ON THE HIGHLY CONNECTED DYADIC, NEAR-REGULAR, AND SIXTH-ROOT-OF-UNITY MATROIDS

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ABSTRACT. Subject to announced results by Geelen, Gerards, and Whittle, we completely characterize the highly connected members of the classes of dyadic, near-regular, and sixth-root-of-unity matroids.

1. INTRODUCTION

We give the definitions of the classes of dyadic, signed-graphic, near-regular, and $\sqrt[6]{1}$-matroids in Section 2; however, unexplained notation and terminology in this paper will generally follow Oxley [15]. One exception is that we denote the vector matroid of a matrix A by $M(A)$ rather than $M[A]$. A matroid M is *vertically k-connected* if, for every set $X \subseteq E(M)$ with $r(X) + r(E - X) - r(M) < k$, either X or $E - X$ is spanning. If M is vertically k-connected, then its dual M^* is *cyclically k-connected*. The matroid U_r is obtained from $M(K_{r+1})$ by adding three specific points to a rank-3 flat; we give the precise definition in Section 4.

Due to the technical nature of Hypotheses 3.1 and 3.2, we delay their statements to Section 3. Subject to these hypotheses, we characterize the highly connected dyadic matroids by proving the following.

Theorem 1.1. *Suppose Hypothesis 3.1 holds. Then there exists $k \in \mathbb{Z}_+$ such that, if M is a k-connected dyadic matroid with at least $2k$ elements, then one of the following holds.*

(1) *Either M or M^* is a signed-graphic matroid.*
(2) *Either M or M^* is a matroid of rank r that is a restriction of U_r.*

Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, if M is a simple, vertically k-connected, dyadic matroid with an $M(K_n)$-minor, then either M is a signed-graphic matroid or M is a restriction of $U_{r(M)}$.

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Let $\mathrm{AG}(2,3)\backslash e$ be the matroid resulting from $\mathrm{AG}(2,3)$ by deleting one point; this matroid is unique up to isomorphism. We prove the following excluded-minor characterization of the highly connected dyadic matroids.

Theorem 1.2. *Suppose Hypothesis 3.1 holds. There exists $k \in \mathbb{Z}_+$ such that, for a k-connected matroid M with at least $2k$ elements, M is dyadic if and only if M is a ternary matroid with no minor isomorphic to either $\mathrm{AG}(2,3)\backslash e$ or $(\mathrm{AG}(2,3)\backslash e)^*$. Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, for a vertically k-connected matroid M with an $M(K_n)$-minor, M is dyadic if and only if M is a ternary matroid with no minor isomorphic to $\mathrm{AG}(2,3)\backslash e$.*

Our final main result characterizes the highly connected near-regular and $\sqrt[6]{1}$-matroids. The matroid T_r^1 is obtained from the complete graphic matroid $M(K_{r+2})$ by adding a point freely to a triangle, contracting that point, and simplifying. We denote the non-Fano matroid by F_7^-.

Theorem 1.3. *Suppose Hypothesis 3.1 holds. There exists $k \in \mathbb{Z}_+$ such that, if M is a k-connected matroid with at least $2k$ elements, the following are equivalent.*

(1) *M is a near-regular matroid,*
(2) *M is a $\sqrt[6]{1}$-matroid,*
(3) *M or M^* is a matroid of rank r that is a restriction of T_r^1, and*
(4) *M is a ternary matroid that has no minor isomorphic to F_7^- or $(F_7^-)^*$.*

Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, if M is a simple, vertically k-connected matroid with an $M(K_n)$-minor, then (1) and (2) are equivalent to each other and also to the following conditions.

(3') *M is a restriction of $T_{r(M)}^1$, and*
(4') *M is a ternary matroid that has no minor isomorphic to F_7^-.*

Theorem 1.3 leads to the following result.

Corollary 1.4. *Suppose Hypothesis 3.1 holds. There exists $k \in \mathbb{Z}_+$ such that, if M is a ternary k-connected matroid with at least $2k$ elements, then M is representable over some field of characteristic other than 3 if and only if M is dyadic. Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, if M is a simple, ternary, vertically k-connected matroid with an $M(K_n)$-minor, then M is representable over some field of characteristic other than 3 if and only if M is dyadic.*

Proof. Whittle [22, Theorem 5.1] showed that a 3-connected ternary matroid that is representable over some field of characteristic other than 3 is either a dyadic matroid or a $\sqrt[6]{1}$-matroid. Therefore, since near-regular matroids are dyadic, Theorem 1.3 immediately implies the first statement in the corollary.

The second statement is proved similarly but also requires the fact that a simple, vertically 3-connected matroid is 3-connected. ■

Hypotheses 3.1 and 3.2 are believed to be true, but their proofs are still forthcoming in future papers by Geelen, Gerards, and Whittle. They are modified versions of a hypothesis given by Geelen, Gerards, and Whittle in [5]. The results announced in [5] rely on the Matroid Structure Theorem by these same authors [4]. We refer the reader to [9] for more details.

Some proofs in this paper involved case checks aided by Version 8.6 of the SageMath software system [18], in particular making use of the *matroids* component [17]. The authors used the CoCalc (formerly SageMathCloud) online interface.

In Section 2, we give some background information about the classes of matroids studied in this paper. In Section 3, we recall results from [7] that will be used to prove our main results. In Section 4, we prove Theorems 1.1 and 1.2, and in Section 5, we prove Theorem 1.3.

2. Preliminaries

We begin this section by clarifying some notation and terminology that will be used throughout the rest of the paper. Let D_r be the $r \times \binom{r}{2}$ matrix such that each column is distinct and such that every column has exactly two nonzero entries—the first a 1 and the second -1. For a field $\mathbb{F}$, we denote by $\mathbb{F}_p$ the prime subfield of $\mathbb{F}$. If $\mathcal{M}$ is a class of matroids, we will denote by $\overline{\mathcal{M}}$ the closure of $\mathcal{M}$ under the taking of minors. The *weight* of a column or row vector of a matrix is its number of nonzero entries. If A is an $m \times n$ matrix and $n' \leq n$, then we call an $m \times n'$ submatrix of A a *column submatrix* of A. In the remainder of this section, we give some background information about the classes of matroids studied in this paper.

The class of *dyadic* matroids consists of those matroids representable by a matrix over $\mathbb{Q}$ such that every nonzero subdeterminant is $\pm 2^i$ for some $i \in \mathbb{Z}$. The class of *sixth-root-of-unity* matroids (or $\sqrt[6]{1}$-matroids) consists of those matroids that are representable by a matrix over $\mathbb{C}$ such that every nonzero subdeterminant is a complex sixth root of unity. Let $\mathbb{Q}(\alpha)$ be the field obtained by extending the rationals $\mathbb{Q}$ by a transcendental α. A matroid is *near-regular* if it can be represented by a matrix over $\mathbb{Q}(\alpha)$ such that every nonzero subdeterminant is contained in the set $\{\pm\alpha^i(\alpha-1)^j : i, j \in \mathbb{Z}\}$.

A matroid is *signed-graphic* if it can be represented by a matrix over GF(3) each of whose columns has at most two nonzero entries. The rows and columns of this matrix can be indexed by the set of vertices and edges, respectively, of a *signed graph*. If the nonzero entries of the column are unequal, then the corresponding edge is a positive edge joining the vertices indexing the rows containing the nonzero entries. If the column has two equal entries, the edge is negative. If the column contains only one nonzero entry, then the corresponding edge is a negative loop at the vertex indexing the row containing the nonzero entry. Every signed-graphic matroid is dyadic. (See, for example, [23, Lemma 8A.3]).

Whittle [22, Theorem 1.4] showed that the following statements are equivalent for a matroid M:

- M is near-regular
- M is representable over GF(3), GF(4), and GF(5)
- M is representable over all fields except possibly GF(2)

He also showed [22, Theorem 1.2] that the class of $\sqrt[6]{1}$-matroids consists exactly of those matroids representable over GF(3) and GF(4) and [22, Theorem 1.1] that the class of dyadic matroids consists exactly of those matroids representable over GF(3) and GF(5). Thus, the class of near-regular matroids is the intersection of the classes of $\sqrt[6]{1}$-matroids and dyadic matroids.

A geometric representation of $\mathrm{AG}(2,3)\backslash e$ is given in Figure 1. It is fairly well

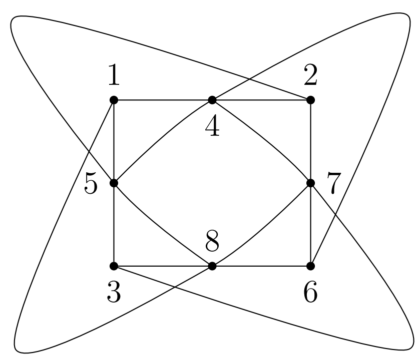

FIGURE 1. A Geometric Representation of $\mathrm{AG}(2,3)\backslash e$

known that $\mathrm{AG}(2,3)\backslash e$ is an excluded minor for the class of dyadic matroids. (See, for example, [15, Section 14.7].) We will use this fact in Section 4.

It is an open problem to determine the complete list of excluded minors for the dyadic matroids; however, the excluded minors for the classes of $\sqrt[6]{1}$-matroids and near-regular matroids have been determined. Geelen, Gerards, and Kapoor [3, Corollary 1.4] showed that the excluded minors for the class of $\sqrt[6]{1}$-matroids are $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, F_7^-, $(F_7^-)^*$, and P_8. (We refer the reader to [3] or [15] for the definitions of these matroids.) Hall, Mayhew, and Van Zwam [10, Theorem 1.2], based on unpublished work by Geelen, showed that the excluded minors for the class of near-regular matroids are $U_{2,5}$, $U_{3,5}$, F_7,

F_7^*, F_7^-, $(F_7^-)^*$, $\mathrm{AG}(2,3)\backslash e$, $(\mathrm{AG}(2,3)\backslash e)^*$, $\Delta_T(\mathrm{AG}(2,3)\backslash e)$, and P_8. Here, $\Delta_T(\mathrm{AG}(2,3)\backslash e)$ is the result of performing a Δ-Y operation on $\mathrm{AG}(2,3)\backslash e$.

If $r \neq 3$, it follows from results of Kung [11, Theorem 1.1] and Kung and Oxley [12, Theorem 1.1] that the largest simple dyadic matroid of rank r is the rank-r ternary Dowling geometry, which is a signed-graphic matroid. Again, suppose $r \neq 3$. Then Oxley, Vertigan, and Whittle [16, Theorem 2.1, Corollary 2.2] showed that T_r^1 is the largest simple $\sqrt[6]{1}$-matroid of rank r and the largest simple near-regular matroid of rank r. We remark without proof that our main results here, combined with [9, Lemmas 4.14, 4.16], show that Hypothesis 3.2 agrees with these known results.

3. Frame Templates

The notion of frame templates was introduced by Geelen, Gerards, and Whittle in [5] to describe the structure of the highly connected members of minor-closed classes of matroids representable over a fixed finite field. Frame templates have been studied further in [8, 14, 9, 7]. In this section, we give several results proved in those papers that we will need to prove the main results in this paper. The results in [5] technically deal with represented matroids—which can be thought of as fixed representation matrices for matroids. However, since we only deal with ternary matroids in this paper, and since ternary matroids are uniquely GF(3)-representable [1], we will state the results in terms of matroids rather than represented matroids.

If $\mathbb{F}$ is a field, let $\mathbb{F}^\times$ denote the multiplicative group of $\mathbb{F}$, and let Γ be a subgroup of $\mathbb{F}^\times$. A Γ-frame matrix is a frame matrix A such that:

- Each column of A with a nonzero entry contains a 1.
- If a column of A has a second nonzero entry, then that entry is $-\gamma$ for some $\gamma \in \Gamma$.

If $\Gamma = \{1\}$, then the vector matroid of a Γ-frame matrix is a graphic matroid. For this reason, we will call the columns of a $\{1\}$-frame matrix *graphic columns*.

A *frame template* over a field $\mathbb{F}$ is a tuple $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ such that the following hold.

(i) Γ is a subgroup of $\mathbb{F}^\times$.
(ii) C, X, Y_0 and Y_1 are disjoint finite sets.
(iii) $A_1 \in \mathbb{F}^{X \times (C \cup Y_0 \cup Y_1)}$.
(iv) Λ is a subgroup of the additive group of $\mathbb{F}^X$ and is closed under scaling by elements of Γ.
(v) Δ is a subgroup of the additive group of $\mathbb{F}^{C \cup Y_0 \cup Y_1}$ and is closed under scaling by elements of Γ.

Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template. Let B and E be finite sets, and let $A' \in \mathbb{F}^{B\times E}$. We say that A' *respects*[1] Φ if the following hold:

(i) $X \subseteq B$ and $C, Y_0, Y_1 \subseteq E$.
(ii) $A'[X, C \cup Y_0 \cup Y_1] = A_1$.
(iii) There exists a set $Z \subseteq E - (C \cup Y_0 \cup Y_1)$ such that $A'[X, Z] = 0$, each column of $A'[B - X, Z]$ is a unit vector or zero vector, and $A'[B - X, E - (C \cup Y_0 \cup Y_1 \cup Z)]$ is a Γ-frame matrix.
(iv) Each column of $A'[X, E - (C \cup Y_0 \cup Y_1 \cup Z)]$ is contained in Λ.
(v) Each row of $A'[B - X, C \cup Y_0 \cup Y_1]$ is contained in Δ.

The structure of A' is shown below.

		Z	Y_0 Y_1 C
X	columns from Λ	0	A_1
	Γ-frame matrix	unit or zero columns	rows from Δ

Now, suppose that A' respects Φ and that $A \in \mathbb{F}^{B\times E}$ satisfies the following conditions:

(i) $A[B, E - Z] = A'[B, E - Z]$
(ii) For each $i \in Z$ there exists $j \in Y_1$ such that the i-th column of A is the sum of the i-th and the j-th columns of A'.

We say that such a matrix A *conforms*[1] to Φ.

Let M be an $\mathbb{F}$-representable matroid. We say that M *conforms*[1] to Φ if there is a matrix A conforming to Φ such that M is isomorphic to $M(A)/C\backslash Y_1$. We denote by $\mathcal{M}(\Phi)$ the set of matroids that conform to Φ. If M^* conforms to a template Φ, we say that M *coconforms* to Φ. We denote by $\mathcal{M}^*(\Phi)$ the set of matroids that coconform to Φ.

We now state the hypotheses on which the main results are based. As stated in Section 1, they are modified versions of a hypothesis given by Geelen, Gerards, and Whittle in [5], and their proofs are forthcoming. In their current forms, these hypotheses were stated in [9].

Hypothesis 3.1 ([9, Hypothesis 4.3])**.** *Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids. Then there exist $k \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that*

(1) *$\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$,*
(2) *$\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$, and*

[1]For simplicity, we will use the terms *respecting* and *conforming* to mean what was called *virtual respecting* and *virtual conforming* in [8] and [7]. The distinction between conforming and virtually conforming is explained in [8]. We can do this since every matroid virtually conforming to a template is a minor of a matroid conforming to a template [8, Lemma 3.4].

(3) *if M is a simple k-connected member of $\mathcal{M}$ with at least $2k$ elements and $\widetilde{M}$ has no $\mathrm{PG}(m-1,\mathbb{F}_p)$-minor, then either M is a member of at least one of the classes $\mathcal{M}(\Phi_1),\dots,\mathcal{M}(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}(\Psi_1),\dots,\mathcal{M}(\Psi_t)$.*

Hypothesis 3.2 ([9, Hypothesis 4.6])**.** *Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids. Then there exist $k,n\in\mathbb{Z}_+$ and frame templates $\Phi_1,\dots,\Phi_s,\Psi_1,\dots,\Psi_t$ such that*

(1) *$\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1),\dots,\mathcal{M}(\Phi_s)$,*
(2) *$\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1),\dots,\mathcal{M}(\Psi_t)$,*
(3) *if M is a simple vertically k-connected member of $\mathcal{M}$ with an $M(K_n)$-minor but no $\mathrm{PG}(m-1,\mathbb{F}_p)$-minor, then M is a member of at least one of the classes $\mathcal{M}(\Phi_1),\dots,\mathcal{M}(\Phi_s)$, and*
(4) *if M is a cosimple cyclically k-connected member of $\mathcal{M}$ with an $M^*(K_n)$-minor but no $\mathrm{PG}(m-1,\mathbb{F}_p)$-minor, then M^* is a member of at least one of the classes $\mathcal{M}(\Psi_1),\dots,\mathcal{M}(\Psi_t)$.*

If Φ and Φ' are frame templates, it is possible that $\mathcal{M}(\Phi)=\mathcal{M}(\Phi')$ even though Φ and Φ' look very different.

Definition 3.3 ([7, Definition 6.3])**.** Let Φ and Φ' be frame templates over a field $\mathbb{F}$, then the pair Φ,Φ' are *strongly equivalent* if $\mathcal{M}(\Phi)=\mathcal{M}(\Phi')$. The pair Φ,Φ' are *minor equivalent* if $\overline{\mathcal{M}(\Phi)}=\overline{\mathcal{M}(\Phi')}$.

There are other notions of template equivalence (namely equivalence, algebraic equivalence, and semi-strong equivalence) given in [7], but all of these imply minor equivalence.

If $\mathbb{F}$ is a field and E is a set, we say that two subgroups U and W of the additive subgroup of the vector space $\mathbb{F}^E$ are *skew* if $U\cap W=\{0\}$. Nelson and Walsh [14] gave Definition 3.4 below.

Definition 3.4. A frame template $\Phi=(\Gamma,C,X,Y_0,Y_1,A_1,\Delta,\Lambda)$ is *reduced* if there is a partition (X_0,X_1) of X such that

- $\Delta=\Gamma(\mathbb{F}_p^C\times\Delta')$ for some additive subgroup Δ' of $\mathbb{F}^{Y_0\cup Y_1}$,
- $\mathbb{F}_p^{X_0}\subseteq\Lambda|X_0$ while $\Lambda|X_1=\{0\}$ and $A_1[X_1,C]=0$, and
- the rows of $A_1[X_1,C\cup Y_0\cup Y_1]$ form a basis for a subspace whose additive group is skew to Δ.

We will refer to the partition $X=X_0\cup X_1$ given in Definition 3.4 as the *reduction partition* of Φ.

The following definition and theorem are found in [9].

Definition 3.5 ([9, Definition 5.3])**.** Let $\Phi=(\Gamma,C,X,Y_0,Y_1,A_1,\Delta,\Lambda)$ be a reduced frame template , with reduction partition $X=X_0\cup X_1$. If Y_1 spans the matroid $M(A_1[X_1,Y_0\cup Y_1])$, then Φ is *refined.*

Theorem 3.6 ([9, Theorem 5.6])**.** *If Hypothesis 3.1 holds for a class $\mathcal{M}$, then the constant k and the templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ can be chosen so that the templates are refined. Moreover, if Hypothesis 3.2 holds for a class $\mathcal{M}$, then the constants k, n, and the templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ can be chosen so that the templates are refined.*

A few specific templates have been given names. We list some of those now, specifically for the ternary case. Note that $\mathcal{M}(\Phi_2)$ is the class of signed-graphic matroids.

Definition 3.7 ([7, Definition 7.8, ternary case])**.**

- Φ_2 is the template with all sets empty and all groups trivial except that $\Gamma = \{\pm 1\}$.
- Φ_C is the template with all groups trivial and all sets empty except that $|C| = 1$ and $\Delta \cong \mathbb{Z}/3\mathbb{Z}$.
- Φ_X is the template with all groups trivial and all sets empty except that $|X| = 1$ and $\Lambda \cong \mathbb{Z}/3\mathbb{Z}$.
- Φ_{Y_0} is the template with all groups trivial and all sets empty except that $|Y_0| = 1$ and $\Delta \cong \mathbb{Z}/3\mathbb{Z}$.
- Φ_{CX} is the template with $Y_0 = Y_1 = \emptyset$, with $|C| = |X| = 1$, with $\Delta \cong \Lambda \cong \mathbb{Z}/3\mathbb{Z}$, with Γ trivial, and with $A_1 = [1]$.
- Φ_{CX2} is the template with $Y_0 = Y_1 = \emptyset$, with $|C| = |X| = 1$, with $\Delta \cong \Lambda \cong \mathbb{Z}/3\mathbb{Z}$, with Γ trivial, and with $A_1 = [-1]$.

The next lemma follows directly from [7, Lemma 7.9].

Lemma 3.8. *The following are true: $\overline{\mathcal{M}(\Phi_{Y_0})} \subseteq \overline{\mathcal{M}(\Phi_C)}$, and $\overline{\mathcal{M}(\Phi_X)} \subseteq \overline{\mathcal{M}(\Phi_{CX})}$, and $\overline{\mathcal{M}(\Phi_X)} \subseteq \overline{\mathcal{M}(\Phi_{CX2})}$.*

Frame templates where the groups Γ, Λ, and Δ are trivial are studied extensively in [7].

Definition 3.9 ([7, Definitions 6.9–6.10])**.** A *Y-template* is a refined frame template with all groups trivial (so $C = X_0 = \emptyset$). If A_1 has the form below, then $\mathrm{YT}(P_0, P_1)$ is defined to be the Y-template $(\{1\}, \emptyset, X, Y_0, Y_1, A_1, \{0\}, \{0\})$.

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y_1} & \multicolumn{1}{c}{Y_0} \\ \hline \multicolumn{1}{|c|}{I_{|X|}} & P_1 & P_0 \\ \hline \end{array}$$

The next lemma follows from [7, Lemma 7.16–17].

Lemma 3.10. *Let Φ be a frame template such that $\overline{\mathcal{M}(\Phi')} \not\subseteq \overline{\mathcal{M}(\Phi)}$ for each template $\Phi' \in \{\{\Phi_X, \Phi_C, \Phi_{Y_0}, \Phi_{CX}, \Phi_{CX2}\}$. Then Φ is minor equivalent to a template $(\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ with $C = \emptyset$ and with Λ and Δ both trivial.*

The next several results and definitions are found in [7].

Lemma 3.11 ([7, Theorem 7.18])**.** *Let Φ be a refined ternary frame template. Then either $\overline{\mathcal{M}(\Phi')} \subseteq \overline{\mathcal{M}(\Phi)}$ for some $\Phi' \in \{\Phi_X, \Phi_C, \Phi_{Y_0}, \Phi_{CX}, \Phi_{CX2}, \Phi_2\}$, or Φ is a Y-template.*

Definition 3.12 ([7, Definition 9.3])**.** Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a refined frame template with reduction partition $X = X_0 \cup X_1$, with $A_1[X_0, Y_1]$ a zero matrix, and with $A_1[X_1, Y_1]$ an identity matrix. Then Φ is a *lifted* template.

Lemma 3.13 ([7, Lemma 9.6])**.** *Every refined frame template is minor equivalent to a lifted template.*

Definition 3.14 ([7, Definition 9.7])**.** A Y-template $\mathrm{YT}(P_0, P_1)$ is *complete* if P_0 contains $D_{|X|}$ as a submatrix.

Definition 3.15 ([7, Definition 9.12])**.** The Y-template $\mathrm{YT}([P_0|D_{|X|}], [\emptyset])$ is the complete, lifted Y-template *determined* by P_0 and is denoted by Φ_{P_0}.

Definition 3.16 ([7, Definition 6.12])**.** Let Φ_{P_0} be a complete, lifted Y-template. The *rank-r universal matroid* for Φ_{P_0} is the matroid represented by the following matrix.

$$\left[\begin{array}{c|c|c} I_r & D_r & \begin{array}{c} P_0 \\ \hline 0 \end{array} \end{array}\right]$$

It is shown in [7, Section 9] that every matroid conforming to Φ_{P_0} is a restriction of some universal matroid for Φ_{P_0}.

We refer to [15, Section 11.4] for the definition of generalized parallel connections of matroids.

Lemma 3.17 ([7, Lemma 9.13])**.** *The rank-r universal matroid of Φ_{P_0} is the generalized parallel connection of $M(K_{r+1})$ and $M([I_m|D_m|P_0])$ along $M(K_{m+1})$, where m is the number of rows of P_0.*

Combining [7, Lemma 9.6], [7, Lemma 9.9], and [7, Lemma 9.14], we obtain the following.

Lemma 3.18.

(i) *Every Y-template is minor equivalent to the complete, lifted Y-template determined by a matrix the sum of whose rows is the zero vector.*
(ii) *Conversely, let Φ be the complete, lifted Y-template determined by a matrix P_0 the sum of whose rows is the zero vector. Choose any one row of P_0. Then Φ is minor equivalent to the complete, lifted Y-template determined by the matrix obtained from P_0 by removing that row.*

The next lemma is an easy but useful result.

Lemma 3.19. *Let P_0' be a matrix with a submatrix P_0. Every matroid conforming to Φ_{P_0} is a minor of a matroid conforming to $\Phi_{P_0'}$.*

We use the next lemma to prove the excluded minor characterizations in Theorems 1.2 and 1.3; it is obtained by combining Lemmas 8.1 and 8.2 of [7].

Lemma 3.20. *Let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids, where $\mathbb{F}_p$ is the prime subfield of $\mathbb{F}$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two sets of $\mathbb{F}$-representable matroids such that*

(i) *no member of $\mathcal{E}_1 \cup \mathcal{E}_2$ is contained in $\mathcal{M}$,*
(ii) *some member of $\mathcal{E}_1 \cup \mathcal{E}_2$ is $\mathbb{F}_p$-representable,*
(iii) *for every refined frame template Φ over $\mathbb{F}$ such that $\mathcal{M}(\Phi) \not\subseteq \mathcal{M}$, there is a member of $\mathcal{E}_1$ that is a minor of a matroid conforming to Φ, and*
(iv) *for every refined frame template Ψ over $\mathbb{F}$ such that $\mathcal{M}^*(\Psi) \not\subseteq \mathcal{M}$, there is a member of $\mathcal{E}_2$ that is a minor of a matroid coconforming to Ψ.*

Suppose Hypothesis 3.1 holds; there exists $k \in \mathbb{Z}_+$ such that a k-connected $\mathbb{F}$-representable matroid with at least $2k$ elements is contained in $\mathcal{M}$ if and only if it contains no minor isomorphic to one of the matroids in the set $\mathcal{E}_1 \cup \mathcal{E}_2$.

Moreover, suppose Hypothesis 3.2 holds; there exist $k, n \in \mathbb{Z}_+$ such that a vertically k-connected $\mathbb{F}$-representable matroid with an $M(K_n)$-minor is contained in $\mathcal{M}$ if and only if it contains no minor isomorphic to one of the matroids in $\mathcal{E}_1$ and such that a cyclically k-connected $\mathbb{F}$-representable matroid with an $M^*(K_n)$-minor is contained in $\mathcal{M}$ if and only if it contains no minor isomorphic to one of the matroids in $\mathcal{E}_2$.

The next lemma has not appeared previously, but it will be useful in Section 4. Recall from Definition 3.12 that every lifted template is refined and therefore reduced. Thus, a lifted template has a reduction partition as in Definition 3.4.

Lemma 3.21. *Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a lifted template with reduction partition $X = X_0 \cup X_1$. Let $P_0 = A_1[X_1, Y_0]$, and let S be any Γ-frame matrix with $|X_1|$ rows. Then $M([S|P_0]) \in \overline{\mathcal{M}(\Phi)}$.*

Proof. Let $R = A_1[X_0, Y_0 \cup C]$. Note that the following matrix conforms to Φ, since $\mathbb{F}_p^{X_0} \subseteq \Lambda|X_0$ in a reduced template and since the zero vector is an element of Λ and Δ.

$$
\begin{array}{c|c|c|c|c|c|}
 & H_1 & H_2 & Z & Y_0 & C \\
\hline
X_0 & I_{|X_0|} & 0 & 0 & \multicolumn{2}{c|}{R} \\
\hline
X_1 & 0 & 0 & I_{|X_1|} & P_0 & 0 \\
\hline
 & 0 & S & I_{|X_1|} & 0 & 0 \\
\hline
\end{array}
$$

By contracting $H_1 \cup Z \cup C$ (recalling that C must be contracted to obtain a matroid that conforms to Φ), we obtain the desired matroid. ■

4. Dyadic Matroids

In this section, we characterize the highly connected dyadic matroids by proving Theorems 1.1 and 1.2. First, we will need to define the family of matroids U_r and to prove several lemmas that build on the machinery of Section 3.

Consider the rank-3 ternary Dowling geometry $Q_3(\mathrm{GF}(3)^\times)$. This matroid contains a restriction isomorphic to $M(K_4)$, with the signed-graphic representation given in Figure 2, with negative edges printed in bold. This represen-

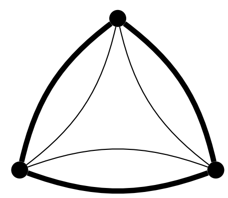

Figure 2. A signed-graphic representation of $M(K_4)$

tation of $M(K_4)$ has been encountered before, for example in [24, 6, 21, 9]. The Dowling geometry $Q_3(\mathrm{GF}(3)^\times)$ can also be represented by the matrix $[I_3|D_3|T]$, where

$$T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

with the columns of T representing the joints of the Dowling geometry.

For $r \geq 3$, we define the matroid U_r to be the generalized parallel connection of the complete graphic matroid $M(K_{r+1})$ with $Q_3(\mathrm{GF}(3)^\times)$ along a common restriction isomorphic $M(K_4)$, where the restriction of $Q_3(\mathrm{GF}(3)^\times)$ has the signed-graphic representation given in Figure 2. (The restriction isomorphic to $M(K_4)$ is a modular flat of $M(K_{r+1})$, which is uniquely representable over any field. Therefore, this generalized parallel connection is well-defined.) By Lemma 3.17, U_r is the rank-r universal matroid for the complete, lifted Y-template Φ_T and therefore has the following representation matrix.

$$\left[\begin{array}{c|c|c} I_r & D_r & \begin{array}{c} T \\ \hline 0 \end{array} \end{array}\right]$$

Lemma 4.1. *The matroid U_r is dyadic for every $r \geq 3$.*

Proof. Since U_3 is the ternary rank-3 Dowling geometry, it is signed-graphic and therefore dyadic. Since U_r is the generalized parallel connection of U_3 and the complete graphic matroid $M(K_{r+1})$ along a common restriction isomorphic to $M(K_4)$, and since every complete graphic matroid is uniquely representable over every field, a result of Brylawski [2, Theorem 6.12] implies that U_r is representable over both GF(3) and GF(5) and therefore dyadic. (The reader familiar with partial fields may also apply [13, Theorem 3.1] to the dyadic partial field.) ■

We remark that, for $r \geq 4$, the matroid U_r is not signed-graphic; this can be easily checked by using SageMath to show that U_4 is not a restriction of the rank-4 ternary Dowling geometry.

Lemma 4.2. *If* $\Phi \in \{\Phi_X, \Phi_C, \Phi_{Y_0}\ \Phi_{CX}, \Phi_{CX2}\}$, *then* $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi)}$.

Proof. Note that the following ternary matrix conforms to Φ_{Y_0}.

$$\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}\\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1\\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1
\end{bmatrix}
\end{array}$$

By contracting 9, pivoting on the first entry, we obtain the following representation of $\mathrm{AG}(2,3)\backslash e$ (with column labels matching the labels in Figure 1).

$$\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}\\
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & -1 & 0 & 1 & -1 & 1\\
0 & 0 & 1 & 0 & -1 & 1 & 1 & -1
\end{bmatrix}
\end{array}$$

Therefore, $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi_{Y_0})}$. Moreover, by Lemma 3.8, we also have $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi_C)}$.

Now, note that the following ternary matrix is a representation of $\mathrm{AG}(2,3)\backslash e$. Also note that the matrix conforms to Φ_X, with the top row indexed by X and the bottom two rows forming a $\{1\}$-frame matrix.

$$\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}\\
\begin{bmatrix}
0 & 0 & -1 & 0 & 1 & 1 & -1 & 1\\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1\\
0 & 1 & 0 & -1 & 0 & 1 & 1 & -1
\end{bmatrix}
\end{array} \tag{4.1}$$

Therefore, $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi_X)}$. Moreover, by Lemma 3.8, we also have $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi_{CX})}$ and $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi_{CX2})}$. ■

Lemma 4.3. *Suppose* $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ *is a refined ternary frame template such that* $\mathrm{AG}(2,3)\backslash e \notin \overline{\mathcal{M}(\Phi)}$. *Then either* $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi_2)$, *or* Φ *is a* Y*-template.*

Proof. Combining Lemmas 4.2 and 3.10, we see that $C = \emptyset$ and that Λ and Δ are both trivial. Since Φ is refined, it is reduced, with reduction partition $X = X_0 \cup X_1$. However, the fact that Λ is trivial implies that $X_0 = \emptyset$. By Lemma 3.13, we may assume that Φ is lifted. Therefore, we may assume that A_1 is of the form $[I|P_0]$, with Y_1 indexing the columns of the identity matrix and Y_0 indexing the columns of P_0.

The only proper subgroup of the multiplicative group of GF(3) is the trivial group $\{1\}$. If Γ is the trivial group $\{1\}$, then Φ is a Y-template. Thus, we may assume that Γ is the entire multiplicative group of GF(3).

Suppose that P_0 contains a column with at least three nonzero entries. Then by column scaling and permuting of rows, we may assume that P_0 contains either $[1,1,1]^T$ or $[1,1,-1]^T$ as a submatrix. Call this submatrix P_0', and let $\Phi' = (\Gamma, \emptyset, X', \{y\}, Y_1', A_1', \{0\}, \{0\})$, where X' and $\{y\}$ index the rows and column of P_0', respectively, and where $A_1' = [I|P_0']$, with Y_1' indexing the columns of the identity matrix. It is not difficult to see that every matroid conforming to Φ' is a minor of some matroid conforming to $(\Gamma, \emptyset, X, Y_0, Y_1, [I|P_0], \{0\}, \{0\})$, which is Φ. Thus, $\mathrm{AG}(2,3)\backslash e$ is not a minor of any matroid conforming to Φ'.

However, if $P_0 = [1,1,-1]^T$, then the representation of $\mathrm{AG}(2,3)\backslash e$ given in Matrix (4.1) is of the form $[S|P_0']$, where S is a Γ-frame matrix. Thus, by Lemma 3.21, $\mathrm{AG}(2,3)\backslash e$ is a minor of a matroid conforming to Φ'. Similarly, if $P_0' = [1,1,1]^T$, we may take Matrix (4.1) and scale by -1 the last row and the columns indexed by 2 and 7. We see that $\mathrm{AG}(2,3)\backslash e$ can be represented by a matrix of the form $[S|P_0']$, where S is a Γ-frame matrix. Again, Lemma 3.21 implies that $\mathrm{AG}(2,3)\backslash e$ is a minor of a matroid conforming to Φ'.

Therefore, we deduce that every column of P_0 has at most two nonzero entries, implying that every column of every matrix conforming to Φ has at most two nonzero entries. Thus, $\mathcal{M}(\Phi)$ consists entirely of signed-graphic matroids, and $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi_2)$. ■

Lemma 4.4. *Let Φ be the complete, lifted Y-template determined by some matrix P_0. If P_0 contains as a submatrix some matrix listed in Table 1, then $\mathrm{AG}(2,3)\backslash e \in \overline{\mathcal{M}(\Phi)}$.*

Proof. Let $M = M([I|D|P_0'])$, where P_0' is some submatrix listed in Table 1. By Lemma 3.19, it suffices to show that $\mathrm{AG}(2,3)\backslash e$ is a minor of M. For each of the matrices A–K, the authors used SageMath to show this. The computations are expedited by contracting some subset of $E(M)$ and simplifying the resulting matroid before testing if $\mathrm{AG}(2,3)\backslash e$ is a minor. If P_0' is an $r \times c$ matrix, then $|E(M)| = \binom{r+1}{2} + c$. Label the elements of M, from left to right, as $\{0, 1, 2, \ldots, \binom{r+1}{2} + c - 1\}$. If S is the set listed in Table 1 corresponding to the matrix P_0', then the authors used SageMath to show that the simplification of

Table 1. Forbidden Matrices

Matrix	Set to Contract	Matrix	Set to Contract
$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$	$\{10\}$	$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\{15, 16\}$
$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\{0, 18, 23\}$	$D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}$	$\{0, 1, 28, 29\}$
$E = \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\{0, 16\}$	$F = \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$	$\{0, 15, 22\}$
$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}$	$\{0, 4, 22\}$	$H = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$	$\{0, 16\}$
$I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$	$\{0\}$	$J = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\{0\}$
$K = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\{0, 15\}$		

M/S contains $\mathrm{AG}(2,3)\backslash e$ as a minor. The code for an example computation is given in Appendix A. ∎

Lemma 4.5. *Let V be a ternary matrix consisting entirely of unit columns, and let W be a ternary matrix each of whose columns has exactly two nonzero entries, both of which are 1s. If Φ is the complete, lifted Y-template determined by the following ternary matrix, then $\mathcal{M}(\Phi)$ is contained in the class of signed-graphic matroids.*

$$\left[\begin{array}{c|c} 1\cdots 1 & -1\cdots -1 \\ 1\cdots 1 & -1\cdots -1 \\ \hline V & W \end{array}\right]$$

Proof. Let P_0 be the matrix obtained from the given matrix by removing the top row. By Lemma 3.18(ii), Φ is minor equivalent to the complete, lifted Y-template Φ_{P_0}. Every matroid M conforming to Φ_{P_0} is a restriction of the vector matroid of a matrix of the following form. (If the rank of M is larger than the number of rows of P_0, then some of the rows of $[V|W]$ below are zero rows.)

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 0\cdots 0 & 1\cdots 1 & 0\cdots 0 & 1\cdots 1 & -1\cdots -1 \\ \hline 0 & & & & & \\ \vdots & I & -I & D & V & W \\ 0 & & & & & \end{array}\right]$$

From the first row, subtract all other rows. The result is the following.

$$\left[\begin{array}{c|c|c|c|c|c} 1 & -1\cdots -1 & -1\cdots -1 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline 0 & & & & & \\ \vdots & I & -I & D & V & W \\ 0 & & & & & \end{array}\right]$$

This is a ternary matrix such that each column has at most two nonzero entries. Therefore, M is a signed-graphic matroid. ∎

Lemma 4.6. *Let Φ be a refined ternary frame template with $\mathrm{AG}(2,3)\backslash e \notin \mathcal{M}(\Phi)$. Either $\mathcal{M}(\Phi)$ is contained in the class of signed-graphic matroids, or Φ is minor equivalent to the complete, lifted Y-template determined by a submatrix of the matrix T given at the beginning of this section.*

Proof. By Lemma 4.3, since $\mathcal{M}(\Phi_2)$ is the class of signed-graphic matroids, we may assume that Φ is a Y-template. Therefore, by Lemma 3.18(i), we may assume that Φ is the complete, lifted Y-template Φ_{P_0} determined by a matrix P_0 the sum of whose rows is the zero vector. The template Φ_{P_0} is $\mathrm{YT}([D_n|P_0],[\emptyset])$, where n is the number of rows of P_0. Since D_n is already included, we may assume that no column of P_0 is graphic. Since the only weight-2 columns whose entries sum to 0 are graphic, every column of P_0 has at least three nonzero entries.

In the remainder of the proof, matrices A–K are the matrices listed in Table 1. By Lemma 4.4, none of matrices A–K may be submatrices of P_0. Since each of matrices C–K has rows whose sum is the zero vector, Lemma 3.18(ii) implies that no matrix obtained from one of matrices C–K by removing one row can be a submatrix of P_0 either.

By Lemma 3.18(ii) the complete, lifted Y-template determined by the matrix $[1, 1, 1, 1, -1]^T$ is minor equivalent to the complete, lifted Y-templates determined by both $[1, 1, 1, -1]^T$ and $[1, 1, 1, 1]^T$, which is matrix A. Since matrix A is forbidden from being a submatrix of P_0, so is $[1, 1, 1, -1]^T$ by minor equivalence. Thus, no column of P_0 can contain four equal nonzero entries, and no column of P_0 with three equal nonzero entries can contain the negative of that nonzero entry. Therefore, by scaling columns of P_0, we may assume that every column of P_0 is of the form $[1, 1, 1, 0, \ldots, 0]^T$ or of the form $[1, 1, -1, -1, 0, \ldots, 0]^T$, up to permuting rows.

Since matrices B, C, and K are forbidden, the intersection of the supports of all of the weight-3 columns of P_0 must be nonempty. Therefore, if P_0 consists entirely of weight-3 columns, then Lemma 3.18(ii) implies that Φ is minor equivalent to the template Φ' determined by the matrix obtained from P_0 by removing the row where every column has a nonzero entry. Every matrix conforming to Φ' has at most two nonzero entries per column. Therefore, every matroid conforming to the template is signed-graphic.

Thus, we may assume that P_0 has a weight-4 column. Let G' be the matrix obtained by removing the row of matrix G with two nonzero entries. If P_0 contains a weight-3 column and a weight-4 column whose supports have an intersection of size 0 or 1, then P_0 contains matrix G' or G, respectively, up to permuting rows and scaling columns. Since matrices G' and G are both forbidden, every pair of columns, one of which has weight 3 and one of which has weight 4, must have supports with an intersection of size at least 2. Moreover, since matrices H, I, and J are forbidden, if P_0 contains both weight-3 and weight-4 columns, then P_0 must be of the form given in Lemma 4.5. By that lemma, $\mathcal{M}(\Phi)$ consists of signed-graphic matroids.

Therefore, we may assume that P_0 consists entirely of weight-4 columns. Let D' be the matrix obtained by removing the row of matrix D with two nonzero entries. A similar argument to the one involving G and G' in the last paragraph shows that every pair of weight-4 columns must have supports whose intersection has size at least 2. Since matrices E and F are forbidden, either $\mathcal{M}(\Phi)$ consists of signed-graphic matroids, by Lemma 4.5, or restricting to the nonzero rows of P_0 results in a column submatrix of the matrix T^+ obtained from matrix T by appending the row $[-1, -1, -1]$. Then by Lemma

3.18(ii), we may assume that restricting to the nonzero rows of P_0 results in a column submatrix of the matrix T. ■

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will prove the statement in Theorem 1.1 that is dependent on Hypothesis 3.1. The statement that is dependent on Hypothesis 3.2 is proved similarly.

Recall that $\mathrm{AG}(2,3)\backslash e$ is an excluded minor for the class of dyadic matroids. Since $\mathrm{AG}(2,3)\backslash e$ is a restriction of $\mathrm{PG}(2,3)$, Hypothesis 3.1, with $\mathbb{F} = \mathrm{GF}(3)$ and $m = 3$, implies that there exist $k \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that every k-connected dyadic matroid M with at least $2k$ elements either is contained in one of $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$ or has a dual M^* contained in one of $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$. Moreover, each of $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s), \mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$ is contained in the class of dyadic matroids. By Theorem 3.6, we may assume that these templates are refined. Let Φ be any template in $\{\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t\}$. By Lemma 4.6, either $\mathcal{M}(\Phi)$ is contained in the class of signed-graphic matroids, or Φ is minor equivalent to the complete, lifted Y-template determined by a submatrix of the matrix T given at the beginning of this section. In the former case, condition (1) of Theorem 1.1 holds. In the latter case, since U_r is the rank-r universal matroid for Φ_T, Lemma 3.19 implies that every matroid conforming to Φ is a minor of some U_r.

If we contract from U_r one of the elements that indexes a column of the matrix T, then the simplification of the resulting matroid has the following representation matrix.

$$\left[\begin{array}{c|c|ccc|c} & & -1 & \cdots & -1 & 1 \\ & & 1 & \cdots & 1 & 1 \\ \cline{3-6} I_{r-1} & D_{r-1} & & & & 0 \\ & & & -I_{r-3} & & \vdots \\ & & & & & 0 \end{array}\right]$$

We see that this matroid is signed-graphic by adding to the second row the negatives of all other rows. Therefore, every simple matroid $M \in \overline{\mathcal{M}(\Phi)}$ is either signed-graphic or a restriction of some U_r. (Since we are dealing with highly connected matroids, they must also be simple. Therefore, we are only concerned with simple matroids.) We also see that we may choose $r = r(M)$ because otherwise there are rows of the matrix representing M that can be removed without changing the matroid. ■

Now we prove Theorem 1.2.

Proof of Theorem 1.2. We will use Lemma 3.20 to prove Theorem 1.2. Let $\mathcal{E}_1 = \{\mathrm{AG}(2,3)\backslash e\}$, and let $\mathcal{E}_2 = \{(\mathrm{AG}(2,3)\backslash e)^*\}$. Since $\mathrm{AG}(2,3)\backslash e$ is ternary but not dyadic, and since the classes of ternary and dyadic matroids are closed under duality, conditions (i) and (ii) of Lemma 3.20 are satisfied. Since every signed-graphic matroid is dyadic and U_r is dyadic for every r, Lemma 4.6 implies that condition (iii) of Lemma 3.20 is satisfied. Then (iv) follows from (iii) and duality. The result follows. ∎

5. Near-Regular and $\sqrt[6]{1}$-Matroids

In this section, we prove Theorem 1.3 after proving several lemmas.

Lemma 5.1. *Let $\Phi \in \{\Phi_2, \Phi_X, \Phi_C, \Phi_{Y_0}\, \Phi_{CX}, \Phi_{CX2}\}$. Then the non-Fano matroid F_7^- is a minor of some member of $\mathcal{M}(\Phi)$. Therefore, $\mathcal{M}(\Phi)$ is not contained in the class of $\sqrt[6]{1}$-matroids.*

Proof. It is well known (see [15, Proposition 6.4.8]) that F_7^- is $\mathbb{F}$-representable if and only if the characteristic of $\mathbb{F}$ is not 2. Thus, F_7^- is not representable over GF(4) and is therefore not a $\sqrt[6]{1}$-matroid. Therefore, the last statement of the lemma follows from the first part of the lemma.

We saw in Section 4 that $M(K_4)$ can be obtained from the rank-3 ternary Dowling geometry by deleting the three joints. If we leave one of the joints in place, then it is contained in the closures of exactly two pairs of points that are not collinear in $M(K_4)$. The result is the non-Fano matroid F_7^-. Therefore, the F_7^- is signed-graphic, implying that $F_7^- \in \mathcal{M}(\Phi_2)$.

The following ternary matrix is a representation of F_7^- that conforms to both Φ_X and Φ_{Y_0}.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 \end{bmatrix} \tag{5.1}$$

Therefore, $F_7^- \in \mathcal{M}(\Phi_X)$ and $F_7^- \in \mathcal{M}(\Phi_{Y_0})$. By Lemma 3.8, we also have that F_7^- is a minor of matroids conforming to Φ_C, Φ_{CX}, and Φ_{CX2}. ∎

Recall that the matroid T_r^1 is obtained from the complete graphic matroid $M(K_{r+2})$ by adding a point freely to a triangle, contracting that point, and simplifying. For $r \geq 2$, Semple (see [20, Section 2] and [19, Proposition 3.1]) showed that T_r^1 is representable over a field $\mathbb{F}$ if and only if $\mathbb{F} \neq \mathrm{GF}(2)$. Therefore, every T_r^1 is near-regular. The following matrix represents T_r^1 over every field of characteristic other than 2.

$$\left[\begin{array}{c|c|c} I_r & D_r & \begin{array}{c} 1 \cdots 1 \\ \hline I_{r-1} \end{array} \end{array}\right] \tag{5.2}$$

Lemma 5.2. *Let Φ be the complete, lifted Y-template determined by some ternary matrix P_0. Either F_7^- is a minor of some matroid conforming to Φ, or every member of $\mathcal{M}(\Phi)$ has a simplification that is a restriction of some T_r^1.*

Proof. Suppose $P_0 = [1,1,1,1]^T$. Consider the vector matroid of $[I_4|D_4|P_0]$. By contracting the element represented by P_0, we obtain a matroid containing F_7^- as a restriction. Therefore, no column of P_0 can contain four equal nonzero entries. If $P_0 = [1,1,-1]^T$, then $[I_3|D_3|P_0]$ is (up to column scaling) the representation of F_7^- given in Matrix (5.1). Therefore, if a column of P_0 contains unequal nonzero entries, then it can contain no other nonzero entry. Thus, the column is a graphic column, which is already assumed to be included in a complete Y-template.

Thus, every column of P_0 contains at most three nonzero entries, all of which are equal. Consider the following matrices.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

If P_0 is either of these matrices, then $M([I|D|P_0])$ contains F_7^- as a minor. (This can be easily checked with SageMath or by hand. If P_0 is the first matrix, contracting one of the elements represented by a column of P_0 and simplifying results in the rank-3 ternary Dowling geometry. We saw in the proof of Lemma 5.1 that F_7^- is signed-graphic, implying that it is a restriction of the rank-3 ternary Dowling geometry. If P_0 is the second matrix, then $M([I|D|P_0])$ is itself the rank-3 ternary Dowling geometry.)

It is routine to check that the class of matroids whose simplifications are restrictions of some T_r^1 is minor-closed. Therefore, it suffices to consider a template that is minor equivalent to Φ. Thus, by Lemma 3.18(i), we may assume that the sum of the rows of P_0 is the zero vector. Therefore, we may assume that every column has exactly three nonzero entries all of which are equal. Because the two matrices above are forbidden, P_0 is of the following form, where V consists entirely of unit columns.

$$\left[\begin{array}{c} 1 \cdots 1 \\ 1 \cdots 1 \\ \hline V \end{array}\right]$$

Now, Lemma 3.18(ii) implies that Φ is minor equivalent to the complete, lifted Y-template Φ' determined by the matrix obtained by removing the top row from P_0. Every member of $\mathcal{M}(\Phi')$ has a simplification that is a restriction of some T_r^1. ∎

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We prove the statement in Theorem 1.3 that is dependent on Hypothesis 3.1. The statement dependent on Hypothesis 3.2 is proved similarly.

Every near-regular matroid is also a $\sqrt[6]{1}$-matroid; therefore, (1) implies (2). We will now show that (2) implies (3). By Hypothesis 3.1, there exist ternary frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ and a positive integer k_1 such that every matroid conforming to these templates is a $\sqrt[6]{1}$-matroid and such that every simple k_1-connected $\sqrt[6]{1}$-matroid with at least $2k_1$ elements either conforms or coconforms to one of these templates. By Theorem 3.6, we may assume that these templates are refined. Lemmas 5.1 and 3.11 imply that each of these templates is a Y-template. Then Lemmas 3.18 and 5.2 imply that every matroid conforming to these templates has a simplification isomorphic to a restriction of T_r^1 for some r (because the class of matroids whose simplifications are restrictions of T_r^1 is minor-closed). We see that we may choose $r = r(M)$ because otherwise there are rows of the matrix representing M that can be removed without changing the matroid. By taking $k \geq k_1$, we see that (2) implies (3).

Since T_r^1 is near-regular for every r, (3) implies (1). We complete the proof of the theorem by showing the equivalence of (2) and (4) using Lemma 3.20. In that lemma, let $\mathcal{M}$ be the class of $\sqrt[6]{1}$-matroids, let $\mathcal{E}_1 = \{F_7^-\}$, and let $\mathcal{E}_2 = \{(F_7^-)^*\}$. Since F_7^- and $(F_7^-)^*$ are ternary matroids that are not $\sqrt[6]{1}$-matroids, conditions (i) and (ii) of Lemma 3.20 are satisfied. Combining Lemmas 5.1 and 3.11, we see that for every refined frame template Φ that is not a Y-template, $F_7^- \in \overline{\mathcal{M}(\Phi)}$. Combining Lemmas 3.18(i), and 5.2, we see that $F_7^- \in \overline{\mathcal{M}(\Phi)}$ for every Y-template Φ such that $\overline{\mathcal{M}(\Phi)}$ is not contained in the class of matroids whose simplifications are restrictions of some T_r^1. Since T_r^1 is a $\sqrt[6]{1}$-matroid for every r, condition (iii) of Lemma 3.20 holds. Finally, condition (iv) of Lemma 3.20 holds because of condition (iii) and the fact that the class of $\sqrt[6]{1}$-matroids is closed under duality. Therefore, by Lemma 3.20, there is a positive integer k_2 such that a k_2-connected ternary matroid with at least $2k_2$ elements is a $\sqrt[6]{1}$-matroid if and only if it contains no minor isomorphic to either F_7^- or $(F_7^-)^*$. By taking $k \geq k_2$, we see that (2) is equivalent to (4), completing the proof. ■

Appendix A. SageMath Code

We give here an example of the code for the computations used to prove Lemma 4.4. The function `complete_Y_template_matrix` takes as input a matrix P_0 and returns the matrix $[I|D|P_0]$. The code below returns `True`, showing that if P_0 is matrix C from Table 1, then $M([I|D|P_0])/\{0, 18, 23\}$ contains $\mathrm{AG}(2,3)\backslash e$ as a minor.

```
N=matroids.named_matroids.AG23minus()
# N is the matroid AG(2,3)\e

def complete_Y_template_matrix(P0):
    k=P0.nrows()
    num_elts=k+k*(k-1)/2+P0.ncols()
    F=P0.base_ring()
    A = Matrix(F, k, num_elts)
    i = 0
    # identity in front
    for j in range(k):
        A[j,j] = 1
    i = k
    # all pairs
    for S in Subsets(range(k),2):
        A[S[0],i]=1
        A[S[1],i]=-1
        i = i + 1
    # Columns from Y0
    for l in range(P0.ncols()):
        for j in range(k):
            A[j, i] = P0[j, l]
        i=i+1
    return A

P0 = Matrix(GF(3), [[1,1,0],
                    [1,0,1],
                    [0,1,1],
                    [1,0,0],
                    [0,1,0],
                    [0,0,1]])
A=complete_Y_template_matrix(P0)
M=Matroid(field=GF(3), matrix=A)
((M/0/18/23).simplify()).has_minor(N)
```

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