# MATROIDS WITH MANY SMALL CIRCUITS AND COCIRCUITS

JAMES OXLEY, SIMON PFEIL, CHARLES SEMPLE, AND GEOFF WHITTLE

ABSTRACT. Tutte proved that a non-empty 3-connected matroid with every element in a 3-element circuit and a 3-element cocircuit is either a whirl or the cycle matroid of a wheel. This result led to the Splitter Theorem. More recently, Miller proved that a matroid of sufficient size with every pair of elements in a 4-element circuit and a 4-element cocircuit is a tipless spike. Here we investigate matroids having similar restrictions on their small circuits and cocircuits. In particular, we completely determine the 3-connected matroids with every pair of elements in a 4-element circuit and every element in a 3-element cocircuit, as well as the 4-connected matroids with every pair of elements in a 4-element circuit and every element in a 4-element cocircuit.

## 1. INTRODUCTION

The study of matroids with many small circuits and cocircuits begins with Tutte's well-known Wheels-and-Whirls Theorem [6]. This theorem was originally stated in terms of *essential* elements of a 3-connected matroid $M$, that is, elements $e$ of $M$ with the property that neither $M\backslash e$ nor $M/e$ is 3-connected. We present it here in terms of 3-circuits and 3-cocircuits, where, as in the rest of the paper, a $k$-element circuit and a $k'$-element cocircuit is denoted as a $k$-circuit and $k'$-cocircuit, respectively.

**Theorem 1.1.** *Let $M$ be a non-empty 3-connected matroid. Then every element of $M$ is in a 3-circuit and a 3-cocircuit if and only if $M$ has rank at least three and is isomorphic to a wheel or a whirl.*

Theorem 1.1 and its well-known extension, Seymour's Splitter Theorem [5], has been instrumental in the analysis of 3-connected matroids. More recently, Miller [2] proved the following result which has conditions similar to those in Tutte's theorem. For all $r \geq 3$, a rank-$r$ *tipless spike* is a matroid

*Date*: November 27, 2018.

1991 *Mathematics Subject Classification.* 05B35.

*Key words and phrases.* Wheels-and-Whirls Theorem, 4-circuits, 3-cocircuits, 4-cocircuits.

The third and fourth authors were supported by the New Zealand Marsden Fund.

$M$ with ground set $E = \{x_1, y_1, x_2, y_2, \dots, x_r, y_r\}$ whose circuits consist of the following sets:

(i) all sets of the form $\{x_i, y_i, x_j, y_j\}$ with $1 \leq i < j \leq r$,
(ii) a subset of $\{\{z_1, z_2, \dots, z_r\} : \ z_i \in \{x_i, y_i\} \text{ for all } i\}$ such that no two members of this subset have more than $r-2$ common elements, and
(iii) all $(r+1)$-element subsets of $E$ that contain none of the sets in (i) and (ii).

**Theorem 1.2.** *Let $M$ be a matroid with $|E(M)| \geq 13$. Then every pair of elements of $M$ is in a 4-circuit and a 4-cocircuit if and only if $M$ is a tipless spike.*

In this paper, we continue along a similar line of inquiry. A matroid $M$ has *property* (P1) if every pair of elements is in a 4-circuit and every element is in a 3-cocircuit. Furthermore, a matroid $M$ has *property* (P2) if every pair of elements is in a 4-circuit and every element is in a 4-cocircuit. The next two theorems are the main results of this paper. We denote the rank-3 whirl, the Fano matroid, and the non-Fano matroid by $\mathcal{W}^3$, $F_7$, and $F_7^-$, respectively. Also, up to isomorphism, we denote the rank-3 simple matroid with ground set $\{1, 2, \dots, 7\}$ and whose 3-circuits are $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 6\}$, and $\{3, 5, 7\}$ by $P_7$.

**Theorem 1.3.** *Let $M$ be a non-empty 3-connected matroid. Then $M$ has property* (P1) *if and only if*

(i) *$|E(M)| \leq 8$ and $M$ is isomorphic to one of the matroids $U_{3,5}$, $M(K_4)$, $\mathcal{W}^3$, $F_7$, $(F_7^-)^*$, and $P_7^*$, or*
(ii) *$|E(M)| \geq 9$ and $M$ is isomorphic to $M(K_{3,n})$ for some $n \geq 3$.*

**Theorem 1.4.** *Let $M$ be a non-empty 4-connected matroid. Then $M$ has property* (P2) *if and only if*

(i) *$|E(M)| \leq 15$ and $M$ is isomorphic to one of the thirty-five matroids listed in the appendix, or*
(ii) *$|E(M)| \geq 16$ and $M$ is isomorphic to $M(K_{4,n})$ for some $n \geq 4$.*

It is clear that $M(K_{3,n})$, where $n \geq 3$, and $M(K_{4,n})$, where $n \geq 4$, satisfy (P1) and (P2), respectively. For $|E(M)| \geq 9$ and $|E(M)| \geq 16$, the necessary directions of the proofs of Theorems 1.3 and 1.4 are given in Sections 2 and 3, respectively. The connectivity conditions in Theorems 1.3 and 1.4 prevent an $i$-element subset of $E(M)$ from being both an $i$-circuit and an $i$-cocircuit, for $i = 3$ and $i = 4$, respectively. For the proof of Theorem 1.3 when $|E(M)| \leq 8$ and the proof of Theorem 1.4 when $|E(M)| \leq 15$, we refer the interested reader to Pfeil's PhD thesis [4]. We end the introduction with some preliminaries.

Throughout the paper, notation and terminology follows Oxley [3]. Let $M$ be a matroid. Two subsets $X$ and $Y$ of $E(M)$ *meet* if $X \cap Y$ is non-empty. Referred to as *orthogonality*, it is well known that if $C$ is a circuit and $D$ is a cocircuit of $M$, then $|C \cap D| \neq 1$.

Lastly, let $M_1$ and $M_2$ be two matroids with ground sets $E_1$ and $E_2$, respectively, and let $\varphi : E_1 \to E_2$ be a bijection. Then $\varphi$ is a *weak map* from $M_1$ to $M_2$ if, for every independent set $I$ in $M_2$, we have $\varphi^{-1}(I)$ is independent in $M_1$, in which case, $M_2$ is a *weak-map image* of $M_1$. Equivalently, it is easily checked that, $\varphi$ is a weak map from $M_1$ to $M_2$ if, for every circuit $C$ of $M_1$, we have $\varphi(C)$ contains a circuit in $M_2$. As in this paper, it is typical to assume that $E_1$ and $E_2$ are the same sets and $\varphi$ is the identity map. The following theorem is due to Lucas [1].

**Theorem 1.5.** *Let $M_2$ be the weak-map image of a binary matroid $M_1$, and suppose that $r(M_2) = r(M_1)$. Then $M_2$ is binary. Moreover, if $M_2$ is connected, then $M_2 \cong M_1$.*

## 2. Matroids with Property (P1) and at Least 9 Elements

Throughout this section, $M$ is a 3-connected matroid satisfying (P1) and with ground set $E(M) = \{x_1, x_2, \ldots, x_t\}$, where $t \geq 4$. Our ability to determine $M$, for when $|E(M)| \geq 9$, explicitly relies on showing that $E(M)$ can be partitioned into blocks in which each block is a 3-cocircuit and $M$ restricted to any two of these blocks is isomorphic to $M(K_{2,3})$. We first prove that if $M$ has two distinct 3-cocircuits that meet in two elements, then $M$ is isomorphic to $U_{3,5}$.

**Lemma 2.1.** *Let $D_1$ and $D_2$ be two 3-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. Then $M \cong U_{3,5}$.*

*Proof.* Without loss of generality, let $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_2, x_4\}$. Then $M^*|(D_1 \cup D_2) \cong U_{2,4}$ since $M$ is 3-connected. This implies that if $|E(M)| = 4$, then $M$ has no 4-circuits; a contradiction, so $|E(M)| \geq 5$. Furthermore, by orthogonality, any circuit meeting $D_1 \cup D_2$ does so in at least three elements. By (P1), $M$ has a 4-circuit $C_1$ containing $\{x_1, x_5\}$, and there is a unique element $x_i$ in $\{x_2, x_3, x_4\}$ that is not in $C_1$. Then, similarly, $M$ has a 4-circuit $C_2$ containing $\{x_i, x_5\}$. Now $C_1 \cup C_2 = \{x_1, x_2, x_3, x_4, x_5\}$ and $r(C_1 \cup C_2) = 3$. Also $r^*(C_1 \cup C_2) \leq 3$. Therefore

$$r(C_1 \cup C_2) + r^*(C_1 \cup C_2) - |C_1 \cup C_2| \leq 3 + 3 - 5 = 1,$$

and so $|E(M)| \leq 6$ as $M$ is 3-connected. Using the fact that $M$ satisfies (P1), a routine check shows that $|E(M)| \leq 5$, and so $M \cong U_{3,5}$. □

The next three lemmas concern disjoint 3-cocircuits. The first shows that $M$ restricted to two such 3-cocircuits is isomorphic to $M(K_{2,3})$, while the second and third accumulate in showing that if $|E(M)| \geq 9$, then $M$ has three pairwise-disjoint 3-cocircuits.

**Lemma 2.2.** *Let $D_1$ and $D_2$ be two disjoint 3-cocircuits of $M$. Then $M|(D_1 \cup D_2) \cong M(K_{2,3})$.*

*Proof.* Without loss of generality, let $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$. By (P1), $M$ has a 4-circuit $C_1$ containing $\{x_1, x_4\}$. By orthogonality, we may assume $C_1 = \{x_1, x_2, x_4, x_5\}$. Similarly, $M$ has a 4-circuit $C_2$ containing $\{x_3, x_6\}$. By symmetry, we may assume $C_2 = \{x_1, x_3, x_4, x_6\}$. Lastly, $M$ has a 4-circuit $C_3$ containing $\{x_2, x_6\}$. We next show that $C_3$ does not meet either $C_1$ or $C_2$ in three elements.

Say $|C_1 \cap C_3| = 3$. Then $C_3 \subseteq (D_1 \cup D_2) - x_3$. As $M$ is 3-connected, $M|(C_1 \cup C_3) \cong U_{3,5}$, and so there exists a 4-circuit in $M$ meeting $D_1$ in exactly one element; a contradiction. Thus $|C_1 \cap C_3| \neq 3$ and, similarly, $|C_2 \cap C_3| \neq 3$.

By orthogonality with $D_1$ and $D_2$, it now follows that neither $x_1$ nor $x_4$ is in $C_3$, and so $C_3 = \{x_2, x_3, x_5, x_6\}$. We now apply Theorem 1.5 to complete the proof. Since $M|(D_1 \cup D_2) = M|(C_1 \cup C_2 \cup C_3)$, we have $r(M|(D_1 \cup D_2)) = 4$. Next, consider $K_{2,3}$, and label its edges so that

$$\big\{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\big\}$$

is a partition of $E(K_{2,3})$, where each block is a bond of $K_{2,3}$, and $\{x_1, x_2, x_4, x_5\}$, $\{x_1, x_3, x_4, x_6\}$, and $\{x_2, x_3, x_5, x_6\}$ are the 4-cycles of $K_{2,3}$. Then the identity map from $E(M(K_{2,3}))$ to $E(M|(D_1 \cup D_2))$ is a weak map from $M(K_{2,3})$ to $M|(D_1 \cup D_2)$. Moreover, as $M|(D_1 \cup D_2)$ is connected, Theorem 1.5 implies that $M|(D_1 \cup D_2) \cong M(K_{2,3})$. □

**Lemma 2.3.** *If $|E(M)| \geq 9$, then $M$ has two disjoint 3-cocircuits.*

*Proof.* Suppose $|E(M)| \geq 9$ and $M$ has no pair of disjoint 3-cocircuits. Let $D_1$ and $D_2$ be distinct 3-cocircuits of $M$. Then, by Lemma 2.1, $|D_1 \cap D_2| = 1$ and so, without loss of generality, we may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$. We first show that $M$ has an element contained in three 3-cocircuits.

Assume $M$ has no such element. By (P1), $M$ has a 3-cocircuit $D_3$ containing $x_6$. By assumption, $D_3$ meets each of $D_1$ and $D_2$ and so, without loss of generality, $D_3 = \{x_2, x_4, x_6\}$. But $M$ also has a 3-cocircuit containing $x_7$, and such a cocircuit cannot meet each of $D_1$, $D_2$, and $D_3$ without using an element shared by two of them. Thus $M$ has an element contained in three 3-cocircuits.

By Lemma 2.1, we may now assume that $M$ has a 3-cocircuit $D_3 = \{x_1, x_6, x_7\}$. Consider a 3-cocircuit $D_4$ of $M$ containing $x_8$. Since $D_4$ meets each of $D_1$, $D_2$, and $D_3$, we have $x_1 \in D_4$ and so, by Lemma 2.1, we may assume $D_4 = \{x_1, x_8, x_9\}$. However, by (P1), $M$ has a 4-circuit $C$ containing $\{x_1, x_2\}$. By orthogonality, each of $|C \cap D_2|$, $|C \cap D_3|$, and $|C \cap D_4|$ is at least 2 which is impossible as $|C| = 4$. This contradiction establishes the lemma. □

**Lemma 2.4.** *If $|E(M)| \geq 9$, then $M$ has three pairwise-disjoint 3-cocircuits.*

*Proof.* Suppose $|E(M)| \geq 9$. By Lemma 2.3, $M$ has disjoint 3-cocircuits, $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$ say. By Lemma 2.2, we have $M|(D_1 \cup D_2) \cong M(K_{2,3})$. Therefore, without loss of generality, we may assume that $M$ has circuits $C_1 = \{x_1, x_2, x_4, x_5\}$, $C_2 = \{x_1, x_3, x_4, x_6\}$, and $C_3 = \{x_2, x_3, x_5, x_6\}$. By (P1), $M$ has a 3-cocircuit $D_3$ containing $x_7$. If $M$ does not contain three pairwise-disjoint 3-cocircuits, then $D_3$ meets $D_1 \cup D_2$ and, by orthogonality, it must do so in one of the pairs $\{x_1, x_4\}$, $\{x_2, x_5\}$, $\{x_3, x_6\}$. Therefore, by symmetry, we may assume $D_3 = \{x_1, x_4, x_7\}$. Similarly, if $D_4$ is a 3-cocircuit of $M$ containing $x_8$, then, by Lemma 2.1, we may assume $D_4 = \{x_2, x_5, x_8\}$. Finally, applying the same argument again, if $D_5$ is a 3-cocircuit of $M$ containing $x_9$, we have $D_5 = \{x_3, x_6, x_9\}$. But then $D_3$, $D_4$, and $D_5$ are disjoint, thereby completing the proof of the lemma. □

We next show that $E(M)$ can be partitioned into 3-cocircuits provided $|E(M)| \geq 9$.

**Lemma 2.5.** *If $|E(M)| \geq 9$, then $E(M)$ can be partitioned into 3-element blocks, where each block is a 3-cocircuit.*

*Proof.* Suppose $|E(M)| \geq 9$, and let $S = \{D_1, D_2, \ldots, D_n\}$ be the largest collection of pairwise-disjoint 3-cocircuits of $M$. By Lemma 2.4, we have $n \geq 3$. Suppose there is an element $x$ in $M$ not in any of the sets $D_1, D_2, \ldots, D_n$. By (P1), $M$ has a 3-cocircuit $D$ containing $x$. Now $D$ has a non-empty intersection with a 3-cocircuit in $S$; otherwise, $S$ is not of maximum size. Without loss of generality, $D \cap D_1 \neq \emptyset$ and so, by Lemma 2.1, $|D \cap D_1| = 1$. By Lemma 2.2, $M|(D_1 \cup D_i) \cong M(K_{2,3})$ for all $i \in \{2, 3, \ldots, n\}$. Thus, by orthogonality, $D$ meets each of $D_2, D_3, \ldots, D_n$. But then $|D| \geq 4$ as $n \geq 3$; a contradiction. Thus, the lemma is proved. □

We are now ready to prove the necessary direction of Theorem 1.3 when $|E(M)| \geq 9$.

*Proof of Theorem 1.3 for $|E(M)| \geq 9$.* Suppose $|E(M)| \geq 9$. Then, by Lemma 2.5, there is a partition of $E(M)$ into 3-cocircuits $D_1, D_2, \ldots, D_n$

where $D_i = \{x_i, y_i, z_i\}$ for all $i$. By Lemma 2.2, $M|(D_1 \cup D_i) \cong M(K_{2,3})$ for all $i \in \{2, 3, \dots, n\}$, so we may assume that $M$ has 4-circuits $\{x_1, x_i, y_1, y_i\}$, $\{x_1, x_i, z_1, z_i\}$, and $\{y_1, y_i, z_1, z_i\}$ for all such $i$. Consider the circuits $\{x_1, x_i, y_1, y_i\}$ and $\{x_1, x_j, y_1, y_j\}$, where $i$ and $j$ are distinct. By circuit elimination and orthogonality, $\{x_i, y_i, x_j, y_j\}$ is a 4-circuit of $M$. Similarly, for all distinct $i, j \in \{2, 3, \dots, n\}$, we have $\{x_i, z_i, x_j, z_j\}$ and $\{y_i, z_i, y_j, z_j\}$ are 4-circuits of $M$.

We next show that each set of the form

$$\{x_i, y_i, y_j, z_j, z_k, x_k\},$$

where $i$, $j$, and $k$ are distinct elements in $\{1, 2, \dots, n\}$, is a 6-circuit of $M$ Using circuit elimination on $\{x_i, y_i, x_j, y_j\}$ and $\{x_j, z_j, x_k, z_k\}$, it follows that $\{x_i, y_i, y_j, x_k, z_j, z_k\}$ contains a circuit of $M$. By orthogonality and as each of $M|(D_i \cup D_j)$, $M|(D_i \cup D_k)$, and $M|(D_j \cup D_k)$ is isomorphic to $M(K_{2,3})$, it is easily checked that $\{x_i, y_i, y_j, x_k, z_j, z_k\}$ is itself a 6-circuit of $M$.

Now consider $K_{3,n}$, where $n \geq 3$. Label the edge set of $K_{3,n}$ so that

$$\big\{\{x_1, y_1, z_1\}, \{x_2, y_2 . z_2\}, \dots, \{x_n, y_n, z_n\}\big\}$$

is a partition of $E(K_{3,n})$, where each block is a bond of $K_{3,n}$, and $\{x_i, y_i, x_j, y_j\}$, $\{x_i, z_i, x_j, z_j\}$, and $\{y_i, z_i, y_j, z_j\}$ are 4-cycles of $K_{3,n}$ for all distinct $i, j \in \{1, 2, \dots, n\}$. Then the identity map $\varphi$ from $E(M(K_{3,n}))$ to $E(M)$ is a weak map from $M(K_{3,n})$ to $M$ since, for each circuit $C$ of $M(K_{3,n})$, we have $\varphi(C)$ is a circuit of $M$ by above.

We next prove that $r(M) = r(M(K_{3,n}))$. To do this, we show, by induction, that for all 3-connected matroids $M'$ satisfying (P1) and whose ground set can be partitioned into $m$ 3-cocircuits, where $m \geq 3$, we have $r(M') = r(M(K_{3,m}))$. If $n = 3$, then, by Lemma 2.2 and the 4-circuits established above, $r(M) = r(M(K_{3,3}))$. Therefore suppose $n \geq 4$ and that, for all matroids $M'$ as described above, with $3 \leq m \leq n-1$, we have $r(M') = r(K_{3,m})$. Let $M^-$ denote the matroid $M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1})$. We first show that $M^-$ satisfies (P1). Evidently, every element of $M^-$ is in a 3-cocircuit. Let $x$ and $y$ be distinct elements of $M^-$. If $x$ and $y$ are in distinct 3-cocircuits $D_i$ and $D_j$ of $M^-$, then, by orthogonality and $M$ satisfying (P1), $M^-$ has a 4-circuit containing $\{x, y\}$. Say $x$ and $y$ are in the same 3-cocircuit, $D_i$ say, of $M^-$. By considering $D_i$ with either $D_1$ if $i \neq 1$ or $D_2$ if $i = 1$, it follows by Lemma 2.2 that $M^-$ has a 4-circuit containing $\{x, y\}$. Lastly, it remains to show that $M^-$ is 3-connected. If $M^-$ is not 3-connected, then it has a 2-separation $(A, B)$. Since $n - 1 \geq 3$, it follows that, for some $i \in \{1, 2, \dots, n-1\}$, there is a 3-cocircuit $D_i$ such that for one of $A$ and $B$, say $A$, we have $D_i \subseteq A$, or $|D_i \cap A| = 2$ and $|B| \geq 3$. Thus, we may assume that $D_1 \subseteq A$. But then, by the 4-circuits above,

$r(A \cup D_n) = r(A) + 1$. Therefore

$$r(A \cup D_n) + r(B) - r(M) = r(A) + 1 + r(B) - (r(M^-) + 1)$$
$$= r(A) + r(B) - r(M^-),$$

and so $(A \cup D_n, B)$ is a 2-separation in $M$; a contradiction. Thus $M^-$ is 3-connected. By induction,

$$r(M^-) = r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1}) = r(M(K_{3,n-1})),$$

and so, as $D_n$ is a cocircuit of $M$,

$$r(M) = r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1}) + 1 = r(M(K_{3,n})).$$

Finally, $M$ is connected and so, by Theorem 1.5, $M \cong M(K_{3,n})$. This completes the proof of Theorem 1.3. $\square$

## 3. Matroids with Property (P2) and at Least 16 Elements

Throughout this section, $M$ is a 4-connected matroid satisfying (P2). Unless stated otherwise, $M$ has ground set $E(M) = \{x_1, x_2, \ldots, x_t\}$, where $t \geq 4$. The approach is similar to that of the last section. In particular, most of the work is in establishing that if $|E(M)| \geq 16$, then there is partition of $E(M)$ into blocks in which each block is a 4-cocircuit. However, because of the freedom of 4-cocircuits in comparison to 3-cocircuits, the case analysis is much more involved. We begin with a lemma analogous to Lemma 2.1.

**Lemma 3.1.** *Let $D_1$ and $D_2$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 3$. Then $M \cong U_{3,6}$.*

*Proof.* Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_3, x_5\}$. Then $M^*|(D_1 \cup D_2) \cong U_{3,5}$ as $M$ is 4-connected. Therefore, if $|E(M)| = 5$, then $M$ has no 4-circuits; a contradiction, so $|E(M)| \geq 6$. Furthermore, by orthogonality, any circuit meeting $D_1 \cup D_2$ does so in at least three elements.

By (P2), $M$ has a 4-circuit $C_1$ containing $\{x_1, x_6\}$. Similarly, $M$ has a 4-circuit $C_2$ containing $\{x_i, x_6\}$, where $x_i \in (D_1 \cup D_2) - C_1$. Since $C_1 - x_6 \subseteq D_1 \cup D_2$ and $C_2 - x_6 \subseteq D_1 \cup D_2$, it follows by circuit elimination that $M$ has a circuit $C_3 \subseteq D_1 \cup D_2$. Since $M$ is 4-connected and $|E(M)| \geq 6$, we have $|C_3| \in \{4, 5\}$. Now

$$r(C_3) + r^*(C_3) - |C_3| = 2,$$

so, as $M$ is 4-connected, $|E(M)| \leq 7$. As $M$ satisfies (P2), a routine check shows that $|E(M)| \leq 6$, and so $M \cong U_{3,6}$. $\square$

We next establish an analogue of Lemma 2.2. In particular, Lemma 3.5 states that if $M$ has two disjoint 4-cocircuits, then $M$ restricted to these 4-cocircuits is isomorphic to $M(K_{2,4})$. This lemma requires three preliminary results. In each of these preliminary results as well as Lemma 3.5, we suppose that $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ are disjoint 4-cocircuits of $M$. Observe that orthogonality and the 4-connectedness of $M$ imply that every 4-circuit contained in $X \cup Y$ meets each of $X$ and $Y$ in exactly two elements.

**Lemma 3.2.** *Let $C_1$ and $C_2$ be distinct 4-circuits of $M$ contained in $X \cup Y$ such that $|C_1 \cap C_2 \cap X| \geq 1$. Then $|C_1 \cap C_2 \cap X| = 1$.*

*Proof.* Since each 4-circuit contained in $X \cup Y$ meets each of $X$ and $Y$ in exactly two elements, it suffices to show that $|C_1 \cap C_2 \cap X| \neq 2$. Suppose $|C_1 \cap C_2 \cap X| = 2$. Then $|C_1 \cap C_2| \in \{2, 3\}$. If $|C_1 \cap C_2| = 3$, then we may assume that $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_1, y_3\}$. By circuit elimination, $M$ has a circuit contained in $\{x_1, y_1, y_2, y_3\}$, but such a circuit contradicts either the 4-connectivity of $M$ or orthogonality. If $|C_1 \cap C_2| = 2$, then we may assume that $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_3, y_4\}$. By circuit elimination, $M$ has a circuit contained in $\{x_1, y_1, y_2, y_3, y_4\}$. But again, such a circuit contradicts either the 4-connectivity of $M$ or orthogonality. Thus $|C_1 \cap C_2 \cap X| \neq 2$, thereby completing the proof of the lemma. □

**Lemma 3.3.** *Let $C_1$, $C_2$, and $C_3$ be distinct 4-circuits of $M$ such that $C_1 \cup C_2 \cup C_3 \subseteq X \cup Y$ and $X \subseteq C_1 \cup C_2 \cup C_3$. Then $Y \subseteq C_1 \cup C_2 \cup C_3$.*

*Proof.* Suppose $Y - (C_1 \cup C_2 \cup C_3)$ is non-empty. Then, by Lemma 3.2, we may assume that $C_1 \cap Y = \{y_2, y_3\}$, $C_2 \cap Y = \{y_1, y_3\}$, and $C_3 \cap Y = \{y_1, y_2\}$. Furthermore, by Lemma 3.2, we may also assume that $C_1 \cap X = \{x_1, x_2\}$ and $C_2 \cap X = \{x_1, x_3\}$, in which case, $\{x_1, y_1, y_2, y_3\}$ spans $X$. Now $X$ is independent as $M$ is 4-connected, and so $\{y_1, y_2, y_3\} \subseteq \mathrm{cl}(X)$. But then $M$ has a circuit that contains $y_1$ and is contained in $X \cup y_1$. This contradiction to orthogonality completes the proof of the lemma. □

**Lemma 3.4.** *Let $C_1$ and $C_2$ be distinct 4-circuits of $M$ in $X \cup Y$ such that $|C_1 \cap C_2| \geq 1$. Then $|C_1 \cap C_2| = 2$.*

*Proof.* Assume $|C_1 \cap C_2| \neq 2$. Since

$$|C_1 \cap C_2| = |C_1 \cap C_2 \cap X| + |C_1 \cap C_2 \cap Y|,$$

it follows by Lemma 3.2 and symmetry that we may assume $|C_1 \cap C_2 \cap X| = 1$ and $|C_1 \cap C_2 \cap Y| = 0$. Without loss of generality, let $C_1 = \{x_1, x_2, y_1, y_2\}$ and $C_2 = \{x_1, x_3, y_3, y_4\}$. By Lemma 3.3, any additional 4-circuit of $M$ contained in $X \cup Y$ includes $x_4$. By (P2), $M$ has a 4-circuit $C_3$ containing $\{x_2, y_3\}$. By orthogonality and Lemma 3.2, we may assume $C_3 = \{x_2, x_4, y_1, y_3\}$.

Similarly, $M$ has a 4-circuit $C_4$ containing $\{x_2, y_4\}$. But then $x_4 \in C_4$ and $|C_3 \cap C_4 \cap X| = 2$, contradicting Lemma 3.2. The lemma now follows. $\square$

**Lemma 3.5.** *The restriction $M|(X \cup Y) \cong M(K_{2,4})$.*

*Proof.* By (P2), $M$ has a 4-circuit $C_1$ containing $x_1$ and $y_1$. By orthogonality, we may assume $C_1 = \{x_1, x_2, y_1, y_2\}$. Furthermore, $M$ has a 4-circuit $C_2$ containing $x_1$ and $y_3$. By orthogonality and Lemmas 3.2 and 3.4, we may assume $C_2 = \{x_1, x_3, y_1, y_3\}$. Similarly, $M$ has a 4-circuit $C_3$ containing $x_1$ and $y_4$ and, by Lemmas 3.2 and 3.4, $C_3 = \{x_1, x_4, y_1, y_4\}$.

Continuing this process, $M$ has a 4-circuit $C_4$ containing $x_2$ and $y_3$. Since $x_2 \in C_1 \cap C_4$, we have $x_1 \notin C_4$ by Lemma 3.2. Therefore, as $y_3 \in C_2 \cap C_4$, Lemma 3.4 implies that $x_3 \in C_4$. Since $x_2 \in C_1 \cap C_4$ and $x_3, y_3 \in C_2 \cap C_4$, it follows by Lemma 3.4 that $y_2 \in C_4$. Hence $C_4 = \{x_2, x_3, y_2, y_3\}$. Similarly, $M$ has a unique 4-circuit containing $x_2$ and $y_4$ and it is $C_5 = \{x_2, x_4, y_2, y_4\}$, and $M$ has a unique 4-circuit containing $x_3$ and $y_4$ and it is $C_6 = \{x_3, x_4, y_3, y_4\}$.

We now show that $\mathcal{C}(M|(X \cup Y)) = \{C_1, C_2, \ldots, C_6\}$. First observe that, since every 2-element subset of each of $X$ and $Y$ is in one of $C_1, C_2, \ldots, C_6$, Lemma 3.2 implies that $M|(X \cup Y)$ has no other 4-circuits. Clearly, $r(X \cup Y) = 5$. Suppose there is a circuit $C \in \mathcal{C}(M|(X \cup Y)) - \{C_1, C_2, \ldots, C_6\}$. If $|C| = 6$, then $C$ contains $C_i$ for some $i \in \{1, 2, \ldots, 6\}$; a contradiction. Therefore, $|C| = 5$. To maintain orthogonality, either $|C \cap X| = 2$ or $|C \cap Y| = 2$. Thus to avoid containing one of the six 4-circuits, we may assume that $C = \{x_1, x_2, y_2, y_3, y_4\}$. But then, $\mathrm{cl}(\{x_1, y_2, y_3, y_4\}) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$, so $r(X \cup Y) = 4$; a contradiction. Thus $\mathcal{C}(M|(X \cup Y)) = \{C_1, C_2, \ldots, C_6\}$. It is now easily checked that $M|(X \cup Y) \cong M(K_{2,4})$. $\square$

The next main step in the proof of Theorem 1.4 is to show that if $|E(M)| \geq 11$, then $M$ has two disjoint 4-cocircuits. Stated as Lemma 3.15, its proof is long and consists of a sequence of preliminary lemmas. Except for the first, these preliminary lemmas concern the way 4-cocircuits intersect if $M$ has no two disjoint 4-cocircuits.

**Lemma 3.6.** *Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2 \cap D_3| = 1$ and $|D_i \cap D_j| = 1$ for all distinct $i, j \in \{1, 2, 3\}$. Then $E(M) = D_1 \cup D_2 \cup D_3$, that is, $|E(M)| = 10$.*

*Proof.* Suppose that $E(M) - (D_1 \cup D_2 \cup D_3) \neq \emptyset$. Let $y \in E(M) - (D_1 \cup D_2 \cup D_3)$ and $D_1 \cap D_2 \cap D_3 = \{x\}$. By (P2), $M$ has a 4-circuit $C$ containing $\{x, y\}$ and, by orthogonality, $|C \cap D_i| \geq 2$ for all $i$. But then $|C| \geq 5$; a contradiction. $\square$

The next two lemmas show that if $|E(M)| \geq 11$ and $M$ has no two disjoint 4-cocircuits, then $M$ has two 4-cocircuits meeting in exactly two elements and that every other 4-cocircuit of $M$ meets the union of two such 4-cocircuits in at least two elements. These two lemmas underlie the approach taken to establish Lemma 3.15.

**Lemma 3.7.** *Let $|E(M)| \geq 11$, and suppose that $M$ has no two disjoint 4-cocircuits. Then $M$ has 4-cocircuits $D_1$ and $D_2$ such that $|D_1 \cap D_2| = 2$.*

*Proof.* Suppose the lemma does not hold. By (P2), $M$ has a 4-cocircuit $D_1$ containing $x_1$. Without loss of generality, we may assume $D_1 = \{x_1, x_2, x_3, x_4\}$. Also, $M$ has a 4-cocircuit $D_2$ that contains $x_5$ and, as $M$ has no two disjoint 4-cocircuits, meets $D_1$. By Lemma 3.1, $|D_1 \cap D_2| = 1$, and so we may assume $D_2 = \{x_1, x_5, x_6, x_7\}$. Similarly, $M$ has a 4-cocircuit $D_3$ that contains $x_8$ and $|D_1 \cap D_3| = |D_2 \cap D_3| = 1$. As $|E(M)| \geq 11$, it follows by Lemma 3.6 that $x_1 \notin D_3$. Therefore, without loss of generality, $D_3 = \{x_2, x_5, x_8, x_9\}$. Lastly, $M$ has a 4-cocircuit $D_4$ containing $x_{10}$ and

$$|D_1 \cap D_4| = |D_2 \cap D_4| = |D_3 \cap D_4| = 1.$$

By Lemma 3.6, we may assume $D_4 = \{x_3, x_6, x_8, x_{10}\}$. But then, a similar argument implies that $M$ has the 4-cocircuit $D_5 = \{x_4, x_7, x_9, x_{11}\}$, in which case $D_4$ and $D_5$ are disjoint; a contradiction. □

**Lemma 3.8.** *Let $|E(M)| \geq 10$, and suppose that $M$ has no two disjoint 4-cocircuits. Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. Then $|D_3 \cap (D_1 \cup D_2)| \geq 2$.*

*Proof.* If the lemma does not hold, then $|D_3 \cap (D_1 \cup D_2)| = 1$. More specifically, as $M$ has no two disjoint 4-cocircuits, $|D_1 \cap D_2 \cap D_3| = 1$. Let $\{x\} = D_1 \cap D_2 \cap D_3$. By circuit elimination, $M$ has a cocircuit $D_4 \subseteq (D_1 \cup D_2) - \{x\}$. Since $D_3 \cap D_4 = \emptyset$, it follows that $|D_4| \neq 4$. Therefore, as $M$ is 4-connected, $D_4 = (D_1 \cup D_2) - \{x\}$.

Since $|E(M)| \geq 10$, we have $|E(M) - (D_1 \cup D_2 \cup D_3)| \geq 1$. Let $y \in E(M) - (D_1 \cup D_2 \cup D_3)$, and let $C$ be a 4-circuit containing $\{x, y\}$. To preserve orthogonality, $C$ contains an element in $D_3 - \{x\}$ and the unique element in $(D_1 \cap D_2) - \{x\}$. But then $|C \cap D_4| = 1$. This contradiction to orthogonality proves the lemma. □

**Lemma 3.9.** *Let $|E(M)| \geq 9$, and suppose that $M$ has no two disjoint 4-cocircuits. Let $D_1$, $D_2$, and $D_3$ be distinct 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. Then $D_1 \cap D_2 \not\subseteq D_3$.*

*Proof.* Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$. Suppose that $\{x_1, x_2\} \subseteq D_3$. By Lemma 3.1, we may assume that $D_3 = \{x_1, x_2, x_7, x_8\}$. Using circuit elimination on each pair of cocircuits in $\{D_1, D_2, D_3\}$ and eliminating $x_2$, we find that each of $\{x_1, x_3, x_4, x_5, x_6\}$, $\{x_1, x_3, x_4, x_7, x_8\}$, and $\{x_1, x_5, x_6, x_7, x_8\}$ contains a cocircuit. Noting that $M$ has no cocircuits of size at most three, each such cocircuit must contain $x_1$; otherwise, $M$ has two disjoint 4-cocircuits. Moreover, for each of these 5-element sets, every 4-element subset containing $x_1$ meets $D_1$, $D_2$, or $D_3$ in exactly three elements. Thus, by Lemma 3.1, none of these subsets is a 4-cocircuit. Hence each of these 5-element sets is a cocircuit, which we refer to as $D_5$, $D_6$, and $D_7$, respectively.

By (P2), $M$ has a 4-circuit $C_1$ containing $\{x_1, x_9\}$. By considering the intersection of $C_1$ with each of $D_1$, $D_2$, and $D_3$, we see that $x_2 \in C_1$. But then, regardless of the choice for the remaining element in $C_1$, it follows that $C_1$ meets one of $D_5$, $D_6$, and $D_7$ in exactly one element, contradicting orthogonality. This contradiction proves the lemma. □

**Lemma 3.10.** *Let $|E(M)| \geq 11$, and suppose that $M$ has no two disjoint 4-cocircuits. Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2 \cap D_3| = 1$. Then $|D_i \cap D_j| = 1$ for some distinct elements $i, j \in \{1, 2, 3\}$.*

*Proof.* Suppose the lemma does not hold. Then, by Lemmas 3.1 and 3.9, we may assume, without loss of generality, that $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_1, x_3, x_5, x_7\}$. Let $C_1$ be a 4-circuit of $M$ containing $\{x_8, x_9\}$. If $C_1$ meets $D_1 \cup D_2 \cup D_3$, then, by orthogonality, it does so in at least three elements. Therefore $C_1 \cap (D_1 \cup D_2 \cup D_3) = \emptyset$, so $|E(M)| \geq 11$ and we may assume $C_1 = \{x_8, x_9, x_{10}, x_{11}\}$.

Now let $D_4$ be a 4-cocircuit of $M$ containing $x_8$. By orthogonality, we may assume $x_9 \in D_4$. Since $M$ has no two disjoint 4-cocircuits, $D_4$ meets each of $D_1$, $D_2$, and $D_3$. Furthermore, by Lemma 3.8, $D_4$ contains at least two elements from each of $D_1 \cup D_2$, $D_1 \cup D_3$, and $D_2 \cup D_3$. If $x_1 \in D_4$, then, by Lemma 3.9, none of $x_2$, $x_3$, and $x_5$ are in $D_4$. It follows that $x_1 \notin D_4$. Therefore, without loss of generality, $D_4 = \{x_2, x_3, x_8, x_9\}$.

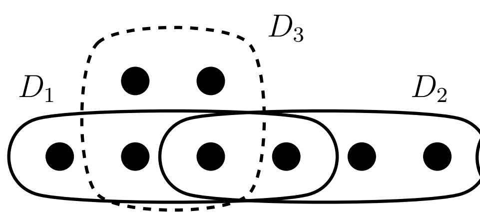
Finally, let $C_2$ be a 4-circuit of $M$ containing $\{x_4, x_{10}\}$. By orthogonality, $|C_2 \cap D_1| \geq 2$. If $x_1 \notin C_2$, then, without loss of generality, we may assume that $x_2 \in C_2$. But then $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_4 \neq \emptyset$, and it follows by orthogonality that $|C_2 \cap D_2| \geq 2$ and $|C_2 \cap D_4| \geq 2$, which is not possible. Thus $x_1 \in C_2$. Therefore $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_3 \neq \emptyset$, and so $C_2 = \{x_1, x_4, x_5, x_{10}\}$. Similarly, $M$ has a unique 4-circuit $C_3$ containing $\{x_4, x_{11}\}$ and it is $C_3 = \{x_1, x_4, x_5, x_{11}\}$. As $M$ is 4-connected, $M|(C_2 \cup C_3)$ is isomorphic to $U_{3,5}$. In turn, this implies that $M$ has a circuit, namely, $\{x_4, x_5, x_{10}, x_{11}\}$ meeting $D_1$ in exactly one element. This contradiction completes the proof of the lemma. □

For the rest of the lemmas leading to the proof that $M$ has two disjoint 4-cocircuits if $|E(M)| \geq 11$, we frequently refer to the way in which a 4-cocircuit intersects two other 4-cocircuits which share two elements. For ease of reading, we introduce the following terminology.

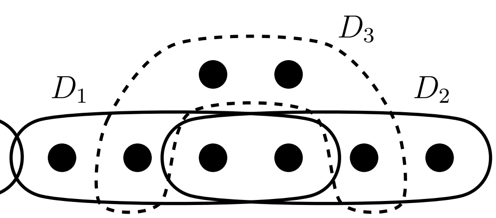
Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. With respect to $(D_1, D_2)$, we say that $D_3$ is

(i) *Type*-1 if $|D_3 \cap (D_1 \cap D_2)| = 1$, and $|D_3 \cap (D_1 - D_2)| = 1$, and $|D_3 \cap (D_2 - D_1)| = 0$,
(ii) *Type*-2 if $|D_3 \cap (D_1 \cap D_2)| = 0$, and $|D_3 \cap D_1| = |D_3 \cap D_2| = 1$, and
(iii) *Type*-3 if $|D_3 \cap (D_1 \cap D_2)| = 0$, and $|D_3 \cap D_1| = 2$, and $|D_3 \cap D_2| = 1$.

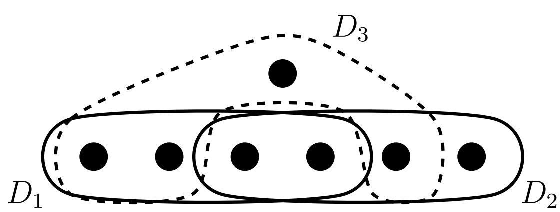
Set diagrams of the three types are shown in Fig. 1.




(A) $(D_1, D_2)$-Type-1




(B) $\{D_1, D_2\}$-Type-2




(C) $(D_1, D_2)$-Type-3

FIGURE 1. Set diagrams of Types-1, -2, and -3 intersections.

Note that Type-2 intersections are symmetric, and therefore we will denote this intersection by $\{D_1, D_2\}$-Type-2. There will be occasions in which it is sufficient to specify that $D_3$ is either $(D_1, D_2)$-Type-$i$ or $(D_2, D_1)$-Type-$i$ for a fixed $i \in \{1, 3\}$. In these instances, we will say that $D_3$ is $\{D_1, D_2\}$-Type-$i$. The previous lemmas ensure that any 4-cocircuit not contained in $D_1 \cup D_2$ intersects $D_1 \cup D_2$ in one of the above types if $M$ has no two disjoint 4-cocircuits and $|E(M)| \geq 11$. We prove this in the following lemma.

**Lemma 3.11.** *Let $|E(M)| \geq 11$, and suppose that $M$ has no two disjoint 4-cocircuits. Let $D_1$ and $D_2$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. If $D_3$ is a 4-cocircuit of $M$ such that $D_3 \not\subseteq D_1 \cup D_2$, then $D_3$ is $\{D_1, D_2\}$-Type-$i$ for some $i \in \{1, 2, 3\}$.*

*Proof.* Let $D_3$ be a 4-cocircuit of $M$ not contained in $D_1 \cup D_2$. By Lemma 3.9, $|D_3 \cap (D_1 \cap D_2)| \in \{0,1\}$. Suppose that $|D_3 \cap (D_1 \cap D_2)| = 1$. By Lemma 3.8, $|D_3 \cap (D_1 \cup D_2)| \geq 2$, so we may assume $|D_3 \cap (D_1 - D_2)| = 1$. Since $|D_1 \cap D_2| = 2$ and $|D_1 \cap D_3| = 2$, it follows by Lemma 3.10 that $|D_2 \cap D_3| = 1$. Therefore $|D_3 \cap (D_2 - D_1)| = 0$, and $D_3$ is $(D_1, D_2)$-Type-1.

Now suppose that $|D_3 \cap (D_1 \cap D_2)| = 0$. As $M$ has no two disjoint 4-cocircuits, we have $D_1 \cap D_3 \neq \emptyset$ and $D_2 \cap D_3 \neq \emptyset$. Therefore, without loss of generality, as $D_3 \not\subseteq D_1 \cup D_2$, either $|D_1 \cap D_3| = |D_2 \cap D_3| = 1$, or $|D_1 \cap D_3| = 2$ and $|D_2 \cap D_3| = 1$. In particular, $D_3$ is $\{D_1, D_2\}$-Type-2 or $(D_1, D_2)$-Type-3, respectively. □

For when $|E(M)| \geq 11$, the next three lemmas show that if $M$ has no two disjoint 4-cocircuits, and $D_1$, $D_2$, and $D_3$ are 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$ and $D_3 \not\subseteq D_1 \cup D_2$, then $D_3$ is neither $\{D_1, D_2\}$-Type-2 nor $\{D_1, D_2\}$-Type-3.

**Lemma 3.12.** *Let $|E(M)| \geq 11$, and suppose that $M$ has no two disjoint 4-cocircuits. Let $D_1$, $D_2$, $D_3$, and $D_4$ be distinct 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$ and $D_3$ is $\{D_1, D_2\}$-Type-2. If $D_4 \not\subseteq D_1 \cup D_2 \cup D_3$, then $D_4$ is $\{D_1, D_2\}$-Type-1.*

*Proof.* Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_3, x_5, x_7, x_8\}$, and suppose $x_9 \in D_4$. If $D_4$ is not $\{D_1, D_2\}$-Type-1, then, by Lemma 3.11, it is either $\{D_1, D_2\}$-Type-2 or $\{D_1, D_2\}$-Type-3. First assume that $D_4$ is $\{D_1, D_2\}$-Type-3. Then, without loss of generality, either $D_4 = \{x_3, x_4, x_5, x_9\}$ or $D_4 = \{x_3, x_4, x_6, x_9\}$. If $D_4 = \{x_3, x_4, x_5, x_9\}$, then $|D_3 \cap D_4| = 2$ and $|D_2 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 3.8. Similarly, if $D_4 = \{x_3, x_4, x_6, x_9\}$, then $|D_1 \cap D_4| = 2$ and $|D_3 \cap (D_1 \cup D_4)| < 2$, again contradicting Lemma 3.8. Thus $D_4$ is not $\{D_1, D_2\}$-Type-3.

Now assume that $D_4$ is $\{D_1, D_2\}$-Type-2. Then $|D_4 \cap \{x_3, x_5\}| \leq 1$; otherwise, $\{x_3, x_5\} \subseteq D_4$ and $|D_1 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 3.8. If $|D_4 \cap \{x_3, x_5\}| = 1$, then, without loss of generality, $x_3 \in D_4$. Since $D_4$ is $\{D_1, D_2\}$-Type-2, we have $x_6 \in D_4$. Furthermore, either $x_7$ or $x_8$ is in $D_4$; otherwise, $|D_1 \cap D_3 \cap D_4| = 1$ and $|D_i \cap D_j| = 1$ for all distinct $i, j \in \{1, 3, 4\}$, and so we contradict Lemma 3.6 as $|E(M)| \geq 11$. Hence, by Lemma 3.1, we may assume $D_4 = \{x_3, x_6, x_7, x_9\}$. But then $|D_3 \cap D_4| = 2$ and $|D_1 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 3.8.

It now follows that $D_4$ avoids $\{x_3, x_5\}$, and so $x_4, x_6 \in D_4$. Furthermore, as $M$ has no disjoint 4-cocircuits, we may assume $x_7 \in D_4$. Thus $D_4 = \{x_4, x_6, x_7, x_9\}$. By (P2), $M$ has a 4-cocircuit $D_5$ containing $x_{10}$.

By Lemma 3.11, $D_5$ is $\{D_1, D_2\}$-Type-$i$ for some $i \in \{1, 2, 3\}$. By applying the argument that showed $D_4$ is not $\{D_1, D_2\}$-Type-3 to $D_5$, we have that $D_5$ is not $\{D_1, D_2\}$-Type-3. If $D_5$ is $\{D_1, D_2\}$-Type-2, then, by the analysis of the previous paragraph, $\{x_4, x_6\} \subseteq D_5$ and $\{x_7, x_8\} \cap D_5 \neq \emptyset$. If $D_5 = \{x_4, x_6, x_7, x_{10}\}$, then $|D_4 \cap D_5| = 3$, and so, by Lemma 3.1, $M$ is isomorphic to $U_{3,6}$; a contradiction. If $D_5 = \{x_4, x_6, x_8, x_{10}\}$, then $|D_4 \cap D_5| = 2$ and $|D_1 \cap (D_4 \cup D_5)| < 2$, contradicting Lemma 3.8. Therefore $D_5$ is $\{D_1, D_2\}$-Type-1. It is easily checked that, by symmetry, we may assume that $D_5$ is $(D_1, D_2)$-Type-1.

By symmetry, we may assume $\{x_1, x_3\} \subseteq D_5$. Furthermore, $D_5$ contains either $x_7$ or $x_9$; otherwise, $D_4 \cap D_5 = \emptyset$. But, if $D_5 = \{x_1, x_3, x_7, x_{10}\}$, then $|D_3 \cap D_5| = 2$ and $|D_4 \cap (D_3 \cup D_5)| < 2$, contradicting Lemma 3.8. Similarly, if $D_5 = \{x_1, x_3, x_9, x_{10}\}$, then $|D_1 \cap D_5| = 2$ and $|D_3 \cap (D_1 \cup D_5)| < 2$, again contradicting Lemma 3.8. This completes the proof of the lemma. □

**Lemma 3.13.** *Let $|E(M)| \geq 11$, and suppose $M$ has no two disjoint 4-cocircuits. Let $D_1$, $D_2$ and $D_3$ be distinct 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$ and $D_3 \not\subseteq D_1 \cup D_2$. Then $D_3$ is not $\{D_1, D_2\}$-Type-2.*

*Proof.* Suppose $D_3$ is $\{D_1, D_2\}$-Type-2. Then, without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_3, x_5, x_7, x_8\}$. By (P2), $M$ has a 4-cocircuit $D_4$ containing $x_9$. By symmetry and Lemma 3.12, we may assume that $D_4$ is $(D_1, D_2)$-Type-1, in which case, $D_4$ meets $\{x_3, x_4\}$ but avoids $\{x_5, x_6\}$. Since $|D_1 \cap D_4| = 2$, it follows by Lemma 3.8 that $|D_3 \cap (D_1 \cup D_4)| \geq 2$, so $D_4 \cap \{x_7, x_8\} \neq \emptyset$. Hence, without loss of generality, either $D_4 = \{x_1, x_3, x_7, x_9\}$ or $D_4 = \{x_1, x_4, x_7, x_9\}$. First assume that $D_4 = \{x_1, x_3, x_7, x_9\}$.

**3.13.1.** *Let $D$ be a 4-cocircuit of $M$ such that $D \not\subseteq D_1 \cup D_2 \cup D_3 \cup D_4$. Then $|\{x_1, x_3\} \cap D| = 1$.*

By Lemma 3.12, $D$ is $\{D_1, D_2\}$-Type-1. Furthermore, as $|D_3 \cap D_4| = 2$ and $D_2$ is $(D_3, D_4)$-Type-2, it follows by Lemma 3.12 that $D$ is $\{D_3, D_4\}$-Type-1. Also, as $D_1 \cap D_4 = \{x_1, x_3\}$, Lemma 3.9 implies that $\{x_1, x_3\} \not\subseteq D$.

If $\{x_1, x_3\} \cap D = \emptyset$, then, as $D$ is $\{D_1, D_2\}$-Type-1 and $\{D_3, D_4\}$-Type-1, we have $\{x_2, x_7\} \subseteq D$ as well as $x_5 \in D$. Hence $D \cap (D_1 \cup D_2 \cup D_3 \cup D_4) = \{x_2, x_5, x_7\}$. Now $|D_2 \cap D| = 2$ and $|D_3 \cap D| = 2$. Furthermore, $D_4$ is $\{D_2, D\}$-Type-2, $D_1$ is $\{D_3, D\}$-Type-2, and $D$ is $\{D_1, D_4\}$-Type-2. Therefore, by Lemma 3.12, if $D'$ is a 4-cocircuit of $M$ such that $D' \not\subseteq (D_1 \cup D_2 \cup D_3 \cup D_4 \cup D)$, then $D'$ is $\{D_2, D\}$-Type-1, $\{D_3, D\}$-Type-1, and $\{D_1, D_4\}$-Type-1. As $|E(M)| \geq 11$, $M$ has such a cocircuit $D'$. By Lemma 3.9, if $x_1 \in D'$, then $x_2 \notin D'$ and $x_3 \notin D'$. Since $D'$ is a $\{D_2, D\}$-Type-1 and $\{D_3, D\}$-Type-1, we have $x_5 \in D'$ and, further, $x_7 \notin D'$. Since $D'$ is $\{D_3, D\}$-Type-1 and $\{D_1, D_4\}$-Type-1, we also have $|D' \cap ((D_3 \cup D) -$

$(D_3 \cap D))| = 1$ and $|D' \cap ((D_1 \cup D_4) - (D_1 \cap D_4))| = 1$. But $((D_3 \cup D) - (D_3 \cap D)) \cap ((D_1 \cup D_4) - (D_1 \cap D_4)) = \{x_2, x_3, x_7\}$, so $D' \subseteq (D_1 \cup D_2 \cup D_3 \cup D_4 \cup D)$; a contradiction. Thus $x_1 \notin D'$. A similar argument shows $x_3 \notin D'$. But then $D'$ is not $\{D_1, D_4\}$-Type-1; a contradiction. Hence 3.13.1 holds.

Let $D_5$ be a 4-cocircuit of $M$ that contains $x_{10}$. Now let $\varphi$ be the permutation of $\{x_1, x_2, \ldots, x_9\}$ defined by

$$(x_1, x_3)\,(x_2, x_7)\,(x_4, x_9)\,(x_5)\,(x_6, x_8).$$

Noting that $\varphi(D_1) = D_4$, $\varphi(D_2) = D_3$, $\varphi(D_3) = D_2$, and $\varphi(D_4) = D_1$, it follows by 3.13.1 that we may assume $x_1 \in D_5$ and $x_3 \notin D_5$. Since $D_2$ is $(D_3, D_4)$-Type-2, it follows by Lemma 3.12 that $D_5$ is $\{D_3, D_4\}$-Type-1, and so $x_7 \in D_5$ but $x_5 \notin D_5$. Therefore, as $D_5$ is $\{D_1, D_2\}$-Type-1, it follows by Lemma 3.8 that $D_5$ contains one of $x_4$ and $x_6$.

If $x_4 \in D_5$, then $|D_1 \cap D_4 \cap D_5| = 1$ and

$$|D_1 \cap D_4| = |D_1 \cap D_5| = |D_4 \cap D_5| = 2,$$

contradicting Lemma 3.10. Thus $x_6 \in D_5$ and so $D_5 = \{x_1, x_6, x_7, x_{10}\}$. By (P2), $M$ has a 4-cocircuit $D_6$ containing $x_{11}$. By 3.13.1, either $x_1 \in D_6$ or $x_3 \in D_6$. If $x_1 \in D_6$, then, by the previous argument concerning $D_5$ and now applied to $D_6$, we get $D_6 = \{x_1, x_6, x_7, x_{11}\}$. But then $|D_5 \cap D_6| = 3$ and so, by Lemma 3.1, $M \cong U_{3,6}$; a contradiction. Therefore $x_1 \notin D_6$ and so $x_3 \in D_6$. Observe that $|D_2 \cap D_5| = 2$ and $D_3$ is $\{D_2, D_5\}$-Type-2 and so, by Lemma 3.12, $D_6$ is $\{D_2, D_5\}$-Type-1. But $D_6$ is also $\{D_1, D_2\}$-Type-1 and $\{D_3, D_4\}$-Type 1 by Lemma 3.12. Therefore $D_6$ contains an element from each of the sets $\{1, 6\}$, $\{1, 2\}$, and $\{1, 5, 8, 9\}$. This is impossible as $D_6$ has exactly four elements and $x_1 \notin D_6$. We conclude that $D_4 \neq \{x_1, x_3, x_7, x_9\}$.

We may now assume that $D_4 = \{x_1, x_4, x_7, x_9\}$. Now $|D_1 \cap D_4| = 2$ and $D_3$ is $\{D_1, D_4\}$-Type-2. Therefore, by Lemma 3.12,

**3.13.2.** *if $D$ is a 4-cocircuit of $M$ such that $D \not\subseteq D_1 \cup D_3 \cup D_4$, then $D$ is $\{D_1, D_4\}$-Type-1.*

We next show that

**3.13.3.** *$M$ has a 4-cocircuit containing $x_1$ and an element not in $\{x_1, x_2, \ldots, x_9\}$.*

Let $D_5$ be a cocircuit containing $x_{10}$. If $x_1 \notin D_5$, then, as $D_5$ is $\{D_1, D_2\}$-Type-1 and, by 3.13.2, $\{D_1, D_4\}$-Type-1, it follows that $\{x_2, x_4\} \subseteq D_5$ and $\{x_3, x_5, x_6, x_7, x_9\} \cap D_5 = \emptyset$. Further, as $M$ has no disjoint 4-cocircuits, $D_5 \cap D_3 \neq \emptyset$. Therefore, $D_5 = \{x_2, x_4, x_8, x_{10}\}$. As $|E(M)| \geq 11$, $M$ has a 4-cocircuit $D_6$ containing $x_{11}$. By the same reasoning, $\{x_2, x_4, x_8\} \subseteq D_6$, so $|D_5 \cap D_6| = 3$; a contradiction. Thus 3.13.3 holds.

By 3.13.3, we may assume that $M$ has a 4-cocircuit $D_5$ containing $x_1$ and $x_{10}$. We show that

**3.13.4.** $x_3 \notin D_5$.

If $x_3 \in D_5$, then, as $D_5$ is $\{D_1, D_2\}$-Type-1 and $\{D_1, D_4\}$-Type-1, we have $\{x_2, x_4, x_5, x_6, x_7, x_9\} \cap D_5 = \emptyset$. Furthermore, $|D_1 \cap D_5| = 2$ and so, by Lemma 3.8, the existence of the cocircuit $D_3$ implies that $x_8 \in D_5$. Therefore $D_5 = \{x_1, x_3, x_8, x_{10}\}$. By (P2), $M$ has a 4-cocircuit $D_6$ containing $x_{11}$. Since $|D_3 \cap D_5| = 2$ and $D_4$ is $\{D_3, D_5\}$-Type-2, it follows by Lemma 3.12 that $D_6$ is $\{D_3, D_5\}$-Type-1. As $D_6$ is also $\{D_1, D_2\}$-Type-1 and, by 3.13.2, $\{D_1, D_4\}$-Type-1, it is easily checked that $x_1 \in D_6$, in which case $\{x_2, x_4, x_5, x_7, x_{10}\} \cap D_6 = \emptyset$. This implies that each of $D_6 \cap \{3, 8\}$, $D_6 \cap \{3, 6\}$, and $D_6 \cap \{3, 9\}$ is non-empty, and so $x_3 \in D_6$. But then $D_1 \cap D_5 \cap D_6 = \{1, 3\}$, contradicting Lemma 3.9. Thus $x_3 \notin D_5$, thereby establishing 3.13.4.

Since $D_5$ is $\{D_1, D_2\}$-Type-1 and $\{D_1, D_4\}$-Type-1, but does not contain $x_3$, it follows that $|\{x_5, x_6\} \cap D_5| = 1$ and $|\{x_7, x_9\} \cap D_5| = 1$. In turn this implies $D_5$ contains either $x_5$ or $x_7$; otherwise, it is disjoint from $D_3$. If $x_6 \in D_5$, then $x_7 \in D_5$, in which case, $|D_4 \cap D_5| = 2$. But then $|D_3 \cap (D_4 \cup D_5)| = 1$, contradicting Lemma 3.8. Therefore $x_6 \notin D_5$, so $x_5 \in D_5$. Then $|D_2 \cap D_5| = 2$, and so, by Lemma 3.8, $|D_3 \cap (D_2 \cup D_5)| \geq 2$, which implies $x_7 \in D_5$. By (P2), $M$ has a 4-cocircuit $D_6$ containing $x_{11}$. If $x_1 \in D_6$, then, by an argument analogous to that which determined $D_5$, we have $D_6 = \{x_1, x_5, x_7, x_{11}\}$ and so $|D_5 \cap D_6| = 3$; a contradiction. Thus $x_1 \notin D_6$. Since $D_6$ is $\{D_1, D_2\}$-Type-1 and $\{D_1, D_4\}$-Type 1, it is easily checked that $D_6 = \{x_2, x_3, x_4, x_{11}\}$. But then $|D_1 \cap D_6| = 3$; a contradiction to Lemma 3.1. This completes the proof of Lemma 3.13. □

**Lemma 3.14.** *Let $|E(M)| \geq 11$, and suppose $M$ has no two disjoint 4-cocircuits. Let $D_1$, $D_2$, and $D_3$ be distinct 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$ and $D_3 \not\subseteq D_1 \cup D_2$. Then $D_3$ is not $\{D_1, D_2\}$-Type-3.*

*Proof.* Suppose that $D_3$ is $\{D_1, D_2\}$-Type-3. Then, without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_3, x_4, x_5, x_7\}$. Note that $D_2$ is $(D_1, D_3)$-Type-3.

**3.14.1.** *Let $D$ be a 4-cocircuit of $M$ such that $D \not\subseteq D_1 \cup D_2 \cup D_3$. Then $D$ is neither $\{D_1, D_2\}$-Type-3 nor $\{D_1, D_3\}$-Type-3.*

Without loss of generality, we may assume $x_8 \in D$. First suppose that $D$ is $\{D_1, D_2\}$-Type-3. If $D$ is $(D_1, D_2)$-Type-3, then $D_1 \cap D_3 \cap D = \{x_3, x_4\}$, contradicting Lemma 3.9. Therefore assume that $D$ is $(D_2, D_1)$-Type-3. By symmetry, we may assume $D = \{x_3, x_5, x_6, x_8\}$.

By (P2), $M$ has a 4-cocircuit $D'$ containing $x_9$. Since $D'$ is not $\{D_1, D_2\}$-Type-2 by Lemma 3.13, it follows by Lemmas 3.9 and 3.13 that $D'$ is neither $\{D_1, D_2\}$-Type-3 nor $\{D_1, D_2\}$-Type-2. Therefore, by Lemma 3.11, $D'$ is $\{D_1, D_2\}$-Type-1. By considering the way in which $D_1$, $D_2$, $D_3$, and $D$ relate to each other, we may assume, by symmetry, that $x_1 \in D'$ and that $|D' \cap \{x_3, x_4\}| = 1$ and $|D' \cap \{x_5, x_6\}| = 0$. If $x_4 \in D'$, then $x_8 \in D'$; otherwise, $D \cap D' = \emptyset$. But then, $D'$ is $\{D_3, D\}$-Type-2, contradicting Lemma 3.13 as $|D_3 \cap D| = 2$.

Therefore $x_3 \in D'$. Now $x_8 \in D'$; otherwise $D'$ is $\{D_2, D\}$-Type-2, contradicting Lemma 3.13 as $|D_2 \cap D| = 2$. Hence $D' = \{x_1, x_3, x_8, x_9\}$. By (P2), $M$ has a 4-cocircuit $D''$ containing $x_{10}$. As above, $D''$ is $\{D_1, D_2\}$-Type-1, and $|\{x_1, x_2\} \cap D''| = 1$ by Lemma 3.9. By Lemma 3.13, $D''$ is not $\{D_3, D\}$-Type-2. Furthermore, as either $\{x_1, x_{10}\} \subseteq D''$ or $\{x_2, x_{10}\} \subseteq D''$, it follows that $D''$ is not $\{D_3, D\}$-Type-3. Thus, by Lemma 3.11, $D''$ is $\{D_3, D\}$-Type-1, and so $|\{x_3, x_5\} \cap D''| = 1$. Say $x_1 \in D''$. Then, by Lemma 3.9, $x_3 \notin D''$, so $x_5 \in D''$. But then $D''$ is $\{D_1, D_3\}$-Type-3 and $\{D, D'\}$-Type-3, by Lemma 3.11 and Lemma 3.13, and so $|\{x_2, x_7\} \cap D''| = 1$ and $|\{x_6, x_9\} \cap D''| = 1$; a contradiction. Thus $x_1 \notin D''$ and so $x_2 \in D''$. If $x_3 \in D''$, then $D''$ is $\{D_2, D\}$-Type-3, and so $D'' = \{x_2, x_3, x_8, x_{10}\}$. But then $D \cap D' \cap D'' = \{x_3, x_8\}$, contradicting Lemma 3.9. Therefore $x_3 \notin D''$ and $x_5 \in D''$. So $D''$ is $\{D_1, D'\}$-Type-3, in which case $|\{x_4, x_8, x_9\} \cap D''| = 2$; a contradiction. Hence $D$ is not $\{D_1, D_2\}$-Type-3. Since $D_2$ is $(D_1, D_3)$-Type-3, it follows by symmetry that $D$ is not $\{D_1, D_3\}$-Type-3. Thus 3.14.1 holds.

By Lemma 3.13 and 3.14.1, every 4-cocircuit $D$ of $M$ such that $D \not\subseteq D_1 \cup D_2 \cup D_3$ is both $\{D_1, D_2\}$-Type-1 and $\{D_1, D_3\}$-Type-1. In fact, we show that

**3.14.2.** *$D$ is both $(D_1, D_2)$-Type-1 and $(D_1, D_3)$-Type-1.*

Without loss of generality, we may assume that $x_8 \in D$. Note that $D$ is $(D_1, D_2)$-Type-1 if and only if it is $(D_1, D_3)$-Type-1. Suppose $D$ is neither $(D_1, D_2)$-Type-1 nor $(D_1, D_3)$-Type-1. Then $D$ is $(D_2, D_1)$-Type-1 and $(D_3, D_1)$-Type-1. But the former implies that $D \cap \{x_3, x_4\} = \emptyset$, while the latter implies $D \cap \{x_3, x_4\} \neq \emptyset$; a contradiction. Thus 3.14.2 holds.

By (P2), $M$ has a 4-cocircuit $D_4$ that contains $x_8$. By 3.14.2, we may assume $D_4 = \{x_1, x_3, x_8, x_9\}$. Furthermore, $M$ has a 4-cocircuit $D_5$ containing $x_{10}$. By 3.14.2, $D_5$ is $(D_1, D_2)$-Type-1 and $(D_1, D_3)$-Type-1. This implies $|\{x_1, x_2\} \cap D_5| = 1$ and $|\{x_3, x_4\} \cap D_5| = 1$. By Lemma 3.9, $\{x_1, x_3\} \not\subseteq D_5$ and so $D_1 \cap D_5$ is one of $\{x_1, x_4\}$, $\{x_2, x_3\}$, and $\{x_2, x_4\}$.

Say $\{x_1, x_4\} \subseteq D_5$. Then $\{x_8, x_9\} \cap D_5 \neq \emptyset$; otherwise, $|D_2 \cap D_4 \cap D_5| = 1$ and $|D_2 \cap D_4| = |D_2 \cap D_5| = |D_4 \cap D_5| = 1$, and so, by Lemma 3.6,

$|E(M)| = 10$. Therefore, we may assume $D_5 = \{x_1, x_4, x_8, x_{10}\}$. But then $|D_1 \cap D_4 \cap D_5| = 1$ and $|D_1 \cap D_4| = |D_1 \cap D_5| = |D_4 \cap D_5| = 2$, contradicting Lemma 3.10. Similarly $\{x_2, x_3\} \not\subseteq D_5$, and therefore $\{x_2, x_4\} \subseteq D_5$. Now, $D_5 \cap \{x_8, x_9\} \neq \emptyset$; otherwise, $D_4$ and $D_5$ are disjoint. Hence, without loss of generality, $D_5 = \{x_2, x_4, x_8, x_{10}\}$. By (P2), $M$ has a 4-cocircuit $D_6$ containing $x_{11}$. As the restrictions on $D_5$ also apply to $D_6$, we have $\{x_2, x_4\} \subseteq D_6$, which contradicts Lemma 3.9 as $D_1 \cap D_5 = \{x_2, x_4\}$. This completes the proof of Lemma 3.14. □

At last we show that $M$ has two disjoint 4-cocircuits if $|E(M)| \geq 11$.

**Lemma 3.15.** *Let $|E(M)| \geq 11$. Then $M$ has two disjoint 4-cocircuits.*

*Proof.* Suppose that $M$ has no two disjoint 4-cocircuits. By Lemma 3.7, $M$ has 4-cocircuits $D_1$ and $D_2$ with $|D_1 \cap D_2| = 2$. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$. By (P2), $M$ has a 4-cocircuit $D_3$ containing $x_7$. Lemmas 3.13 and 3.14 together with Lemma 3.11 imply that $D_3$ is $\{D_1, D_2\}$-Type-1. Therefore, without loss of generality, $D_3 = \{x_1, x_3, x_7, x_8\}$. Let $D$ be a 4-cocircuit of $M$ such that $D \not\subseteq D_1 \cup D_2 \cup D_3$. Since $|D_1 \cap D_3| = 2$, it again follows by Lemmas 3.11, 3.13, and 3.14 that $D$ is $\{D_1, D_2\}$-Type-1 as well as $\{D_1, D_3\}$-Type-1. We next show

**3.15.1.** *$D$ is not both $(D_2, D_1)$-Type-1 and $(D_3, D_1)$-Type-1.*

If $D$ is both $(D_2, D_1)$-Type-1 and $(D_3, D_1)$-Type-1, then, without loss of generality, $\{x_5, x_7\} \subseteq D$. In turn, this implies $x_1 \in D$, so we may assume $D = \{x_1, x_5, x_7, x_9\}$. By (P2), $M$ has a 4-cocircuit $D'$ containing $x_{10}$. As $|D_2 \cap D| = 2$ and $|D_3 \cap D| = 2$, it follows by Lemmas 3.11, 3.13, and 3.14 that $D'$ is $\{D_1, D_2\}$-Type-1, $\{D_1, D_3\}$-Type-1, $\{D_2, D\}$-Type-1, and $\{D_3, D\}$-Type-1. This implies that $x_1 \in D'$, and it is easily checked that either $\{x_4, x_9\} \subseteq D'$ or $\{x_6, x_8\} \subseteq D'$. If $\{x_4, x_9\} \subseteq D'$, then $D' = \{x_1, x_4, x_9, x_{10}\}$. But then $|D_2 \cap D_3 \cap D'| = 1$ and $|D_2 \cap D_3| = |D_2 \cap D'| = |D_3 \cap D'| = 1$, and so, by Lemma 3.6, $|E(M)| = 10$; a contradiction. Similarly, if $\{x_6, x_8\} \subseteq D'$, then $|D_1 \cap D \cap D'| = 1$ and $|D_1 \cap D| = |D_1 \cap D'| = |D \cap D'| = 1$ and we contradict Lemma 3.6. This proves 3.15.1.

In addition to 3.15.1, we also have

**3.15.2.** *$\{x_2, x_3\} \subseteq D$.*

By 3.15.1, $D$ is at least one of $(D_1, D_2)$-Type-1 and $(D_1, D_3)$-Type-1. If $D$ is $(D_1, D_2)$-Type-1, then, since $D$ is $\{D_1, D_3\}$-Type-1, we have $|\{x_1, x_3\} \cap D| = 1$. If $x_1 \in D$, then $x_4 \in D$ and $D \cap (D_1 \cup D_2 \cup D_3) = \{x_1, x_4\}$, and so $|D_2 \cap D_3 \cap D| = 1$ and $|D_2 \cap D_3| = |D_2 \cap D| = |D_3 \cap D| = 1$. But then, by Lemma 3.6, $|E(M)| = 10$; a contradiction. Therefore $x_1 \notin D$, and so

$\{x_2, x_3\} \subseteq D$. Similarly, if $D$ is $(D_1, D_3)$-Type-1, we have $\{x_2, x_3\} \subseteq D$. Thus 3.15.2 holds.

By (P2), $M$ has a 4-cocircuit $D_4$ containing $x_9$. By 3.15.2, $\{x_2, x_3\} \subseteq D_4$. Therefore, as $|E(M)| \geq 11$, we deduce that $M$ has a 4-cocircuit $D_5$ such that $D_5 \not\subseteq D_1 \cup D_2 \cup D_3 \cup D_4$. But then, by 3.15.2, we have $\{x_2, x_3\} \subseteq D_5$. Therefore $D_1 \cap D_4 \cap D_5 = \{x_2, x_3\}$, contradicting Lemma 3.9. This last contradiction completes the proof of Lemma 3.15. □

Having established that $M$ has two disjoint 4-cocircuits if $|E(M)| \geq 11$, the last step before proving the necessary direction of Theorem 1.4 for $|E(M)| \geq 16$ is to show that $E(M)$ can be partitioned into 4-cocircuits if $|E(M)| \geq 16$. Before showing this, we prove two preliminary results.

**Lemma 3.16.** *Let $X \subseteq E(M)$ such that $M|X \cong M(K_{2,4})$, and let $D$ be a 4-cocircuit of $M$ meeting $X$. Then either $D$ contains exactly one element from each of the four series pairs of $M|X$, or $D \cap X$ is a series pair of $M|X$.*

*Proof.* Suppose the lemma does not hold. For all $i \in \{1, 2, 3, 4\}$, let $\{x_i, y_i\}$ denote the series pairs of $M|X$. Since $M$ is 4-connected, $D \cap X \neq \{x_i, y_i, x_j, y_j\}$ for distinct $i, j \in \{1, 2, 3, 4\}$. Therefore, for some $i$ and $j$, we have $|D \cap \{x_i, y_i\}| = 1$ and $|D \cap \{x_j, y_j\}| = 0$. But $\{x_i, x_j, y_i, y_j\}$ is a circuit; a contradiction. Thus the lemma holds. □

**Lemma 3.17.** *If $|E(M)| \geq 13$, then $M$ has three pairwise-disjoint 4-cocircuits.*

*Proof.* Suppose that $|E(M)| \geq 13$ and $M$ has no three pairwise-disjoint 4-cocircuits. By Lemma 3.15, $M$ has two disjoint 4-cocircuits, $D_1$ and $D_2$ say. Moreover, by Lemma 3.5, $M|(D_1 \cup D_2) \cong M(K_{2,4})$. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_5, x_6, x_7, x_8\}$, and let $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the series pairs in $M|(D_1 \cup D_2)$. By (P2), $M$ has a 4-cocircuit $D_3$ containing $x_9$. Since $D_3 \cap (D_1 \cup D_2)$ is nonempty, it follows by Lemma 3.16 that $D_3 \cap (D_1 \cup D_2)$ is a series pair of $M|(D_1 \cup D_2)$. Thus, without loss of generality, $D_3 = \{x_1, x_5, x_9, x_{10}\}$. Let $D$ be a 4-cocircuit of $M$ such that $D \not\subseteq D_1 \cup D_2 \cup D_3$. We next show that

**3.17.1.** *$D_3 \cap D \neq \emptyset$.*

If $D_3 \cap D = \emptyset$, then, as $D \cap (D_1 \cup D_2)$ is nonempty, we may assume by Lemma 3.16 that $D = \{x_2, x_6, x_{11}, x_{12}\}$. By Lemma 3.5, $M|(D_3 \cup D) \cong M(K_{2,4})$ and so, by orthogonality, $\{x_1, x_2\}$ and $\{x_5, x_6\}$ are series pairs in $M|(D_3 \cup D)$. Thus, without loss of generality, we may assume $\{x_9, x_{11}\}$ and $\{x_{10}, x_{12}\}$ are also series pairs in $M|(D_3 \cup D)$.

Now $M$ has a 4-cocircuit $D'$ containing $x_{13}$. Furthermore, $D' \cap (D_1 \cup D_2)$ and $D' \cap (D_3 \cup D)$ are both nonempty. By Lemma 3.16,

$$D' \cap (D_1 \cup D_2) \in \{\{x_1, x_5\}, \{x_2, x_6\}, \{x_3, x_7\}, \{x_4, x_8\}\}$$

and

$$D' \cap (D_3 \cup D) \in \{\{x_1, x_2\}, \{x_5, x_6\}, \{x_9, x_{11}\}, \{x_{10}, x_{12}\}\}.$$

As $|D'| = 4$, the intersections $D' \cap (D_1 \cup D_2)$ and $D' \cap (D_3 \cup D)$ are not disjoint. But then $D'$ meets a circuit of $M|(D_1 \cup D_2)$ in exactly one element; a contradiction. Thus 3.17.1 holds.

We also have

**3.17.2.** $\{x_1, x_5\} \cap D = \emptyset$.

If $x_1 \in D$, then either $x_5 \in D$, or $D$ meets each of $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$. In the latter case, $D \subseteq D_1 \cup D_2 \cup D_3$; a contradiction. Then $x_9, x_{10} \notin D$ by Lemma 3.1, so we may assume that $D = \{x_1, x_5, x_{11}, x_{12}\}$. Now $(D_3 \cup D) - x_1$ contains a cocircuit and, by orthogonality, this cocircuit avoids $x_5$. Hence, as $M$ is 4-connected, $\{x_9, x_{10}, x_{11}, x_{12}\}$ is a 4-cocircuit of $M$ disjoint from $D_1$ and $D_2$; a contradiction. Thus $x_1 \notin D$ and, similarly, $x_5 \notin D$, and 3.17.2 holds.

By (P2), $M$ has a 4-cocircuit $D_4$ containing $x_{11}$. By 3.17.1 and 3.17.2, we may assume $x_9 \in D_4$. Furthermore, as $D_4$ meets $D_1 \cup D_2$, we may assume that by Lemma 3.16 that $D_4 = \{x_2, x_6, x_9, x_{11}\}$. Now $M$ has a 4-cocircuit $D_5$ containing $x_{12}$. By 3.17.1 and 3.17.2, $D_3 \cap D_5 \neq \emptyset$ and $\{x_1, x_5\} \cap D_5 = \emptyset$. Moreover, replacing $D_3$ with $D_4$ in the above argument shows that $D_4 \cap D_5 \neq \emptyset$ and $\{x_2, x_6\} \cap D_5 = \emptyset$. Therefore we may assume that $D_5 = \{x_3, x_7, x_9, x_{12}\}$. Now $M$ has a 4-circuit $C$ containing $\{x_4, x_9\}$. By orthogonality, $C$ meets each of $\{x_1, x_5, x_{10}\}$, $\{x_2, x_6, x_{11}\}$, and $\{x_3, x_7, x_{12}\}$. But then $|C| \geq 5$; a contradiction. This completes the proof of Lemma 3.17. □

The next lemma extends Lemmas 3.15 and 3.17.

**Lemma 3.18.** *If $|E(M)| \geq 16$, then $E(M)$ can be partitioned into 4-element blocks, where each block is a 4-cocircuit.*

*Proof.* Suppose that $|E(M)| \geq 16$. We first show that $M$ has four pairwise-disjoint 4-cocircuits. By Lemma 3.17, $M$ has three pairwise-disjoint 4-cocircuits, $D_1$, $D_2$, and $D_3$ say. Moreover, by Lemma 3.5, we have $M|(D_i \cup D_j) \cong M(K_{2,4})$ for all distinct $i, j \in \{1, 2, 3\}$. Let $z_1$, $z_2$, $z_3$, and $z_4$ be distinct elements of $E(M) - (D_1 \cup D_2 \cup D_3)$. By (P2), each of these elements is in a 4-cocircuit, $Z_1$, $Z_2$, $Z_3$, and $Z_4$ say, of $M$. If $Z_i \cap (D_1 \cup D_2 \cup D_3) = \emptyset$ for some $i$, then $M$ has four pairwise-disjoint

4-cocircuits. Therefore assume $Z_i \cap (D_1 \cup D_2 \cup D_3)$ is nonempty for all $i$. Then, by Lemma 3.16, we have $|Z_i \cap (D_1 \cup D_2 \cup D_3)| = 3$ and $|Z_i \cap D_1| = |Z_i \cap D_2| = |Z_i \cap D_3| = 1$ for all $i$. If, for distinct $i$ and $j$, we have $Z_i \cap Z_j \neq \emptyset$, then it is easily checked that $|Z_i \cap Z_j| = 3$, contradicting Lemma 3.1. It now follows that $Z_1$, $Z_2$, $Z_3$, and $Z_4$ are four pairwise-disjoint 4-cocircuits of $M$.

Now suppose that $E(M)$ cannot be partitioned into 4-cocircuits. Let $\{D_1, D_2, \dots, D_n\}$ be a maximum-sized set of pairwise-disjoint 4-cocircuits of $M$. Then, by above, $n \geq 4$. Let $x$ be an element of $E(M) - (D_1 \cup D_2 \cup \cdots \cup D_n)$. By (P2), $M$ has a 4-cocircuit $D$ containing $x$. Furthermore, $D \cap (D_1 \cup D_2 \cup \cdots \cup D_n) \neq \emptyset$. Without loss of generality, we may assume that $D \cap D_1 \neq \emptyset$, and so $D \cap (D_1 \cup D_2 \cup D_3 \cup D_4) \neq \emptyset$. But, for all distinct $i, j \in \{1, 2, 3, 4\}$, we have $M|(D_i \cup D_j) \cong M(K_{2,4})$ and so, by Lemma 3.16, $|D \cap (D_1 \cup D_2 \cup D_3 \cup D_4| \geq 4$; a contradiction. The lemma now follows. □

We are now ready to prove the necessary direction of Theorem 1.4 when $|E(M)| \geq 16$.

*Proof of Theorem 1.4 for $|E(M)| \geq 16$.* Suppose $|E(M)| \geq 16$. Then, by Lemma 3.18, there is a partition of $E(M)$ into 4-cocircuits $D_1, D_2, \dots, D_n$, where $D_i = \{w_i, x_i, y_i, z_i\}$ for all $i$. By Lemma 3.5, $M|(D_1 \cup D_i) \cong M(K_{2,4})$ for all $i \in \{2, 3, \dots, n\}$, so we may assume $M$ has 4-circuits $\{w_1, x_1, w_i, x_i\}$, $\{w_1, y_1, w_i, y_i\}$, $\{w_1, z_1, w_i, z_i\}$, $\{x_1, y_1, x_i, y_i\}$, $\{x_1, z_1, x_i, z_i\}$, and $\{y_1, z_1, y_i, z_i\}$ for all such $i$. Consider the 4-circuits $\{w_1, x_1, w_i, x_i\}$ and $\{w_1, x_1, w_j, x_j\}$ for some distinct $i, j \in \{2, 3, \dots, n\}$. By circuit elimination and orthogonality, $\{w_i, x_i, w_j, x_j\}$ is a 4-circuit of $M$. Similarly, for all distinct $i, j \in \{2, 3, \dots, n\}$, we have $\{w_i, y_i, w_j, y_j\}$, $\{w_i, z_i, w_j, z_j\}$, $\{x_i, y_i, x_j, y_j\}$, $\{x_i, z_i, x_j, y_j\}$, and $\{y_i, z_i, y_j, z_j\}$ are 4-circuits of $M$. In turn, as $M|(D_i \cup D_j) \cong M(K_{2,4})$ for all distinct $i$ and $j$, we have

**3.18.1.** *$\{w_i, w_j\}$, $\{x_i, x_j\}$, $\{y_i, y_j\}$, and $\{z_i, z_j\}$ are the series pairs in $M|(D_i \cup D_j)$ for all distinct $i$ and $j$.*

Now consider $K_{4,n}$, where $n \geq 4$. Label the edge set of $K_{4,n}$ so that

$$\big\{\{w_1, x_1, y_1, z_1\}, \{w_2, x_2, y_2, z_2\}, \dots, \{w_n, x_n, y_n, z_n\}\big\}$$

is a partition of $E(K_{4,n})$, where each block is a bond of $K_{4,n}$, and $\{w_i, x_i, w_j, x_j\}$, $\{w_i, y_i, w_j, y_j\}$, $\{w_i, z_i, w_j, z_j\}$, $\{x_i, y_i, x_j, y_j\}$, $\{x_i, z_i, x_j, z_j\}$, and $\{y_i, z_i, y_j, z_j\}$ are 4-cycles of $K_{4,n}$ for distinct $i, j \in \{1, 2, \dots, n\}$. We next show that the identity map $\varphi$ from $E(M(K_{4,n}))$ to $E(M)$ is a weak map from $M(K_{4,n})$ to $M$. Let $C$ be a circuit of $M(K_{4,n})$. Then $|C| \in \{4, 6, 8\}$. If $C$ is a 4-circuit, then, by above, $\varphi(C)$ is a 4-circuit of $M$. Now assume that $|C| = 6$. Then, without loss of generality, we may assume

$$C = \{w_i, x_i, x_j, y_j, y_k, w_k\},$$

where $i$, $j$, and $k$ are distinct elements in $\{1, 2, \dots, n\}$. Using circuit elimination on the 4-circuits $\{w_i, x_i, w_j, x_j\}$ and $\{w_j, y_j, w_k, y_k\}$ of $M$, it follows that $\{w_i, x_i, x_j, y_j, y_k, w_k\}$ contains a circuit of $M$. By orthogonality and 3.18.1, it is easily checked that

$$\{w_i, x_i, x_j, y_j, y_k, w_k\}$$

is a 6-circuit of $M$. Thus if $C$ is a 6-circuit of $M(K_{4,n})$, then $\varphi(C)$ is a 6-circuit of $M$. Lastly, assume that $|C| = 8$. Then, without loss of generality, we may assume

$$C = \{w_i, w_j, x_j, x_k, y_k, y_l, z_l, z_i\},$$

where $i$, $j$, $k$, and $l$ are distinct elements in $\{1, 2, \dots, n\}$. Now $\{w_i, w_j, x_j, x_k, y_k, y_i\}$ and $\{y_i, y_l, z_l, z_i\}$ are circuits of $M$. By circuit elimination, $\{w_i, w_j, x_j, x_k, y_k, y_l, z_l, z_i\}$ contains a circuit of $M$. If this last set is not a circuit, then, by orthogonality and 3.18.1, it contains a 6-circuit of $M$. Without loss of generality, we may assume that this 6-circuit is $\{w_i, w_j, x_j, x_k, y_k, z_i\}$. But then, as $\{x_j, y_j, x_k, y_k\}$ is a 4-circuit of $M$, it follows by circuit elimination that

$$X = \{w_i, w_j, x_j, x_k, z_i, y_j\}$$

contains a circuit of $M$. By orthogonality and 3.18.1, $X$ contains no circuit of $M$. Thus $C$ is an 8-circuit of $M$. It now follows that if $C$ is a circuit of $M(K_{4,n})$, then $\varphi(C)$ is a circuit of $M$. Hence $M$ is a weak-map image of $M(K_{4,n})$ under $\varphi$.

We next show that

**3.18.2.** *$M|(D_i \cup D_j \cup D_k \cup D_l) \cong M(K_{4,4})$ for all distinct $i, j, k, l \in \{1, 2, \dots, n\}$.*

By above, $M|(D_i \cup D_j \cup D_k \cup D_l)$ is a weak-map image of $M(K_{4,4})$. Furthermore, as $M|(D_i \cup D_j \cup D_k \cup D_l)$ has an 8-circuit, it follows that $r(M|(D_i \cup D_j \cup D_k \cup D_l)) \geq 7$. Since $M|(D_i \cup D_j) \cong M(K_{2,4})$, we have $r(M|(D_i \cup D_j)) = 5$ and so, by the 4-circuits of $M$ established above,

$$r(M|(D_i \cup D_j \cup D_k)) \leq 6.$$

In turn, as $D_l$ is a cocircuit of $M|(D_i \cup D_j \cup D_k \cup D_l)$, we deduce that $r(M|(D_i \cup D_j \cup D_k \cup D_l)) \leq 7$. Thus $r(M|(D_i \cup D_j \cup D_k \cup D_l)) = 7$, that is, $r(M|(D_i \cup D_j \cup D_k \cup D_l)) = r(M(K_{4,4}))$. Since $M|(D_i \cup D_j \cup D_k \cup D_l)$ is connected, it follows by Theorem 1.5 that $M|(D_i \cup D_j \cup D_k \cup D_l) \cong M(K_{4,4})$. Thus 3.18.2 holds.

We next prove that $r(M) = r(M(K_{4,n}))$. To do this, we show, by induction, that for all 4-connected matroids $M'$ satisfying (P2) and whose ground set can be partitioned into $m$ 4-cocircuits, where $m \geq 4$, we have $r(M') = r(M(K_{4,m}))$. If $n = 4$, then, by 3.18.2, $r(M) = r(M(K_{4,4}))$. Therefore suppose that $n \geq 5$ and that, for all matroids $M'$ as described above, with $4 \leq m \leq n-1$, we have $r(M') = r(M(K_{4,m}))$. Let $M^-$ denote the matroid $M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1})$. We first show that $M^-$ satisfies (P2). Evidently, every element of $M^-$ is in a 4-cocircuit. Let $x$ and $y$ be distinct elements of $M^-$. If $x$ and $y$ are in distinct 4-cocircuits $D_i$ and $D_j$ of $M^-$, then, by orthogonality and $M$ satisfying (P2), $M^-$ has a 4-circuit containing $\{x, y\}$. Thus assume $x$ and $y$ are in the same 4-cocircuit $D_i$ of $M^-$. By considering $D_i$ with $D_1$ if $i \neq 1$ or $D_2$ if $i = 1$, it follows that $M^-$ has a 4-circuit containing $\{x, y\}$. Lastly, if $M^-$ is not 4-connected, then it has a 2- or 3-separation $(A, B)$. Since $n \geq 5$, it is easily checked that, for distinct $i$, $j$, $k$, and $l$, there are four 4-cocircuits $D_i$, $D_j$, $D_k$, and $D_l$ of $M^-$ such that $|A \cap (D_i \cup D_j \cup D_k \cup D_l)| \geq 3$ and $|B \cap (D_i \cup D_j \cup D_k \cup D_l) \geq 3$. Now, by [3, Lemma 8.2.3],

$$\begin{aligned} 2 &\geq r(A) + r(B) - r(M^-) \\ &\geq r(A \cap (D_i \cup D_j \cup D_k \cup D_l)) + r(B \cap (D_i \cup D_j \cup D_k \cup D_l)) \\ &\qquad - r(D_i \cup D_j \cup D_k \cup D_l). \end{aligned}$$

But then $M|(D_i \cup D_j \cup D_k \cup D_l)$ is not 4-connected, contradicting 3.18.2. It follows that $M^-$ is 4-connected, and so $M^-$ satisfies (P2). By induction,

$$r(M^-) = r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1})) = r(M(K_{4,n-1}))$$

and so, as $D_n$ is a cocircuit of $M$,

$$r(M) = r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1})) + 1 = r(M(K_{4,n})).$$

Finally, as $M$ is connected, it now follows by Theorem 1.5 that $M \cong M(K_{4,n})$, thereby completing the proof of Theorem 1.4. □

## Acknowledgements

The authors would like to thank the reviewer for their careful reading of this article and thoughtful suggestions regarding its improvement.

## References

[1] D. Lucas, Weak maps of combinatorial geometries, Trans. Amer. Math. Soc. 206 (1975) 247–279.
[2] J. Miller, Matroids in which Every Pair of Elements Belongs to Both a 4-Circuit and a 4-Cocircuit, MSc thesis, Victoria University of Wellington, 2013.
[3] J. Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.
[4] S. Pfeil, On Properties of Matroid Connectivity, PhD thesis, Louisiana State University, 2016.
[5] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory Ser. B 28 (1980) 305–359.
[6] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.

## Appendix

Let $M$ be a 4-connected matroid satisfying (P2) with $|E(M)| \leq 15$. Then $M$ is one of thirty-five matroids. These thirty-five matroids comprise of $U_{3,6}$, twenty-one 8-element paving matroids, ten 9-element paving matroids, $R_{10}$, a 12-element matroid, and a 14-element matroid. The matroid $R_{10}$ is the unique splitter for the class of regular matroids and for which

$$\left[\begin{array}{c|ccccc} & -1 & 1 & 0 & 0 & 1 \\ & 1 & -1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & -1 & 1 & 0 \\ & 0 & 0 & 1 & -1 & 1 \\ & 1 & 0 & 0 & 1 & -1 \end{array}\right]$$

is a representation of it over all fields. Precise descriptions of the 8-, 9-, 12-, and 14-element matroids are given below. For ease of reference, the notation is in keeping with the notation in [4],

**8-Element Matroids.** If $|E(M)| = 8$, then $M$ is one of twenty-one rank-4 paving matroids. Let $E(M) = \{1, 2, \ldots, 8\}$. Up to isomorphism, to describe $M$, it is sufficient, to list the 4-circuits of $M$. The first table consists of those matroids $M$ having the property that, for every 4-circuit $C$, there is another 4-circuit of $M$ meeting $C$ in exactly one element.

| $M$ | 4-Circuits of $M$ | | | | |
|---|---|---|---|---|---|
| $M_{8,1}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{3,4,5,8\}$, | $\{2,3,6,7\}$, |
| | $\{2,4,5,6\}$, | $\{1,4,7,8\}$, | $\{1,3,6,8\}$ | | |
| $M_{8,2}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{3,4,5,8\}$, | $\{2,3,6,7\}$, |
| | $\{2,4,5,6\}$, | $\{1,3,6,8\}$, | $\{4,6,7,8\}$ | | |
| $M_{8,3}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{3,4,5,8\}$, | $\{2,3,6,7\}$, |
| | $\{1,4,6,8\}$, | $\{2,4,7,8\}$ | | | |
| $M_{8,3^+}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{3,4,5,8\}$, | $\{2,3,6,7\}$, |
| | $\{1,4,6,8\}$, | $\{2,4,7,8\}$, | $\{1,3,7,8\}$ | | |
| $M_{8,4}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{3,4,5,8\}$, | $\{2,3,6,7\}$, |
| | $\{4,6,7,8\}$ | | | | |
| $M_{8,4+}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{3,4,5,8\}$, | $\{2,3,6,7\}$, |
| | $\{4,6,7,8\}$, | $\{1,3,6,8\}$ | | | |
| $M_{8,5}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{2,3,6,8\}$, | $\{3,4,5,6\}$, |
| | $\{1,4,7,8\}$, | $\{3,5,7,8\}$, | $\{2,4,6,8\}$ | | |
| $M_{8,6}$ | $\{1,2,3,4\}$, | $\{1,5,6,7\}$, | $\{1,2,5,8\}$, | $\{2,3,6,7\}$, | $\{3,4,5,6\}$, |
| | $\{2,4,7,8\}$, | $\{1,3,6,8\}$ | | | |

The second table consists of those matroids $M$ having a 4-circuit $C$ such that every other 4-circuit of $M$ meets $C$ in exactly two elements. Note that, in the table, $F_7^+$ denotes the free coextension of $F_7$.

| $M$ | 4-Circuits of $M$ |
|---|---|
| $F_7^+$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$ $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{1,4,6,7\}$ |
| $M_{8,7}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{2,4,6,7\}$ |
| $M_{8,7^+}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{2,4,6,7\}$, $\{3,4,6,7\}$ |
| $M_{8,8a}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{2,4,5,8\}$, $\{3,4,6,7\}$ |
| $M_{8,8b}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{2,4,5,8\}$, $\{2,4,6,7\}$ |
| $M_{8,9a}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,6,8\}$, $\{2,4,6,7\}$ |
| $M_{8,9b}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,6,8\}$, $\{3,4,6,7\}$ |
| $M_{8,9b^+}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,6,8\}$, $\{3,4,6,7\}$, $\{2,4,5,7\}$ |
| $M_{8,10}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{3,4,6,7\}$ |
| $M_{8,10^+}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{3,4,6,7\}$, $\{2,3,6,8\}$ |
| $M_{8,10^{++}}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{3,4,6,7\}$, $\{2,3,6,8\}$, $\{2,4,5,7\}$ |
| $M_{8,11}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{2,4,5,7\}$ |
| $M_{8,12}$ | $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,7,8\}$, $\{2,4,6,7\}$, $\{3,4,6,8\}$ |

9**-Element Matroids.** If $|E(M)| = 9$, then $M$ is one of ten rank-4 paving matroids. Let $E(M) = \{1, 2, \ldots, 9\}$. Again, to describe $M$, it suffices, up to isomorphism, to list the 4-circuits of $M$. Here, if $M$ is such a matroid, then its set of 4-circuits contains every 4-element subset of each the sets $\{1,2,3,4,5\}$, $\{4,5,7,8,9\}$, and $\{2,3,6,8,9\}$. The remaining 4-circuits of $M$ are given in the next table.

| $M$ | Remaining 4-Circuits of $M$ |
|---|---|
| $M_{9,1}$ | $\{1,2,6,7\}$, $\{1,3,7,8\}$, $\{1,4,6,9\}$, $\{1,5,6,8\}$ |
| $M_{9,1a}$ | $\{1,2,6,7\}$, $\{1,3,7,8\}$, $\{1,4,6,9\}$, $\{1,5,6,8\}$, $\{3,4,6,7\}$ |
| $M_{9,1b}$ | $\{1,2,6,7\}$, $\{1,3,7,8\}$, $\{1,4,6,9\}$, $\{1,5,6,8\}$, $\{3,5,6,7\}$ |
| $M_{9,2}$ | $\{1,2,6,7\}$, $\{1,3,7,8\}$, $\{1,4,6,9\}$ $\{3,5,6,7\}$ |
| $M_{9,3}$ | $\{1,4,6,8\}$, $\{1,2,7,8\}$, $\{1,5,6,9\}$, $\{1,3,7,9\}$, $\{2,4,6,7\}$ |
| $M_{9,3^+}$ | $\{1,4,6,8\}$, $\{1,2,7,8\}$, $\{1,5,6,9\}$, $\{1,3,7,9\}$, $\{2,4,6,7\}$, $\{3,5,6,7\}$ |
| $M_{9,4}$ | $\{1,4,6,8\}$, $\{1,2,7,8\}$, $\{1,5,6,9\}$, $\{1,3,7,9\}$, $\{2,5,6,7\}$ |
| $M_{9,4^+}$ | $\{1,4,6,8\}$, $\{1,2,7,8\}$, $\{1,5,6,9\}$, $\{1,3,7,9\}$, $\{2,5,6,7\}$, $\{3,4,6,7\}$ |
| $M_{9,5}$ | $\{1,4,6,8\}$, $\{1,2,7,8\}$, $\{1,5,6,9\}$, $\{3,4,6,7\}$ |
| $M_{9,6}$ | $\{1,4,6,8\}$, $\{1,2,7,9\}$, $\{3,5,6,7\}$ |

**12- and 14-Element Matroids.** The unique 4-connected 12-element matroid satisfying (P2) and the unique 4-connected 14-element matroid satisfying (P2) have $GF(4)$-representations

$$\left[\begin{array}{c|ccccccc} & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ I_5 & 0 & 1 & 0 & 1 & \alpha & 1 & \alpha \\ & 0 & 0 & 1 & 1 & \alpha^2 & 1 & \alpha^2 \\ & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{c|cccccccc} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ I_6 & 0 & 1 & 0 & 0 & 1 & 0 & \alpha & \alpha \\ & 0 & 0 & 1 & 0 & 0 & 1 & \alpha^2 & \alpha^2 \\ & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right],$$

respectively, where $\alpha^2 + \alpha + 1 = 0$.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, USA

*E-mail address*: oxley@math.lsu.edu

Department of Mathematics, Angelo State University, San Angelo, Texas, USA

*E-mail address*: spfeil@angelo.edu

School of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand

*E-mail address*: charles.semple@canterbury.ac.nz

School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, Wellington, New Zealand

*E-mail address*: geoff.whittle@mcs.vuw.ac.nz