ON THE HIGHLY CONNECTED DYADIC, NEAR-REGULAR, AND SIXTH-ROOT-OF-UNITY MATROIDS

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Abstract. Subject to announced results by Geelen, Gerards, and Whittle, we completely characterize the highly connected members of the classes of dyadic, near-regular, and sixth-root-of-unity matroids.

1. Introduction

We give the definitions of the classes of dyadic, signed-graphic, near-regular, and \( \sqrt[6]{1} \)-matroids in Section 2; however, unexplained notation and terminology in this paper will generally follow Oxley [15]. One exception is that we denote the vector matroid of a matrix \( A \) by \( M(A) \) rather than \( M[A] \). A matroid \( M \) is vertically \( k \)-connected if, for every set \( X \subseteq E(M) \) with \( r(X) + r(E - X) - r(M) < k \), either \( X \) or \( E - X \) is spanning. If \( M \) is vertically \( k \)-connected, then its dual \( M^* \) is cyclically \( k \)-connected. The matroid \( U_r \) is obtained from \( M(K_{r+1}) \) by adding three specific points to a rank-3 flat; we give the precise definition in Section 4.

Due to the technical nature of Hypotheses 3.1 and 3.2, we delay their statements to Section 3. Subject to these hypotheses, we characterize the highly connected dyadic matroids by proving the following.

Theorem 1.1. Suppose Hypothesis 3.1 holds. Then there exists \( k \in \mathbb{Z}_+ \) such that, if \( M \) is a \( k \)-connected dyadic matroid with at least \( 2k \) elements, then one of the following holds.

1. Either \( M \) or \( M^* \) is a signed-graphic matroid.
2. Either \( M \) or \( M^* \) is a matroid of rank \( r \) that is a restriction of \( U_r \).

Moreover, suppose Hypothesis 3.2 holds. There exist \( k, n \in \mathbb{Z}_+ \) such that, if \( M \) is a simple, vertically \( k \)-connected, dyadic matroid with an \( M(K_n) \)-minor, then either \( M \) is a signed-graphic matroid or \( M \) is a restriction of \( U_{r(M)} \).

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Let $AG(2, 3)\setminus e$ be the matroid resulting from $AG(2, 3)$ by deleting one point; this matroid is unique up to isomorphism. We prove the following excluded-minor characterization of the highly connected dyadic matroids.

**Theorem 1.2.** Suppose Hypothesis 3.1 holds. There exists $k \in \mathbb{Z}_+$ such that, for a $k$-connected matroid $M$ with at least $2k$ elements, $M$ is dyadic if and only if $M$ is a ternary matroid with no minor isomorphic to either $AG(2, 3)\setminus e$ or $(AG(2, 3)\setminus e)^*$. Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, for a vertically $k$-connected matroid $M$ with an $M(K_n)$-minor, $M$ is dyadic if and only if $M$ is a ternary matroid with no minor isomorphic to $AG(2, 3)\setminus e$.

Our final main result characterizes the highly connected near-regular and $\sqrt[6]{6}$-matroids. The matroid $T^1_r$ is obtained from the complete graphic matroid $M(K_{r+2})$ by adding a point freely to a triangle, contracting that point, and simplifying. We denote the non-Fano matroid by $F_7^-$. 

**Theorem 1.3.** Suppose Hypothesis 3.1 holds. There exists $k \in \mathbb{Z}_+$ such that, if $M$ is a $k$-connected matroid with at least $2k$ elements, the following are equivalent.

1. $M$ is a near-regular matroid,
2. $M$ is a $\sqrt[6]{6}$-matroid,
3. $M$ or $M^*$ is a matroid of rank $r$ that is a restriction of $T^1_r$, and
4. $M$ is a ternary matroid that has no minor isomorphic to $F_7^-$ or $(F_7^-)^*$.

Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, if $M$ is a simple, vertically $k$-connected matroid with an $M(K_n)$-minor, then (1) and (2) are equivalent to each other and also to the following conditions.

1. $M$ is a restriction of $T^1_r(M)$, and
2. $M$ is a ternary matroid that has no minor isomorphic to $F_7^-$. 

**Corollary 1.4.** Suppose Hypothesis 3.1 holds. There exists $k \in \mathbb{Z}_+$ such that, if $M$ is a ternary $k$-connected matroid with at least $2k$ elements, then $M$ is representable over some field of characteristic other than $3$ if and only if $M$ is dyadic. Moreover, suppose Hypothesis 3.2 holds. There exist $k, n \in \mathbb{Z}_+$ such that, if $M$ is a simple, ternary, vertically $k$-connected matroid with an $M(K_n)$-minor, then $M$ is representable over some field of characteristic other than $3$ if and only if $M$ is dyadic.

**Proof.** Whittle [22, Theorem 5.1] showed that a 3-connected ternary matroid that is representable over some field of characteristic other than $3$ is either a dyadic matroid or a $\sqrt[6]{6}$-matroid. Therefore, since near-regular matroids are dyadic, Theorem 1.3 immediately implies the first statement in the corollary.
The second statement is proved similarly but also requires the fact that a simple, vertically 3-connected matroid is 3-connected.

Hypotheses 3.1 and 3.2 are believed to be true, but their proofs are still forthcoming in future papers by Geelen, Gerards, and Whittle. They are modified versions of a hypothesis given by Geelen, Gerards, and Whittle in [5]. The results announced in [5] rely on the Matroid Structure Theorem by these same authors [4]. We refer the reader to [9] for more details.

Some proofs in this paper involved case checks aided by Version 8.6 of the SageMath software system [18], in particular making use of the matroids component [17]. The authors used the CoCalc (formerly SageMathCloud) online interface.

In Section 2, we give some background information about the classes of matroids studied in this paper. In Section 3, we recall results from [7] that will be used to prove our main results. In Section 4, we prove Theorems 1.1 and 1.2, and in Section 5, we prove Theorem 1.3.

2. Preliminaries

We begin this section by clarifying some notation and terminology that will be used throughout the rest of the paper. Let \( D_r \) be the \( r \times \binom{r}{2} \) matrix such that each column is distinct and such that every column has exactly two nonzero entries—the first a 1 and the second \(-1\). For a field \( \mathbb{F} \), we denote by \( \mathbb{F}_p \) the prime subfield of \( \mathbb{F} \). If \( M \) is a class of matroids, we will denote by \( \overline{M} \) the closure of \( M \) under the taking of minors. The weight of a column or row vector of a matrix is its number of nonzero entries. If \( A \) is an \( m \times n \) matrix and \( n' \leq n \), then we call an \( m \times n' \) submatrix of \( A \) a column submatrix of \( A \).

In the remainder of this section, we give some background information about the classes of matroids studied in this paper.

The class of dyadic matroids consists of those matroids representable by a matrix over \( \mathbb{Q} \) such that every nonzero subdeterminant is \( \pm 2^i \) for some \( i \in \mathbb{Z} \). The class of sixth-root-of-unity matroids (or \( \sqrt[6]{T} \)-matroids) consists of those matroids that are representable by a matrix over \( \mathbb{C} \) such that every nonzero subdeterminant is a complex sixth root of unity. Let \( \mathbb{Q}(\alpha) \) be the field obtained by extending the rationals \( \mathbb{Q} \) by a transcendental \( \alpha \). A matroid is near-regular if it can be represented by a matrix over \( \mathbb{Q}(\alpha) \) such that every nonzero subdeterminant is contained in the set \( \{\pm \alpha^i(\alpha - 1)^j : i, j \in \mathbb{Z}\} \).

A matroid is signed-graphic if it can be represented by a matrix over GF(3) each of whose columns has at most two nonzero entries. The rows and columns of this matrix can be indexed by the set of vertices and edges, respectively, of a signed graph. If the nonzero entries of the column are unequal, then the corresponding edge is a positive edge joining the vertices indexing the
rows containing the nonzero entries. If the column has two equal entries, the edge is negative. If the column contains only one nonzero entry, then the corresponding edge is a negative loop at the vertex indexing the row containing the nonzero entry. Every signed-graphic matroid is dyadic. (See, for example, [23, Lemma 8A.3]).

Whittle [22, Theorem 1.4] showed that the following statements are equivalent for a matroid $M$:

- $M$ is near-regular
- $M$ is representable over $GF(3)$, $GF(4)$, and $GF(5)$
- $M$ is representable over all fields except possibly $GF(2)$

He also showed [22, Theorem 1.2] that the class of $\sqrt{1}$-matroids consists exactly of those matroids representable over $GF(3)$ and $GF(4)$ and [22, Theorem 1.1] that the class of dyadic matroids consists exactly of those matroids representable over $GF(3)$ and $GF(5)$. Thus, the class of near-regular matroids is the intersection of the classes of $\sqrt{1}$-matroids and dyadic matroids.

A geometric representation of $AG(2, 3) \setminus e$ is given in Figure 1. It is fairly well known that $AG(2, 3) \setminus e$ is an excluded minor for the class of dyadic matroids. (See, for example, [15, Section 14.7].) We will use this fact in Section 4.

It is an open problem to determine the complete list of excluded minors for the dyadic matroids; however, the excluded minors for the classes of $\sqrt{1}$-matroids and near-regular matroids have been determined. Geelen, Gerards, and Kapoor [3, Corollary 1.4] showed that the excluded minors for the class of $\sqrt{1}$-matroids are $U_{2,5}$, $U_{3,5}$, $F_7$, $F_7^*$, $F_7^-$, $(F_7^-)^*$, and $P_8$. (We refer the reader to [3] or [15] for the definitions of these matroids.) Hall, Mayhew, and Van Zwam [10, Theorem 1.2], based on unpublished work by Geelen, showed that the excluded minors for the class of near-regular matroids are $U_{2,5}$, $U_{3,5}$, $F_7$,
Here, $\Delta_T(AG(2,3)\setminus e)$ is the result of performing a $\Delta-Y$ operation on $AG(2,3)\setminus e$.

If $r \neq 3$, it follows from results of Kung [11, Theorem 1.1] and Kung and Oxley [12, Theorem 1.1] that the largest simple dyadic matroid of rank $r$ is the rank-$r$ ternary Dowling geometry, which is a signed-graphic matroid. Again, suppose $r \neq 3$. Then Oxley, Vertigan, and Whittle [16, Theorem 2.1, Corollary 2.2] showed that $T^r_1$ is the largest simple $\sqrt{\gamma}$-matroid of rank $r$ and the largest simple near-regular matroid of rank $r$. We remark without proof that our main results here, combined with [9, Lemmas 4.14, 4.16], show that Hypothesis 3.2 agrees with these known results.

3. Frame Templates

The notion of frame templates was introduced by Geelen, Gerards, and Whittle in [5] to describe the structure of the highly connected members of minor-closed classes of matroids representable over a fixed finite field. Frame templates have been studied further in [8, 14, 9, 7]. In this section, we give several results proved in those papers that we will need to prove the main results in this paper. The results in [5] technically deal with represented matroids—which can be thought of as fixed representation matrices for matroids. However, since we only deal with ternary matroids in this paper, and since ternary matroids are uniquely $GF(3)$-representable [1], we will state the results in terms of matroids rather than represented matroids.

If $F$ is a field, let $F^\times$ denote the multiplicative group of $F$, and let $\Gamma$ be a subgroup of $F^\times$. A $\Gamma$-frame matrix is a frame matrix $A$ such that:

- Each column of $A$ with a nonzero entry contains a 1.
- If a column of $A$ has a second nonzero entry, then that entry is $-\gamma$ for some $\gamma \in \Gamma$.

If $\Gamma = \{1\}$, then the vector matroid of a $\Gamma$-frame matrix is a graphic matroid. For this reason, we will call the columns of a $\{1\}$-frame matrix graphic columns.

A frame template over a field $F$ is a tuple $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ such that the following hold.

(i) $\Gamma$ is a subgroup of $F^\times$.
(ii) $C$, $X$, $Y_0$ and $Y_1$ are disjoint finite sets.
(iii) $A_1 \in F^{X \cup (C \cup Y_0 \cup Y_1)}$.
(iv) $\Delta$ is a subgroup of the additive group of $F^X$ and is closed under scaling by elements of $\Gamma$.
(v) $\Delta$ is a subgroup of the additive group of $F^{C \cup Y_0 \cup Y_1}$ and is closed under scaling by elements of $\Gamma$. 


Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template. Let $B$ and $E$ be finite sets, and let $A' \in \mathbb{F}^{B \times E}$. We say that $A'$ respects $\Phi$ if the following hold:

(i) $X \subseteq B$ and $C, Y_0, Y_1 \subseteq E$.
(ii) $A'[X, C \cup Y_0 \cup Y_1] = A_1$.
(iii) There exists a set $Z \subseteq E - (C \cup Y_0 \cup Y_1)$ such that $A'[X, Z] = 0$, each column of $A'[X, E - (C \cup Y_0 \cup Y_1 \cup Z)]$ is a $\Gamma$-frame matrix.
(iv) Each column of $A'[X, E - (C \cup Y_0 \cup Y_1 \cup Z)]$ is contained in $\Lambda$.
(v) Each row of $A'[B - X, C \cup Y_0 \cup Y_1]$ is contained in $\Delta$.

The structure of $A'$ is shown below.

<table>
<thead>
<tr>
<th>$X$</th>
<th>columns from $\Lambda$</th>
<th>$Y_0$</th>
<th>$Y_1$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$-frame matrix</td>
<td>unit or zero columns</td>
<td></td>
<td></td>
<td>$A_1$</td>
</tr>
</tbody>
</table>

Now, suppose that $A'$ respects $\Phi$ and that $A \in \mathbb{F}^{B \times E}$ satisfies the following conditions:

(ii) For each $i \in Z$ there exists $j \in Y_1$ such that the $i$-th column of $A$ is the sum of the $i$-th and the $j$-th columns of $A'$.

We say that such a matrix $A$ conforms$^1$ to $\Phi$.

Let $M$ be an $\mathbb{F}$-representable matroid. We say that $M$ conforms$^1$ to $\Phi$ if there is a matrix $A$ conforming to $\Phi$ such that $M$ is isomorphic to $M(A)/C \setminus Y_1$. We denote by $\mathcal{M}(\Phi)$ the set of matroids that conform to $\Phi$. If $M^*$ conforms to a template $\Phi$, we say that $M$ coconforms to $\Phi$. We denote by $\mathcal{M}^*(\Phi)$ the set of matroids that coconform to $\Phi$.

We now state the hypotheses on which the main results are based. As stated in Section 1, they are modified versions of a hypothesis given by Geelen, Gerards, and Whittle in [5], and their proofs are forthcoming. In their current forms, these hypotheses were stated in [9].

**Hypothesis 3.1** ([9, Hypothesis 4.3]). Let $\mathbb{F}$ be a finite field, let $m$ be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids. Then there exist $k \in \mathbb{Z}_+$ and frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that

1. $\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s),$
2. $\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t),$

---

$^1$For simplicity, we will use the terms respecting and conforming to mean what was called virtual respecting and virtual conforming in [8] and [7]. The distinction between conforming and virtually conforming is explained in [8]. We can do this since every matroid virtually conforming to a template is a minor of a matroid conforming to a template [8, Lemma 3.4].
(3) if $M$ is a simple $k$-connected member of $\mathcal{M}$ with at least $2k$ elements and $\tilde{M}$ has no PG$(m-1, F_p)$-minor, then either $M$ is a member of at least one of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$, or $M^*$ is a member of at least one of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$.

**Hypothesis 3.2** ([9, Hypothesis 4.6]). Let $F$ be a finite field, let $m$ be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $F$-representable matroids. Then there exist $k, n \in \mathbb{Z}_+$ and frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that

1. $\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$,
2. $\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$,
3. if $M$ is a simple vertically $k$-connected member of $\mathcal{M}$ with an $M(K_n)$-minor but no PG$(m-1, F_p)$-minor, then $M$ is a member of at least one of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$, and
4. if $M$ is a cosimple cyclically $k$-connected member of $\mathcal{M}$ with an $M^*(K_n)$-minor but no PG$(m-1, F_p)$-minor, then $M^*$ is a member of at least one of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$.

If $\Phi$ and $\Phi'$ are frame templates, it is possible that $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$ even though $\Phi$ and $\Phi'$ look very different.

**Definition 3.3** ([7, Definition 6.3]). Let $\Phi$ and $\Phi'$ be frame templates over a field $F$, then the pair $\Phi, \Phi'$ are strongly equivalent if $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$. The pair $\Phi, \Phi'$ are minor equivalent if $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$.

There are other notions of template equivalence (namely equivalence, algebraic equivalence, and semi-strong equivalence) given in [7], but all of these imply minor equivalence.

If $F$ is a field and $E$ is a set, we say that two subgroups $U$ and $W$ of the additive subgroup of the vector space $F^E$ are *skew* if $U \cap W = \{0\}$. Nelson and Walsh [14] gave Definition 3.4 below.

**Definition 3.4.** A frame template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is reduced if there is a partition $(X_0, X_1)$ of $X$ such that

- $\Delta = \Gamma(F_p^C \times \Delta')$ for some additive subgroup $\Delta'$ of $F_{Y_0 \cup Y_1}$,
- $F_p^{X_0} \subseteq \Lambda|X_0$ while $\Lambda|X_1 = \{0\}$ and $A_1[X_1, C] = 0$, and
- the rows of $A_1[X_1, C \cup Y_0 \cup Y_1]$ form a basis for a subspace whose additive group is skew to $\Delta$.

We will refer to the partition $X = X_0 \cup X_1$ given in Definition 3.4 as the reduction partition of $\Phi$.

The following definition and theorem are found in [9].

**Definition 3.5** ([9, Definition 5.3]). Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a reduced frame template, with reduction partition $X = X_0 \cup X_1$. If $Y_1$ spans the matroid $M(A_1[X_1, Y_0 \cup Y_1])$, then $\Phi$ is refined.
Theorem 3.6 ([9, Theorem 5.6]). If Hypothesis 3.1 holds for a class \( \mathcal{M} \), then the constant \( k \) and the templates \( \Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t \) can be chosen so that the templates are refined. Moreover, if Hypothesis 3.2 holds for a class \( \mathcal{M} \), then the constants \( k, n \), and the templates \( \Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t \) can be chosen so that the templates are refined.

A few specific templates have been given names. We list some of those now, specifically for the ternary case. Note that \( \mathcal{M}(\Phi_2) \) is the class of signed-graphic matroids.

Definition 3.7 ([7, Definition 7.8, ternary case]).

- \( \Phi_2 \) is the template with all sets empty and all groups trivial except that \( \Gamma = \{ \pm 1 \} \).
- \( \Phi_C \) is the template with all groups trivial and all sets empty except that \( |C| = 1 \) and \( \Delta \cong \mathbb{Z}/3\mathbb{Z} \).
- \( \Phi_X \) is the template with all groups trivial and all sets empty except that \( |X| = 1 \) and \( \Lambda \cong \mathbb{Z}/3\mathbb{Z} \).
- \( \Phi_{Y_0} \) is the template with all groups trivial and all sets empty except that \( |Y_0| = 1 \) and \( \Delta \cong \mathbb{Z}/3\mathbb{Z} \).
- \( \Phi_{CX} \) is the template with \( Y_0 = Y_1 = \emptyset \), with \( |C| = |X| = 1 \), with \( \Delta \cong \Lambda \cong \mathbb{Z}/3\mathbb{Z} \), with \( \Gamma \) trivial, and with \( A_1 = [1] \).
- \( \Phi_{CX_2} \) is the template with \( Y_0 = Y_1 = \emptyset \), with \( |C| = |X| = 1 \), with \( \Delta \cong \Lambda \cong \mathbb{Z}/3\mathbb{Z} \), with \( \Gamma \) trivial, and with \( A_1 = [-1] \).

The next lemma follows directly from [7, Lemma 7.9].

Lemma 3.8. The following are true: \( \mathcal{M}(\Phi_{Y_0}) \subseteq \mathcal{M}(\Phi_C) \), and \( \mathcal{M}(\Phi_X) \subseteq \mathcal{M}(\Phi_{CX}) \), and \( \mathcal{M}(\Phi_{CX}) \subseteq \mathcal{M}(\Phi_{CX_2}) \).

Frame templates where the groups \( \Gamma, \Lambda, \) and \( \Delta \) are trivial are studied extensively in [7].

Definition 3.9 ([7, Definitions 6.9–6.10]). A \( Y \)-template is a refined frame template with all groups trivial (so \( C = X_0 = \emptyset \)). If \( A_1 \) has the form below, then \( YT(P_0, P_1) \) is defined to be the \( Y \)-template \( (\{1\}, \emptyset, X, Y_0, Y_1, A_1, \{0\}, \{0\}) \).

\[
\begin{array}{c|c|c|c|c|c|c}
Y_1 & Y_0 & I_{|X|} & P_1 & P_0 \\
\end{array}
\]

The next lemma follows from [7, Lemma 7.16–17].

Lemma 3.10. Let \( \Phi \) be a frame template such that \( \overline{\mathcal{M}(\Phi')} \not\subseteq \mathcal{M}(\Phi) \) for each template \( \Phi' \in \{ \Phi_X, \Phi_C, \Phi_{Y_0}, \Phi_{CX}, \Phi_{CX_2} \} \). Then \( \Phi \) is minor equivalent to a template \( (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda) \) with \( C = \emptyset \) and with \( \Lambda \) and \( \Delta \) both trivial.

The next several results and definitions are found in [7].
Lemma 3.11 ([7, Theorem 7.18]). Let $\Phi$ be a refined ternary frame template. Then either $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi)$ for some $\Phi' \in \{\Phi_X, \Phi_C, \Phi_{Y_0}, \Phi_{CX}, \Phi_{C2}, \Phi_2\}$, or $\Phi$ is a $Y$-template.

Definition 3.12 ([7, Definition 9.3]). Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a refined frame template with reduction partition $X = X_0 \cup X_1$, with $A_1[X_0, Y_1]$ a zero matrix, and with $A_1[X_1, Y_1]$ an identity matrix. Then $\Phi$ is a lifted template.

Lemma 3.13 ([7, Lemma 9.6]). Every refined frame template is minor equivalent to a lifted template.

Definition 3.14 ([7, Definition 9.7]). A $Y$-template $YT(P_0, P_1)$ is complete if $P_0$ contains $D_{|X|}$ as a submatrix.

Definition 3.15 ([7, Definition 9.12]). The $Y$-template $YT([P_0|D_{|X|}], [0])$ is the complete, lifted $Y$-template determined by $P_0$ and is denoted by $\Phi_{P_0}$.

Definition 3.16 ([7, Definition 6.12]). Let $\Phi_{P_0}$ be a complete, lifted $Y$-template. The rank-$r$ universal matroid for $\Phi_{P_0}$ is the matroid represented by the following matrix.

$$
\begin{bmatrix}
I_r & D_r & P_0 \\
0 & 0 & 0
\end{bmatrix}
$$

It is shown in [7, Section 9] that every matroid conforming to $\Phi_{P_0}$ is a restriction of some universal matroid for $\Phi_{P_0}$.

We refer to [15, Section 11.4] for the definition of generalized parallel connections of matroids.

Lemma 3.17 ([7, Lemma 9.13]). The rank-$r$ universal matroid of $\Phi_{P_0}$ is the generalized parallel connection of $M(K_{r+1})$ and $M([I_m|D_m|P_0])$ along $M(K_{m+1})$, where $m$ is the number of rows of $P_0$.

Combining [7, Lemma 9.6], [7, Lemma 9.9], and [7, Lemma 9.14], we obtain the following.

Lemma 3.18.

(i) Every $Y$-template is minor equivalent to the complete, lifted $Y$-template determined by a matrix the sum of whose rows is the zero vector.

(ii) Conversely, let $\Phi$ be the complete, lifted $Y$-template determined by a matrix $P_0$ the sum of whose rows is the zero vector. Choose any one row of $P_0$. Then $\Phi$ is minor equivalent to the complete, lifted $Y$-template determined by the matrix obtained from $P_0$ by removing that row.

The next lemma is an easy but useful result.
Lemma 3.19. Let $P'_0$ be a matrix with a submatrix $P_0$. Every matroid conforming to $\Phi_{P_0}$ is a minor of a matroid conforming to $\Phi_{P'_0}$.

We use the next lemma to prove the excluded minor characterizations in Theorems 1.2 and 1.3; it is obtained by combining Lemmas 8.1 and 8.2 of [7].

Lemma 3.20. Let $M$ be a minor-closed class of $\mathbb{F}$-representable matroids, where $\mathbb{F}$ is the prime subfield of $\mathbb{F}$. Let $E_1$ and $E_2$ be two sets of $\mathbb{F}$-representable matroids such that

(i) no member of $E_1 \cup E_2$ is contained in $M$,
(ii) some member of $E_1 \cup E_2$ is $\mathbb{F}_p$-representable,
(iii) for every refined frame template $\Phi$ over $\mathbb{F}$ such that $M(\Phi) \not\subseteq M$, there is a member of $E_1$ that is a minor of a matroid conforming to $\Phi$, and
(iv) for every refined frame template $\Psi$ over $\mathbb{F}$ such that $M^*(\Psi) \not\subseteq M$, there is a member of $E_2$ that is a minor of a matroid coconforming to $\Psi$.

Suppose Hypothesis 3.1 holds; there exists $k \in \mathbb{Z}_+$ such that a $k$-connected $\mathbb{F}$-representable matroid with at least $2k$ elements is contained in $M$ if and only if it contains no minor isomorphic to one of the matroids in the set $E_1 \cup E_2$.

Moreover, suppose Hypothesis 3.2 holds; there exist $k,n \in \mathbb{Z}_+$ such that a vertically $k$-connected $\mathbb{F}$-representable matroid with an $M(K_n)$-minor is contained in $M$ if and only if it contains no minor isomorphic to one of the matroids in $E_1$ and such that a cyclically $k$-connected $\mathbb{F}$-representable matroid with an $M^*(K_n)$-minor is contained in $M$ if and only if it contains no minor isomorphic to one of the matroids in $E_2$.

The next lemma has not appeared previously, but it will be useful in Section 4. Recall from Definition 3.12 that every lifted template is refined and therefore reduced. Thus, a lifted template has a reduction partition as in Definition 3.4.

Lemma 3.21. Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Lambda)$ be a lifted template with reduction partition $X = X_0 \cup X_1$. Let $P_0 = A_1[X_1, Y_0]$, and let $S$ be any $\Gamma$-frame matrix with $|X_1|$ rows. Then $M([S|P_0]) \in M(\Phi)$.

Proof. Let $R = A_1[X_0, Y_0 \cup C]$. Note that the following matrix conforms to $\Phi$, since $\mathbb{F}^{X_0}_p \subseteq \Lambda|X_0$ in a reduced template and since the zero vector is an element of $\Lambda$ and $\Delta$.

\[
\begin{array}{cccccc}
  & H_1 & H_2 & Z & Y_0 & C \\
 X_0 & I_{|X_0|} & 0 & 0 & R & \ \\
 X_1 & 0 & 0 & I_{|X_1|} & P_0 & 0 \\
 & 0 & S & I_{|X_1|} & 0 & 0
\end{array}
\]

By contracting $H_1 \cup Z \cup C$ (recalling that $C$ must be contracted to obtain a matroid that conforms to $\Phi$), we obtain the desired matroid. \qed
4. Dyadic Matroids

In this section, we characterize the highly connected dyadic matroids by proving Theorems 1.1 and 1.2. First, we will need to define the family of matroids $U_r$ and to prove several lemmas that build on the machinery of Section 3.

Consider the rank-3 ternary Dowling geometry $Q_3(GF(3)^\times)$. This matroid contains a restriction isomorphic to $M(K_4)$, with the signed-graphic representation given in Figure 2, with negative edges printed in bold. This representation of $M(K_4)$ has been encountered before, for example in [24, 6, 21, 9].

The Dowling geometry $Q_3(GF(3)^\times)$ can also be represented by the matrix $[I_3|D_3|T]$, where

$$T = \begin{bmatrix}
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix},$$

with the columns of $T$ representing the joints of the Dowling geometry.

For $r \geq 3$, we define the matroid $U_r$ to be the generalized parallel connection of the complete graphic matroid $M(K_{r+1})$ with $Q_3(GF(3)^\times)$ along a common restriction isomorphic $M(K_4)$, where the restriction of $Q_3(GF(3)^\times)$ has the signed-graphic representation given in Figure 2. (The restriction isomorphic to $M(K_4)$ is a modular flat of $M(K_{r+1})$, which is uniquely representable over any field. Therefore, this generalized parallel connection is well-defined.) By Lemma 3.17, $U_r$ is the rank-$r$ universal matroid for the complete, lifted $Y$-template $\Phi_T$ and therefore has the following representation matrix.

$$\begin{bmatrix}
I_r & D_r & T \\
& & 0
\end{bmatrix}$$

**Lemma 4.1.** The matroid $U_r$ is dyadic for every $r \geq 3$.

**Proof.** Since $U_3$ is the ternary rank-3 Dowling geometry, it is signed-graphic and therefore dyadic. Since $U_r$ is the generalized parallel connection of $U_3$ and the complete graphic matroid $M(K_{r+1})$ along a common restriction isomorphic to $M(K_4)$, and since every complete graphic matroid is uniquely representable
over every field, a result of Brylawski [2, Theorem 6.12] implies that \( U_r \) is representable over both \( GF(3) \) and \( GF(5) \) and therefore dyadic. (The reader familiar with partial fields may also apply [13, Theorem 3.1] to the dyadic partial field.)

We remark that, for \( r \geq 4 \), the matroid \( U_r \) is not signed-graphic; this can be easily checked by using SageMath to show that \( U_4 \) is not a restriction of the rank-4 ternary Dowling geometry.

**Lemma 4.2.** If \( \Phi \in \{ \Phi_X, \Phi_C, \Phi_{Y_0} \Phi_{C,X}, \Phi_{C*X} \} \), then \( AG(2,3) \backslash e \in \overline{M}(\Phi) \).

**Proof.** Note that the following ternary matrix conforms to \( \Phi_{Y_0} \).

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1
\end{bmatrix}
\]

By contracting 9, pivoting on the first entry, we obtain the following representation of \( AG(2,3) \backslash e \) (with column labels matching the labels in Figure 1).

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & 0 & -1 & 1 & 1 & -1
\end{bmatrix}
\]

Therefore, \( AG(2,3) \backslash e \in \overline{M}(\Phi_{Y_0}) \). Moreover, by Lemma 3.8, we also have \( AG(2,3) \backslash e \in \overline{M}(\Phi_C) \).

Now, note that the following ternary matrix is a representation of \( AG(2,3) \backslash e \). Also note that the matrix conforms to \( \Phi_X \), with the top row indexed by \( X \) and the bottom two rows forming a \( \{1\} \)-frame matrix.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 0 & -1 & 0 & 1 & 1 & -1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 & 1 & -1
\end{bmatrix}
\]

Therefore, \( AG(2,3) \backslash e \in \overline{M}(\Phi_X) \). Moreover, by Lemma 3.8, we also have \( AG(2,3) \backslash e \in \overline{M}(\Phi_{C*X}) \) and \( AG(2,3) \backslash e \in \overline{M}(\Phi_{C*X^2}) \).

\[\Box\]

**Lemma 4.3.** Suppose \( \Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda) \) is a refined ternary frame template such that \( AG(2,3) \backslash e \not\in \overline{M}(\Phi) \). Then either \( \mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi_2) \), or \( \Phi \) is a \( Y \)-template.
Lemma 4.4. Let $\Phi$ be the complete, lifted $Y$-template determined by some matrix $P_0$. If $P_0$ contains as a submatrix some matrix listed in Table 1, then $\text{AG}(2, 3) \setminus e \in \mathcal{M}(\Phi)$.

Proof. Let $M = M([I \mid D | P_0'])$, where $P_0'$ is some submatrix listed in Table 1. By Lemma 3.19, it suffices to show that $\text{AG}(2, 3) \setminus e$ is a minor of $M$. For each of the matrices $A–K$, the authors used SageMath to show this. The computations are expedited by contracting some subset of $E(M)$ and simplifying the resulting matroid before testing if $\text{AG}(2, 3) \setminus e$ is a minor. If $P_0'$ is an $r \times c$ matrix, then $|E(M)| = \binom{r+1}{2} + c$. Label the elements of $M$, from left to right, as $\{0, 1, 2, \ldots, \binom{r+1}{2} + c - 1\}$ If $S$ is the set listed in Table 1 corresponding to the matrix $P_0'$, then the authors used SageMath to show that the simplification of
<table>
<thead>
<tr>
<th>Matrix</th>
<th>Set to Contract</th>
<th>Matrix</th>
<th>Set to Contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix}$</td>
<td>${10}$</td>
<td>$B = \begin{bmatrix} 1 &amp; 0 \ 1 &amp; 0 \ 1 &amp; 0 \ 0 &amp; 1 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>${15, 16}$</td>
</tr>
<tr>
<td>$C = \begin{bmatrix} 1 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>${0, 18, 23}$</td>
<td>$D = \begin{bmatrix} 1 &amp; 0 \ 1 &amp; 0 \ -1 &amp; 0 \ -1 &amp; 1 \ 0 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>${0, 1, 28, 29}$</td>
</tr>
<tr>
<td>$E = \begin{bmatrix} -1 &amp; 1 \ -1 &amp; -1 \ 1 &amp; -1 \ 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>${0, 16}$</td>
<td>$F = \begin{bmatrix} -1 &amp; 1 \ -1 &amp; -1 \ 1 &amp; 0 \ 1 &amp; 0 \ 0 &amp; 1 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>${0, 15, 22}$</td>
</tr>
<tr>
<td>$G = \begin{bmatrix} 1 &amp; 0 \ 1 &amp; 0 \ 1 &amp; 1 \ 0 &amp; 1 \ 0 &amp; -1 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>${0, 4, 22}$</td>
<td>$H = \begin{bmatrix} 1 &amp; 0 \ 1 &amp; 1 \ 1 &amp; -1 \ 0 &amp; 1 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>${0, 16}$</td>
</tr>
<tr>
<td>$I = \begin{bmatrix} 1 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; -1 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; -1 \end{bmatrix}$</td>
<td>${0}$</td>
<td>$J = \begin{bmatrix} -1 &amp; -1 &amp; 1 \ -1 &amp; 1 &amp; 1 \ 1 &amp; -1 &amp; 1 \ 1 &amp; 1 &amp; 0 \end{bmatrix}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>$K = \begin{bmatrix} 1 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 1 \end{bmatrix}$</td>
<td>${0, 15}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lemma 4.5. Let \( V \) be a ternary matrix consisting entirely of unit columns, and let \( W \) be a ternary matrix each of whose columns has exactly two nonzero entries, both of which are 1s. If \( \Phi \) is the complete, lifted \( Y \)-template determined by the following ternary matrix, then \( M(\Phi) \) is contained in the class of signed-graphic matroids.

\[
\begin{bmatrix}
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
V & & W
\end{bmatrix}
\]

Proof. Let \( P_0 \) be the matrix obtained from the given matrix by removing the top row. By Lemma 3.18(ii), \( \Phi \) is minor equivalent to the complete, lifted \( Y \)-template \( \Phi_{P_0} \). Every matroid \( M \) conforming to \( \Phi_{P_0} \) is a restriction of the vector matroid of a matrix of the following form. (If the rank of \( M \) is larger than the number of rows of \( P_0 \), then some of the rows of \( [V|W] \) below are zero rows.)

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & -1 & \cdots & -1 \\
0 & I & -I & D & V & W
\end{bmatrix}
\]

From the first row, subtract all other rows. The result is the following.

\[
\begin{bmatrix}
1 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & I & -I & D & V & W
\end{bmatrix}
\]

This is a ternary matrix such that each column has at most two nonzero entries. Therefore, \( M \) is a signed-graphic matroid. 

Lemma 4.6. Let \( \Phi \) be a refined ternary frame template with \( \text{AG}(2,3) \backslash e \notin M(\Phi) \). Either \( M(\Phi) \) is contained in the class of signed-graphic matroids, or \( \Phi \) is minor equivalent to the complete, lifted \( Y \)-template determined by a submatrix of the matrix \( T \) given at the beginning of this section.

Proof. By Lemma 4.3, since \( M(\Phi_2) \) is the class of signed-graphic matroids, we may assume that \( \Phi \) is a \( Y \)-template. Therefore, by Lemma 3.18(i), we may assume that \( \Phi \) is the complete, lifted \( Y \)-template \( \Phi_{P_0} \) determined by a matrix \( P_0 \) the sum of whose rows is the zero vector. The template \( \Phi_{P_0} \) is \( YT([D_n|P_0],[\emptyset]) \), where \( n \) is the number of rows of \( P_0 \). Since \( D_n \) is already included, we may assume that no column of \( P_0 \) is graphic. Since the only
weight-2 columns whose entries sum to 0 are graphic, every column of $P_0$ has at least three nonzero entries.

In the remainder of the proof, matrices $A$–$K$ are the matrices listed in Table 1. By Lemma 4.4, none of matrices $A$–$K$ may be submatrices of $P_0$. Since each of matrices $C$–$K$ has rows whose sum is the zero vector, Lemma 3.18(ii) implies that no matrix obtained from one of matrices $C$–$K$ by removing one row can be a submatrix of $P_0$ either.

By Lemma 3.18(ii) the complete, lifted Y-template determined by the matrix $[1, 1, 1, 1, -1]^T$ is minor equivalent to the complete, lifted Y-templates determined by both $[1, 1, 1, -1]^T$ and $[1, 1, 1, 1]^T$, which is matrix $A$. Since matrix $A$ is forbidden from being a submatrix of $P_0$, so is $[1, 1, 1, -1]^T$ by minor equivalence. Thus, no column of $P_0$ can contain four equal nonzero entries, and no column of $P_0$ with three equal nonzero entries can contain the negative of that nonzero entry. Therefore, by scaling columns of $P_0$, we may assume that every column of $P_0$ is of the form $[1, 1, 1, 0, \ldots, 0]^T$ or of the form $[1, 1, -1, -1, 0, \ldots, 0]^T$, up to permuting rows.

Since matrices $B$, $C$, and $K$ are forbidden, the intersection of the supports of all of the weight-3 columns of $P_0$ must be nonempty. Therefore, if $P_0$ consists entirely of weight-3 columns, then Lemma 3.18(ii) implies that $\Phi$ is minor equivalent to the template $\Phi'$ determined by the matrix obtained from $P_0$ by removing the row where every column has a nonzero entry. Every matrix conforming to $\Phi'$ has at most two nonzero entries per column. Therefore, every matroid conforming to the template is signed-graphic.

Thus, we may assume that $P_0$ has a weight-4 column. Let $G'$ be the matrix obtained by removing the row of matrix $G$ with two nonzero entries. If $P_0$ contains a weight-3 column and a weight-4 column whose supports have an intersection of size 0 or 1, then $P_0$ contains matrix $G'$ or $G$, respectively, up to permuting rows and scaling columns. Since matrices $G'$ and $G$ are both forbidden, every pair of columns, one of which has weight 3 and one of which has weight 4, must have supports with an intersection of size at least 2. Moreover, since matrices $H$, $I$, and $J$ are forbidden, if $P_0$ contains both weight-3 and weight-4 columns, then $P_0$ must be of the form given in Lemma 4.5. By that lemma, $\mathcal{M}(\Phi)$ consists of signed-graphic matroids.

Therefore, we may assume that $P_0$ consists entirely of weight-4 columns. Let $D'$ be the matrix obtained by removing the row of matrix $D$ with two nonzero entries. A similar argument to the one involving $G$ and $G'$ in the last paragraph shows that every pair of weight-4 columns must have supports whose intersection has size at least 2. Since matrices $E$ and $F$ are forbidden, either $\mathcal{M}(\Phi)$ consists of signed-graphic matroids, by Lemma 4.5, or restricting to the nonzero rows of $P_0$ results in a column submatrix of the matrix $T^+$ obtained from matrix $T$ by appending the row $[-1, -1, -1]$. Then by Lemma
We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will prove the statement in Theorem 1.1 that is dependent on Hypothesis 3.1. The statement that is dependent on Hypothesis 3.2 is proved similarly.

Recall that AG(2, 3) \( \setminus \{ \} \) is an excluded minor for the class of dyadic matroids. Since AG(2, 3) \( \setminus \{ \} \) is a restriction of PG(2, 3), Hypothesis 3.1, with \( \mathbb{F} = \text{GF}(3) \) and \( m = 3 \), implies that there exist \( k \in \mathbb{Z}_+ \) and frame templates \( \Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t \) such that every \( k \)-connected dyadic matroid \( M \) with at least \( 2k \) elements either is contained in one of \( \mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s) \) or has a dual \( M^* \) contained in one of \( \mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t) \). Moreover, each of \( \mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s), \mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t) \) is contained in the class of dyadic matroids. By Theorem 3.6, we may assume that these templates are refined. Let \( \Phi \) be any template in \( \{ \Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t \} \). By Lemma 4.6, either \( \mathcal{M}(\Phi) \) is contained in the class of signed-graphic matroids, or \( \Phi \) is minor equivalent to the complete, lifted \( Y \)-template determined by a submatrix of the matrix \( T \) given at the beginning of this section. In the former case, condition (1) of Theorem 1.1 holds. In the latter case, since \( U_r \) is the rank-\( r \) universal matroid for \( \Phi_T \), Lemma 3.19 implies that every matroid conforming to \( \Phi \) is a minor of some \( U_r \).

If we contract from \( U_r \) one of the elements that indexes a column of the matrix \( T \), then the simplification of the resulting matroid has the following representation matrix.

\[
\begin{bmatrix}
I_{r-1} & D_{r-1} \\
-1 & \cdots & -1 & 1 \\
1 & \cdots & 1 & 1 \\
-1 & \cdots & 0 \\
-1 & \cdots & 0
\end{bmatrix}
\]

We see that this matroid is signed-graphic by adding to the second row the negatives of all other rows. Therefore, every simple matroid \( M \in \mathcal{M}(\Phi) \) is either signed-graphic or a restriction of some \( U_r \). (Since we are dealing with highly connected matroids, they must also be simple. Therefore, we are only concerned with simple matroids.) We also see that we may choose \( r = r(M) \) because otherwise there are rows of the matrix representing \( M \) that can be removed without changing the matroid.

Now we prove Theorem 1.2.
Proof of Theorem 1.2. We will use Lemma 3.20 to prove Theorem 1.2. Let
$E_1 = \{AG(2,3)\setminus e\}$, and let $E_2 = \{\{AG(2,3)\setminus e\}^*\}$. Since $AG(2,3)\setminus e$ is ternary but not dyadic, and since the classes of ternary and dyadic matroids are closed under duality, conditions (i) and (ii) of Lemma 3.20 are satisfied. Since every signed-graphic matroid is dyadic and $U_r$ is dyadic for every $r$, Lemma 4.6 implies that condition (iii) of Lemma 3.20 is satisfied. Then (iv) follows from (iii) and duality. The result follows. ■

5. Near-Regular and $\sqrt{1}$-Matroids

In this section, we prove Theorem 1.3 after proving several lemmas.

Lemma 5.1. Let $\Phi \in \{\Phi_2, \Phi_X, \Phi_C, \Phi_{Y_0}, \Phi_{CX}, \Phi_{CX_2}\}$. Then the non-Fano matroid $F_7^-$ is a minor of some member of $M(\Phi)$. Therefore, $M(\Phi)$ is not contained in the class of $\sqrt{1}$-matroids.

Proof. It is well known (see [15, Proposition 6.4.8]) that $F_7^-$ is $F$-representable if and only if the characteristic of $F$ is not 2. Thus, $F_7^-$ is not representable over $GF(4)$ and is therefore not a $\sqrt{1}$-matroid. Therefore, the last statement of the lemma follows from the first part of the lemma.

We saw in Section 4 that $M(K_4)$ can be obtained from the rank-3 ternary Dowling geometry by deleting the three joints. If we leave one of the joints in place, then it is contained in the closures of exactly two pairs of points that are not collinear in $M(K_4)$. The result is the non-Fano matroid $F_7^-$. Therefore, the $F_7^-$ is signed-graphic, implying that $F_7^- \in M(\Phi_2)$.

The following ternary matrix is a representation of $F_7^-$ that conforms to both $\Phi_X$ and $\Phi_{Y_0}$.

\[
\begin{bmatrix}
1 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & -1 & -1 \\
\end{bmatrix}
\]

Therefore, $F_7^- \in M(\Phi_X)$ and $F_7^- \in M(\Phi_{Y_0})$. By Lemma 3.8, we also have that $F_7^-$ is a minor of matroids conforming to $\Phi_C$, $\Phi_{CX}$, and $\Phi_{CX_2}$. ■

Recall that the matroid $T_r^1$ is obtained from the complete graphic matroid $M(K_{r+2})$ by adding a point freely to a triangle, contracting that point, and simplifying. For $r \geq 2$, Semple (see [20, Section 2] and [19, Proposition 3.1]) showed that $T_r^1$ is representable over a field $F$ if and only if $F \neq GF(2)$. Therefore, every $T_r^1$ is near-regular. The following matrix represents $T_r^1$ over every field of characteristic other than 2.

\[
\begin{bmatrix}
I_r & D_r \\
\hline
1 \cdots 1 \\
I_{r-1}
\end{bmatrix}
\]
Lemma 5.2. Let $\Phi$ be the complete, lifted $Y$-template determined by some ternary matrix $P_0$. Either $F_7^-$ is a minor of some matroid conforming to $\Phi$, or every member of $\mathcal{M}(\Phi)$ has a simplification that is a restriction of some $T^1_r$.

Proof. Suppose $P_0 = [1, 1, 1, 1]^T$. Consider the vector matroid of $[I_4|D_4|P_0]$. By contracting the element represented by $P_0$, we obtain a matroid containing $F_7^-$ as a restriction. Therefore, no column of $P_0$ can contain four equal nonzero entries. If $P_0 = [1, 1, -1]^T$, then $[I_3|D_3|P_0]$ is (up to column scaling) the representation of $F_7^-$ given in Matrix (5.1). Therefore, if a column of $P_0$ contains unequal nonzero entries, then it can contain no other nonzero entry. Thus, the column is a graphic column, which is already assumed to be included in a complete $Y$-template.

Thus, every column of $P_0$ contains at most three nonzero entries, all of which are equal. Consider the following matrices.

$$
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
$$

If $P_0$ is either of these matrices, then $M([I|D|P_0])$ contains $F_7^-$ as a minor. (This can be easily checked with SageMath or by hand. If $P_0$ is the first matrix, contracting one of the elements represented by a column of $P_0$ and simplifying results in the rank-3 ternary Dowling geometry. We saw in the proof of Lemma 5.1 that $F_7^-$ is signed-graphic, implying that it is a restriction of the rank-3 ternary Dowling geometry. If $P_0$ is the second matrix, then $M([I|D|P_0])$ is itself the rank-3 ternary Dowling geometry.)

It is routine to check that the class of matroids whose simplifications are restrictions of some $T^1_r$ is minor-closed. Therefore, it suffices to consider a template that is minor equivalent to $\Phi$. Thus, by Lemma 3.18(i), we may assume that the sum of the rows of $P_0$ is the zero vector. Therefore, we may assume that every column has exactly three nonzero entries all of which are equal. Because the two matrices above are forbidden, $P_0$ is of the following form, where $V$ consists entirely of unit columns.

$$
\begin{bmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
V
\end{bmatrix}
$$

Now, Lemma 3.18(ii) implies that $\Phi$ is minor equivalent to the complete, lifted $Y$-template $\Phi'$ determined by the matrix obtained by removing the top row from $P_0$. Every member of $\mathcal{M}(\Phi')$ has a simplification that is a restriction of some $T^1_r$. ■
We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We prove the statement in Theorem 1.3 that is dependent on Hypothesis 3.1. The statement dependent on Hypothesis 3.2 is proved similarly.

Every near-regular matroid is also a $\sqrt{1}$-matroid; therefore, (1) implies (2). We will now show that (2) implies (3). By Hypothesis 3.1, there exist ternary frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ and a positive integer $k_1$ such that every matroid conforming to these templates is a $\sqrt{1}$-matroid and such that every simple $k_1$-connected $\sqrt{1}$-matroid with at least $2k_1$ elements either conforms or coconforms to one of these templates. By Theorem 3.6, we may assume that these templates are refined. Lemmas 5.1 and 3.11 imply that each of these templates is a $Y$-template. Then Lemmas 3.18 and 5.2 imply that every matroid conforming to these templates has a simplification isomorphic to a restriction of $T^1_r$ for some $r$ (because the class of matroids whose simplifications are restrictions of $T^1_r$ is minor-closed). We see that we may choose $r = r(M)$ because otherwise there are rows of the matrix representing $M$ that can be removed without changing the matroid. By taking $k \geq k_1$, we see that (2) implies (3).

Since $T^1_r$ is near-regular for every $r$, (3) implies (1). We complete the proof of the theorem by showing the equivalence of (2) and (4) using Lemma 3.20. In that lemma, let $\mathcal{M}$ be the class of $\sqrt{1}$-matroids, let $\mathcal{E}_1 = \{F^-_7\}$, and let $\mathcal{E}_2 = \{(F^-_7)^*\}$. Since $F^-_7$ and $(F^-_7)^*$ are ternary matroids that are not $\sqrt{1}$-matroids, conditions (i) and (ii) of Lemma 3.20 are satisfied. Combining Lemmas 5.1 and 3.11, we see that for every refined frame template $\Phi$ that is not a $Y$-template, $F^-_7 \in \mathcal{M}(\Phi)$. Combining Lemmas 3.18(i), and 5.2, we see that $F^-_7 \in \mathcal{M}(\Phi)$ for every $Y$-template $\Phi$ such that $\mathcal{M}(\Phi)$ is not contained in the class of matroids whose simplifications are restrictions of some $T^1_r$. Since $T^1_r$ is a $\sqrt{1}$-matroid for every $r$, condition (iii) of Lemma 3.20 holds. Finally, condition (iv) of Lemma 3.20 holds because of condition (iii) and the fact that the class of $\sqrt{1}$-matroids is closed under duality. Therefore, by Lemma 3.20, there is a positive integer $k_2$ such that a $k_2$-connected ternary matroid with at least $2k_2$ elements is a $\sqrt{1}$-matroid if and only if it contains no minor isomorphic to either $F^-_7$ or $(F^-_7)^*$. By taking $k \geq k_2$, we see that (2) is equivalent to (4), completing the proof.

Appendix A. SageMath Code

We give here an example of the code for the computations used to prove Lemma 4.4. The function $\text{complete}_Y\text{template}_\text{matrix}$ takes as input a matrix $P_0$ and returns the matrix $[I|D|P_0]$. The code below returns True,
showing that if $P_0$ is matrix $C$ from Table 1, then $M([I|D|P_0])\{0,18,23\}$ contains $AG(2,3)\{\epsilon\}$ as a minor.

N=matroids.named_matroids.AG23minus()
# N is the matroid AG(2,3)\{\epsilon\}

def complete_Y_template_matrix(P0):
    k=P0.nrows()
    num_elts=k+k*(k-1)/2+P0.ncols()
    F=P0.base_ring()
    A = Matrix(F, k, num_elts)
    i = 0
    # identity in front
    for j in range(k):
        A[j,j] = 1
    i = k
    # all pairs
    for S in Subsets(range(k),2):
        A[S[0],i]=1
        A[S[1],i]=-1
        i = i + 1
    # Columns from $Y_0$
    for l in range(P0.ncols()):
        for j in range(k):
            A[j, i] = P0[j, l]
        i=i+1
    return A

P0 = Matrix(GF(3), [[1,1,0],
                    [1,0,1],
                    [0,1,1],
                    [1,0,0],
                    [0,1,0],
                    [0,0,1]])
A=complete_Y_template_matrix(P0)
M=Matroid(field=GF(3), matrix=A)
((M/0/18/23).simplify()).has_minor(N)

References


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