroid M.*

(i) *M is $GF(2)$-chordal.*
(ii) *M has no member of $\{M(C_4), M(K_4)\}$ as an induced minor.*
(iii) *M has no member of $\{M(C_n) : n \geq 4\} \cup \{M(K_4), M^*(K_{3,3})\}$ as an induced restriction.*

In Section 3, we also prove the following analog of Theorem 1.1 for all other primes.

Theorem 1.2. *For each prime $p > 2$, the following are equivalent for a $GF(p)$-representable matroid M.*

(i) *M is $GF(p)$-chordal.*
(ii) *M has no member of $\{U_{2,k} : 3 \leq k \leq p\}$ as an induced minor.*
(iii) *M has no member of $\{U_{n,n+1} : n \geq 2\} \cup \{U_{2+t,k+t} : 4 \leq k \leq p \text{ and } 0 \leq t \leq p+1-k\}$ as an induced restriction.*

Chordal graphs have been characterized in several other ways apart from Dirac's [6] description. A *perfect elimination ordering of a graph* G is an ordering of $V(G)$ such that, for every vertex v, the graph induced by v and all of its neighbors that occur after v in the ordering is a clique. In 1970, Rose [11] proved the following characterization.

Theorem 1.3. *A graph G chordal if and only if G has a perfect elimination ordering.*

A *perfect elimination ordering of cocircuits* of a matroid M is a collection $C_1^*, C_2^*, \dots, C_r^*$ such that, for all i in $[r]$, the set C_i^* is a cocircuit of the matroid M_i, which is $M \backslash (C_1^* \cup C_2^* \cup \cdots \cup C_{i-1}^*)$, and $M|\mathrm{cl}_{M_i}(C_i^*)$ is a projective geometry. In Section 4, we prove the following analog of Theorem 1.3 for $GF(q)$-chordal matroids.

Theorem 1.4. *A matroid M is $GF(q)$-chordal if and only M has a perfect elimination ordering of cocircuits.*

2. Binary chordal matroids

In this section, we will show that the class of binary chordal matroids coincides with the class of matroids with no $M(C_4)$ as an induced minor. We then give a constructive characterization of such matroids.

Lemma 2.1. *Let n be the size of a largest circuit that is an induced minor of M. Then M has an n-element circuit as an induced restriction.*

Proof. We may assume that, for some independent set I of M, the matroid $\mathrm{si}(M/I)$ has an n-element circuit as a flat. If $|I| = 0$, then the result holds. Assume the result holds for $|I| < k$, and let $|I| = k \geq 1$. Take $e \in I$. Then $\mathrm{si}((M/e)/(I - e))$ has an n-element circuit as a flat. Certainly a largest circuit that occurs as an induced minor of $\mathrm{si}(M/e)$ has n elements. Thus, by the induction assumption, $\mathrm{si}(M/e)$ has, as an induced restriction, an n-element circuit C where $C = \{e_1, e_2, \dots, e_n\}$. Then C is a circuit of $M/e\backslash Y$ for some set Y. Thus C or $C \cup e$ is a circuit of M. We may assume that no n-element circuit is an induced restriction of M. Now view M as a restriction of the binary projective geometry P_r where $r = r(M)$. For each i in $[n]$, let f_i be the third point on the projective line $\mathrm{cl}_{P_r}(\{e, e_i\})$. For each i in $[n]$, the set $\{e_1, e_2, \dots, e_n\} - \{e_i\}$ is an independent set I_i of M/e and hence of M. Now $\mathrm{cl}_M(I_i)$ does not contain e otherwise I_i contains a circuit of M/e. Moreover, $\mathrm{cl}_M(I_i)$ does not contain e_i as M does not have an n-element circuit as an induced restriction. The projective flat $\mathrm{cl}_{P_r}(I_i)$ must meet $\{e, f_i, e_i\}$ in P_r, so this intersection is f_i. If f_i is in $E(M)$, then M has $I_i \cup f_i$ as an induced restriction that is an n-element circuit. Thus $\{f_1, f_2, \dots, f_n\}$ avoids $E(M)$. We deduce that $C \cup e$ is an $(n+1)$-element circuit of M that is an induced minor of M, a contradiction. □

In the next theorem, the equivalence of (i) and (iii) is an immediate consequence of the definition. In [7, Lemma 3.8], we had asserted the equivalence of (i) and (ii). Our proof of this relies on Lemma 2.1 and is more subtle than we originally thought, so we have included it.

Theorem 2.2. *The following are equivalent for a binary matroid M.*

(i) *M is chordal.*
(ii) *M has no $M(C_4)$ as an induced minor.*
(iii) *M has no member of $\{M(C_n) : n \geq 4\}$ as an induced restriction.*

Proof. Clearly (iii) implies (ii). Now suppose that M has $M(C_4)$ as an induced minor. Let n be the size of a largest circuit that is an induced minor of M. Since M has $M(C_4)$ as an induced minor, $n \geq 4$. Then, by Lemma 2.1, M has an n-element circuit as an induced restriction, that is, M has $M(C_n)$ as an induced restriction for some $n \geq 4$. □

We now give a constructive characterization of binary chordal matroids. In a matroid M, denote a vertical k-separation (X, Y) by (X, G, Y) where $G = \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$. For a rank-$r$ binary matroid M that is viewed as a restriction of P_r, if $X \subseteq E(P_r) - E(M)$, we denote by $M + X$ the matroid $P_r|(E(M) \cup X)$.

Lemma 2.3. *For some $k \geq 2$, let (X, G, Y) be a vertical k-separation of a binary matroid M where $G = \emptyset$. Then M has $M(C_4)$ as an induced minor.*

Proof. Let C be a smallest circuit of $\mathrm{cl}_{P_r}(M)$ such that $|C \cap (\mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y))| = 1$ and $C - (\mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y)) \subseteq X$. Let $\{z\} = C \cap (\mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y))$, let a and b be distinct elements in $C - z$, and let $M' = (M + z)/(C - \{a, b, z\})$ where we do not simplify here after doing this contraction. Then $\{a, b, z\}$ is a triangle of M', and, for some $k' \leq k$, the matroid M' has a vertical k'-separation (X', G', Y) where $X' = X - (C - \{a, b, z\})$ and $G' = \{z\}$. To see this, observe that if $|G'| > 1$, then there is a circuit $C' \cup z'$ in $\mathrm{cl}_{P_r}(M)$ where C' is contained in $C - \{a, b, z\}$, and $\{z'\}$ is in $\mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y)$. This contradicts the choice of C as a smallest circuit with these properties. Moreover, there is no element parallel to z in M'. Let D be a smallest circuit of M' such that D contains z and $D - z$ is contained in Y. Note that $D \cup \{a, b\}$ is a circuit of M'. Moreover, $\mathrm{cl}_{M'}(D \cup \{a, b, z\}) = D \cup \{a, b, z\}$. Therefore, $M' \backslash \{z\}$ has an n-element circuit as an induced restriction for some $n \geq 4$. By Theorem 2.2, $M' \backslash \{z\}$ has $M(C_4)$ as an induced minor. It follows that M has $M(C_4)$ as an induced minor. □

Lemma 2.4. *For some $k \geq 2$, suppose (X, G, Y) is a vertical k-separation of a binary chordal matroid M. Then $r_M(G) = k - 1$.*

Proof. Suppose $r(G) < k - 1$. Then, for $k' = k - r(G)$, we see that $k' \geq 2$ and M/G has a vertical k'-separation (X', G', Y') with $G' = \emptyset$. By Lemma 2.3, M has $M(C_4)$ as an induced minor, a contradiction to Theorem 2.2. Therefore, $r(G) = k - 1$. □

In the next two proofs, we allow the matroids to be non-simple and we do not simplify after contracting. The next result seems unlikely to be new, but we include a proof for completeness.

Lemma 2.5. *If G is a flat of a matroid N and $G - g$ is a modular flat of N/g for some g in G, then G is modular flat of N.*

Proof. The result is immediate if g is a loop of N, so assume $r(\{g\}) = 1$. For all flats F of N/g,

$$r_{N/g}(F) + r_{N/g}(G - g) = r_{N/g}(F \cap (G - g)) + r_{N/g}(F \cup (G - g)). \quad (2.1)$$

We shall show that, if H is a flat of N, then

$$r_N(H) + r_N(G) - r_N(H \cap G) - r_N(H \cup G) = 0. \quad (2.2)$$

Suppose $g \in H$. Then $r_{N/g}(H - g) = r_N(H) - 1$ and $r_{N/g}((H \cup G) - g) = r_N(H \cup G) - 1$. Therefore, since (2.1) holds, so does (2.2).

Now suppose $g \notin H$. Then $r_{N/g}(H) = r_N(H \cup g) - 1$ and $r_{N/g}(H \cap G) = r_N((H \cap G) \cup g) - 1$. Since H and G are flats of N, it follows that $H \cap G$ is a flat of N. Therefore, $r_N(H \cup g) = r_N(H) + 1$ and $r_N((H \cap G) \cup g) = r_N(H \cap G) + 1$. Thus (2.2) holds and the lemma is proved. □

Lemma 2.6. *For some $k \geq 2$, let (X, G, Y) be a vertical k-separation of a binary chordal matroid M. Then G is a modular flat of $M|\text{cl}(X)$ or of $M|\text{cl}(Y)$.*

Proof. If $k = 2$, then, by Lemma 2.4, $r(G) = 1$. Moreover, M is a parallel connection of $M|\text{cl}(X)$ and $M|\text{cl}(Y)$. By Theorem 2.2, both of these matroids are chordal. As G is a single point, it is a modular flat in both $M|\text{cl}(X)$ and $M|\text{cl}(Y)$ and the result holds. Suppose the result holds for all $k < n$ and let (X, G, Y) be a vertical n-separation of M. Then, for any g in G, the matroid M/g has a vertical $(n-1)$-separation $(X-g, G-g, Y-g)$. By the induction assumption, $G-g$ is a modular flat in either $(M/g)|\text{cl}(X-g)$ or $(M/g)|\text{cl}(Y-g)$. Therefore, by Lemma 2.5, it follows that G is a modular flat of $M|\text{cl}(X)$ or of $M|\text{cl}(Y)$. □

Theorem 2.7. *All binary chordal matroids can be obtained by starting with round binary chordal matroids and repeatedly taking generalized parallel connections of two previously constructed matroids across a set that is a modular flat of one of them.*

Proof. Let M be a binary chordal matroid. If M has no vertical k-separations for any k, then M is round and the result holds. If M has a vertical k-separation (X, G, Y) for some $k > 1$, then, by Lemma 2.6, M is a generalized parallel connection of $M|\text{cl}(X)$ and $M|\text{cl}(Y)$ across $M|\text{cl}(G)$, and G is a modular flat of $M|\text{cl}(X)$ or of $M|\text{cl}(Y)$. □

3. $GF(q)$-CHORDAL MATROIDS

The goal of this section is to prove Theorem 1.1. Recall that a matroid is $GF(2)$-chordal if it can be built from binary projective geometries by repeated generalized parallel connections across projective geometries. In [7, Theorem 1.3], we showed that the class of $GF(2)$-chordal matroids is closed under taking induced minors and therefore, the class is also closed under taking induced restrictions. For a binary matroid M, we may uniquely specify M by describing its complement in the projective geometry $P_{r(M)}$. In general, Brylawski and Lucas [3] (see also [10, Proposition 10.1.7]) showed that the complement of any uniquely $GF(q)$-representable matroid M is well defined in any projective geometry of rank at least $r(M)$. We will first examine the matroids that have $M(K_4)$ as an induced minor.

Lemma 3.1. *If M is a simple binary matroid with $r(M) = 4$ and $|E(M)| > 9$, then either M has $M(C_4)$ or $M(K_4)$ as an induced restriction, or M does not contain $M(K_4)$ as an induced minor.*

Proof. Let P_4 be the rank-4 binary projective geometry. Suppose T is a largest subset of $E(P_4)$ such that $P_4|T$ contains $M(K_4)$ as an induced minor, but T does not contain one of $M(C_4)$ or $M(K_4)$ as an induced restriction. Certainly, P_4 does not contain $M(K_4)$ as an induced minor and so $|T| < 15$. If $|T| \in \{13, 14\}$, then $P_4|T$ certainly has $M(K_4)$ as an induced restriction. If $|T| = 12$, then $P_4\backslash T$ is either P_2 or $U_{3,3}$. If $P_4\backslash T$ is P_2, then $P_4|T$ has $M(C_4)$ as an induced restriction. If $P_4\backslash T$ is $U_{3,3}$, then $P_4|T$ will again contain $M(K_4)$ as an induced restriction. If $|T| = 11$, then $P_4\backslash T$ is $M(C_4)$, $U_{4,4}$, or $P_2 \oplus U_{1,1}$. When $P_4\backslash T$ is $M(C_4)$, we see that $P_4|T \cong P_{P_2}(P_3, P_3)$, so it has no element whose contraction gives $M(K_4)$. If $P_4\backslash T$ is $U_{4,4}$, then $P_4|T$ has a plane with exactly six points, so $P_4|T$ has $M(K_4)$ as an induced restriction. If $P_4\backslash T$ is $P_2 \oplus U_{1,1}$, then $P_4|T$ has $M(C_4)$ as an induced restriction. If $|T| = 10$, then $P_4\backslash T$ is $M(K_4 - e)$, $M(C_4) \oplus U_{1,1}$, or $P_2 \oplus U_{2,2}$. In each case, $P_4|T$ has $M(K_4)$ as an induced restriction. □

Lemma 3.2. *If M/f has $M(K_4)$ as an induced restriction for some f in $E(M)$, then M has $M(C_4), M(K_4)$, or $M^*(K_{3,3})$ as an induced restriction.*

Proof. It suffices to show the result holds for $r(M) = 4$ since we may restrict to a rank-4 flat F such that $(M|F)/f \cong M(K_4)$. Assume M does not have any of $M(C_4), M(K_4)$, or $M^*(K_{3,3})$ as an induced restriction. Since M/f is isomorphic to $M(K_4)$, it follows that $|E(M)| \geq 7$, and, by Lemma 3.1, we have $|E(M)| \leq 9$. Certainly M is connected. If M is not 3-connected, then M is isomorphic to the 2-sum or the parallel connection of $U_{2,3}$ and $M(K_4)$. In the former case, M has $M(C_4)$ as an induced restriction, and, in the latter case, M has $M(K_4)$ as an induced restriction. Thus we may assume M is 3-connected. Suppose M is graphic. Then M is obtained from $M(K_5)$ by deleting at most two edges. If two incident edges have been deleted from K_5, then M is not 3-connected. Therefore M is isomorphic to $M(K_5\backslash e)$ for some edge e, or M is isomorphic to $M(\mathcal{W}_4)$. In the first case, M has $M(K_4)$ as an induced restriction; in the second case, M has $M(C_4)$ as an induced restriction.

We may now assume that M is not graphic. Then M has one of $M^*(K_5)$, $M^*(K_{3,3})$, F_7, or F_7^* as a minor by a theorem of Tutte [13] (see also [10, Theorem 6.6.7]). Since $|E(M)| \leq 9$, we have $M \ncong M^*(K_5)$. If $M \cong M^*(K_{3,3})$, the result holds. If $r^*(M) = 3$, then $M \cong F_7^*$, and thus M has $M(C_4)$ as an induced restriction, a contradiction. If $r^*(M) = 4$, then, by a result of Seymour [12] (see also [10, Lemma 12.2.4]), M is isomorphic to either $AG(3,2)$ or S_8, and, in each case, M has $M(C_4)$ as an induced restriction, a contradiction. By the Splitter Theorem, if $|E(M)| = 9$, then M is an extension of $AG(3,2)$ or S_8. Since the rank-4 complements of $AG(3,2)$ and S_8 are F_7 and $M(K_4) \oplus U_{1,1}$, respectively, there are exactly two possible 9-element matroids as extensions of $AG(3,2)$ and S_8, that is, M is the rank-4 complement of $M(K_4)$ or the rank-4 complement of $M(K_4\backslash e) \oplus U_{1,1}$. In each case, M has $M(C_4)$ as an induced restriction, a contradiction. □

To complete the proof of Theorem 1.1, we only need to consider the matroids with $M^*(K_{3,3})$ as an induced minor.

Lemma 3.3. *If M/e has $M^*(K_{3,3})$ as an induced minor, then M contains one of $M(C_4), M(K_4)$, or $M^*(K_{3,3})$ as an induced restriction.*

Proof. It suffices to show the result holds when $r(M) = 5$ since we may restrict to the relevant rank-5 flat. Assume M does not have any member of $\{M(C_4), M(K_4), M^*(K_{3,3})\}$ as an induced restriction. Let e_1, e_2, e_3, e_4 be the standard basis for P_4. Then we may assume that the ground set Z of the $M^*(K_{3,3})$-restriction of M/e is $\{e_1, e_2, e_3, e_4, e_1 + e_2, e_2 + e_3, e_3 + e_4, e_1 + e_4, e_1 + e_2 + e_3 + e_4\}$. Then the ground set of M may only contain e, elements of Z, and elements of the form $e + f$ where f is an element of Z. Let T a rank-4 flat of M such that $(M/e)|\text{cl}_{M/e}(T) \cong M^*(K_{3,3})$ and $|T|$ is maximal with this property. Certainly $|T| < 9$. Observe that, for every rank-4 flat F of M that avoids e, we have $(M/e)|\text{cl}_{M/e}(F) \cong M^*(K_{3,3})$.

3.3.1. $|T| < 8$.

If $|T| = 8$, then $M|T$ is isomorphic to a 4-wheel, which has $M(C_4)$ as an induced restriction, a contradiction. Thus 3.3.1 holds.

3.3.2. $|T| < 7$.

If $|T| = 7$, then, to avoid having $M(C_4)$ as an induced restriction, we may assume $T = \{e_1, e_3, e_4, e_1+e_4, e_2+e_3, e_3 + e_4, e_1 + e_2 + e_3 + e_4\}$. Therefore, $E(M)$ must contain $\{e, e + e_2, e + e_1 + e_2\}$. Look at the flat containing $\{e_3, e_2 + e_3, e + e_2, e\}$, noting that e_2 is absent. Since this flat is not isomorphic to $M(C_4)$, it must contain either $e + e_3$ or $e + e_2 + e_3$. Suppose that $e + e_2 + e_3$ is present. Then M contains $\{e_1, e_1+e_4, e_4, e_3+e_4, e_3, e+e_2+e_3, e+e_2, e+e_1+e_2\}$ as eight elements of a rank-4 flat F that avoids e. By 3.3.1, $|F| \leq 7$, a contradiction. We deduce that $e+e_2+e_3$ is absent, and hence $e+e_3$ must be present. In this case, M has $\{e_1, e_4, e_1+e_4, e_1+e_2+e_3+e_4, e_2+e_3, e+e_3, e+e_2, e+e_1+e_2\}$ as eight elements of a rank-4 flat that avoids e, and again we contradict 3.3.1. We conclude that, 3.3.2 holds.

3.3.3. *The set $Z - T$ does not contain a triangle.*

Without loss of generality, suppose $Z - T$ contains the triangle $\{e_1, e_2, e_1 + e_2\}$. Then M must contain the 4-element circuit $\{e, e + e_1, e + e_2, e + e_1 + e_2\}$ as a flat, a contradiction. Therefore, 3.3.3 holds.

3.3.4. $|T| < 6$.

If $|T| = 6$, then, since M does not have $M(C_4)$ as an induced restriction and $Z-T$ does not contain a triangle, we may assume T is missing $\{e_2, e_1 + e_2, e_2 + e_3\}$. Then M contains $\{e + e_2, e + e_1 + e_2, e + e_2 + e_3\}$. Thus M contains $\{e_1, e+e_1+e_2, e+e_2, e+e_2+e_3, e_3, e_3+e_4, e_4, e_1+e_4\}$ as eight elements of a rank-4 flat of M that avoids e, a contradiction to 3.3.1. Therefore, 3.3.4 holds.

3.3.5. $|T| < 5$.

If $|T| = 5$, then, since M does not have $M(C_4)$ as an induced restriction and $Z - T$ does not contain a triangle, we may assume T is missing one of the following four sets of points.

(a) $\{e_2, e_3, e_3 + e_4, e_1 + e_2\}$
(b) $\{e_2, e_2 + e_3, e_3 + e_4, e_1 + e_2\}$
(c) $\{e_1, e_2, e_3 + e_4, e_1 + e_2 + e_3 + e_4\}$

In case (a), M has $\{e_1, e + e_1 + e_2, e + e_2, e + e_3, e_2 + e_3, e_1 + e_2 + e_3 + e_4, e_1 + e_4, e_4\}$ as eight elements of a rank-4 flat of M that avoids e, which contradicts 3.3.1. In case (b), $\{e_1, e + e_1 + e_2, e + e_2, e + e_2 + e_3, e_3, e_4, e_1 + e_4\}$ spans a rank-4 flat F of M that avoids e and has at least seven elements, which contradicts 3.3.2. In case (c), M has $\{e + e_2, e + e_2 + e_3, e_3, e_4, e + e_1, e_1 + e_2\}$ contained in a rank-4 flat of M avoiding e, which contradicts 3.3.4. Therefore, 3.3.5 holds.

If $|T| = 4$, then, as T must be independent and $Z - T$ contains no triangles, we may assume that $T = \{e_1, e_1 + e_2, e_3, e_1 + e_2 + e_3 + e_4\}$.

This implies that M has $\{e_1, e+e_2, e+e_2+e_3, e_3, e+e_3+e_4\}$ spanning a rank-4 flat of M that avoids e, a contradiction to 3.3.5. $\square$

The next result establishes a connection between the excluded induced minors of a class of matroids and the excluded induced restrictions of that class of matroids.

Lemma 3.4. *Suppose $\mathcal{N}$ is a class of matroids such that if a matroid M has an element e such that M/e is isomorphic to a member of $\mathcal{N}$, then M has a member of $\mathcal{N}$ as an induced restriction. Let $\mathcal{N}'$ be the class of induced-minor-minimal members of $\mathcal{N}$. Then the class of matroids with no member of $\mathcal{N}'$ as an induced minor coincides with the class of matroids with no member of $\mathcal{N}$ as an induced restriction.*

Proof. Clearly if M has a member of $\mathcal{N}$ as an induced restriction, then M has a member of $\mathcal{N}'$ as an induced minor. Conversely, suppose M has a member of $\mathcal{N}'$ as an induced minor. We may assume that M has no member of $\mathcal{N}'$ as an induced restriction. Then M has a flat F and a nonempty independent set T such that $(M|F)/T$ is isomorphic to a member of $\mathcal{N}'$ and hence a member of $\mathcal{N}$. We argue by induction on $|T|$ that M has a member of $\mathcal{N}$ as an induced restriction.

When $|T| = 1$, we see that the assertion holds by the definition of $\mathcal{N}$. Now assume the result holds for $|T| < n$, and let $|T| = n \geq 2$. Let $M_1 = M|F$, and take e in T. Then $(M_1/e)/(T - e)$ is isomorphic to a member of $\mathcal{N}$. Thus, by the induction assumption, M_1/e has a member of $\mathcal{N}$ as an induced restriction. Then, by the induction assumption again, M_1 has a member of $\mathcal{N}$ as an induced restriction, and hence M has a member of $\mathcal{N}$ as an induced restriction. $\square$

Proof of Theorem 1.1. Let $\mathcal{N} = \{M(C_n) : n \geq 4\} \cup \{M(K_4), M^*(K_{3,3})\}$. Then, by Lemmas 2.1, 3.2, and 3.3, the set $\mathcal{N}$ has the property that if M/e is isomorphic to a member of $\mathcal{N}$, then M has a member of $\mathcal{N}$ as an induced minor. Since $\{M(C_4), M(K_4)\}$ is the set of induced-minor-minimal members of $\mathcal{N}$, the result holds by Lemma 3.4. $\square$

Next we prove the analog of Theorem 1.1 when $q > 2$. Although the result is rather unattractive, we include it for completeness. The equivalence of (i) and (ii) was shown in [7]. Because $U_{3,q+2}$ is $GF(q)$-representable if and only if q is even, when q is odd, we can omit $U_{3,q+2}$ from (ii). This corrects a small error in Theorem 1.4 of [7]. Note that some matroids in (iii) may be excluded as they fail to be $GF(q)$-representable. But the problem of determining precisely which uniform matroids are $GF(q)$-representable is open. For results on exactly which uniform matroids are known to be $GF(q)$-representable, we direct the reader to [1, 2, 8, 9] (see also [10, Conjecture 6.5.20]). Recall that the class of $GF(q)$-chordal matroids is the class of matroids that can be obtained by a sequence of generalized parallel connections across projective geometries over $GF(q)$ starting with projective geometries over $GF(q)$.

Theorem 3.5. *When $q > 2$, the following are equivalent for a $GF(q)$-representable matroid M.*
- (i) *M is $GF(q)$-chordal.*
- (ii) *M has no member of $\{U_{2,k} : 2 < k \leq q\} \cup \{U_{3,q+2}\}$ as an induced minor.*
- (iii) *M has no member of $\{U_{n,n+1} : n \geq 2\} \cup \{U_{2+t,k+t} : 4 \leq k \leq q$ and $0 \leq t \leq q-3\} \cup \{U_{3,q+2}\}$ as an induced restriction.*

Lemma 3.6. *Suppose M is $GF(q)$-representable. If M/e has $U_{2,n}$ as an induced minor for some n with $3 \leq n \leq q$, then M has a member of $\{U_{2,k} : 3 \leq k \leq q\} \cup \{U_{3,n+1}\}$ as an induced restriction.*

Proof. It suffices to show that the result holds for $r(M) = 3$ since we may restrict to the relevant rank-3 flat. Let $E(M/e) = \{x_1, x_2, \dots, x_k\}$. If M has a $(q+1)$-point line, then e must be on that line, otherwise, M/e would be $U_{2,q+1}$. Without loss of generality, suppose $\text{cl}(\{e, x_k\})$ is a $(q+1)$-point line, L_1. Then the line $L_2 = \text{cl}(\{x_1, x_2\})$ must intersect L_1 at a point different from e. Therefore, L_2 is either isomorphic to $U_{2,k}$ for some k with $3 \leq k \leq q$, in which case the result holds, or L_2 is a $(q+1)$-point line, and M/e is isomorphic to $U_{2,q+1}$, a contradiction. Therefore, we may assume that M has no $(q+1)$-point lines. Let X be a 3-element subset of $E(M)$. Since M has no member of $\{U_{2,k} : 3 \leq k \leq q\}$ as an induced restriction, and M has no $(q+1)$-point lines, it follows that X is independent. Therefore, M is isomorphic to $U_{3,n+1}$. □

To complete the proof Theorem 3.5, we consider the following general result for uniform matroids.

Lemma 3.7. *Let M be $GF(q)$-representable for some $q > 2$ and suppose $M/e \cong U_{r,n}$ for some r and n with $2 < r < n$. Then M has a member of $\{U_{2+t,k+t} : 3 \leq k \leq q$ and $0 \leq t \leq q-3\} \cup \{U_{r,n}, U_{r+1,n+1}\}$ as an induced restriction.*

Proof. Assume the result fails. Observe that, since $U_{2,q+2}$ is not $GF(q)$-representable, we must have $n \leq q + r - 1$. Next note that if M has a hyperplane H_0 that avoids e, then $M|H_0$ must be isomorphic to a uniform matroid for if H_0 contains a non-spanning circuit C, then M/e must also contain C as a non-spanning circuit. Moreover, H_0 can contain at most n elements as $|E(M/e)| = n$. Therefore, $M|H_0$

is isomorphic to $U_{r,s}$ for some s with $r \leq s \leq n$. Suppose that H_0 can be chosen so that $s > r$. If $s = n$, then M has $U_{r,n}$ as an induced restriction, a contradiction. Thus we may suppose that $s < n$. Observe that if $r > q-1$, then, since $U_{q,q+2}$ is not $GF(q)$-representable, we must have $n = r+1$, so $s = n$, a contradiction. Thus we may assume that $r \leq q-1$. Then we obtain a contradiction by taking $t = r-2$ and $k = s-r+2$ for then $t \leq q-3$ and $k < n-r+2 \leq (q+r-1)-r+2 = q+1$. Hence we may assume that every hyperplane that avoids e is an independent set of size r.

Let $E(M/e) = \{x_1, x_2, \dots, x_n\}$. If M has a $(q+1)$-point line, this line must contain e as M/e does not have a $(q+1)$-point line. Without loss of generality, suppose $\mathrm{cl}_M(\{e, x_n\})$ is a $(q+1)$-point line, L. Then $\mathrm{cl}_M(\{x_1, x_2, \dots, x_r\})$ is a hyperplane H of M that meets L at a point different from e. Therefore H is a hyperplane that avoids e and contains a circuit of M, a contradiction. We conclude that M has no $(q+1)$-point lines. Therefore, either e is contained in a $U_{2,k}$ induced restriction for some k with $3 \leq k \leq q$, and the result holds, or every circuit containing e has size $r+2$. Let X be a subset of $E(M)-e$. If $|X| < r+1$, then X is certainly independent as X must be independent in M/e. Suppose $|X| = r+1$ and that X is dependent. Then $\mathrm{cl}_M(X)$ is a hyperplane that avoids e and contains a circuit of M, a contradiction. Therefore, X is an independent set of size $r+1$, and M is isomorphic to $U_{r+1,n+1}$. □

Proof of Theorem 3.5. It suffices to show the equivalence of (ii) and (iii). Let $\mathcal{N} = \{U_{n,n+1} : n \geq 2\} \cup \{U_{2+t,k+t} : 4 \leq k \leq q \text{ and } 0 \leq t \leq q-3\} \cup \{U_{3,q+2}\}$. Then, since $U_{4,q+3}$ is not $GF(q)$-representable for any q, it follows by Lemmas 3.6 and 3.7 that $\mathcal{N}$ has the property that if M/e is isomorphic to a member of $\mathcal{N}$, then M has a member of $\mathcal{N}$ as an induced minor. Since $\{U_{2,k} : 3 \leq k \leq q\} \cup \{U_{3,q+2}\}$ is the set of induced-minor-minimal members of $\mathcal{N}$, the result holds by Lemma 3.4.

Note if M has $U_{2+t,k+t}$ as an induced restriction for some t and k with $t \geq q-2$ and $k \geq 4$, then M has $U_{q,q+2}$ as a minor, so M is not $GF(q)$-representable, a contradiction. □

Theorem 1.2 follows from Theorem 3.5 and a result of Ball [1] (see also [10, Theorem 6.5.21]), which establishes precisely when a uniform matroid is $GF(p)$-representable for p a prime.

When $p = 3$, Theorem 1.2 gives the following.

Corollary 3.8. *The following are equivalent for a ternary matroid M.*

(i) *M is $GF(3)$-chordal.*
(ii) *M has no $M(C_3)$ induced minor.*

(iii) *M has no member of $\{M(C_n) : n \geq 3\}$ as an induced restriction.*

Similarly, Theorem 3.5 gives the following when $q = 4$.

Corollary 3.9. *The following are equivalent for a matroid M that is representable over $GF(4)$.*
(i) *M is $GF(4)$-chordal.*
(ii) *M has no member of $\{U_{2,3}, U_{2,4}, U_{3,6}\}$ as an induced minor.*
(iii) *M has no member of $\{U_{n,n+1} : n \geq 2\} \cup \{U_{2,4}, U_{3,5}, U_{3,6}\}$ as an induced restriction.*

4. Perfect Elimination Ordering

In this section, we will prove Theorem 1.4. Recall that a perfect elimination ordering of cocircuits of a matroid M is a collection of sets $C_1^*, C_2^*, \dots, C_r^*$ such that, for all i in $[r]$, the set C_i^* is a cocircuit of the matroid $M_i = M \backslash (C_1^* \cup C_2^* \cup \cdots \cup C_{i-1}^*)$ and $M|\text{cl}_{M_i}(C_i^*)$ is a projective geometry. Observe that $M|\text{cl}_{M_i}(C_i^*) = M_i|\text{cl}_{M_i}(C_i^*)$ and that $E(M_i)$ is a flat of M of rank $r(M) - i + 1$. We omit the straightforward proof of the next result.

Lemma 4.1. *Projective geometries have a perfect elimination ordering of cocircuits.*

We now prove the second main result of the paper.

Proof of Theorem 1.4. Suppose M is a $GF(q)$-chordal matroid that does not have a perfect elimination ordering of cocircuits and has $|E(M)|$ a minimum among such matroids. Since M is $GF(q)$-chordal, M may be written as a $P_N(M_1, M_2)$ where N is a projective geometry and M_2 is a projective geometry. Therefore, M_2 has a perfect elimination ordering of cocircuits. Choose C_1^* to be a cocircuit of M_2 that avoids $E(N)$. By the minimality of M, the matroid $M \backslash C_1^*$ has a perfect elimination ordering of cocircuits $C_2^*, C_3^*, \dots, C_r^*$. Therefore, M has a perfect elimination ordering.

Suppose M has a perfect elimination ordering of cocircuits labeled $C_1^*, C_2^*, \dots, C_r^*$. Let $M_1 = M$, and let $M_i = M \backslash (C_1^* \cup C_2^* \cup \cdots \cup C_{i-1}^*)$ for each i with $2 \leq i \leq r$. Certainly, M_r is a projective geometry, and therefore is $GF(q)$-chordal. Let k be the smallest integer such that M_k is not $GF(q)$-chordal. Then M_{k+1} is $GF(q)$-chordal, $M_k|\text{cl}_{M_k}(C_k^*)$ is a projective geometry, and M_{k+1} is a hyperplane of M_k. Thus $E(M_{k+1}) \cap E(M_k|\text{cl}_{M_k}(C_k^*))$ is a flat, N, of a projective geometry and so $M|N$ is a projective geometry. Hence M_k is a generalized parallel connection of $GF(q)$-chordal matroids across a projective geometry, namely $P_{M|N}(M_{k+1}, M|\text{cl}_{M_k}(C_k^*))$. Thus, M_k is $GF(q)$-chordal, a contradiction. □

On combining Theorems 1.1 and 1.4, we obtain the following.

Corollary 4.2. *The following are equivalent for a binary matroid M.*

(i) *M is $GF(2)$-chordal.*

(ii) *M has no member of $\{M(C_4), M(K_4)\}$ as an induced minor.*

(iii) *M has no member of $\{M(C_n) : n \geq 4\} \cup \{M(K_4), M^*(K_{3,3})\}$ as an induced restriction.*

(iv) *M has a perfect elimination ordering of cocircuits.*

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