On Crapo's Beta Invariant for Matroids

By *James G. Oxley*

In this paper we characterize the matroids for which Crapo's beta invariant does not exceed 4. In addition, we relate the beta invariant of a matroid to its chromatic number and to its connectivity.

1. Introduction

The terminology used here for matroids and graphs will, in general, follow Welsh [13] and Bondy and Murty [1], respectively. The ground set of a matroid M will be denoted $E(M)$, and if $T \subseteq E(M)$, we denote the rank and closure of T by $\text{rk}\, T$ and $\bar{T}$, respectively. The rank of M will be written $\text{rk}\, M$.

For a matroid M, the invariant $\beta(M)$, which was introduced by Crapo [5], is defined as follows:

$$\beta(M) = (-1)^{\text{rk}\, M} \sum_{A \subseteq E(M)} (-1)^{|A|} \text{rk}\, A. \tag{1.1}$$

This invariant is related to the chromatic polynomial $P(M; \lambda)$ [13, p. 262] of M by the identity

$$\beta(M) = (-1)^{\text{rk}\, M+1} \left. \frac{dP(M;\lambda)}{d\lambda} \right|_{\lambda=1}. \tag{1.2}$$

From this and the well-known deletion-contraction formula for the chromatic polynomial of a matroid [13, p. 263], it follows that

$$\beta(M) = \beta(M \setminus e) + \beta(M/e) \tag{1.3}$$

for any element e which is neither a loop nor a coloop of M [5, Theorem I].

From (1.1) we get immediately that

$$\beta(M) = 0 \qquad \text{if} \qquad M \text{ is a loop,} \tag{1.4}$$

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and

$$\beta(M) = 1 \quad \text{if} \quad M \text{ is a coloop.} \tag{1.5}$$

Using these results and (1.3), an induction argument gives that

$$\beta(M) \geqslant 0 \quad \text{for all matroids } M. \tag{1.6}$$

In fact, as Crapo [5, Theorem II] has shown, provided $|E(M)| \geqslant 2$,

$$\beta(M) > 0 \quad \text{if and only if} \quad M \text{ is connected.} \tag{1.7}$$

Two further fundamental properties of β are as follows:

(1.8) [5, Theorem IV]. $\beta(M)=\beta(M^*)$ for all matroids M having at least two elements.

(1.9) [3, Corollary 6.9]. If N is a minor of the connected matroid M, then $\beta(N) \leqslant \beta(M)$.

For $n \geqslant 1$, the matroid M is *n-connected* [11, p. 1303] provided that, for all positive integers $k<n$, there is no subset T of $E(M)$ such that $|T| \geqslant k$, $|E(M)\backslash T| \geqslant k$ and $\operatorname{rk} T + \operatorname{rk}[E(M)\backslash T] - \operatorname{rk} M = k-1$. Thus a matroid is 2-connected precisely when it is connected [13, p. 69]. Moreover,

(1.10) M is n-connected if and only if M^* is n-connected.

If M is n-connected for all positive integers n, then the *connectivity* $\kappa(M)$ of M is ∞; otherwise $\kappa(M)=\max\{k : M \text{ is } k\text{-connected}\}$.

We shall assume that the reader is familiar with the operations of series connection and parallel connection for matroids, these operations having been discussed in detail in [2]. For matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$, we shall denote the parallel connection of M_1 and M_2 with respect to the basepoint p as $P((M_1, p), (M_2, p))$, or sometimes as just $P(M_1, M_2)$. Brylawski [2, Theorem 6.16(vi)] has shown that, provided p is neither a loop nor a coloop of M_1 or M_2,

$$\beta(P(M_1, M_2)) = \beta(M_1)\beta(M_2). \tag{1.11}$$

The following link between 3-connection and parallel connection was proved by Seymour and will be a fundamental tool in this paper.

(1.12) THEOREM [10, (2.5)]. *A connected matroid M is not 3-connected if and only if there are matroids M_1 and M_2 each having at least three elements such that $M = P((M_1, p), (M_2, p))\backslash p$, where p is not a loop or a coloop of M_1 or M_2.*

When M decomposes as in the preceding theorem, we call M the *2-sum* of M_1 and M_2. It is routine to check, using the properties of parallel connection, that

(1.13) If $|E(M_i)| \geqslant 2$ for $i=1,2$, then $P((M_1, p), (M_2, p))\backslash p$ is connected if and only if both M_1 and M_2 are connected. Hence, in particular, the 2-sum of M_1 and M_2 is connected if and only if both M_1 and M_2 are connected.

If $\{x, y\}$ is a cocircuit of the matroid M, we say that x and y are *in series* in M. If, instead, $\{x, y\}$ is a circuit, then x and y are *in parallel*. A *series class* of M is a maximal subset A of $E(M)$ such that if a and b are distinct elements of A, then a and b are in series. *Parallel classes* are defined analogously. We call a series or parallel class *nontrivial* if it contains at least two elements. The matroid M' if a *series extension* of M if $M' = M/x$, and x is in series with some element y of M'. If, instead, $M' = M \setminus x$ and x is in parallel with y, then M' is a *parallel extension* of M. We call M'' a *series-parallel extension* of M if M'' can be obtained from M by repeated application of the operations of series and parallel extension. The next two statements are straightforward to check.

(1.14) [5, Propositions 4 and 5]. If M_1 is a series-parallel extension of the loopless matroid M, then $\beta(M_1) = \beta(M)$.

(1.15) [14, Lemma 3.1]. If $|E(M)| \geq 4$ and M has a nontrivial series or parallel class, then M is not 3-connected.

2. The matroids with $\beta < 4$

The problem of characterizing the matroids for which $\beta = 1$ was solved by Brylawski.

(2.1) THEOREM [2, Theorem 7.6]. *For a connected matroid M on a set of at least two elements, the following statements are equivalent:*

(i) $\beta(M) = 1$.
(ii) *M is a series-parallel extension of $U_{1,1}$.*
(iii) *M has no minor isomorphic to $U_{2,4}$ or $M(K_4)$.*

A connected matroid having at least two elements and satisfying one, and hence all, of the above conditions is called a *series-parallel network*.

In this section we shall prove the following two results. The wheel on $r+1$ vertices, the whirl of rank r, and the Fano matroid will be denoted $\mathcal{W}_r$, $\mathcal{W}^r$, and F_7, respectively.

(2.2) THEOREM. *For a matroid M, $\beta(M) = 2$ if and only if M is a series-parallel extension of $U_{2,4}$ or $M(K_4)$.*

(2.3) THEOREM. *For a matroid M, $\beta(M) = 3$ if and only if M is a series-parallel extension of $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, $M(\mathcal{W}_4)$, or $\mathcal{W}^3$.*

The proofs of these results will require several preliminaries. We note first that

(2.4) [5, Proposition 10]. $\beta(M(\mathcal{W}_r)) = r - 1$ and $\beta(\mathcal{W}^r) = r$.

(2.5) PROPOSITION. *Let M be a matroid and suppose that $\beta(M) = k > 1$. Then either*

(i) *M is a series-parallel extension of a 3-connected matroid N such that $\beta(N) = k$, or*
(ii) *M is the 2-sum of two matroids each having $\beta < k$.*

Proof: We argue by induction on $|E(M)|$. If M has a nontrivial parallel class K and $e \in K$, then $\beta(M) = \beta(M \setminus e)$. Now, by the induction assumption, (i) or (ii) holds for $M\backslash e$. Hence, as M is a parallel extension of $M\backslash e$, (i) or (ii) holds for M. It follows that we may assume that M has no nontrivial parallel classes, and similarly, that M has no nontrivial series classes.

We now suppose that (ii) does not occur and show that (i) must hold. If M is 3-connected, the result is immediate. Thus suppose that M is not 3-connected. Then, as $\beta(M)>0$, we have by (1.7) that M is connected. Therefore, by Theorem 1.12, $M=P((M_1,p),(M_2,p))\backslash p$, where $|E(M_i)|\geq 3$ for $i=1,2$. Thus, by (1.3),

$$\beta(M)=\beta(P(M_1,M_2))-\beta(P((M_1,p),(M_2,p))/p).$$

But $P((M_1,p),(M_2,p))/p$ is disconnected [2, Proposition 5.8] and so has $\beta=0$. Therefore, by (1.11) and (1.13), $\beta(M)=\beta(M_1)\beta(M_2)$. Since (ii) does not occur, we may assume that $\beta(M_1)=k$ and $\beta(M_2)=1$. Thus, M_2 is a series-parallel network having at least three elements. But now a routine induction argument, using the properties of series-parallel networks [2, §7], shows that M_2 has a 2-element circuit or cocircuit not containing p. Thus M has a 2-element circuit or cocircuit, and hence M has a nontrivial series or parallel class. This contradiction completes the proof. □

(2.6) Corollary. *Suppose that $\beta(M)$ is prime. Then M is a series-parallel extension of a 3-connected matroid N for which $\beta(N)=\beta(M)$.*

(2.7) Lemma. *Let N be a minor of the matroid M, and suppose that $\beta(N)=\beta(M)>0$. If N is 3-connected, then M is a series-parallel extension of N.*

Proof: If $\beta(N)=1$, then, as N is 3-connected, $N\cong U_{1,1}$, $U_{1,2}$, $U_{1,3}$, or $U_{2,3}$ and the result is easy to check. Now suppose that $\beta(N)\geq 2$. By Theorem 2.1 and (1.15), $|E(N)|\geq 4$ and N has no nontrivial series or parallel classes. We argue by induction on $|E(M)|$, noting that the result holds trivially for $|E(M)|=4$. Suppose then that the result holds for $|E(M)|<k$ and let $|E(M)|=k>4$. If M has a nontrivial series or parallel class, then the required result follows, as in the proof of Proposition 2.5. We may therefore assume that M has no nontrivial series or parallel classes. Thus, by Proposition 2.5, either M is 3-connected, in which case $M=N$, or $M=P((M_1,p),(M_2,p))\backslash p$ where $|E(M_i)|\geq 3$ for $i=1,2$ and $\beta(N)$ exceeds both $\beta(M_1)$ and $\beta(M_2)$. If both $|E(N)\cap E(M_1\backslash p)|$ and $|E(N)\cap E(M_2\backslash p)|$ exceed 1, then as the operation of parallel connection commutes with the operations of deletion and contraction of elements other than the basepoint [2, Proposition 5.6] it follows that N can be expressed as a 2-sum. Thus, by Theorem 1.12, we obtain the contradiction that N is not 3-connected. Therefore, we may assume, without loss of generality, that $|E(N)\cap E(M_2\backslash p)|\leq 1$.

We now show that N is a minor of M_1. This is certainly true if $|E(N)\cap E(M_2\backslash p)|=0$. If $|E(N)\cap E(M_2\backslash p)|=1$, then N can be obtained by deleting p from the parallel connection of (M_1',p) and (M_2',p), where M_i' is a minor of M_i for $i=1,2$, and $|E(M_2')|=2$. Now N is connected, so by (1.13), M_2' is connected and hence $M_2'\cong U_{1,2}$. We conclude that N is a minor of M_1. But, by (1.13) again, M_1 is connected and so, by (1.9), $\beta(N)\leq\beta(M_1)$. This contradiction to the fact that $\beta(N)>\beta(M_1)$ completes the proof. □

We now prove the two main results of this section.

Proof of Theorem 2.2: As $\beta(U_{2,4})=2=\beta(M(K_4))$, if M is a series-parallel extension of $U_{2,4}$ or $M(K_4)$, then, by (1.14), $\beta(M)=2$.

Now suppose that $\beta(M)=2$. Then M is a connected matroid having at least two elements. Thus, by Theorem 2.1, M has a minor isomorphic to $U_{2,4}$ or $M(K_4)$. Since each of these matroids is 3-connected having $\beta=2$, the result follows by Lemma 2.7. □

Proof of Theorem 2.3: It is not difficult to check that $\beta(U_{2,5})=\beta(F_7)=\beta(M(\mathcal{W}_4))=\beta(\mathcal{W}^3)=3$. It follows, by (1.8) and (1.14), that if M is a series-parallel extension of one of the six matroids listed in (2.3), then $\beta(M)=3$.

Now assume that $\beta(M)=3$. Then, by Corollary 2.6, M is a series-parallel extension of a 3-connected matroid N such that $\beta(N)=3$. By Theorems 2.1 and 2.2, $|E(N)|\geqslant 5$. Suppose that for all elements e of N, neither $N\backslash e$ nor N/e is 3-connected. Then, by [11, 8.3], N is isomorphic to a whirl or the cycle matroid of a wheel. Hence, by (2.4), $N\cong M(\mathcal{W}_4)$ or $\mathcal{W}^3$.

We may now assume that for some element e of N, either $N\backslash e$ or N/e is 3-connected. Suppose the former. Then as $|E(N\backslash e)|\geqslant 4$, we have by Theorem 2.1 that $\beta(N\backslash e)\geqslant 2$. But N/e is connected, so $\beta(N/e)\geqslant 1$. Thus, by (1.3), $\beta(N\backslash e)=2$ and $\beta(N/e)=1$. Hence, by Theorem 2.2 and (1.15), $N\backslash e\cong U_{2,4}$ or $M(K_4)$, and, as $\beta(N)=3$ and N is 3-connected, it is straightforward to check that $N\cong U_{2,5}$ or F_7.

If N/e is 3-connected for some element e of N, then, by (1.10), $N^*\backslash e$ is 3-connected. By (1.8), we may argue as above to deduce that $N^*\cong U_{2,5}$ or F_7, and therefore $N\cong U_{3,5}$ or F_7^*. □

3. The matroids with $\beta=4$

The characterization of the matroids having $\beta=4$ resembles in part the proof of Theorem 2.3. However, an additional complication arises here because 4 is not prime.

Let N_1 and N_2 be the rank-three matroids for which affine representations are given in Figure 1(a, b). In addition, let N_3 be the non-Fano matroid [see Figure 1(c)] and K_5^- be the graph obtained by deleting an edge from K_5. Finally, let N_4 be the dependence matroid of the following binary matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then it is straightforward to check that both N_1 and N_4 are isomorphic to their duals and that each of N_1, N_2, N_3, N_4, and $M(K_5^-)$ has $\beta=4$.

(3.1) THEOREM. *Let M be a matroid. Then $\beta(M)=4$ if and only if either*

(i) *M is a series-parallel extension of one of the matroids $U_{2,6}$, $U_{4,6}$, $\mathcal{W}^4$, $M(\mathcal{W}_5)$, N_1, N_2, N_2^*, N_3, N_3^*, N_4, $M(K_5^-)$, or $M^*(K_5^-)$; or*

(ii) *M is a 2-sum of matroids M_1 and M_2 each of which is a series-parallel extension of $M(K_4)$ or $U_{2,4}$.*

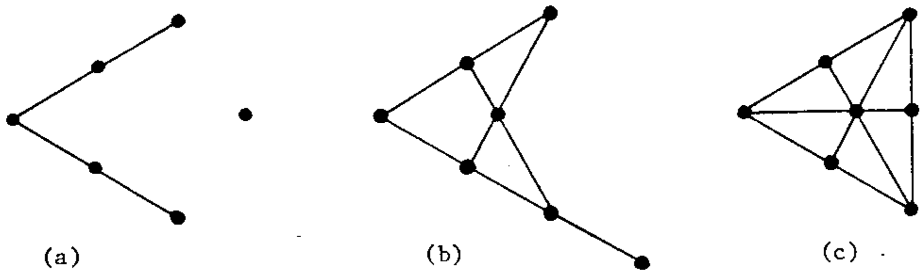

Figure 1. (a) N_1, (b) N_2, (c) N_3.

Proof: It is not difficult to show that if (i) or (ii) holds, then $\beta(M)=4$. Now assume that $\beta(M)=4$. Then, by Proposition 2.5, either M is a series-parallel extension of a 3-connected matroid N having $\beta=4$, or $M=P((M_1,p),(M_2,p))\backslash p$, where each of M_1 and M_2 has $\beta<4$. In the latter case, since M is connected, we have by (1.13) that both M_1 and M_2 are connected. It follows, by (1.11) and Theorem 2.2, that (ii) holds. We may now assume that M is a series-parallel extension of a 3-connected matroid N having $\beta=4$. If for every element e of N, neither $N\backslash e$ nor N/e is 3-connected, then N is a whirl or the cycle matroid of a wheel [11, 8.3], and so, by (2.4), $N\cong\mathcal{W}^4$ or $M(\mathcal{W}_5)$.

Now suppose that N has an element e such that $N\backslash e$ is 3-connected. As N/e is connected, $\beta(N/e)\geqslant 1$, and so, by (1.3), $\beta(N\backslash e)\leqslant 3$. But $N\backslash e$ is 3-connected and Theorems 2.1 and 2.2 imply that $|E(N\backslash e)|\geqslant 4$. Hence, by Theorems 2.2 and 2.3, $N\backslash e$ is isomorphic to one of $U_{2,4}$, $U_{2,5}$, $U_{3,5}$, $\mathcal{W}^3$, $M(K_4)$, F_7, F_7^*, and $M(\mathcal{W}_4)$. We now check each of these possibilities. Firstly, we note that $N\backslash e\not\cong U_{2,4}$, for $U_{2,4}$ has no single-element extension with β equal to 4. If $N\backslash e\cong U_{2,5}$, then, as N is 3-connected, $N\cong U_{2,6}$. If $N\backslash e\cong U_{3,5}$, then it is straightforward to check that $N\cong N_1$. Every 3-connected single-element extension of F_7 has β exceeding four, hence $N\backslash e\not\cong F_7$. If $N\backslash e\cong\mathcal{W}^3$, then $N\cong N_3$, while if $N\backslash e\cong M(K_4)$, then $N\cong N_2$ or N_3.

The two remaining possibilities are that $N\backslash e\cong F_7^*$ or $N\backslash e\cong M(\mathcal{W}_4)$. We shall show in each case that N is isomorphic to one of the matroids listed in (i). Suppose that $N\backslash e\cong M(\mathcal{W}_4)$. In the affine representation for $M(\mathcal{W}_4)$ shown in Figure 2, the elements 1, 2, 3, and 4 are the vertices of a tetrahedron. By (1.3), $\beta(N\backslash e)=1$. Now assume that in N, $e\notin\overline{\{1,2,3\}}$. Then N/e has a minor isomorphic to $N|\{1,2,3,5,6\}$ and $N|\{1,2,3,5,6\}\cong P(U_{2,3},U_{2,3})$. Thus, as $\beta(N/e)=1$, it is not difficult to check that the simple matroid associated with N/e is isomorphic to $P(U_{2,3},U_{2,3})$. Hence, for each of the points 4, 7, and 8, there is a line containing e and one of the points in $\{1,2,3,5,6\}$. It is straightforward to show that the only possibility here is that $e\in\overline{\{1,4\}}\cap\overline{\{5,8\}}\cap\overline{\{6,7\}}$. It then follows that $N\cong M(K_5^-)$.

Clearly e avoids one of the planes $\overline{\{1,2,3\}}$, $\overline{\{1,2,4\}}$, $\overline{\{1,3,4\}}$ and $\overline{\{2,3,4\}}$. By symmetry, we may apply the above argument in each case. We conclude that if $N\backslash e\cong M(\mathcal{W}_4)$, then $N\cong M(K_5^-)$.

Finally, suppose that $N\backslash e\cong F_7^*$. We shall first show that N is binary. Suppose not. Then N has $U_{2,4}$ as an upper minor and hence there is an element x of N such that $U_{2,4}$ is a minor of N/x. We shall show that $x=e$. As N is 3-connected, for all elements f of N, both $N\backslash f$ and N/f are connected; hence β is positive for each.

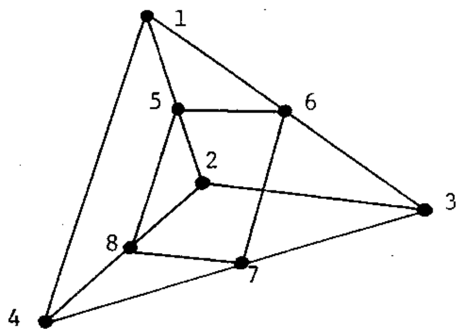

Figure 2.

Thus $\beta(N/f) \leqslant 3$. If $f \neq e$, then consider $N/f \backslash e = N \backslash e/f$. As $N \backslash e \cong F_7^*$, we have $N/f \backslash e \cong M(K_4)$, and so $\beta(N/f) \geqslant 2$. Thus N/f has rank 3 and $\beta(N/f) = 2$ or 3. In the latter case, by Theorem 2.3, $N/f \cong F_7$, while in the former, as $M(K_4)$ is a minor of N/f, Lemma 2.7 implies that N/f is a parallel extension of $M(K_4)$. Thus N/f is binary for all elements $f \neq e$. Hence $x = e$, and so, by (1.9), $\beta(N/e) \geqslant 2$. But $\beta(N \backslash e) = 3$. Thus $\beta(N) \geqslant 5$, a contradiction. It follows that we may assume that N is binary. Then as $N \backslash e \cong F_7^*$, N is obtained by deleting a point from $AG(3,2)$. Therefore, $N \backslash e$ is the complement in $PG(3,2)$ of the direct sum of F_7 and a coloop. By a result of Brylawski and Lucas [4, pp. 93–94], it follows that $N \cong AG(3,2)$ or N_4. But $\beta(AG(3,2)) = 6$; hence $N \cong N_4$.

We may now assume that N has an element e such that N/e is 3-connected. Then, as $(N/e)^* \cong N^* \backslash e$, we have, by (1.8), that $\beta(N^* \backslash e) = \beta(N/e)$. On applying the above argument with N^* in place of N, we obtain that N^* is isomorphic to one of $U_{2,6}$, N_1, N_2, N_3, N_4, and $M(K_5^-)$. Hence N is isomorphic to $U_{4,6}$, N_1, N_2^*, N_3^*, N_4, or $M^*(K_5^-)$, and so N is included in the list under (i). □

The technique that was used above to determine the 3-connected matroids with $\beta = 4$ may be applied to find those 3-connected matroids with $\beta = 5$. Then, by Corollary 2.6, every matroid with β equal to 5 is a series-parallel extension of one of these 3-connected matroids. Evidently, one can continue to apply this technique to determine the matroids with β equal to 6, 7, and so on. However, even for β equal to 5, the technique is rather cumbersome.

For each positive integer n, there is a set $\mathbb{S}_n$ of matroids each of which is irreducible with respect to series-parallel extensions and such that $\beta(M) = n$ if and only if M is a series-parallel extension of some member of $\mathbb{S}_n$. Theorems 2.1,

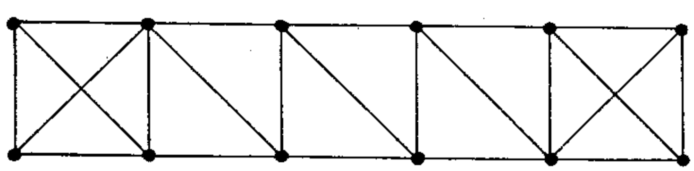

Figure 3. G_1.

2.2, and 2.3, respectively, assert that $\mathbb{S}_1=\{U_{1,1}\}$, $\mathbb{S}_2=\{U_{2,4}, M(K_4)\}$, and $\mathbb{S}_3=\{U_{2,5}, U_{3,5}, F_7, F_7^*, M(\mathcal{W}_4), \mathcal{W}^3\}$. By (3.1)(ii), if G_1 is the graph in Figure 3, then $\beta(M(G_1))=4$. Evidently $M(G_1)\in\mathbb{S}_4$, and it is easy to generalize this example to show that $\mathbb{S}_4$ is infinite. The next result follows by extending the above observations and using the fact that, for $n\geq 2$, a 3-connected member of $\mathbb{S}_n$ has at most $2n+2$ elements. The latter is a consequence of an easy induction argument.

(3.2) PROPOSITION. *$\mathbb{S}_n$ is finite if and only if $n=1$ or n is prime.*

4. The beta invariant, the chromatic number, and connectivity

If M is a loopless matroid, then as in [8, p. 15; 12], we define $\pi(M)=\min\{k\in\mathbb{Z}^+ : P(M; k+j)>0 \text{ for all } j=0,1,2,\ldots\}$. Thus, if G is a loopless graph, then $\pi(M(G))$ is the chromatic number of G.

(4.1) [2, Theorem 7.9]. If $\beta(M)=1$, then $\pi(M)\leq 3$.

The results of the last two sections may be used to extend this as follows.

(4.2) THEOREM. *If $\beta(M)=k$, where $1\leq k\leq 4$, then $\pi(M)\leq k+2$.*

Since $P(U_{2,k+2};\lambda)=(\lambda-1)(\lambda-k-1)$, we have $\beta(U_{2,k+2})=k$ and $\pi(U_{2,k+2})=k+2$. Therefore the bound in Theorem 4.2 is best possible. The author knows of no example to contradict the statement of (4.2) when the restriction on k is removed. Indeed, if M is regular, a stronger result holds for arbitrary k, since for such matroids, $\pi(M)=\min\{k\in\mathbb{Z}^+ : P(M; k)>0\}$ (see, for example, [8, Theorem 2.9]).

The proof of Theorem 4.2 requires the following preliminaries, the first of which is a technical lemma.

(4.3) LEMMA. *Let M' be a series-parallel extension of a matroid M. If $x\in E(M')$, then there is an element y of M such that M'/x is isomorphic to a series-parallel extension of the direct sum of M or M/y with a matroid of rank zero.*

Proof: We argue by induction on $|E(M')\backslash E(M)|$, noting that the result is immediate if this cardinality is zero. Now assume the result true for $|E(M')\backslash E(M)|<k$ and let $|E(M')\backslash E(M)|=k$. As M' is a series-parallel extension of M, we may assume, without loss of generality, that $x\in E(M)$. It will follow from this assumption that we may take $y=x$. Now there is an element e of M' such that $e\notin E(M)$ and either

(i) $\{e, f_1, f_2,\ldots,f_m\}$ is a nontrivial parallel class of M', or
(ii) $\{e, f_1, f_2,\ldots,f_m\}$ is a nontrivial series class of M'.

In case (i), $M'\backslash e$ is a series-parallel extension of M, so, by the induction assumption, $M'\backslash e/x$ has the required property. Now $\{f_1, f_2,\ldots,f_m\}$ is contained in a parallel class in $M'\backslash e/x$ unless $x=f_i$ for some i. In the former case, as $M'\backslash e/x$ has the required property, so does M'/x. In the latter case, e is a loop of M'/x, and again, as $M'\backslash e/x$ has the required property, so does M'/x.

In case (ii), consider M'/e. This is a series-parallel extension of M and so, by the induction assumption, $M'/e/x$ has the required property. The result follows easily unless $\{e, f_1, f_2,\ldots,f_m\}=\{e, x\}$, but in that case, $M'/e\cong M'/x$ and the result again follows. □

(4.4) LEMMA. *Let M' be a series-parallel extension of a loopless matroid M. Suppose that $\pi(N) \leq m$ for all loopless contractions N of M. Then $\pi(M') \leq m$ unless $m=2$, in which case $\pi(M') \leq m+1$.*

Proof: We argue by induction on $|E(M') \setminus E(M)|$, noting that the result is trivially true if $|E(M') \setminus E(M)|=0$. Assume it true for $|E(M') \setminus E(M)|<k$, and let $|E(M') \setminus E(M)|=k$. Then there are elements e and f of M' such that $e \notin E(M)$ and $\{e,f\}$ is a 2-element circuit or a 2-element cocircuit of M'. In the former case, it is clear that $\pi(M')=\pi(M)$. In the latter case, we have $P(M';\lambda)= P(M' \setminus e;\lambda)-P(M' \setminus e;\lambda)$ and hence, as f is a coloop of $M' \setminus e$,

$$P(M';\lambda)=(\lambda-1)P(M' \setminus e/f;\lambda)-P(M'/e;\lambda).$$

Now $P(M' \setminus e/f;\lambda)=P(M'/f \setminus e;\lambda)=P(M'/f;\lambda)+P(M'/e,f;\lambda)$ and $M'/e \cong M'/f$; thus

$$P(M';\lambda)=(\lambda-2)P(M'/e;\lambda)+(\lambda-1)P(M'/e,f;\lambda). \tag{4.5}$$

As $e \notin E(M)$, it is clear that M'/e is a series-parallel extension of M. Therefore, by the induction assumption, $\pi(M'/e) \leq \max\{3,m\}$. Moreover, by Lemma 4.3, there is an element g of M such that $M'/e,f$ is isomorphic to a series-parallel extension of the direct sum of M or M/g and a matroid of rank zero. By (4.1), Theorem 2.1, and the induction assumption, it follows that either $M'/e,f$ has a loop, or $\pi(M'/e,f) \leq \max\{m,3\}$. We conclude, from (4.5), that $\pi(M') \leq \max\{m,3\}$. □

Proof of Theorem 4.2: By (4.1), the result is true for $\beta(M)=1$. The proof for $\beta(M)=2$, 3, or 4 involves first checking the chromatic numbers of the loopless contractions of the matroids in Theorems 2.2 and 2.3 and in (3.1)(i). In these cases the result follows by Lemma 4.4. Having established the result when $\beta(M)=2$, the case of the matroids in (3.1)(ii) follows by Lemma 4.3. The details of this straightforward argument are omitted. □

We now consider briefly the relationship between $\beta(M)$ and the connectivity of M. We shall concentrate on n-connected matroids having at least $2(n-1)$ elements, as for such matroids every single-element deletion and contraction is $(n-1)$-connected. The only n-connected matroids with fewer than $2(n-1)$ elements have infinite connectivity and hence are uniform matroids whose rank and corank differ by at most 1 [9, Lemma 2; 6, Theorem 1]. The β invariant for such matroids can be determined from Brylawski's result [3, Corollary 7.14] that

$$\beta(U_{r,k})=\binom{k-2}{r-1}.$$

For each integer $n \geq 2$, define

$$b(n)=\min\{\beta(M): M \text{ is } n\text{-connected and } |E(M)| \geq 2(n-1)\}.$$

An immediate consequence of this definition is that

$$b(n+1) \geq 2b(n) \quad \text{for all} \quad n \geq 2. \tag{4.6}$$

As $b(2)=1$, it follows that

$$(4.7) \qquad b(n) \geqslant 2^{n-2} \quad \text{for} \quad n \geqslant 2.$$

This bound is attained when $n=2$ and $n=3$, but, in general, seems rather weak. Indeed, the author knows of no counterexample to the statement that $b(n) \geqslant \binom{2n-4}{n-2}$. The latter bound is attained by $U_{n-1,2n-2}$ and, for example, when $n=3$, by $M(K_4)$.

We shall now use the results of the preceding sections to determine $b(4)$ and $b(5)$ and thereby to sharpen (4.7) when $n \geqslant 5$.

(4.8) PROPOSITION. *If M is a 4-connected matroid and $|E(M)| \geqslant 6$, then $\beta(M) \geqslant 10$ unless $M \cong U_{3,6}$.*

Proof: Since a 4-connected matroid with at least 6 elements has no circuits or cocircuits with fewer than 4 elements [14, Lemma 3.1], it is not difficult to check that the only 4-connected matroids with 6 or 7 elements are $U_{3,6}$, $U_{3,7}$, and $U_{4,7}$. As $\beta(U_{3,7})=10=\beta(U_{4,7})$, we may assume that $|E(M)| \geqslant 8$. Hence if $x \in E(M)$, then both $M \backslash x$ and M/x are 3-connected and both $|E(M \backslash x)|$ and $|E(M/x)|$ exceed 6. Therefore, by Theorems 2.2, 2.3, and 3.1, both $\beta(M \backslash x)$ and $\beta(M/x)$ exceed 4 unless $M \backslash x$ or M/x is in $\{F_7, F_7^*, M(\mathcal{W}_4), \mathcal{W}^4, M(\mathcal{W}_5), N_2, N_2^*, N_3, N_3^*, N_4, M(K_5^-), M^*(K_5^-)\}$. Now $M \backslash x$ has no 3-element circuits and M/x has no 3-element cocircuits. Thus $\beta(M \backslash x) \geqslant 5$ unless $M \backslash x \in \{F_7^*, N_3^*\}$, and $\beta(M/x) \geqslant 5$ unless $M/x \in \{F_7, N_3\}$. By (1.8) and (1.3), it follows that the required result will be established if we can show that for $M/x \in \{F_7, N_3\}$, $\beta(M) \geqslant 10$.

Suppose that $M/x = F_7$. Then, as M is 4-connected and therefore has no 3-element circuits, one can check without difficulty that M is the free lift of F_7 (see, for example, [7, p. 150]). Hence $\beta(M)=13 \geqslant 10$.

If $M/x = N_3$, then again it is not difficult to check that M is either the free lift N_3' of N_3, or the matroid N_3'' for which an affine representation is shown in Figure 4. As $\beta(N_3')=14$ and $\beta(N_3'')=13$, we conclude that $\beta(M) \geqslant 10$. □

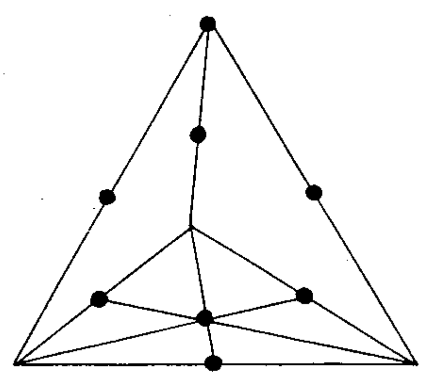

Figure 4. N_3''.

As $\beta(U_{3,6})=6$, an immediate consequence of the preceding result is that

(4.9) $$b(4)=6.$$

Moreover, using (4.8) and its proof technique, it is straightforward to show that $b(5)\geqslant 20$. As $\beta(U_{4,8})=20$, it follows that

(4.10) $$b(5)=20.$$

Thus, by (4.6), we have

(4.11) $$b(n)\geqslant 5\times 2^{n-3} \qquad \text{for all} \quad n\geqslant 5.$$

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