THE SYMMETRIC STRONG CIRCUIT ELIMINATION PROPERTY

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Abstract. If C_1 and C_2 are circuits in a matroid M with e_1 in $C_1 - C_2$ and e in $C_1 \cap C_2$, then M has a circuit C_3 such that $e \in C_3 \subseteq (C_1 \cup C_2) - e$. This strong circuit elimination axiom is inherently asymmetric. A matroid M has the symmetric strong circuit elimination property (SSCE) if, when the above conditions hold and $e_2 \in C_2 - C_1$, there is a circuit C_3' with $\{e_1, e_2\} \subseteq C_3' \subseteq (C_1 \cup C_2) - e$. We prove that a connected matroid has this property if and only if it has no two skew circuits. We also characterize such matroids in terms of forbidden series minors, and we give a new matroid axiom system that is built around a modification of SSCE.

1. Introduction

A matroid M has the *symmetric strong circuit elimination property* (SSCE) if, whenever C_1 and C_2 are circuits and e_1, e_2, and e are elements such that $e_1 \in C_1 - C_2$, $e_2 \in C_2 - C_1$, and $e \in C_1 \cap C_2$, there is a circuit C_3 that contains $\{e_1, e_2\}$ and is contained in $(C_1 \cup C_2) - e$. Sets X and Y in a matroid are *skew* if $r(X) + r(Y) = r(X \cup Y)$. The next theorem, the main result of this paper, gives several characterizations of matroids satisfying SSCE. A matroid M is *unbreakable* if M is connected and M/F is connected for every flat F of M.

Theorem 1.1. *The following are equivalent for a connected matroid M.*

(i) M has the symmetric strong circuit elimination property;
(ii) M has no pair of skew circuits;
(iii) for all integers k and l exceeding two, M has no series minor isomorphic to $S(U_{k-2,k}, U_{l-2,l})$; and
(iv) M^* is unbreakable.

To see that not every connected matroid has the symmetric strong circuit elimination property, consider the matroid N_5 that is obtained

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from a 3-circuit $\{e_1, e_2, e\}$ by adding f_i in parallel to e_i for each i in $\{1, 2\}$. Consider the circuits $C_1 = \{e_1, f_2, e\}$ and $C_2 = \{e_2, f_1, e\}$. Then $e_1 \in C_1 - C_2$ and $e_2 \in C_2 - C_1$, but N_5 has no circuit contained in $(C_1 \cup C_2) - e$ that contains $\{e_1, e_2\}$. Clearly N_5 has $\{e_1, f_1\}$ and $\{e_2, f_2\}$ as skew circuits.

The equivalence between (ii) and (iii) in Theorem 1.1 extends a result of Drummond, Fife, Grace, and Oxley [2, Proposition 15], which shows that a connected binary matroid M has a pair of skew circuits if and only if M has a series minor isomorphic to N_5. Note that N_5 is isomorphic to the series connection of two copies of $U_{1,3}$. A matroid M is *circuit-difference* if $C_1 \Delta C_2$ is a circuit for every distinct intersecting pair of circuits C_1 and C_2 of M. Drummond et al. [2, Theorem 1] proved the following.

Theorem 1.2. *A connected regular matroid is circuit difference if and only if it has no pair of skew circuits.*

Combining Theorems 1.1 and 1.2, we immediately obtain the following.

Corollary 1.3. *A connected regular matroid has the symmetric strong circuit elimination property if and only if M is circuit-difference.*

Oxley and Pfeil [5, Theorem 1.1] proved that a loopless matroid M is unbreakable if and only if M^* has no pair of skew circuits. Thus (ii) and (iv) in Theorem 1.1 are equivalent. Oxley and Pfeil gave several other characterizations of connected matroids with no skew circuits. Thus the list of equivalent statements in Theorem 1.1 could be extended.

Theorem 1.1 gives several characterizations of when a connected matroid does not have a pair of skew circuits. The next theorem characterizes when a connected binary matroid has three skew circuits where, for any integer k exceeding one, a matroid M has k *skew circuits* if M has a set $\{C_1, C_2, \ldots, C_k\}$ of k circuits such that

$$M\big| \left(\cup_{i=1}^{k} C_i\right) = (M|C_1) \oplus (M|C_2) \oplus \ldots \oplus (M|C_k).$$

Theorem 1.4. *Let M be a connected binary matroid with three skew circuits. Then M has a series minor isomorphic to $M(G)$, where G is one of the graphs shown in Figure 1.*

After a section of preliminaries, the main theorem is proved in Section 3. Section 4 proves Theorem 1.4. Finally, in Section 5, we give a new circuit axiom system for matroids that is built around a modification of SSCE.

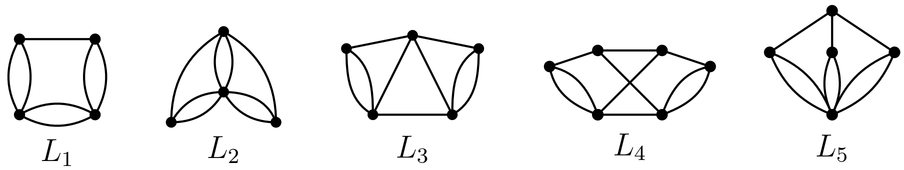

FIGURE 1

2. PRELIMINARIES

The terminology and notation used here will follow [4]. The class of matroids satisfying SSCE is not closed under taking contractions. To see this, recall that N_5 has a pair of skew circuits. Although $M(K_4)$ has no pair of skew circuits, every single-element contraction of $M(K_4)$ is isomorphic to N_5. We now show that the class of matroids satisfying SSCE is closed under series contraction.

Lemma 2.1. *The class of matroids satisfying the symmetric strong circuit elimination property is closed under series minors.*

Proof. Clearly the class of matroids satisfying SSCE is closed under deletion. Now suppose that M satisfies SSCE and has $\{f, g\}$ as a cocircuit. Then, for every circuit C of M, either C or $C - g$ is a circuit of M/g. Let C_1 and C_2 be circuits of M/g with e in $C_1 \cap C_2$ such that $e_1 \in C_1 - C_2$ and $e_2 \in C_2 - C_1$. Since $\{f, g\}$ is a cocircuit of M, we have $\mathcal{C}(M/g) = \mathcal{D}_1 \cup \mathcal{D}_2$ where $\mathcal{D}_1 = \{C \in \mathcal{C}(M) : \{f, g\} \cap C = \emptyset\}$ and $\mathcal{D}_2 = \{C - g : g \in C \in \mathcal{C}(M)\}$.

If $C_1, C_2 \in \mathcal{D}_1$, then M has a circuit C_3 such that $\{e_1, e_2\} \subseteq C_3 \subseteq (C_1 \cup C_2) - e$. Thus $g \notin C_3$ so $C_3 \in \mathcal{C}(M/g)$. Now suppose $C_1 \in \mathcal{D}_1$ and $C_2 \in \mathcal{D}_2$. Then $C_2 \cup g$ is a circuit of M and M has a circuit C_3 such that $\{e_1, e_2\} \subseteq C_3 \subseteq (C_1 \cup C_2 \cup g) - e$. If $g \in C_3$, then $C_3 - g$ is a circuit of M/g containing $\{e_1, e_2\}$ such that $C_3 - g \subseteq (C_1 \cup C_2) - e$. If $g \notin C_3$, then C_3 is a circuit of M containing $\{e_1, e_2\}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$. Similarly, if $C_1, C_2 \in \mathcal{D}_2$, then M has a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2 \cup g) - e$ and $\{e_1, e_2\} \subseteq C_3$. Either C_3 or $C_3 - g$ is a circuit of M/g contained in $(C_1 \cup C_2) - e$ and containing $\{e_1, e_2\}$. □

The next three lemmas will be used to prove the main results.

Lemma 2.2. *Let M be a connected matroid with ground set $D_1 \cup D_2 \cup e$ where D_1 and D_2 are skew circuits of M. Then either M has a 2-cocircuit avoiding e, or $M \cong S((U_{k-2,k}; e), (U_{l-2,l}; e))$ for some integers k and l exceeding two.*

Proof. Since $M\backslash e = M|D_1 \oplus M|D_2$, it follows that M is equal to $S((M_1;e),(M_2;e))$ where $M_1 = M/D_2$ and $M_2 = M/D_1$. Evidently, $r(M) = r(D_1) + r(D_2) = |D_1| + |D_2| - 2$, so $r(M^*) = 3$ and M^* has $D_1 \cup e$ and $D_2 \cup e$ as hyperplanes. Thus either M^* has a 2-circuit avoiding e, or $M^* \cong P((U_{2,k};e),(U_{2,l};e))$ for some integers k and l exceeding two. The lemma follows immediately. □

The proof of the next lemma uses the well-known fact (see, for example, [4, Exercise 2.1.7]) that $\{x, y\}$ is a circuit of a connected matroid if and only if every circuit that contains x also contains y.

Lemma 2.3. *Let $S((M_1;e),(M_2;e))$ be a connected matroid M having a circuit contained in $E(M_2) - e$. Then M has $S((M_1;e),(U_{k-2,k};e))$ as a series minor for some k exceeding two. Moreover, $E(U_{k-2,k}) - e$ is skew to $E(M_1) - e$ in M.*

Proof. Clearly, if M has $S((M_1;e),(U_{k-2,k};e))$ as a series minor, then $E(U_{k-2,k}) - e$ is skew to $E(M_1) - e$ since $S((M_1;e),(U_{k-2,k};e))\backslash e = (M_1\backslash e) \oplus (U_{k-2,k}\backslash e)$. To see that M has $S((M_1;e),(U_{k-2,k};e))$ as a series minor, let M be a minimal counterexample. Let C be a circuit contained in $E(M_2) - e$. Choose a circuit D of M_2 that contains e and meets C so that $|D - C|$ is a minimum. Then $E(M_2) = D \cup C$, otherwise $S((M_1;e),(M_2|(D \cup C);e))$ is a proper series minor of M, contradicting the minimality of M.

We now show that $D-C$ is a series class of M_2. Take $x \in (D-C)-e$. Suppose M_2 has a circuit D' containing e and avoiding x. Then $D' \subseteq (D - x) \cup C$, so $|D' - C| < |D - C|$. Therefore x is in every circuit of M_2 containing e, so $\{x, e\}$ is a cocircuit of M_2. Thus $D - C$ is a series class of M_2.

Since M is series-minor-minimal, it follows that $E(M_2) \cap (D - C) = \{e\}$. Then $r(M_2) = r(C) = |C| - 1$ so $r^*(M_2) = 2$. Since M_2^* is connected of rank two having $\{e\}$ as a flat, it follows that $M_2^* \cong U_{2,k}$ for some k exceeding two. Note that M_2^* has no nontrivial parallel class, since M_2^* is parallel-minor-minimal. Hence $M_2 \cong U_{k-2,k}$, so $M \cong S((M_1;e),(U_{k-2,k},e))$, a contradiction. □

For the next result, recall that N_5 is the series connection of two copies of $U_{1,3}$.

Lemma 2.4. *Let M be a connected binary matroid containing an element e and a pair of skew circuits both avoiding e. Then, for some i in $\{1, 2, 3, 4, 5\}$, M contains $M(G_i)$ as series minor, where G_i is one of the five graphs in Figure 2. Each such graphic matroid $M(G_i)$ has a unique pair of skew circuits avoiding e, indicated in bold in Figure 2.*

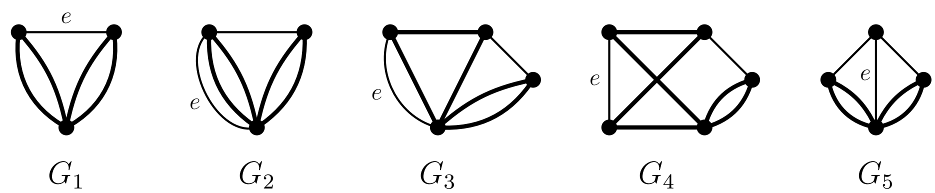

FIGURE 2

Proof. Let M be a minimal counterexample. If $E(M) = C_1 \cup C_2 \cup \{e\}$, then $M = M(G_1) \cong N_5$, since $M \backslash e = M|(C_1 \cup C_2) = M|C_1 \oplus M|C_2$, and M has no nontrivial series class. Now suppose M has an element f contained in $E(M) - (C_1 \cup C_2 \cup e)$ such that $M \backslash f$ is disconnected. Thus $M = S((M_1; f), (M_2; f))$, where M_1 and M_2 are nonempty, connected binary matroids.

Suppose first that C_1 and C_2 are circuits in M_1 and that e is an element of M_2. Take a circuit D of M_2 containing $\{e, f\}$. Then $E(M_2) = D$ by the minimality of M. Further, $\{e, f\}$ is a 2-cocircuit of M, a contradiction. We may now assume that $C_1 \cup e \subseteq E(M_1)$ and $C_2 \subseteq E(M_2)$. Since M is binary, it follows from Lemma 2.3 that $M = S((M_1; f), (U_{1,3}; f))$.

Suppose $E(M_1) = C_1 \cup \{e, f\}$. Then $r(M_1) = r(C_1)$, otherwise $\{e, f\}$ is a cocircuit of M. Hence $r^*(M_1) = 3$. It follows that M_1^* is a rank-3 binary matroid containing C_1 as a cocircuit and $\{e, f\}$ as a hyperplane. Hence $|C_1| \in \{2, 3, 4\}$ and $r(M_1) \in \{1, 2, 3\}$. For each possible value of $r(M_1)$, there is a unique matroid satisfying these conditions, namely $U_{1,4}$ containing C_1 as a 2-circuit, N_5 containing C_1 as a 3-circuit, or $M(K_4)$ containing C_1 as a 4-circuit. Hence M_1 is isomorphic to one of $U_{1,4}$, N_5, and $M(K_4)$. Since $M = S((M_1; f), (U_{1,3}; f))$, it follows that M is isomorphic to $M(G)$ for some $G \in \{G_2, G_3, G_4\}$.

We may now assume that M_1 has an element $g \in E(M_1) - (C_1 \cup \{e, f\})$. Then $M_1 \backslash g$ is disconnected, so $M_1 = S((N_1; g), (N_2; g))$, where N_1 and N_2 are connected, binary matroids. If g is in a series pair in M_1, then g is in a series pair of M, a contradiction. It follows that we may assume that $\{e, f\} \subseteq E(N_1)$, while $C_1 \subseteq E(N_2)$. By Lemma 2.3, we have that $M_1 = S((N_1; g), (U_{1,3}; g)$.

Suppose N_1 has a circuit C avoiding $\{e, f\}$. Then $N_1 \backslash x$ is disconnected for all x in $C - g$. It follows by [3, Lemma 2.3] (see also [4, Lemma 4.3.10]) that $C - g$ contains a 2-cocircuit of N_1. As this is a contradiction, every circuit of N_1 must meet $\{e, f\}$. It follows that $\{e, f\}$ contains a cobasis of N_1, so $r(N_1^*) = 1$, or $r(N_1^*) = 2$. If $r(N_1^*) = 1$, then N_1 is a circuit and $\{e, f, g\}$ is a series class of M_1, a contradiction. Thus $r(N_1^*) = 2$. Since N_1^* is binary and parallel-minor minimal, $N_1^* \cong U_{2,3}$. Therefore $M_1 = S((U_{1,3}; g), (U_{1,3}; g) \cong N_5$, containing 2-circuits C_1 and $\{e, f\}$. We conclude that $M \cong M(G_5)$. □

Note that the graphs G_3 and G_5 are isomorphic up to a relabeling of the special element e that is specified in the statement of Lemma 2.4. The unique pair of skew circuits avoiding e in $M(G_3)$ have sizes two and three, while, in $M(G_5)$, the unique pair of skew circuits both have size two.

3. Proof of the Main Theorem

In this section, we prove Theorem 1.1 by showing that (iii) implies (ii), that (ii) implies (i), and that (i) implies (iii). The equivalence of (ii) and (iv) was proved in [5, Theorem 1.1].

Proof of Theorem 1.1. To see that (iii) implies (ii), let M be a connected matroid satisfying (iii) and suppose that M has C_1 and C_2 as skew circuits but that no proper connected series minor of M has a pair of skew circuits. Let D be a circuit of M that intersects both C_1 and C_2 such that $|D-(C_1 \cup C_2)|$ is minimal. Since $M|(C_1 \cup C_2 \cup D)$ is connected and C_1 and C_2 are skew in this restriction, $M = M|(C_1 \cup C_2 \cup D)$. We will show that $|D-(C_1 \cup C_2)| = 1$. Suppose $\{f, g\} \subseteq D-(C_1 \cup C_2)$. We shall show that f and g are in series in M. Assume they are not. Then M has a circuit K that contains f but not g. By the choice of D, the circuit K cannot meet both C_1 and C_2. Since K is not a proper subset of D, we may assume that $K \cap (C_1 - D) \neq \emptyset$ and $K \cap C_2 = \emptyset$. Take h in $D \cap C_2$. Then M has a circuit C_3 such that $h \in C_3 \subseteq (D \cup K) - f$. Because C_3 is not a proper subset of D, there is an element k in $C_3 - D$. Thus C_3 meets both C_1 and C_2 but avoids f, a contradiction to the choice of D. We conclude that f and g are in series in M. Then M/g is a connected series minor of M having C_1 and C_2 as skew circuits. This contradiction to the choice of M implies that $|D-(C_1 \cup C_2)| = 1$. Let $D - (C_1 \cup C_2) = \{e\}$. Then since M has no 2-cocircuits, it follows from Lemma 2.2 that $M \cong S(U_{k-2,k}, U_{l-2,l})$ for some integers k and l exceeding two, which contradicts (iii). Thus (iii) implies (ii).

To see that (ii) implies (i), let M be a connected matroid satisfying (ii) but not (i). Let C_1 and C_2 be distinct circuits of M with $e \in C_1 \cap C_2$, $e_1 \in C_1 - C_2$, and $e_2 \in C_2 - C_1$. Then M has no circuit D such that $\{e_1, e_2\} \subseteq D \subseteq (C_1 \cup C_2) - e$. For each $i \in \{1, 2\}$, there is a circuit D_i of M such that $e_i \in D_i$ and $D_i \subseteq (C_1 \cup C_2) - e$. By assumption,

D_1 and D_2 are not skew. Then $M|(D_1 \cup D_2)$ is connected, so M has a circuit D containing $\{e_1, e_2\}$ with $D \subseteq D_1 \cup D_2$. It follows that $\{e_1, e_2\} \subseteq D \subseteq (C_1 \cup C_2) - e$, a contradiction. We conclude that (ii) implies (i).

Now assume that M satisfies (i) but not (iii). Let N be the series minor of M that is isomorphic to $S(N_1, N_2)$, with $N_1 \cong U_{k-2,k}$ and $N_2 \cong U_{l-2,l}$ for some k and l exceeding two. Then N satisfies SSCE by Lemma 2.1. Let p be the basepoint of the series connection N. Choose distinct circuits C_1 and C_2 of N_1, and distinct circuits D_1 and D_2 of N_2 so that all of C_1, C_2, D_1, and D_2 contain p. Let $e_1 \in C_1 - C_2$ and $e_2 \in D_2 - D_1$. Then N has circuits K_1 and K_2 such that $K_1 = C_1 \cup D_1$ and $K_2 = C_2 \cup D_2$. Clearly $e_1 \in K_1 - K_2$ and $e_2 \in K_2 - K_1$, while $p \in K_1 \cap K_2$. Since $e_1 \in E(N_1)$ and $e_2 \in E(N_2)$, every circuit of N containing $\{e_1, e_2\}$ must also contain p. Hence N does not satisfy SSCE, a contradiction. Thus (i) implies (iii). We conclude that the theorem holds. □

4. Connected binary matroids with three skew circuits

The goal of this section is to complete the proof of Theorem 1.4. The core of this proof is contained in Lemmas 2.3 and 2.4. After giving this proof, we consider potential extensions of this theorem.

Proof of Theorem 1.4. Let M be a minimal counterexample and let C_1, C_2, and C_3 be skew circuits in M. Since M is connected, there is an element e in $E(M) - (C_1 \cup C_2 \cup C_3)$. Then $M \backslash e$ is disconnected, so $M = S((M_1; e), (M_2; e))$; that is, M is a series connection of connected matroids M_1 and M_2 across the basepoint e.

Suppose first that $\{C_1, C_2, C_3\} \subseteq \mathcal{C}(M_1)$. Take a circuit C of M containing e and let $N = M|(E(M_1) \cup C)$. The matroid N has $C \cap E(M_2)$ contained in a series class. Now $N/((C \cap E(M_2)) - e)$ is a connected, binary, proper series minor of M having C_1, C_2, and C_3 as skew circuits. Thus we contradict the minimality of M.

We may now assume that $C_1, C_2 \in \mathcal{C}(M_1)$ and $C_3 \in \mathcal{C}(M_2)$. Since M is binary, by Lemma 2.3, $M = S((M_1; e), (U_{1,3}; e))$ with $E(U_{1,3}) - e$ skew to $E(M_1) - e$. By Lemma 2.4, the matroid M_1 is isomorphic to the cycle matroid of one of the graphs pictured in Figure 2. Hence $M \cong S((M(G_i); e), (U_{1,3}; e)) \cong M(L_i)$ for some i in $\{1, 2, 3, 4, 5\}$. □

The techniques used in the proof of Theorem 1.4 can be extended to prove analagous results for connected binary matroids containing k skew circuits, for $k \geq 4$. As one may gather from the proof of

Theorem 1.4, the number of cases required to obtain an exhaustive list of connected, binary, series-minor-minimal matroids containing k skew circuits becomes unmanageably large as k increases.

To see why an analagous result for non-binary matroids is not included in this paper, let M' be isomorphic to the direct sum of k circuits. Let M be the matroid obtained by freely adding an element e to M'. Then M is connected and non-binary, containing k skew circuits. The addition of e has the effect of turning every 2-cocircuit of M' into a 3-cocircuit of M containing e. Thus M contains no connected series minor containing k skew circuits.

5. A new circuit axiom system

The symmetric strong circuit elimination property does not hold for all matroids, and is therefore not equivalent to the weak and strong circuit elimination axioms, **(C3)** and **(C3)**$'$ in [4, pp.9 and 29]. By adding an additional hypothesis to the definition of SSCE presented in Section 1, we are able to provide a symmetric variant of the well-known circuit elimination axioms.

Lemma 5.1. *The set $\mathscr{C}$ of circuits of a matroid M obeys the following.*

(C3)$''$ *Let C_1 and C_2 be members of $\mathscr{C}$ with $e_1 \in C_1 - C_2$ and $e_2 \in C_2 - C_1$. If $e \in C_1 \cap C_2$ and $(C_1 - e_1) \cup (C_2 - e_2)$ contains no member of $\mathscr{C}$, then $\mathscr{C}$ contains a member C_3 such that $\{e_1, e_2\} \subseteq C_3 \subseteq (C_1 \cup C_2) - e$.*

Furthermore, C_3 is the unique circuit of M contained in $(C_1 \cup C_2) - e$.

Proof. Certainly $(C_1 \cup C_2) - e$ is dependent. Let C_3 be a circuit contained in this set. We shall show first that $\{e_1, e_2\} \subseteq C_3$. As $(C_1 - e_1) \cup (C_2 - e_2)$ is independent, we may assume that $e_1 \in C_3$. Suppose $e_2 \notin C_3$. Then $e_1 \in C_1 \cap C_3$ and $e \in C_3 - C_1$, so there is a circuit C_4 such that $C_4 \subseteq (C_1 \cup C_3) - e_1$. Thus $C_4 \subseteq (C_1 - e_1) \cup (C_2 - e_2)$, a contradiction. We deduce that $\{e_1, e_2\} \subseteq C_3$.

To see that C_3 is unique, suppose there is a second circuit C_3' contained in $(C_1 \cup C_2) - e$. Then $e_1 \in C_3 \cap C_3'$, so M has a circuit C_5 contained in $(C_3 \cup C_3') - e_1$. As C_5 is contained in $(C_1 \cup C_2) - e$, we deduce that $\{e_1, e_2\} \subseteq C_5$, a contradiction. Hence C_3 is indeed unique. □

The following theorem seems to give a new axiom system for matroids in terms of their circuits. For example, it is absent from the two standard reference books for the subject [4,6] and also does not appear in Brylawski's encyclopedic appendix of matroid cryptomorphisms [1].

Theorem 5.2. *A collection* $\mathscr{C}$ *of nonempty pairwise incomparable subsets of a finite set* E *is the set of circuits of a matroid on* E *if and only if* $\mathscr{C}$ *satisfies* **(C3)**$''$.

Proof. By Lemma 5.1, if $\mathscr{C}$ is the set of circuits of a matroid on E, then $\mathscr{C}$ satisfies **(C3)**$''$. Conversely, assume $\mathscr{C}$ satisfies **(C3)**$''$. Suppose C_1 and C_2 are distinct members of $\mathscr{C}$ with e in $C_1 \cap C_2$. Assume that **(C3)** fails for (C_1, C_2, e) and that $|C_1 \cup C_2|$ is a minimum among such triples. As the members of $\mathscr{C}$ are incomparable, there are elements e_1 and e_2 of $C_1 - C_2$ and $C_2 - C_1$, respectively. By **(C3)**$''$, $(C_1 - e_1) \cup (C_2 - e_2)$ must contain a member C_4 of $\mathscr{C}$, so $e \in C_4$. Then $e \in C_1 \cap C_4$ and $|C_1 \cup C_4| \le |(C_1 \cup C_2) - e_2| < |C_1 \cup C_2|$, so $(C_1 \cup C_4) - e$, and hence $(C_1 \cup C_2) - e$ contains a member of $\mathscr{C}$, a contradiction. □

It is tempting to try to weaken **(C3)**$''$ to require only that $e_1 \in C_1$ and $e_2 \in C_2$. To see that this variant need not hold, consider the cycle matroid of the graph $K_{2,3}$ and let C_1 and C_2 be the circuits $\{e_1, a, e, e_2\}$ and $\{b, c, e, e_2\}$. Then $(C_1 - e_1) \cup (C_2 - e_2)$ does not contain a circuit. But, although $(C_1 \cup C_2) - e$ does contain a circuit, that circuit does not contain e_2.

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