CAPTURING MATROID ELEMENTS IN UNAVOIDABLE 3-CONNECTED MINORS

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ABSTRACT. A result of Ding, Oporowski, Oxley, and Vertigan reveals that a large 3-connected matroid M has unavoidable structure. For every $n > 2$, there is an integer $f(n)$ so that if $|E(M)| > f(n)$, then M has a minor isomorphic to the rank-n wheel or whirl, a rank-n spike, the cycle or bond matroid of $K_{3,n}$, or $U_{2,n}$ or $U_{n-2,n}$. In this paper, we build on this result to determine what can be said about a large structure using a specified element e of M. In particular, we prove that, for every integer n exceeding two, there is an integer $g(n)$ so that if $|E(M)| > g(n)$, then e is an element of a minor of M isomorphic to the rank-n wheel or whirl, a rank-n spike, the cycle or bond matroid of $K_{1,1,1,n}$, a specific single-element extension of $M(K_{3,n})$ or the dual of this extension, or $U_{2,n}$ or $U_{n-2,n}$.

1. INTRODUCTION

In 1993, Oporowski, Oxley, and Thomas [9] showed that any sufficiently large 3-connected graph has a large wheel or a large $K_{3,n}$ as a minor. Ding, Oporowski, Oxley, and Vertigan generalized this graph result to find unavoidable minors of large 3-connected matroids, first in the binary case [4] and later in the general case [5]. The latter result is stated in the next theorem. The matroid terminology used here will follow Oxley [10]. In particular, we use $\mathcal{W}^k$ and $M(\mathcal{W}_k)$ to denote the rank-k whirl and the cycle matroid of the k-spoked wheel. We refer to the latter as the rank-k wheel.

Theorem 1.1. *For every integer $n > 2$, there is an integer $f(n)$ so that every 3-connected matroid with more than $f(n)$ elements has a minor isomorphic to $M(\mathcal{W}_n)$, $\mathcal{W}^n$, a tipless rank-n spike, $M(K_{3,n})$, $M^*(K_{3,n})$, or $U_{2,n}$ or $U_{n-2,n}$.*
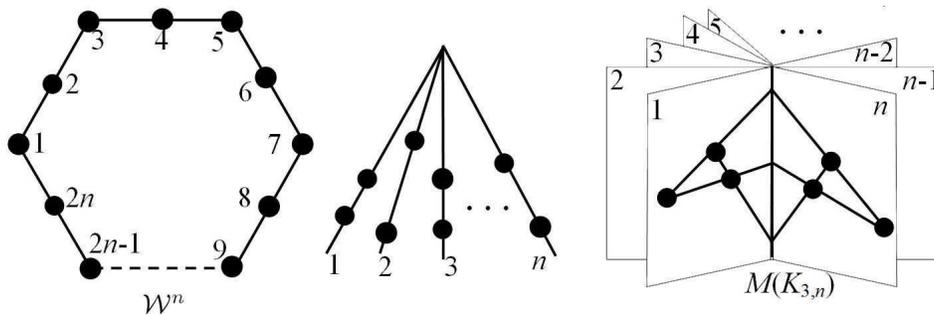

FIGURE 1. Geometric illustrations of three of the unavoidable minors: the rank-n whirl, $\mathcal{W}^n$; the rank-n tipless free spike; and $M(K_{3,n})$, which has rank $n + 2$.

In this paper, we extend this theorem to show that, by slightly modifying the list of unavoidable minors, we can ensure that we capture any specified single element of a large 3-connected matroid in such an unavoidable minor.

Let r be an integer exceeding two. A matroid M is a *rank-r spike with tip t* if and only if M has the following properties [10, p.41]:

(1) $E(M)$ is the union of r lines $L_1, L_2, \ldots, L_r$ each of which is a 3-element circuit containing the point t;

(2) for all k in $\{1, 2, \ldots, r-1\}$, the union of any k of $L_1, L_2, \ldots, L_r$ has rank $k+1$; and

(3) $r(L_1 \cup L_2 \cup \cdots \cup L_r) = r$.

The circuits of such a matroid include $L_1, L_2, \ldots, L_r$ together with all sets of the form $(L_i \cup L_j) - t$ for $i \neq j$. If the matroid has no other non-spanning circuits, it is a *rank-r free spike with a tip*. Evidently, both the Fano and non-Fano matroids are rank-3 spikes with a tip, although neither is a free spike. If we delete the tip t from a spike M, then we obtain a *rank-r tipless spike*. If, instead, we delete an element x from M other then the tip, we get a *rank-r spike with a tip and a cotip*, the *tip* being the element t and the *cotip* being the third element on the line of M spanned by $\{t, x\}$. We denote by $\mathcal{T}_r$ the set of all rank-r spikes having a tip and a cotip. It is easy to see that if N is a member of $\mathcal{T}_r$, then N^* is also in $\mathcal{T}_r$. Moreover, if N has tip t and cotip c, by freely adding an element y on the line $\{t, c\}$ of N, we obtain a rank-r spike with tip t. It is well-known that, for all $r \geq 3$, there is a unique rank-r binary spike with a tip, and there is a unique rank-r binary spike with a tip and a cotip.

Let n be an integer greater than 2. The matroid $M(K_{1,1,1,n})$ has n copies of $M(K_4)$ as restrictions, with one 3-point line common to all these restrictions. We call this common line the *spine* of $M(K_{1,1,1,n})$. We shall denote by $M(K_{3,n})^+$ the matroid obtained by freely adding an element p to the spine of $M(K_{1,1,1,n})$ and deleting every other element from the spine. Notice that deleting p from $M(K_{3,n})^+$ produces $M(K_{3,n})$ (see Figure 2). The following is the main result of the paper.

Theorem 1.2. *Let M be a 3-connected matroid, and let e be an element of M. For every integer $n > 2$, there is an integer $g(n)$ so that if $|E(M)| \geq g(n)$, then e is an element of a minor of M that is isomorphic to $M(\mathcal{W}_n)$, $\mathcal{W}^n$, $M(K_{1,1,1,n})$, $M^*(K_{1,1,1,n})$, $M(K_{3,n})^+$, $(M(K_{3,n})^+)^*$, $U_{2,n}$, $U_{n-2,n}$, or a member of $\mathcal{T}_n$.*
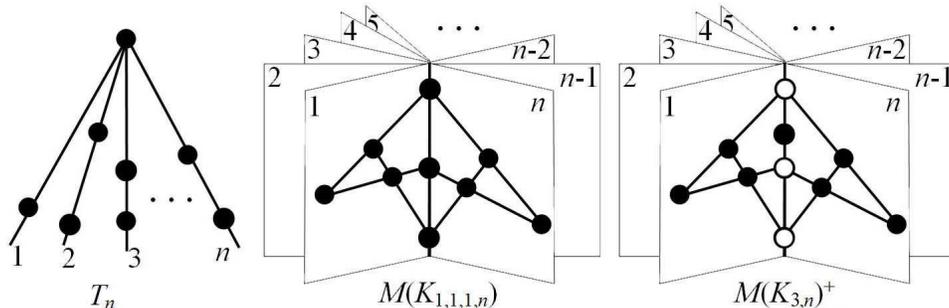

FIGURE 2. Geometric illustrations of three matroids: T_n is the rank-n free spike with a tip and a cotip; $M(K_{1,1,1,n})$ and $M(K_{3,n})^+$ have rank $n+2$ with the white dots representing points that are not present in the matroid.

This theorem shows that not only does every huge 3-connected matroid M contain a large highly structured minor, but a slight modification of such a minor can be chosen to contain any specified element of M. The next two corollaries specialize the main result to the classes of binary and graphic matroids.

Corollary 1.3. *Let M be a 3-connected binary matroid, and let e be an element of M. For every integer $n > 2$, there is an integer $h(n)$ so that if $|E(M)| \geq h(n)$, then e is an element of a minor*

of M that is isomorphic to $M(\mathcal{W}_n)$, the rank-n binary spike with a tip and a cotip, $M(K_{1,1,1,n})$, or $M^*(K_{1,1,1,n})$.

Corollary 1.4. *Let G be a simple 3-connected graph, and let e be an edge of G. For every integer $n > 2$, there is an integer $k(n)$ so that if $|E(G)| \geq k(n)$, then e is an edge of a minor of G that is isomorphic to $\mathcal{W}_n$ or $K_{1,1,1,n}$.*

Following [4] and [5], we make no attempt to find sharp estimates for the functions $g(n), h(n)$, and $k(n)$ in Theorem 1.2, Corollary 1.3, and Corollary 1.4. By contrast, Lemos and Oxley [8] did find sharp bounds for the functions in the corresponding results for connected matroids.

Theorem 1.5. *Let M be a connected matroid having n elements. Then*
  (i) *M has a minor isomorphic to $U_{1,m}$ or $U_{m-1,m}$ for some $m \geq \sqrt{2n}$; and*
  (ii) *for each element e of M, there is a minor of M that uses e and is isomorphic to $U_{1,p}$ or $U_{p-1,p}$ for some $p \geq \sqrt{n-1} + 1$.*

In [3], the first two authors have extended Theorem 1.2 for binary matroids by showing that, for every sufficiently large 3-connected such matroid M, one can capture any two elements of M in the rank-n wheel, the rank-n binary spike with a tip and a cotip, or the cycle or bond matroid of $K_{1,1,1,n}$, these four matroids being the binary members of the list given in Theorem 1.2. It remains open to extend this two-element result to the case when M need not be binary.

2. Background

In this paper, we will rely heavily on the following result of Brylawski [2] and Seymour [12] (see also [10, p.129]).

Theorem 2.1. *Let N be a connected minor of a connected matroid M, and suppose that $e \in E(M) - E(N)$. Then at least one of $M \backslash e$ and M/e is connected and contains N as a minor.*

By Theorem 1.1, a sufficiently large 3-connected matroid has, as a minor, one of the following five matroids:
  (i) an n-element line or its dual;
  (ii) a rank-n spike;
  (iii) a wheel or whirl of rank n;
  (iv) $M(K_{3,n})$; or
  (v) $M^*(K_{3,n})$.
In each of the next five sections, we treat one of these cases identifying an unavoidable minor using the special element. That identification is made possible by using Theorem 2.1. The main theorem is proved in the last section by combining the results from these five sections.

The reader familiar with the matroid concept of roundedness may be reminded of it by the main theorem of this paper. Roundedness was introduced by Seymour [14] (see, for example, [10, p.481]) to encompass certain results relating particular minors of a matroid to specific elements of the matroid. For example, Bixby [1] proved that if x is an element of a connected non-binary matroid M, then M has a $U_{2,4}$-minor using x; and Seymour [15] extended this showing that if x and y are distinct elements of a non-binary 3-connected matroid M, then M has a $U_{2,4}$-minor using x and y. The results of this paper were motivated in part by the idea of roundedness and by the usefulness of the results that relate to it.

In Section 7, we use the following result of Kung [7], which gives an upper bound on the number of elements in a simple matroid that does not contain a long-line minor.

Theorem 2.2. *If M is a simple, rank-r matroid with no $U_{2,q+2}$-minor, then M has at most $\frac{q^r - 1}{q - 1}$ elements.*

Flowers were introduced by Oxley, Semple, and Whittle [11] to describe crossing 3-separations in 3-connected matroids. We will not require a detailed knowledge of flowers here, but the following definitions will be useful. For a positive integer n, we write $[n]$ for $\{1, 2, \ldots, n\}$. Let M be a matroid with rank function r. Its *connectivity function*, λ_M, is defined for all subsets X of $E(M)$ by $\lambda_M(X) = r(X) + r(E - X) - r(M)$; its *local connectivity function*, $\sqcap_M$, is defined for all subsets Y and Z of $E(M)$ by $\sqcap_M(Y, Z) = r(Y) + r(Z) - r(Y \cup Z)$. A detailed discussion of the properties of these two functions can be found in [10, Sections 8.1 and 8.2]. Now suppose that M is 3-connected. Let $(P_1, P_2, \ldots, P_n)$ be an ordered partition Φ of $E(M)$. Consider the following properties.

(1) $|P_i| \geq 2$ for all i in $[n]$.
(2) $\lambda_M(P_i) = 2$ for all i in $[n]$.
(3) $\lambda_M(P_i \cup P_{i+1}) = 2$ for all i in $[n]$ where the indices are considered modulo n.
(4) $\lambda_M(\cup_{i \in S} P_i) = 2$ for all proper non-empty subsets S of $[n]$.
(5) $\sqcap_M(P_i, P_j) = 2$ for all distinct i and j in $[n]$.

If the first three properties hold, then Φ is a *flower* with every set P_i being a *petal*. When the first four properties hold, this flower is an *anemone*. Should all five properties hold, this anemone is a *paddle*.

3. Long Lines

In this section, we examine the case where a connected matroid with an identified element has a long line or its dual as a minor.

Theorem 3.1. *Let M be a connected matroid with a $U_{2,n}$-minor for some $n \geq 3$. If $e \in E(M)$, then e is an element of a connected minor of M that is isomorphic to $U_{2,m}$ for some $m > \sqrt{n}$.*

Proof. The result is immediate if $n = 3$. Thus we may assume that $n \geq 4$. By Theorem 2.1, there is a connected minor N of M so that $N \backslash e \cong U_{2,n}$ or $N/e \cong U_{2,n}$. If $N \backslash e \cong U_{2,n}$, then, as N is connected, $r(N) = r(N \backslash e) = 2$. Thus $N \cong U_{2,n+1}$, or N is obtained from $U_{2,n}$ by adding e parallel to some other element. In either case, we easily identify a $U_{2,n}$-minor of M using e.

Now assume $N/e \cong U_{2,n}$. Thus $r(N) = 3$ and N is as shown in Figure 3 where the possible non-trivial lines through subsets of $\{1, 2, \ldots, n\}$ have not been depicted. Let f be an element of
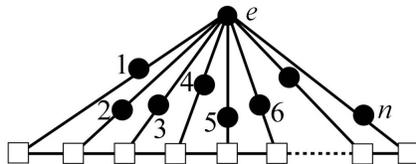

FIGURE 3. Geometric illustration of N in rank 3. Possible lines containing subsets of $[n]$ have not been shown.

N/e. Then N/f has rank 2 and has $\{e\}$ as a rank-1 flat. Simplify N/f without deleting e to produce a minor isomorphic to $U_{2,k}$, for some k. If $k > \sqrt{n}$, then we have identified a desired minor.

Assume $k \leq \sqrt{n}$. Since N has $n+1$ elements, a largest parallel class X of N/f has p elements for some $p \geq \frac{n}{k} \geq \sqrt{n} \geq 2$. As N has no parallel elements, the elements of X are collinear in N, and the matroid $N|(X \cup f) \cong U_{2,p+1}$, so $N|(X \cup f)$ is connected. By Theorem 2.1, N has a connected minor N' with ground set $X \cup \{e, f\}$ such that $N|(X \cup f)$ is a minor of N'. Now $r(N')$ is 2 or 3. In the latter case, e is a coloop of N', a contradiction. Thus $r(N') = 2$, so $N' \cong U_{2,p+2}$, or N is obtained from $U_{2,p+1}$ by adding e parallel to some other element. In either case, we easily identify a $U_{2,p+1}$-minor of M using e. Since $p + 1 \geq \sqrt{n} + 1$, the lemma holds. $\square$

4. Spike Minors

In this section, we examine the case where a connected matroid with an identified element has a large spike as a minor. It is not hard to show that if a non-tip element is contracted from a rank-r spike with a tip and no cotip, the resulting matroid is a rank-$(r-1)$ spike with a tip, no cotip, and an extra element parallel to the tip.

It is easy to see that $M(K_{1,1,n})$ is the parallel connection of n 3-point lines, $L_1, L_2, \ldots, L_n$, across a common basepoint (see Figure 4). Extending this, we have the following.
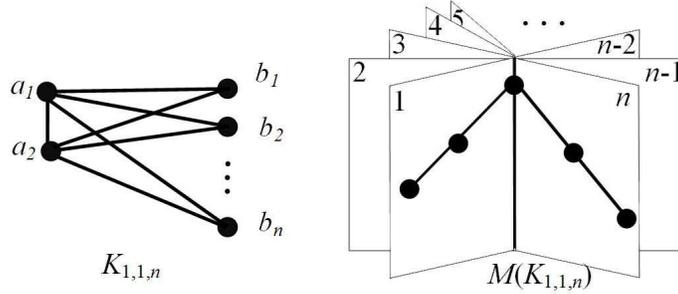

FIGURE 4. The graph of $K_{1,1,n}$ and the geometry of $M(K_{1,1,n})$ in rank $n+1$.

Lemma 4.1. *Let M be a connected matroid so that $M\backslash x \cong M(K_{1,1,n})$ for some $n \geq 3$. Then $M\backslash x$ is the parallel connection of n 3-point lines, $L_1, L_2, \ldots, L_n$, across a common basepoint, t. If x is not contained in the closure of any proper subset of these lines, then M is a rank-$(n+1)$ spike with a tip and a cotip.*

Proof. We can freely add a point y on the line of M containing t and x. Let L_{n+1} be the line $\{t, x, y\}$. By the definition of a spike given above, the result is a rank-$(n+1)$ tipped spike. Hence, without y, the matroid is a rank-$(n+1)$ spike with a tip and a cotip. $\square$

Using this characterization, we prove the main theorem of this section.

Theorem 4.2. *Let M be a connected matroid with an element e so that $M\backslash e$ is isomorphic to a matroid in $\mathcal{T}_n$ for some $n \geq 6$. Then e is an element of a minor of M that is isomorphic to a matroid in $\mathcal{T}_m$ for some $m \geq \frac{n}{2} \geq 3$.*

Proof. Let t and c be the tip and cotip of $M\backslash e$. If e lies on the line joining c and t, then we can easily find the desired minor. If not, M/c is connected, and $M/c\backslash e$ is a rank-$(n-1)$ spike with a tip t and no cotip. By definition, this matroid is the union of $n-1$ lines $L_1, L_2, \ldots, L_{n-1}$, each of which is a 3-element circuit containing the point t so that, for all j in $[n-2]$, the union of any j of $L_1, L_2, \ldots, L_{n-1}$ has rank $j+1$, and $r(L_1 \cup L_2 \cup \cdots \cup L_{n-1}) = n-1$. Let $\{L_1, L_2, \ldots, L_k\}$ be a smallest set of these lines for which $e \in \mathrm{cl}_M(L_1 \cup L_2 \cup \cdots \cup L_k)$.

Suppose $k \leq \frac{n}{2}$. Let $\{s_1, s_2, \ldots, s_k\}$ be a transversal of $\{L_1, L_2, \ldots, L_k\}$ avoiding t. The matroid $M\backslash e/\{c, s_1, s_2, \ldots, s_{k-1}\}$ is a spike with a tip, no cotip, and $k-1$ extra elements parallel to t. In the loopless matroid $M/\{c, s_1, s_2, \ldots, s_{k-1}\}$, the element e is in the closure of $(L_1 \cup L_2 \cup \cdots \cup L_k) - \{s_1, s_2, \ldots, s_{k-1}\}$, so e is in the closure of L_k. Without deleting e, simplify the last matroid. From this simplification, we can remove some set consisting of all but two elements of the closure of L_k to produce a member of $\mathcal{T}_m$ with e as the tip or cotip and with $m = n - 1 - (k-1) \geq \frac{n}{2}$.

We may now assume that $k > \frac{n}{2}$. Notice that $k \leq n - 2$, since the union of any $n-2$ lines has rank $n-1$, which is the rank of $M/c\backslash e$. Moreover, the restriction $(M/c)|(L_1 \cup L_2 \cup \cdots \cup L_k)$ is isomorphic to $M(K_{1,1,k})$. By Lemma 4.1, $(M/c)|(L_1 \cup L_2 \cup \cdots \cup L_k \cup e)$ is a rank-m spike with a tip t and a cotip x and with $m = k + 1 > \frac{n}{2} + 1$. $\square$

5. Wheels and Whirls

In this section, we consider the case where a connected matroid with an identified element has a large wheel or a large whirl as a minor. First, we define a fan, which can be thought of as a partial wheel or whirl. In a simple, cosimple matroid M, consider a sequence $(s_0, r_1, s_1, \ldots, s_{n-1}, r_n, s_n)$ of distinct elements of M so that every set $\{s_{i-1}, r_i, s_i\}$ with $0 < i \leq n$ is a triangle of M and every set $\{r_j, s_j, r_{j+1}\}$ with $0 < j < n$ is a triad of M. Here we call such a sequence a *fan*, noting that this specializes the terminology used in [10], where two other related structures are also called fans. The following result of Seymour shows how closely related fans are to wheels and whirls [13] (see also [10, p. 339]).

Theorem 5.1. *Let M be a connected, simple, cosimple matroid having $(s_0, r_1, s_1, \ldots, s_{n-1}, r_n, s_n)$ as a fan and having another element r_0 so that $\{r_0, s_0, r_1\}$ and $\{r_n, s_n, r_0\}$ are triads and $\{s_0, r_0, s_n\}$ is a triangle. Then M is a wheel or whirl of rank $n + 1$.*

Viewing a fan as a substructure of a wheel makes it natural to refer to each s_i as a *spoke* element and each r_i as a *rim element*. The following is a technical lemma.

Lemma 5.2. *Let M be a 3-connected matroid with an element e so that $M \backslash e$ is 3-connected having a fan $F = (s_0, r_1, s_1, \ldots, s_{n-1}, r_n, s_n)$ with $n \geq 3$. Let $E(M) - (F \cup e)$ be a set A having at least two elements. If no triad of F is a triad in M, and M has no $U_{q-2,q}$-minor, then there is a set X of at least $\frac{n}{q-1}$ elements of $\{r_1, r_2, \ldots, r_n\}$ so that $e \in \mathrm{cl}(A \cup (\{r_1, r_2, \ldots r_n\} - X))$.*

Proof. We begin by establishing the following.

5.2.1. $\{s_0, s_n\} \subseteq \mathrm{cl}_M(A)$.

Suppose $s_0 \notin \mathrm{cl}(A)$. Then, as the disjoint union of A and $F - s_0$ is $E(M \backslash e) - s_0$, it follows that $s_0 \in \mathrm{cl}_{M \backslash e}^*(F - s_0)$. Thus, in $(M \backslash e)^*$, the set F is spanned by $\{r_1, r_2, \ldots, r_n, s_n\}$, so $r_{M \backslash e}^*(F) \leq n + 1$. But $r_{M \backslash e}(F) \leq n + 1$, so $r_{M \backslash e}(F) + r_{M \backslash e}^*(F) - |F| \leq 1$. Hence (F, A) is a 2-separation of $M \backslash e$. This contradiction and symmetry imply that (5.2.1) holds.

Let $N = M \backslash \{e, s_1, s_2, \ldots, s_{n-1}\}$ and $R = \{r_1, r_2, \ldots, r_n\}$. Next we show that

5.2.2. $R \cup \{s_0, s_n\}$ *is a circuit of M and $r_N^*(R) = 1$.*

In $M \backslash e$, every set $\{r_i, s_i, r_{i+1}\}$ with i in $[n-1]$ is a triad. In $M \backslash \{e, s_i\}$, then, $\{r_i, r_{i+1}\}$ is a series pair, and so has corank 1. In N, it follows that every set $\{r_i, r_{i+1}\}$ has corank at most 1. A straightforward induction argument establishes that the set $R \cup \{s_0, s_n\}$ is a circuit of M and so is a circuit of $M \backslash \{e, s_1, s_2, \ldots, s_{n-1}\}$. Hence, by orthogonality, no element of $R \cup \{s_0, s_n\}$ is a coloop. Thus $r_N^*(\{r_1\}) = 1$ and $r_N^*(\{r_i, r_{i+1}\}) = 1$ for all i in $[n-1]$. Hence $\mathrm{cl}_{N^*}(\{r_1\}) = R$ and (5.2.2) follows.

Suppose $i \in [n-1]$. By hypothesis, $\{r_i, s_i, r_{i+1}\}$ is not a triad of M, so M has $\{e, r_i, s_i, r_{i+1}\}$ as a cocircuit. Let $M_1 = M \backslash \{s_1, s_2, \ldots, s_{n-1}\}$. We will show next that

5.2.3. $\{e, r_i, r_{i+1}\}$ *is a triad of M_1.*

Every set $\{e, r_i, r_{i+1}\}$ with i in $[n-1]$ is a union of cocircuits of M_1. By orthogonality with the circuit $R \cup \{s_0, s_n\}$, it follows that a cocircuit contained in $\{e, r_i, r_{i+1}\}$ contains r_i if and only if it contains r_{i+1}. Thus $\{e, r_i, r_{i+1}\}$ is a triad of M_1 unless e is a coloop of M_1. In the exceptional case, $\{e, s_1, s_2, \ldots, s_{n-1}\}$ contains a cocircuit of M containing e. This contradicts orthogonality with the triangles of F.

In M_1, the set $R \cup e$ has corank at least 2, since it contains a triad. Now $N = M_1 \backslash e$ so, by (5.2.2), $r_{M_1 \backslash e}^*(R) = 1$. Therefore, $r_{M_1}^*(R) \leq 2$, so

5.2.4. $r_{M_1}^*(R \cup e) = 2$.

Consider M_1^*. In this matroid, $R \cup e$ has rank 2 and, by (5.2.3), every set $\{e, r_i, r_{i+1}\}$ with i in $[n-1]$ is a triangle. Since M_1^* has no $U_{2,q}$-minor, M_1^* has fewer than q parallel classes. As one of these classes is $\{e\}$, and since the n elements of R are contained in the other parallel classes, there is a parallel class X with at least $\frac{n}{q-1}$ elements. In $M_1^*|(R \cup e)$, the set X is a hyperplane, and its complement, $(R \cup e) - X$, is a cocircuit. Moreover, $R - X \neq \emptyset$. Thus, in M_1, the element e is contained in the closure of $E(M_1) - e - X$. Hence $e \in \mathrm{cl}(E(M) - \{s_1, s_2, \ldots, s_{n-1}\} - e - X)$. Recall that $A = E(M) - (F \cup e) = E(M) - \{s_0, s_1, s_2, \ldots, s_{n-1}, s_n\} - e - R$. By (5.2.1), $\{s_0, s_n\} \subseteq \mathrm{cl}(A)$, so $e \in \mathrm{cl}(E(M) - \{s_0, s_1, s_2, \ldots, s_{n-1}, s_n\} - e - X) = \mathrm{cl}(A \cup (R - X))$. $\qquad \square$

We are now ready to prove the main theorem of this section.

Theorem 5.3. *Let M be a connected matroid with no $U_{q-2,q}$-minor and no $U_{2,q}$-minor for some $q \geq 4$. If M has an element e and a minor isomorphic to $M(\mathcal{W}_{q^3})$ or $\mathcal{W}^{q^3}$, then M has a minor containing e isomorphic to $M(\mathcal{W}_q)$ or $\mathcal{W}^q$.*

Proof. Let $n = q^3$. By Theorem 2.1 and by switching to the dual if necessary, we may assume $M \backslash e = N$ where N is a rank-n wheel or whirl. Let $\{s_0, s_1, \ldots, s_{n-1}\}$ be the set S of spokes of N, and $\{r_1, r_2, \ldots, r_n\}$ be the set R of rim elements of N, where every set $\{s_{i-1}, r_i, s_i\}$ is a triangle of N and every set $\{r_j, s_j, r_{j+1}\}$ is a triad of N with all subscripts being interpreted modulo n.

The matroid M is connected, and $M \backslash e$ is 3-connected. If e is parallel to another element x in M, then $M \backslash x$ is a minor of the desired type. Hence we may assume that e is not parallel to another element. Thus M is 3-connected.

The set S is a basis of M, so $S \cup e$ contains a unique circuit C containing e. Let X be a largest set of consecutive spokes avoiding C. Without loss of generality, when X is non-empty, we may assume that $X = \{s_1, s_2, \ldots, s_k\}$. Every set $\{s_{i-1}, r_i, s_i\}$ with i in $[k+1]$ is a triangle in $M \backslash e$ and so is a triangle in M. Now suppose $i \in [k]$. Then either $\{r_i, s_i, r_{i+1}\}$ or $\{e, r_i, s_i, r_{i+1}\}$ is a cocircuit of M. Since the circuit C is a subset of $(S - X) \cup e$, the spoke $s_i \notin C$. By orthogonality, $\{e, r_i, s_i, r_{i+1}\}$ is not a cocircuit of M. Thus $\{r_i, s_i, r_{i+1}\}$ is a triad of M, and M contains the fan $F = (s_0, r_1, s_1, r_2, s_2, \ldots, r_k, s_k)$.

Assume that $k \geq q$. The circuit C is contained in the complement of F, so we will remove this complement to produce a wheel or whirl minor. It is clear that we may contract rim elements of a wheel or whirl and simplify to produce a smaller wheel or whirl. Let $R' = \{r_{k+1}, r_{k+2}, \ldots, r_n\}$. The matroid N/R' has $\{s_k, s_{k+1}, \ldots, s_{n-1}, s_0\}$ as a parallel class, and the elements of $C - e$ are contained in this parallel class. The matroid M/R' has e in the closure of the set $C - e$ and so $e \in \mathrm{cl}(\{s_k, s_{k+1}, \ldots, s_{n-1}, s_0\})$. Either M/R' has $\{e, s_k, s_{k+1}, \ldots, s_{n-1}, s_0\}$ as a parallel class or has e as a loop. In the latter case, contract the elements of R' from M one at a time until e becomes parallel to some remaining element of R'. In both cases, by simplifying the resulting matroid without removing e, we obtain a wheel or whirl minor with at least q spokes.

Now assume that $k \leq q - 1$, noting that this includes the case when X is empty. Then every set of q consecutive spokes of N contains an element of C. Let $|C - e| = m$. Then $m \geq \frac{q^3}{q} = q^2$. Suppose $s_i \in S - C$. The set $\{r_i, s_i, r_{i+1}\}$ is a triad of $M \backslash e$. By orthogonality with the circuit C of M, it follows that $\{r_i, s_i, r_{i+1}\}$ is a cocircuit of M. In M, the complement of $\{r_i, s_i, r_{i+1}\}$ is a hyperplane H_i containing C (see Figure 5). The element $s_i \notin \mathrm{cl}(C)$, so $M \backslash s_i / r_i$ has C as a circuit. Notice that $(M \backslash s_i / r_i) \backslash e$ is a wheel or whirl of rank $n - 1$. In this way, we may remove s_i and r_i for all s_i in $S - C$ to produce a matroid M_1 in which C is a circuit and $M_1 \backslash e$ is a wheel or whirl of rank m having $C - e$ as its set of spokes.

Reindex both the set of spokes and the set of rim elements of $M_1 \backslash e$ so that $M_1 \backslash e$ has $\{s_{i-1}, r_i, s_i\}$ as a triangle and $\{r_i, s_i, r_{i+1}\}$ as a triad for all i in $[m]$, where $s_m = s_0$ and $r_{m+1} = r_1$. By orthogonality with C, it follows that each $\{e, r_i, s_i, r_{i+1}\}$ is a cocircuit of M_1. Now $(s_0, r_1, s_1, \ldots, r_{m-2}, s_{m-2})$ is a fan F_1 of $M_1 \backslash e$ and no triad of F_1 is a triad of M_1. Let $A_1 = E(M_1 \backslash e) - F_1$. Then

FIGURE 5. The triad $T_i^* = \{r_i, s_i, r_{i+1}\}$ and the complementary hyperplane H_i.

$A_1 = \{r_{m-1}, s_{m-1}, r_m\}$. Thus $|A_1| \geq 2$ and we may apply Lemma 5.2 to get a subset Y of $\{r_1, r_2, \ldots, r_{m-2}\}$ having at least $\frac{m-2}{q-1}$ elements so that $e \in \mathrm{cl}(A_1 \cup (\{r_1, r_2, \ldots, r_{m-2}\} - Y))$. Observe that

$$|Y| \geq \frac{q^2 - 2}{q - 1} = \frac{q^2 - 1}{q - 1} - \frac{1}{q - 1} = q + 1 - \frac{1}{q - 1}.$$

As $q \geq 4$, it follows that $|Y| \geq q + 1$.

Let $M_2 = M_1/(\{r_1, r_2, \ldots, r_{m-2}\} - Y)$. In this matroid, e is in the closure of A_1. Suppose e is a loop of M_2. Then contracting some proper subset of $\{r_1, r_2, \ldots, r_{m-2}\} - Y$ from M_1 makes e parallel to some other element of the set. On the other hand, when e is not a loop of M_2, it is spanned by the set A_1, which equals $\{r_{m-1}, s_{m-1}, r_m\}$. In this case, contracting some subset of $\{r_{m-1}, r_m\}$ from M_2 makes e parallel to some element of $\{r_{m-1}, s_{m-1}, r_m\}$. In either case, by simplifying the resulting matroid without deleting e, we obtain a wheel or whirl that uses e and has rank at least $|Y|$. We conclude that the theorem holds. $\square$

6. $M(K_{3,n})$

In this section, we consider the case where a connected matroid with an identified element has a minor isomorphic to a large $M(K_{3,n})$. We give a lower bound on the rank of a similar minor containing the identified element. It is easy to show that the elements of the matroid $M(K_{3,n})$ can be partitioned into a rank-$(n + 2)$ paddle with n petals of rank 3 (see Figure 6). Two related matroids are $M(K_{1,1,1,n})$ and $M(K_{3,n})^+$ (see Figure 7). If a matroid has an $M(K_{3,n})$-minor, then, by contracting a set of two elements from this minor and simplifying, we obtain a minor isomorphic to $M(K_{1,1,1,n-2})$. The matroid $M(K_{3,n})^+$ was formally defined in the introduction. Recall that $\mathcal{T}_n$ denotes the set of rank-n spikes with a tip and a cotip.

FIGURE 6. The graph $K_{3,n}$ and the geometry of $M(K_{3,n})$ with white dots representing points that are not present in the matroid.
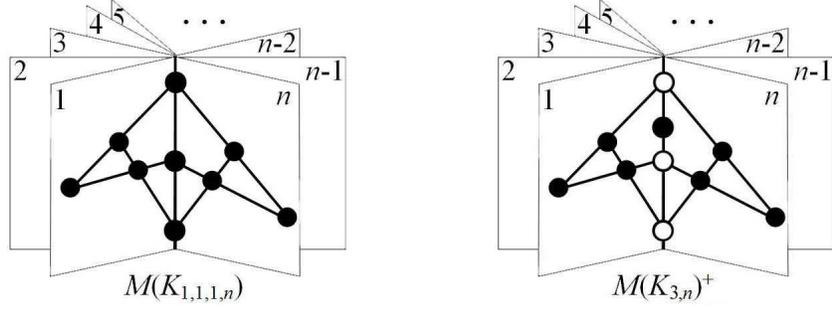
FIGURE 7. Geometric representations of $M(K_{1,1,1,n})$ and $M(K_{3,n})^+$.

Theorem 6.1. *Let M be a connected matroid with $M\backslash e = N \cong M(K_{1,1,1,n})$ for some $n \geq 3$. Then e is an element of a minor of M isomorphic to $M(K_{1,1,1,m})$, $M(K_{3,m})^+$, or some member of $\mathcal{T}_m$ for some $m \geq \frac{n}{2}$.*

Proof. The matroid N has exactly n triads $\{P_1, P_2, P_3, \ldots, P_n\}$. These sets are disjoint and their union is $E(N) - S$, where S is the spine of N. Moreover, $(P_1, P_2, P_3, \ldots, P_n)$ is a rank-$(n+2)$ paddle Φ in $N\backslash S$. After possibly relabelling, we may assume that $\{P_1, P_2, P_3, \ldots, P_k\}$ is a minimal set P of petals of Φ whose closure contains e in M.

Suppose $k \leq \frac{n}{2}$. Let X be a transversal of P. Contract elements of X from M one at a time until the first time that either e becomes parallel to an element of N, or $e \in \mathrm{cl}(S)$. Simplify the resulting matroid without deleting e to form M'. Either $M' \cong M(K_{1,1,1,m})$ for some $m \geq n - k \geq \frac{n}{2}$, or $e \in \mathrm{cl}(S)$ and this line has four elements. In the latter case, delete the elements other than e on this line to form an $M(K_{3,m})^+$-minor of M for some $m \geq n - k \geq \frac{n}{2}$.

Now suppose $k > \frac{n}{2}$ and consider $M|(P_1 \cup P_2 \cup \cdots \cup P_k \cup S \cup e)$. Let Y be a transversal of P so that $M\backslash(Y \cup e)$ is a set of $k + 1$ 3-point lines all intersecting at some point a of S. Then $M\backslash(Y \cup e) \cong M(K_{1,1,k+1})$ (see Figure 8). For each j in $[k]$, let $L_j = P_j - Y$. By the choice of P, the
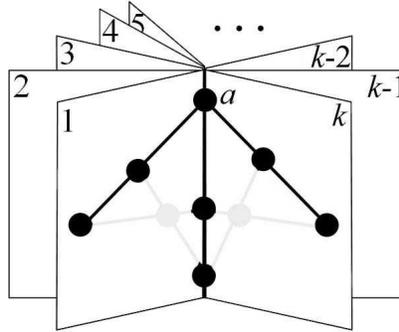

FIGURE 8. Geometric representation of $M(K_{1,1,k+1})$ in rank $k + 1$.

element e is in the closure of $L_1 \cup L_2 \cup \cdots \cup L_k \cup S$, but it is not in $\mathrm{cl}((L_1 \cup L_2 \cup \cdots \cup L_k \cup S) - L_i)$ for any i in $[k]$. This leaves the possibility that $e \in \mathrm{cl}(L_1 \cup L_2 \cup \cdots \cup L_k)$. Thus, for some m in $\{k, k+1\}$, there are m lines in $\{L_1, L_2, \ldots, L_k, S\}$ whose union spans e, and no proper subset of these m lines spans e. By Lemma 4.1, the restriction of M to the union of e with the elements of these m lines is a member of $\mathcal{T}_{m+1}$ with a as the tip and e as the cotip. $\square$

7. $M^*(K_{3,n})$

In this section, we consider the case where a connected matroid with an identified element has a minor isomorphic to a large $M^*(K_{3,n})$. We give a lower bound on the rank of a similar minor, $M^*(K_{1,1,1,m})$, containing the identified element. This result relies on work of Geelen and Whittle. In particular, we extend arguments in [6, Theorem 9.43 and 9.44] to prove the following theorem. The dual of a paddle is a *copaddle*.

Theorem 7.1. *Let M be a connected matroid with $M\backslash e = N \cong M^*(K_{3,m^5})$ for some $m \geq 4$. If M has no $U_{m,m+2}$-minor, then e is an element of a minor of M that is isomorphic to $M^*(K_{1,1,1,m-1})$.*

Proof. The minor N has a copaddle $\Phi = \{T_1, T_2, \ldots, T_{m^5}\}$, with each petal T_i being a triangle $\{\alpha_i, \beta_i, \gamma_i\}$. The set $\{\alpha_1, \alpha_2, \ldots, \alpha_{m^5}\}$ is a transversal A of the petals. Let i be an element of $[m^5]$. The matroid N has rank $2m^5 - 2$ and is spanned by $E(N) - T_i$. Since $\{\beta_j, \gamma_j\}$ spans T_j for all j in $[m^5]$, the set $E(N\backslash A) - \{\beta_i, \gamma_i\}$ is a basis B_i of N and is therefore a basis of M. Let C_i be the fundamental circuit $C(e, B_i)$ and let $\mathcal{Q}_i$ be the set of petals of Φ that meet C_i.

We show next that

7.1.1. *$\mathcal{Q}_i$ is a minimal set of petals of Φ whose closure contains e.*

Suppose that $T_j \in \mathcal{Q}_i$ but that $e \in \text{cl}(\cup_{T_l \in \mathcal{Q}_i} T_l - T_j)$. Then $e \in \text{cl}(\cup_{T_l \in \mathcal{Q}_i} T_l - (T_j \cup A))$, so M has a circuit that is contained in $B_i \cup e$ but differs from C_i; a contradiction. Thus (7.1.1) holds.

7.1.2. *If $T_j \cap C_i = \emptyset$, then $C_j = C_i$, so $\mathcal{Q}_j = \mathcal{Q}_i$.*

To see this, observe that, since $T_j \cap C_i = \emptyset$, we have $C_i \subseteq B_j \cup e$. But C_j is the unique circuit contained in $B_j \cup e$. Hence $C_j = C_i$ and (7.1.2) holds.

By relabelling if necessary, we may assume that the set $\mathbf{Q}$ of distinct sets $\mathcal{Q}_i$ with $i \in [m^5]$ is $\{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_k\}$. Then $\mathbf{Q}$ is the set of all distinct minimal sets of petals whose closure contains e.

7.1.3. *If i and j are distinct elements of $[k]$, then every petal of Φ is in $\mathcal{Q}_i$ or $\mathcal{Q}_j$.*

Suppose that some petal T_s is in neither $\mathcal{Q}_i$ nor $\mathcal{Q}_j$. Then $T_s \cap C_i = \emptyset = T_s \cap C_j$ so, by (7.1.2), $\mathcal{Q}_i = \mathcal{Q}_j$; a contradiction. Thus (7.1.3) holds.

For each $\mathcal{Q}_i$ in $\mathbf{Q}$, let $\mathcal{X}_i$ be the set of petals of Φ that are not in $\mathcal{Q}_i$. By (7.1.3), if $i \neq j$, then $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$. By construction, for all s in $[k]$, the petal T_s is not in $\mathcal{Q}_s$, so $T_s \in \mathcal{X}_s$. It follows from (7.1.2) that $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k\}$ is a partition $\mathbf{X}$ of the petals of Φ.

Since Φ has m^5 petals, it follows by the pigeonhole principle that either

(i) $k \geq m^4$, or
(ii) $|\mathcal{X}_i| \geq m$ for some i.

First, assume that (ii) holds. Then, without loss of generality, $|\mathcal{X}_1| \geq m$. Thus $\mathcal{Q}_1$ avoids at least m petals of Φ. Now $\mathcal{Q}_1$ is a minimal set of petals of Φ whose closure contains e. Choose one petal T_j in $\mathcal{Q}_1$. For each T_i in $\mathcal{Q}_1 - T_j$, delete α_i and contract $\{\beta_i, \gamma_i\}$. In the resulting matroid N', the element e is in the closure of T_j. We observe that $N'\backslash e \cong M^*(K_{3,m'})$ for some $m' \geq m + 1$.

Suppose that e is parallel to another element f of N'. Then $N'\backslash f$ contains e and is isomorphic to $M^*(K_{3,m'})$. The last matroid contains e in a minor isomorphic to $M^*(K_{1,1,1,m'-2})$ and $m' - 2 \geq m - 1$, so the theorem holds in this case. Thus we may assume that e is not parallel to any other element of N'. Then $\text{cl}_{N'}(T_j)$ is a 4-point line containing e. As $N'\backslash e$ is cographic, $N'\backslash e/\alpha_i$ is cographic. Since N'/α_i has $\{e, \beta_i, \gamma_i\}$ as a parallel class, N'/α_i is cographic. Without deleting e, take the simplification of N'/α_i. This matroid is isomorphic to $M^*(K'_{3,m'-1})$, where the graph $K'_{3,m'-1}$ is shown in Figure 9. Contracting the edge $a_3 b_1$ in this graph produces a $K_{1,1,1,m'-2}$-minor using e. Hence N' has an $M^*(K_{1,1,1,m'-2})$-minor using e. As $m' - 2 \geq m - 1$, we conclude that the theorem holds in case (ii).

We may now assume that (i) holds, that is, $k \geq m^4$. In addition, we assume that some $|\mathcal{X}_i| > 1$. In particular, we suppose that $\mathcal{X}_1$ contains T_i and at least one other petal. Consider the matroid $M/T_i = M'$. Observe that $M' = M/\{\beta_i, \gamma_i\}\backslash\alpha_i$ as α_i is a loop of $M/\{\beta_i, \gamma_i\}$. Since $e \notin \mathrm{cl}(T_i)$, the matroid M' is connected. Notice that $M'\backslash e \cong M^*(K_{3,m^5-1})$. Let $\mathbf{X}' = \{\mathcal{X}_1 - T_i, \mathcal{X}_2, \ldots, \mathcal{X}_k\}$. It is clear that $\mathbf{X}'$ partitions the set of triangles of $M^*(K_{3,m^5-1})$, and these triangles are the petals of a copaddle Φ' in $M'\backslash e$.

In the partition $\mathbf{X}$, the set $\mathcal{X}_1$ contains T_i. Hence $T_i \in \mathcal{Q}_j$ for all $j \neq 1$. Now let $\mathcal{Q}'_t = \mathcal{Q}_t - \{T_i\}$ for all t in $[k]$. We will show that the set $\mathbf{Q}' = \{\mathcal{Q}'_1, \mathcal{Q}'_2, \ldots, \mathcal{Q}'_k\}$ has the same properties in M' as the set $\mathbf{Q}$ has in M.

Suppose $1 < j \leq m^5 - 1$. The set $E(N\backslash A) - T_j$ is a basis B_j of M. Since $T_i \in \mathcal{Q}_j$, it follows from (7.1.1) that the fundamental circuit C_j of e with respect to B_j contains β_i or γ_i. Let $C'_j = C_j - \{\beta_i, \gamma_i\}$. We show next that

7.1.4. C'_j *is a circuit of* $M/\{\beta_i, \gamma_i\}\backslash\alpha_i$.

This is certainly true if $\{\beta_i, \gamma_i\} \subseteq C_j$. Thus, we may assume, by symmetry, that C_j contains β_i but not γ_i. Then M/β_i has C'_j as a circuit. Clearly C'_j is contained in $(B_j \cup e) - \{\beta_i, \gamma_i\}$. In M/β_i, the set $B_j - \beta_i$ is a basis and $(B_j \cup e) - \beta_i$ contains a unique circuit, namely C'_j. Thus $\gamma_i \notin \mathrm{cl}_{M/\beta_i}(C'_j)$, so $M/\{\beta_i, \gamma_i\}$ has C'_j as a circuit. Hence so does $M/\{\beta_i, \gamma_i\}\backslash\alpha_i$. Thus (7.1.4) holds.

7.1.5. *In* M', *the set* $\mathbf{Q}'$ *is precisely the set of minimal sets of petals of* Φ *whose closure contains* e.

To see this, observe that, in M', the set $E(N'\backslash A) - T_j$ is a basis B'_j, and $B'_j \cup e$ contains a unique circuit. As $B'_j \cup e$ contains C'_j, that circuit is C'_j. The set of triangles intersecting this circuit is exactly $\mathcal{Q}'_j$, and (7.1.5) follows without difficulty.

7.1.6. *The members of* $\mathbf{Q}'$ *are distinct.*

Suppose $1 < s < t \leq k$. As $\mathcal{Q}_s \neq \mathcal{Q}_t$, clearly $\mathcal{Q}_s - \{T_i\} \neq \mathcal{Q}_t - \{T_i\}$, that is, $\mathcal{Q}'_s \neq \mathcal{Q}'_t$. Now suppose $\mathcal{Q}'_1 = \mathcal{Q}'_s$. Then $\mathcal{Q}_1$ is a proper subset of $\mathcal{Q}_s$, contradicting (7.1.1).

We have now shown that when $T_i \in \mathcal{X}_1$ and $|\mathcal{X}_1| > 1$, we can construct a new matroid M' from M so that $M'\backslash e$ has a copaddle Φ' whose petals are all the petals of the copaddle Φ of $M\backslash e$ except for T_i. In particular, $M' = M/T_i$. Moreover, $(\mathcal{X}_1 - \{T_i\}, \mathcal{X}_2, \ldots, \mathcal{X}_k)$ partitions the set of petals of $M'\backslash e$ and $(\mathcal{Q}_1 - \{T_i\}, \mathcal{Q}_2 - \{T_i\}, \ldots, \mathcal{Q}_k - \{T_i\})$ is a collection of distinct sets coinciding with the set of minimal sets of petals of Φ' whose union spans e. We may repeat this process of shrinking the size of the matroid we are dealing with until each $\mathcal{X}_i$ is reduced to containing a single petal, that is, $|\mathcal{X}_i| = 1$ for all i in $[k]$. We now consider this case, letting the matroid in which it occurs be M_0. Then $M_0\backslash e = N_0 \cong M^*(K_{3,k})$. Thus N_0 has a copaddle Φ_0 whose petals are triangles. By relabelling if necessary, we may assume that these triangles are $T_1, T_2, \ldots, T_k$ where $\mathcal{X}_i = \{T_i\}$ for all i in $[k]$.
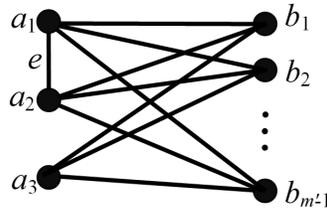

FIGURE 9. The graph $K'_{3,m'-1}$.

Let $\mathbf{P}_0$ be the set of petals of Φ_0. By construction, the minimal sets of petals of Φ_0 whose closure contains e are precisely the k sets $\mathbf{P}_0 - \{T_i\}$ for all i in $[k]$. As before, let $T_i = \{\alpha_i, \beta_i, \gamma_i\}$ for all i in $[k]$, and let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

Suppose i and j are distinct elements of $[k]$. We show next that

7.1.7. $r^*_{M_0\backslash A}(\{\beta_i, \gamma_i, \beta_j, \gamma_j\}) = 3$.

Let $Y = \{\beta_i, \gamma_i, \beta_j, \gamma_j\}$. The set $E(M_0\backslash A) - (Y \cup e)$ is independent and does not span e. Hence $r_{M_0\backslash A}(E(M_0\backslash A) - Y) = 2(k-2) + 1$, so

$$
\begin{aligned}
r^*_{M_0\backslash A}(Y) &= |Y| + r_{M_0\backslash A}(E(M_0\backslash A) - Y) - r(M_0\backslash A) \\
&= 4 + 2(k-2) + 1 - 2(k-1) \\
&= 3.
\end{aligned}
$$

Thus (7.1.7) holds.

Consider M_0^*. It has rank $k + 3$ since $M_0^*/e = (M_0\backslash e)^* \cong M(K_{3,k})$. Now $M(K_{3,k})$ has $T_1, T_2, \ldots, T_k$ as triads and, by orthogonality, has A as an independent set. Thus $r_{M_0^*}(A) = k$, so $r(M_0^*/A) = r(M_0^*) - k = 3$.

By (7.1.7), the lines of M_0^*/A spanned by $\{\beta_i, \gamma_i\}$ and $\{\beta_j, \gamma_j\}$ are distinct for all distinct i and j in $[k]$. Thus the k lines spanned by $\{\beta_1, \gamma_1\}, \{\beta_2, \gamma_2\}, \ldots, \{\beta_k, \gamma_k\}$ are all distinct.

Next observe that

7.1.8. M_0^*/A *has at least* $(2k)^{1/2} + \frac{1}{2}$ *distinct parallel classes.*

Suppose that there are exactly p points in the simplification of the rank-3 matroid M_0^*/A. The number of distinct lines determined by these points is at most $\binom{p}{2}$, that is, $\frac{1}{2}(p^2 - p)$. But M_0^*/A has at least k distinct lines, so $k \leq \frac{1}{2}(p^2 - p)$. Thus $2k + \frac{1}{4} \leq p^2 - p + \frac{1}{4}$. Hence $2k + \frac{1}{4} \leq (p - \frac{1}{2})^2$, and (7.1.8) holds.

By Theorem 2.2, as $r(M_0^*/A) = 3$ and M_0^*/A has no $U_{2,m+2}$-minor, M_0^*/A has at most $\frac{m^3-1}{m-1}$ distinct parallel classes. Thus

$$
(2k)^{1/2} + \tfrac{1}{2} \leq \frac{m^3 - 1}{m - 1} = m^2 + m + 1,
$$

so

$$
(2k)^{1/2} \leq m^2 + m + \tfrac{1}{2} < (m + 2^{-1/2})^2.
$$

Hence

$$
(2k)^{1/4} - 2^{-1/2} < m.
$$

But $k \geq m^4$, so $2^{1/4}m - 2^{-1/2} < m$ and it follows that $m < 2^{-1/2}(2^{-1/4} - 1)^{-1} < 4$. This contradiction completes the proof of the theorem. $\qquad\square$

8. The Main Theorem

In this section, we prove the main result. Recall that $\mathcal{T}_n$ denotes the set of rank-n spikes having a tip and a cotip. We let $\mathcal{S}_n$ denote the set of rank-n spikes having neither a tip nor a cotip.

Theorem 8.1. *Let M be a 3-connected matroid, and let e be an element of M. For every integer $n > 2$, there is an integer $g(n)$ so that if $|E(M)| \geq g(n)$, then e is an element of a minor of $E(M)$ that is isomorphic to the rank-n wheel or whirl, the cycle or bond matroid of $K_{1,1,1,n}$, $M(K_{3,n})^+$ or its dual, $U_{2,n}$ or $U_{n-2,n}$, or a member of $\mathcal{T}_n$.*

Proof. By Theorem 1.1, there is a function f so that if $|E(M)| \geq f(n^{10})$, then M has a minor isomorphic to a member of the set $\mathcal{M} = \mathcal{S}_{n^{10}} \cup \{M(\mathcal{W}_{n^{10}}), \mathcal{W}^{n^{10}}, M(K_{3,n^{10}}), M^*(K_{3,n^{10}}), U_{2,n^{10}}, U_{n^{10}-2,n^{10}}\}$. By Theorem 2.1, M has a connected minor N using e so that N/e or $N\backslash e$ is a member of $\mathcal{M}$.

If N has a U_{2,n^2}-minor, then, by Theorem 3.1, as $n^2 \geq 3$ by assumption, there is an $m \geq n$ so that N has a $U_{2,m}$-minor containing e. Dually, if N has a U_{n^2-2,n^2}-minor, then N has a $U_{n-2,n}$-minor containing e. Therefore, we will assume that M has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2}.

Consider the case when $N/e \in \mathcal{M}$. Then, since $\mathcal{M}$ is closed under duality, $N^*\backslash e \in \mathcal{M}$. In the theorem statement, the list of potential minors of M containing e is also closed under duality, so we may assume that $N\backslash e \in \mathcal{M}$. As N has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2}, we deduce that $N\backslash e$ is a member of $\mathcal{S}_{n^{10}} \cup \{M(\mathcal{W}_{n^{10}}), \mathcal{W}^{n^{10}}, M(K_{3,n^{10}}), M^*(K_{3,n^{10}})\}$.

Suppose first that $N\backslash e \in \mathcal{S}_{n^{10}}$. Choose an element x of $E(N) - e$ that is not parallel to e in N. Then $N/x\backslash e$ is a rank-$(n^{10} - 1)$ spike with a tip and no cotip. Hence N/x is connected. Let y be an element of $N/x\backslash e$ other than the tip. Then $N/x\backslash y\backslash e \in \mathcal{T}_{n^{10}-1}$, and $N/x\backslash y$ is connected. By Theorem 4.2, as $n^{10} - 1 \geq 6$ by assumption, there is an $m \geq \frac{n^{10}-1}{2}$ so that $N/x\backslash y$ has a $\mathcal{T}_m$-minor containing e.

Next suppose that $N\backslash e \in \{M(\mathcal{W}_{n^{10}}), \mathcal{W}^{n^{10}}\}$. Then, by Theorem 2.1 and duality, we may assume that N has a connected minor N' containing e so that $N'\backslash e \in \{M(\mathcal{W}_{n^6}), \mathcal{W}^{n^6}\}$. We have assumed that N has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2}. Thus, by Theorem 5.3, there is an $m \geq n^2$ so that N' and hence N has a minor that contains e and is isomorphic to $M(\mathcal{W}_m)$ or $\mathcal{W}^m$.

Now let $N\backslash e = M(K_{3,n^{10}})$. Then $N\backslash e$ is a paddle whose petals are triads. As $n^{10} \geq 4$, we can find petals P_1, P_2, and P_3 of $N\backslash e$ none of whose elements is parallel to e in N. Moreover, there are elements e_1, e_2, and e_3 of P_1, P_2, and P_3, respectively, such that $\mathrm{si}(N\backslash e/e_1, e_2) \cong M(K_{1,1,1,n^{10}-2}) \cong \mathrm{si}(N\backslash e/e_1, e_3)$. Thus N has a connected minor N' containing e so that $N'\backslash e = M(K_{1,1,1,n^{10}-2})$ unless both $\{e_1, e_2, e\}$ and $\{e_1, e_3, e\}$ are circuits of N. The exceptional case cannot arise since it implies the contradiction that $\{e_1, e_2, e_3\}$ is a circuit of $N\backslash e$. Therefore we can apply Theorem 6.1 to N' to get that there is an $m \geq \frac{n^{10}}{2}$ so that N' and hence N has a minor that contains e and is isomorphic to $M(K_{1,1,1,m}), M(K_{3,m})^+$, or some member of $\mathcal{T}_m$.

Finally, suppose that $N\backslash e = M^*(K_{3,n^{10}})$. Since N has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2} and $n^2 \geq 4$, it follows by Theorem 7.1 that there is an $m \geq n^2$ so that N has a minor that contains e and is isomorphic to $M^*(K_{1,1,1,m-1})$. $\square$

References

[1] R. E. Bixby, *l*-matrices and a characterization of non-binary matroids, *Discrete Math.* **8** (1974), 139–145.
[2] T. H. Brylawski, A decomposition for combinatorial geometries. *Trans. Amer. Math. Soc.* **171** (1972), 235–282.
[3] D. Chun and J. Oxley, Capturing two elements in unavoidable minors of 3-connected binary matroids, submitted. http://www.math.lsu.edu/~oxley/dbcjgo.pdf
[4] G. Ding, B. Oporowski, J. Oxley, and D. Vertigan, Unavoidable minors of large 3-connected binary matroids, *J. Combin. Theory Ser. B* **66** (1996), 334–360.
[5] G. Ding, B. Oporowski, J. Oxley, and D. Vertigan, Unavoidable minors of large 3-connected matroids, *J. Combin. Theory Ser. B* **71** (1997), 244–293.
[6] J. Geelen and G. Whittle, Inequivalent representations of matroids over prime fields, submitted. http://arxiv.org/abs/1101.4683
[7] J. P. S. Kung, Extremal matroid theory, *Graph Structure Theory*, N. Robertson and P. D. Seymour, eds., Amer. Math. Soc., Providence, Rhode Island, 1993, 21–62.

[8] M. Lemos and J. Oxley, A sharp bound on the size of a connected matroid, *Trans. Amer. Math. Soc.* **353** (2001), 4039–4056.

[9] B. Oporowski, J. Oxley, and R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory Ser. B* **57** (1993), 239–257.

[10] J. Oxley, *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.

[11] J. Oxley, C. Semple, and G. Whittle, The structure of the 3-separations of 3-connected matroids, *J. Combin. Theory Ser. B* **92** (2004), 257-293.

[12] P. D. Seymour, A note on the production of matroid minors, *J. Combin. Theory Ser. B* **22** (1977), 289–295.

[13] P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.

[14] P. D. Seymour, On minors of non-binary matroids, *Combinatorica* **1** (1981), 387–394.

[15] P. D. Seymour, Minors of 3-connected matroids, *European J. Combin.* **6** (1985), 375–382.

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