MATROIDS WITH MANY SMALL CIRCUITS AND COCIRCUITS

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ABSTRACT. Tutte proved that a non-empty 3-connected matroid with every element in a 3-element circuit and a 3-element cocircuit is either a whirl or the cycle matroid of a wheel. This result led to the Splitter Theorem. More recently, Miller proved that a matroid of sufficient size with every pair of elements in a 4-element circuit and a 4-element cocircuit is a tipless spike. Here we investigate matroids having similar restrictions on their small circuits and cocircuits. In particular, we completely determine the 3-connected matroids with every pair of elements in a 4-element circuit and every element in a 3-element cocircuit, as well as the 4-connected matroids with every pair of elements in a 4-element circuit and every element in a 4-element cocircuit.

1. INTRODUCTION

The study of matroids with many small circuits and cocircuits begins with Tutte's well-known Wheels-and-Whirls Theorem [6]. This theorem was originally stated in terms of *essential* elements of a 3-connected matroid M, that is, elements e of M with the property that neither $M\backslash e$ nor M/e is 3-connected. We present it here in terms of 3-circuits and 3-cocircuits, where, as in the rest of the paper, a k-element circuit and a k'-element cocircuit is denoted as a k-circuit and k'-cocircuit, respectively.

Theorem 1.1. *Let M be a non-empty 3-connected matroid. Then every element of M is in a 3-circuit and a 3-cocircuit if and only if M has rank at least three and is isomorphic to a wheel or a whirl.*

Theorem 1.1 and its well-known extension, Seymour's Splitter Theorem [5], has been instrumental in the analysis of 3-connected matroids. More recently, Miller [2] proved the following result which has conditions similar to those in Tutte's theorem. For all $r \geq 3$, a rank-r *tipless spike* is a matroid

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M with ground set $E = \{x_1, y_1, x_2, y_2, \dots, x_r, y_r\}$ whose circuits consist of the following sets:

(i) all sets of the form $\{x_i, y_i, x_j, y_j\}$ with $1 \le i < j \le r$,
(ii) a subset of $\{\{z_1, z_2, \dots, z_r\} : z_i \in \{x_i, y_i\}$ for all $i\}$ such that no two members of this subset have more than $r-2$ common elements, and
(iii) all $(r+1)$-element subsets of E that contain none of the sets in (i) and (ii).

Theorem 1.2. *Let M be a matroid with $|E(M)| \ge 13$. Then every pair of elements of M is in a 4-circuit and a 4-cocircuit if and only if M is a tipless spike.*

In this paper, we continue along a similar line of inquiry. A matroid M has *property* (P1) if every pair of elements is in a 4-circuit and every element is in a 3-cocircuit. Furthermore, a matroid M has *property* (P2) if every pair of elements is in a 4-circuit and every element is in a 4-cocircuit. The next two theorems are the main results of this paper. We denote the rank-3 whirl, the Fano matroid, and the non-Fano matroid by $\mathcal{W}^3$, F_7, and F_7^-, respectively. Also, up to isomorphism, we denote the rank-3 simple matroid with ground set $\{1, 2, \dots, 7\}$ and whose 3-circuits are $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 6\}$, and $\{3, 5, 7\}$ by P_7.

Theorem 1.3. *Let M be a non-empty 3-connected matroid. Then M has property* (P1) *if and only if*

(i) *$|E(M)| \le 8$ and M is isomorphic to one of the matroids $U_{3,5}$, $M(K_4)$, $\mathcal{W}^3$, F_7, $(F_7^-)^*$, and P_7^*, or*
(ii) *$|E(M)| \ge 9$ and M is isomorphic to $M(K_{3,n})$ for some $n \ge 3$.*

Theorem 1.4. *Let M be a non-empty 4-connected matroid. Then M has property* (P2) *if and only if*

(i) *$|E(M)| \le 15$ and M is isomorphic to one of the thirty-five matroids listed in the appendix, or*
(ii) *$|E(M)| \ge 16$ and M is isomorphic to $M(K_{4,n})$ for some $n \ge 4$.*

It is clear that $M(K_{3,n})$, where $n \ge 3$, and $M(K_{4,n})$, where $n \ge 4$, satisfy (P1) and (P2), respectively. For $|E(M)| \ge 9$ and $|E(M)| \ge 16$, the necessary directions of the proofs of Theorems 1.3 and 1.4 are given in Sections 2 and 3, respectively. For the proof of Theorem 1.3 when $|E(M)| \le 8$ and the proof of Theorem 1.4 when $|E(M)| \le 15$, we refer the interested reader to Pfeil's PhD thesis [4]. We end the introduction with some preliminaries.

Throughout the paper, notation and terminology follows Oxley [3]. Let M be a matroid. Two subsets X and Y of $E(M)$ *meet* if $X \cap Y$ is non-empty.

Referred to as *orthogonality*, it is well known that if C is a circuit and D is a cocircuit of M, then $|C \cap D| \neq 1$.

Lastly, let M_1 and M_2 be two matroids with ground sets E_1 and E_2, respectively, and let $\varphi : E_1 \to E_2$ be a bijection. Then φ is a *weak map* from M_1 to M_2 if, for every independent set I in M_2, we have $\varphi^{-1}(I)$ is independent in M_1, in which case, M_2 is a *weak-map image* of M_1. Equivalently, it is easily checked that, φ is a weak map from M_1 to M_2 if, for every circuit C of M_1, we have $\varphi(C)$ contains a circuit in M_2. As in this paper, it is typical to assume that E_1 and E_2 are the same sets and φ is the identity map. The following theorem is due to Lucas [1].

Theorem 1.5. *Let M_2 be the weak-map image of a binary matroid M_1, and suppose that $r(M_2) = r(M_1)$. Then M_2 is binary. Moreover, if M_2 is connected, then $M_2 \cong M_1$.*

2. Matroids with Property (P1) and at Least 9 Elements

Throughout this section, M is a 3-connected matroid satisfying (P1) and with ground set $E(M) = \{x_1, x_2, \ldots, x_t\}$, where $t \geq 4$. Our ability to determine M, for when $|E(M)| \geq 9$, explicitly relies on showing that $E(M)$ can be partitioned into blocks in which each block is a 3-cocircuit and M restricted to any two of these blocks is isomorphic to $M(K_{2,3})$. We first prove that if M has two distinct 3-cocircuits that meet in two elements, then M is isomorphic to $U_{3,5}$.

Lemma 2.1. *Let D_1 and D_2 be two 3-cocircuits of M such that $|D_1 \cap D_2| = 2$. Then $M \cong U_{3,5}$.*

Proof. Without loss of generality, let $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_2, x_4\}$. Then $M^*|(D_1 \cup D_2) \cong U_{2,4}$. This implies that if $|E(M)| = 4$, then M has no 4-circuits; a contradiction, so $|E(M)| \geq 5$. Furthermore, by orthogonality, any circuit meeting $D_1 \cup D_2$ does so in at least three elements. By (P1), M has a 4-circuit C_1 containing $\{x_1, x_5\}$. Similarly, M has a 4-circuit C_2 containing $\{x_i, x_5\}$, where x_i is the unique element in $\{x_2, x_3, x_4\}$ not in C_1. Now $C_1 \cup C_2 = \{x_1, x_2, x_3, x_4, x_5\}$ and $r(C_1 \cup C_2) = 3$. Also $r^*(C_1 \cup C_2) \leq 3$. Therefore

$$r(C_1 \cup C_2) + r^*(C_1 \cup C_2) - |C_1 \cup C_2| \leq 3 + 3 - 5 = 1,$$

and so $|E(M)| \leq 6$ as M is 3-connected. Using the fact that M satisfies (P1), a routine check shows that $|E(M)| \leq 5$, and so $M \cong U_{3,5}$. □

The next three lemmas concern disjoint 3-cocircuits. The first shows that M restricted to two such 3-cocircuits is isomorphic to $M(K_{2,3})$, while the second and third accumulate in showing that if $|E(M)| \geq 9$, then M has three pairwise-disjoint 3-cocircuits.

Lemma 2.2. *Let D_1 and D_2 be two disjoint 3-cocircuits of M. Then $M|(D_1 \cup D_2) \cong M(K_{2,3})$.*

Proof. Without loss of generality, let $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$. By (P1), M has a 4-circuit C_1 containing $\{x_1, x_4\}$. By orthogonality, we may assume $C_1 = \{x_1, x_2, x_4, x_5\}$. Similarly, M has a 4-circuit C_2 containing $\{x_3, x_6\}$. By symmetry, we may assume $C_2 = \{x_1, x_3, x_4, x_6\}$. Lastly, M has a 4-circuit C_3 containing $\{x_2, x_6\}$. We next show that C_3 does not meet either C_1 or C_2 in three elements.

Say $|C_1 \cap C_3| = 3$. Then, as M is 3-connected, $M|(C_1 \cup C_3) \cong U_{3,5}$, and so there exists a 4-circuit in M meeting either D_1 or D_2 in exactly one element; a contradiction. Thus $|C_1 \cap C_3| \neq 3$ and, similarly, $|C_2 \cap C_3| \neq 3$.

It now follows that neither x_1 nor x_4 is in C_3, and so, by orthogonality, $C_3 = \{x_2, x_3, x_5, x_6\}$. We now apply Theorem 1.5 to complete the proof. Since $M|(D_1 \cup D_2) = M|(C_1 \cup C_2 \cup C_3)$, we have $r(M|(D_1 \cup D_2)) = 4$. Next, consider $K_{2,3}$, and label its edges so that

$$\{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$$

is a partition of $E(K_{2,3})$, where each block is a bond of $K_{2,3}$, and $\{x_1, x_2, x_4, x_5\}$, $\{x_1, x_3, x_4, x_6\}$, and $\{x_2, x_3, x_5, x_6\}$ are the 4-cycles of $K_{2,3}$. Then the identity map from $E(M(K_{2,3}))$ to $E(M|(D_1 \cup D_2))$ is a weak map from $M(K_{2,3})$ to $M|(D_1 \cup D_2)$. Moreover, as $M|(D_1 \cup D_2)$ is connected, Theorem 1.5 implies that $M|(D_1 \cup D_2) \cong M(K_{2,3})$. □

Lemma 2.3. *If $|E(M)| \geq 9$, then M has two disjoint 3-cocircuits.*

Proof. Suppose $|E(M)| \geq 9$ and M has no disjoint 3-cocircuits. Let D_1 and D_2 be distinct 3-cocircuits of M. Then, by Lemma 2.1, $|D_1 \cap D_2| = 1$ and so, without loss of generality, we may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$. We first show that M has an element contained in three 3-cocircuits.

Assume M has no such element. By (P1), M has a 3-cocircuit D_3 containing x_6. By assumption, D_3 meets each of D_1 and D_2 and so, without loss of generality, $D_3 = \{x_2, x_4, x_6\}$. But M also has a 3-cocircuit containing x_7, and such a cocircuit cannot meet each of D_1, D_2, and D_3 without using an element shared by two of them. Thus M has an element contained in three 3-cocircuits.

By Lemma 2.1, we may now assume that M has a 3-cocircuit $D_3 = \{x_1, x_6, x_7\}$. Consider a 3-cocircuit D_4 of M containing x_8. Since D_4 meets each of D_1, D_2, and D_3, we have $x_1 \in D_4$ and so, by Lemma 2.1, we may assume $D_4 = \{x_1, x_8, x_9\}$. However, by (P1), M has a 4-circuit C containing $\{x_1, x_2\}$. By orthogonality, each of $|C \cap D_2|$, $|C \cap D_3|$, and $|C \cap D_4|$ is at least 2 which is impossible as $|C| = 4$. This contradiction establishes the lemma. □

Lemma 2.4. *If* $|E(M)| \geq 9$*, then* M *has three pairwise-disjoint* 3*-cocircuits.*

Proof. Suppose $|E(M)| \geq 9$. By Lemma 2.3, M has disjoint 3-cocircuits, $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$ say. By Lemma 2.2, we have $M|(D_1 \cup D_2) \cong M(K_{2,3})$. Therefore, without loss of generality, we may assume that M has circuits $C_1 = \{x_1, x_2, x_4, x_5\}$, $C_2 = \{x_1, x_3, x_4, x_6\}$, and $C_3 = \{x_2, x_3, x_5, x_6\}$. By (P1), M has a 3-cocircuit D_3 containing x_7. If M does not contain three pairwise-disjoint 3-cocircuits, then D_3 meets $D_1 \cup D_2$ and, by orthogonality, it must do so in one of the series pairs of $M|(D_1 \cup D_2)$. Therefore, by symmetry, we may assume $D_3 = \{x_1, x_4, x_7\}$. Similarly, if D_4 is a 3-cocircuit of M containing x_8, then, by Lemma 2.1, we may assume $D_4 = \{x_2, x_5, x_8\}$. Finally, applying the same argument again, if D_5 is a 3-cocircuit of M containing x_9, we have $D_5 = \{x_3, x_6, x_9\}$. But then D_3, D_4, and D_5 are disjoint, thereby completing the proof of the lemma. □

We next show that $E(M)$ can be partitioned into 3-cocircuits provided $|E(M)| \geq 9$.

Lemma 2.5. *If* $|E(M)| \geq 9$*, then* $E(M)$ *can be partitioned into* 3*-element blocks, where each block is a* 3*-cocircuit.*

Proof. Suppose $|E(M)| \geq 9$, and let $S = \{D_1, D_2, \ldots, D_n\}$ be the largest collection of pairwise-disjoint 3-cocircuits of M. By Lemma 2.4, we have $n \geq 3$. Suppose there is an element x in M not in any of the sets $D_1, D_2, \ldots, D_n$. By (P1), M has a 3-cocircuit D containing x. Now D has a non-empty intersection with a 3-cocircuit in S; otherwise, S is not of maximum size. Without loss of generality, $D \cap D_1 \neq \emptyset$ and so, by Lemma 2.1, $|D \cap D_1| = 1$. By Lemma 2.2, $M|(D_1 \cup D_i) \cong M(K_{2,3})$ for all $i \in \{2, 3, \ldots, n\}$. Thus, by orthogonality, D meets each of $D_2, D_3, \ldots, D_n$. But then $|D| \geq 4$ as $n \geq 3$; a contradiction. Thus, the lemma is proved. □

We are now ready to prove the necessary direction of Theorem 1.3 when $|E(M)| \geq 9$.

Proof of Theorem 1.3 for $|E(M)| \geq 9$. Suppose $|E(M)| \geq 9$. Then, by Lemma 2.5, there is a partition of $E(M)$ into 3-cocircuits $D_1, D_2, \ldots, D_n$ where $D_i = \{x_i, y_i, z_i\}$ for all i. By Lemma 2.2, $M|(D_1 \cup D_i) \cong M(K_{2,3})$ for all $i \in \{2, 3, \ldots, n\}$, so we may assume that M has 4-circuits $\{x_1, x_i, y_1, y_i\}$,

$\{x_1, x_i, z_1, z_i\}$, and $\{y_1, y_i, z_1, z_i\}$ for all such i. Consider the circuits $\{x_1, x_i, y_1, y_i\}$ and $\{x_1, x_j, y_1, y_j\}$, where i and j are distinct. By circuit elimination and orthogonality, $\{x_i, y_i, x_j, y_j\}$ is a 4-circuit of M. Similarly, for all distinct $i, j \in \{2, 3, \ldots, n\}$, we have $\{x_i, z_i, x_j, z_j\}$ and $\{y_i, z_i, y_j, z_j\}$ are 4-circuits of M.

We next show that each set of the form

$$\{x_i, y_i, y_j, z_j, z_k, x_k\},$$

where i, j, and k are distinct elements in $\{1, 2, \ldots, n\}$, is a 6-circuit of M Using circuit elimination on $\{x_i, y_i, x_j, y_j\}$ and $\{x_j, z_j, x_k, z_k\}$, it follows that $\{x_i, y_i, y_j, x_k, z_j, z_k\}$ contains a circuit of M. By orthogonality and as each of $M|(D_i \cup D_j)$, $M|(D_i \cup D_k)$, and $M|(D_j \cup D_k)$ is isomorphic to $M(K_{2,3})$, it is easily check that $\{x_i, y_i, y_j, x_k, z_j, z_k\}$ is itself a 6-circuit of M.

Now consider $K_{3,n}$, where $n \geq 3$. Label the edge set of $K_{3,n}$ so that

$$\big\{\{x_1, y_1, z_1\}, \{x_2, y_2.z_2\}, \ldots, \{x_n, y_n, z_n\}\big\}$$

is a partition of $E(K_{3,n})$, where each block is a bond of $K_{3,n}$, and $\{x_i, y_i, x_j, y_j\}$, $\{x_i, z_i, x_j, z_j\}$, and $\{y_i, z_i, y_j, z_j\}$ are 4-cycles of $K_{3,n}$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Then the identity map φ from $E(M(K_{3,n}))$ to $E(M)$ is a weak map from $M(K_{3,n})$ to M since, for each circuit C of $M(K_{3,n})$, we have $\varphi(C)$ is a circuit of M by above.

We next prove by induction on n that $r(M) = r(M(K_{3,n}))$ for all $n \geq 3$. If $n = 3$, then, by Lemma 2.2 and the 4-circuits established above, $r(M) = r(M(K_{3,3}))$. Therefore suppose $n \geq 4$ and that, for all matroids M' satisfying (P1) and whose ground set can be partitioned into m 3-cocircuits, where $3 \leq m \leq n-1$, we have $r(M') = r(K_{3,m})$. Let M' denote the matroid $M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1})$. We first show that M' satisfies (P1). Evidently, every element of M' is in a 3-cocircuit. Let x and y be distinct elements of M'. If x and y are in distinct 3-cocircuits D_i and D_j of M', then, by orthogonality and M satisfying (P1), M' has a 4-circuit containing $\{x, y\}$. Say x and y are in the same 3-cocircuit, D_i say, of M'. By considering D_i with either D_1 if $i \neq 1$ or D_2 if $i = 1$, it follows by Lemma 2.2 that M' has a 4-circuit containing $\{x, y\}$. Lastly, if M' is not 3-connected, then it has a 2-separation (A, B). Since $n - 1 \geq 3$, it follows that, for some $i \in \{1, 2, \ldots, n-1\}$, there is a 3-cocircuit D_i such that for one of A and B, say A, we have $D_i \subseteq A$, or $|D_i \cap A| = 2$ and $|B| \geq 3$. Thus, we may assume that $D_1 \subseteq A$. But then, by the 4-circuits above, $r(A \cup D_n) = r(A) + 1$. Therefore

$$\begin{aligned} r(A \cup D_n) + r(B) - r(M) &= r(A) + 1 + r(B) - (r(M') + 1) \\ &= r(A) + r(B) - r(M'), \end{aligned}$$

and so $(A \cup D_n, B)$ is a 2-separation in M; a contradiction. Thus M' is 3-connected, so M' satisfies (P1). By induction,

$$r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1}) = r(M(K_{3,n-1})),$$

and so, as D_n is a cocircuit of M,

$$r(M) = r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1}) + 1 = r(M(K_{3,n})).$$

Finally, M is connected and so, by Theorem 1.5, $M \cong M(K_{3,n})$. This completes the proof of Theorem 1.3. □

3. Matroids with Property (P2) and at Least 16 Elements

Throughout this section, M is a 4-connected matroid satisfying (P2). Unless stated otherwise, M has ground set $E(M) = \{x_1, x_2, \ldots, x_t\}$, where $t \geq 4$. The approach is similar to that of the last section. In particular, most of the work is in establishing that if $|E(M)| \geq 16$, then there is partition of $E(M)$ into blocks in which each block is a 4-cocircuit. However, because of the freedom of 4-cocircuits in comparison to 3-cocircuits, the case analysis is much more involved. We begin with a lemma analogous to Lemma 2.1.

Lemma 3.1. *Let D_1 and D_2 be 4-cocircuits of M such that $|D_1 \cap D_2| = 3$. Then $M \cong U_{3,6}$.*

Proof. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_3, x_5\}$. Then $M^*|(D_1 \cup D_2) \cong U_{3,5}$ as M is 4-connected. Therefore, if $|E(M)| = 5$, then M has no 4-circuits; a contradiction, so $|E(M)| \geq 6$. Furthermore, by orthogonality, any circuit meeting $D_1 \cup D_2$ does so in at least three elements.

By (P2), M has a 4-circuit C_1 containing $\{x_1, x_6\}$. Similarly, M has a 4-circuit C_2 containing $\{x_i, x_6\}$, where $x_i \in (D_1 \cup D_2) - C_1$. Since $C_1 - x_6 \subseteq D_1 \cup D_2$ and $C_2 - x_6 \subseteq D_1 \cup D_2$, it follows by circuit elimination that M has a circuit $C_3 \subseteq D_1 \cup D_2$. Since M is 4-connected and $|E(M)| \geq 6$, we have $|C_3| \in \{4, 5\}$. Now

$$r(C_3) + r^*(C_3) - |C_3| = 2,$$

so, as M is 4-connected, $|E(M)| \leq 7$. As M satisfies (P2), a routine check shows that $|E(M)| \leq 6$, and so $M \cong U_{3,6}$. □

We next establish an analogue of Lemma 2.2. In particular, Lemma 3.5 states that if M has two disjoint 4-cocircuits, then M restricted to these 4-cocircuits is isomorphic to $M(K_{2,4})$. This lemma requires three preliminary results. In each of these preliminary results as well as Lemma 3.5, we suppose that $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ are disjoint 4-cocircuits of

M. Observe that orthogonality and the 4-connectedness of M implies that every 4-circuit contained in $X \cup Y$ meets each of X and Y in exactly two elements.

Lemma 3.2. *Let C_1 and C_2 be distinct 4-circuits of M contained in $X \cup Y$ such that $|C_1 \cap C_2 \cap X| \geq 1$. Then $|C_1 \cap C_2 \cap X| = 1$.*

Proof. Since each 4-circuit contained in $X \cup Y$ meets each of X and Y in exactly two elements, it suffices to show that $|C_1 \cap C_2 \cap X| \neq 2$. Suppose $|C_1 \cap C_2 \cap X| = 2$. Then $|C_1 \cap C_2| \in \{2, 3\}$. If $|C_1 \cap C_2| = 3$, then we may assume that $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_1, y_3\}$. By circuit elimination, M has a circuit contained in $\{x_1, y_1, y_2, y_3\}$, but such a circuit contradicts either the 4-connectivity of M or orthogonality. If $|C_1 \cap C_2| = 2$, then we may assume that $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_3, y_4\}$. By circuit elimination, M has a circuit contained in $\{x_1, y_1, y_2, y_3, y_4\}$. But again, such a circuit contradicts either the 4-connectivity of M or orthogonality. Thus $|C_1 \cap C_2 \cap X| \neq 2$, thereby completing the proof of the lemma. □

Lemma 3.3. *Let C_1, C_2, and C_3 be distinct 4-circuits of M such that $C_1 \cup C_2 \cup C_3 \subseteq X \cup Y$ and $X \subseteq C_1 \cup C_2 \cup C_3$. Then $Y \subseteq C_1 \cup C_2 \cup C_3$.*

Proof. Suppose $Y - (C_1 \cup C_2 \cup C_3)$ is non-empty. Then, by Lemma 3.2, we may assume that $C_1 \cap Y = \{y_2, y_3\}$, $C_2 \cap Y = \{y_1, y_3\}$, and $C_3 \cap Y = \{y_1, y_2\}$. Furthermore, by Lemma 3.2, we may also assume that $C_1 \cap X = \{x_1, x_2\}$ and $C_2 \cap X = \{x_1, x_3\}$, in which case, $\{x_1, y_1, y_2, y_3\}$ spans X. Now X is independent as M is 4-connected, and so $\{y_1, y_2, y_3\} \subseteq \mathrm{cl}(X)$. But then M has a circuit that contains y_1 and is contained in $X \cup y_1$. This contradiction to orthogonality completes the proof of the lemma. □

Lemma 3.4. *Let C_1 and C_2 be distinct 4-circuits of M in $X \cup Y$ such that $|C_1 \cap C_2| \geq 1$. Then $|C_1 \cap C_2| = 2$.*

Proof. Assume $|C_1 \cap C_2| \neq 2$. Since

$$|C_1 \cap C_2| = |C_1 \cap C_2 \cap X| + |C_1 \cap C_2 \cap Y|,$$

it follows by Lemma 3.2 and symmetry that we may assume $|C_1 \cap C_2 \cap X| = 1$ and $|C_1 \cap C_2 \cap Y| = 0$. Without loss of generality, let $C_1 = \{x_1, x_2, y_1, y_2\}$ and $C_2 = \{x_1, x_3, y_3, y_4\}$. By Lemma 3.3, any additional 4-circuit of M contained in $X \cup Y$ includes x_4. By (P2), M has a 4-circuit C_3 containing $\{x_2, y_3\}$. By orthogonality and Lemma 3.2, we may assume $C_3 = \{x_2, x_4, y_1, y_3\}$. Similarly, M has a 4-circuit C_4 containing $\{x_2, y_4\}$. But then $x_4 \in C_4$ and $|C_3 \cap C_4 \cap X| = 2$, contradicting Lemma 3.2. The lemma now follows. □

Lemma 3.5. *The restriction $M|(X \cup Y) \cong M(K_{2,4})$.*

Proof. By (P2), M has a 4-circuit C_1 containing x_1 and y_1. By orthogonality, we may assume $C_1 = \{x_1, x_2, y_1, y_2\}$. Furthermore, M has a 4-circuit C_2 containing x_1 and y_3. By orthogonality and Lemmas 3.2 and 3.4, we may assume $C_2 = \{x_1, x_3, y_1, y_3\}$. Similarly, M has a 4-circuit C_3 containing x_1 and y_4 and, by Lemmas 3.2 and 3.4, $C_3 = \{x_1, x_4, y_1, y_4\}$.

Continuing this process, M has a 4-circuit C_4 containing x_2 and y_3. Since $x_2 \in C_1 \cap C_4$, we have $x_1 \notin C_4$ by Lemma 3.2. Therefore, as $y_3 \in C_2 \cap C_4$, Lemma 3.4 implies that $x_3 \in C_4$. Since $x_2 \in C_1 \cap C_4$ and $x_3, y_3 \in C_2 \cap C_4$, it follows by Lemma 3.4 that $y_2 \in C_4$. Hence $C_4 = \{x_2, x_3, y_2, y_3\}$. Similarly, M has a unique 4-circuit containing x_2 and y_4 and it is $C_5 = \{x_2, x_4, y_2, y_4\}$, and M has a unique 4-circuit containing x_3 and y_4 and it is $C_6 = \{x_3, x_4, y_3, y_4\}$.

We now show that $\mathcal{C}(M|(X \cup Y)) = \{C_1, C_2, \ldots, C_6\}$. First observe that, since every 2-element subset of each of X and Y is in one of $C_1, C_2, \ldots, C_6$, Lemma 3.2 implies that $M|(X \cup Y)$ has no other 4-circuits. Clearly, $r(X \cup Y) = 5$. Suppose there is a circuit $C \in \mathcal{C}(M|(X \cup Y)) - \{C_1, C_2, \ldots, C_6\}$. If $|C| = 6$, then C contains C_i for some $i \in \{1, 2, \ldots, 6\}$; a contradiction. Therefore, $|C| = 5$. To maintain orthogonality, either $|C \cap X| = 2$ or $|C \cap Y| = 2$. Thus to avoid containing one of the six 4-circuits, we may assume that $C = \{x_1, x_2, y_2, y_3, y_4\}$. But then, $\text{cl}(\{x_1, y_2, y_3, y_4\}) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$, so $r(X \cup Y) = 4$; a contradiction. Thus $\mathcal{C}(M|(X \cup Y)) = \{C_1, C_2, \ldots, C_6\}$. It is now easily checked that $M|(X \cup Y) \cong M(K_{2,4})$. □

The next main step in the proof of Theorem 1.4 is to show that if $|E(M)| \geq 11$, then M has two disjoint 4-cocircuits. Stated as Lemma 3.15, its proof is long and consists of a sequence of preliminary lemmas. Except for the first, these preliminary lemmas concern the way 4-cocircuits intersect if M has no two disjoint 4-cocircuits.

Lemma 3.6. *Let D_1, D_2, and D_3 be 4-cocircuits of M such that $|D_1 \cap D_2 \cap D_3| = 1$ and $|D_i \cap D_j| = 1$ for all distinct $i, j \in \{1, 2, 3\}$. Then $E(M) = D_1 \cup D_2 \cup D_3$, that is, $E(M) = 10$.*

Proof. Suppose that $E(M) - (D_1 \cup D_2 \cup D_3) \neq \emptyset$. Let $y \in E(M) - (D_1 \cup D_2 \cup D_3)$ and $D_1 \cap D_2 \cap D_3 = \{x\}$. By (P2), M has a 4-circuit C containing $\{x, y\}$ and, by orthogonality, $|C \cap D_i| \geq 2$ for all i. But then $|C| \geq 5$; a contradiction. □

The next two lemmas show that if $|E(M)| \geq 11$ and M has no two disjoint 4-cocircuits, then M has two 4-cocircuits meeting in exactly two elements and that every other 4-cocircuit of M meets the union of two such 4-cocircuits in at least two elements. These two lemmas underlie the approach taken to establish Lemma 3.15.

Lemma 3.7. *Let $|E(M)| \geq 11$, and suppose that M has no two disjoint 4-cocircuits. Then M has 4-cocircuits D_1 and D_2 such that $|D_1 \cap D_2| = 2$.*

Proof. Suppose the lemma does not hold. By (P2), M has a 4-cocircuit D_1 containing x_1. Without loss of generality, we may assume $D_1 = \{x_1, x_2, x_3, x_4\}$. Also, M has a 4-cocircuit D_2 that contains x_5 and, as M has no two disjoint 4-cocircuits, meets D_1. By Lemma 3.1, $|D_1 \cap D_2| = 1$, and so we may assume $D_2 = \{x_1, x_5, x_6, x_7\}$. Similarly, M has a 4-cocircuit D_3 that contains x_8 and $|D_1 \cap D_3| = |D_2 \cap D_3| = 1$. As $|E(M)| \geq 11$, it follows by Lemma 3.6 that $x_1 \notin D_3$. Therefore, without loss of generality, $D_3 = \{x_2, x_5, x_8, x_9\}$. Lastly, M has a 4-cocircuit D_4 containing x_{10} and

$$|D_1 \cap D_4| = |D_2 \cap D_4| = |D_3 \cap D_4| = 1.$$

By Lemma 3.6, we may assume $D_4 = \{x_3, x_6, x_8, x_{10}\}$. But then, a similar argument implies that M has the 4-cocircuit $D_5 = \{x_4, x_7, x_9, x_{11}\}$, in which case D_4 and D_5 are disjoint; a contradiction. □

Lemma 3.8. *Let $|E(M)| \geq 10$, and suppose that M has no two disjoint 4-cocircuits. Let D_1, D_2, and D_3 be 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. Then $|D_3 \cap (D_1 \cup D_2)| \geq 2$.*

Proof. If the lemma does not hold, then $|D_3 \cap (D_1 \cup D_2)| = 1$. More specifically, as M has no two disjoint 4-cocircuits, $|D_1 \cap D_2 \cap D_3| = 1$. Let $\{x\} = D_1 \cap D_2 \cap D_3$. By circuit elimination, M has a cocircuit $D_4 \subseteq (D_1 \cup D_2) - \{x\}$. Since $D_3 \cap D_4 = \emptyset$, it follows that $|D_4| \neq 4$. Therefore, as M is 4-connected, $D_4 = (D_1 \cup D_2) - \{x\}$.

Since $|E(M)| \geq 10$, we have $|E(M) - (D_1 \cup D_2 \cup D_3)| \geq 1$. Let $y \in E(M) - (D_1 \cup D_2 \cup D_3)$, and let C be a 4-circuit containing $\{x, y\}$. To preserve orthogonality, C contains an element in $D_3 - \{x\}$ and the unique element in $(D_1 \cap D_2) - \{x\}$. But then $|C \cap D_4| = 1$. This contradiction to orthogonality proves the lemma. □

Lemma 3.9. *Let $|E(M)| \geq 10$, and suppose that M has no two disjoint 4-cocircuits. Let D_1, D_2, and D_3 be distinct 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. Then $D_1 \cap D_2 \not\subseteq D_3$.*

Proof. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$. Suppose that $\{x_1, x_2\} \subseteq D_3$. By Lemma 3.1, we may assume that $D_3 = \{x_1, x_2, x_7, x_8\}$. Using circuit elimination on each pair of cocircuits in $\{D_1, D_2, D_3\}$ and eliminating x_2, we find that each of $\{x_1, x_3, x_4, x_5, x_6\}$, $\{x_1, x_3, x_4, x_7, x_8\}$, and $\{x_1, x_5, x_6, x_7, x_8\}$ contains a cocircuit. Noting that M has no cocircuits of size at most three, each such cocircuit must contain x_1; otherwise, M has two disjoint 4-cocircuits. Moreover, for each of these 5-element sets, every 4-element subset containing x_1 meets D_1, D_2, or D_3 in exactly three elements. Thus, by Lemma 3.1, none of these subsets is a 4-cocircuit. Hence each of these 5-element sets is a cocircuit, which we refer to as D_5, D_6, and D_7, respectively.

By (P2), M has a 4-circuit C_1 containing $\{x_1, x_9\}$. By considering the intersection of C_1 with each of D_1, D_2, and D_3, we see that $x_2 \in C_1$. But then, regardless of the choice for the remaining element in C_1, it follows that C_1 meets one of D_5, D_6, and D_7 in exactly one element, contradicting orthogonality. This contradiction proves the lemma. □

Lemma 3.10. *Let $|E(M)| \geq 10$, and suppose that M has no two disjoint 4-cocircuits. Let D_1, D_2, and D_3 be 4-cocircuits of M such that $|D_1 \cap D_2 \cap D_3| = 1$. Then $|D_i \cap D_j| = 1$ for some distinct elements $i, j \in \{1, 2, 3\}$.*

Proof. Suppose the lemma does not hold. Then, by Lemmas 3.1 and 3.9, we may assume, without loss of generality, that $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_1, x_3, x_5, x_7\}$. Let C_1 be a 4-circuit of M containing $\{x_8, x_9\}$. If C_1 meets $D_1 \cup D_2 \cup D_3$, then, by orthogonality, it does so in at least three elements. Therefore $C_1 \cap (D_1 \cup D_2 \cup D_3) = \emptyset$ and so we may assume $C_1 = \{x_8, x_9, x_{10}, x_{11}\}$.

Now let D_4 be a 4-cocircuit of M containing x_8. By orthogonality, we may assume $x_9 \in D_4$. Since M has no two disjoint 4-cocircuits, D_4 meets each of D_1, D_2, and D_3. Furthermore, by Lemma 3.8, D_4 contains at least two elements from each of $D_1 \cup D_2$, $D_1 \cup D_3$, and $D_2 \cup D_3$. If $x_1 \in D_4$, then, by Lemma 3.9, none of x_2, x_3, and x_5 are in D_4. It follows that $x_1 \notin D_4$. Therefore, without loss of generality, $D_4 = \{x_2, x_3, x_8, x_9\}$.

Finally, let C_2 be a 4-circuit of M containing $\{x_4, x_{10}\}$. By orthogonality, $|C_2 \cap D_1| \geq 2$. If $x_1 \notin C_2$, then, without loss of generality, we may assume that $x_2 \in C_2$. But then $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_4 \neq \emptyset$, and it follows by orthogonality that $|C_2 \cap D_2| \geq 2$ and $|C_2 \cap D_4| \geq 2$, which is not possible. Thus $x_1 \in C_2$. Therefore $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_3 \neq \emptyset$, and so $C_2 = \{x_1, x_4, x_5, x_{10}\}$. Similarly, M has a unique 4-circuit C_3 containing $\{x_4, x_{11}\}$ and it is $C_3 = \{x_1, x_4, x_5, x_{11}\}$. As M is 4-connected, $M|(C_2 \cup C_3)$ is isomorphic to $U_{3,5}$. In turn, this implies that M has a circuit, namely, $\{x_4, x_5, x_{10}, x_{11}\}$ meeting D_1 in exactly one element. This contradiction completes the proof of the lemma. □

For the rest of the lemmas leading to the proof that M has two disjoint 4-cocircuits if $|E(M)| \geq 11$, we frequently refer to the way in which a 4-cocircuit intersects two other 4-cocircuits which share two elements. For ease of reading, we introduce the following terminology.

Let D_1, D_2, and D_3 be 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. With respect to (D_1, D_2), we say that D_3 is

(i) *Type*-1 if $|D_3 \cap (D_1 \cap D_2)| = 1$, and $|D_3 \cap (D_1 - D_2)| = 1$, and $|D_3 \cap (D_2 - D_1)| = 0$,
(ii) *Type*-2 if $|D_3 \cap (D_1 \cap D_2)| = 0$, and $|D_3 \cap D_1| = |D_3 \cap D_2| = 1$, and
(iii) *Type*-3 if $|D_3 \cap (D_1 \cap D_2)| = 0$, and $|D_3 \cap D_1| = 2$, and $|D_3 \cap D_2| = 1$.

Set diagrams of the three types are shown in Fig. 1.

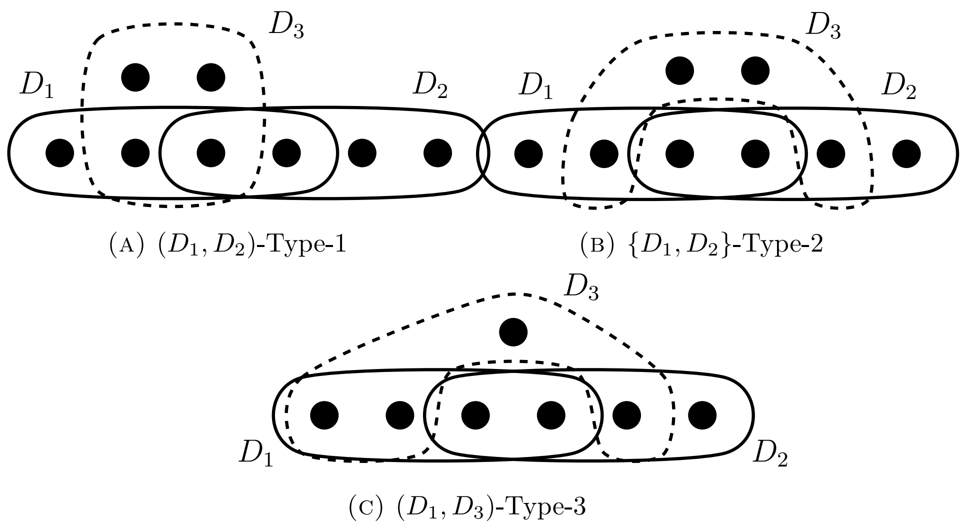

FIGURE 1. Set diagrams of Types-1, -2, and -3 intersections.

Note that Type-2 intersections are symmetric, and therefore we will denote this intersection by $\{D_1, D_2\}$-Type-2. There will be occasions in which it is sufficient to specify that D_3 is either (D_1, D_2)-Type-i or (D_2, D_1)-Type-i for a fixed $i \in \{1, 3\}$. In these instances, we will say that D_3 is $\{D_1, D_2\}$-Type-i. The previous lemmas ensure that any 4-cocircuit not contained in $D_1 \cup D_2$ intersects $D_1 \cup D_2$ in one of the above types if M has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. We prove this in the following lemma.

Lemma 3.11. *Let $|E(M)| \geq 10$, and suppose that M has no two disjoint 4-cocircuits. Let D_1 and D_2 be 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. If D_3 is a 4-cocircuit of M such that $D_3 \not\subseteq D_1 \cup D_2$, then D_3 is $\{D_1, D_2\}$-Type-i for some $i \in \{1, 2, 3\}$.*

Proof. Let D_3 be a 4-cocircuit of M not contained in $D_1 \cup D_2$. By Lemma 3.9, $|D_3 \cap (D_1 \cap D_2)| \in \{0, 1\}$. Suppose that $|D_3 \cap (D_1 \cap D_2)| = 1$. By Lemma 3.8, $|D_3 \cap (D_1 \cup D_2)| \geq 2$, so we may assume $|D_3 \cap (D_1 - D_2)| = 1$. Since $|D_1 \cap D_2| = 2$ and $|D_1 \cap D_3| = 2$, it follows by Lemma 3.10 that $|D_2 \cap D_3| = 1$. Therefore $|D_3 \cap (D_2 - D_1)| = 0$, and D_3 is (D_1, D_2)-Type-1.

Now suppose that $|D_3 \cap (D_1 \cap D_2)| = 0$. As M has no two disjoint 4-cocircuits, we have $D_1 \cap D_3 \neq \emptyset$ and $D_2 \cap D_3 \neq \emptyset$. Therefore, without loss of generality, as $D_3 \not\subseteq D_1 \cup D_2$, either $|D_1 \cap D_3| = |D_2 \cap D_3| = 1$, or $|D_1 \cap D_3| = 2$ and $|D_2 \cap D_3| = 1$. In particular, D_3 is $\{D_1, D_2\}$-Type-2 or (D_1, D_2)-Type-3, respectively. □

For when $|E(M)| \geq 11$, the next three lemmas show that if M has no two disjoint 4-cocircuits, and D_1, D_2, and D_3 are 4-cocircuits of M such that $|D_1 \cap D_2| = 2$ and $D_3 \not\subseteq D_1 \cup D_2$, then D_3 is neither $\{D_1, D_2\}$-Type-2 nor $\{D_1, D_2\}$-Type-3.

Lemma 3.12. *Let $|E(M)| \geq 10$, and suppose that M has no two disjoint 4-cocircuits. Let D_1, D_2, D_3, and D_4 be distinct 4-cocircuits of M such that $|D_1 \cap D_2| = 2$ and D_3 is $\{D_1, D_2\}$-Type-2. If $D_4 \not\subseteq D_1 \cup D_2 \cup D_3$, then D_4 is $\{D_1, D_2\}$-Type-1.*

Proof. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_3, x_5, x_7, x_8\}$, and suppose $x_9 \in D_4$. If D_4 is not $\{D_1, D_2\}$-Type-1, then, by Lemma 3.11, it is either $\{D_1, D_2\}$-Type-2 or $\{D_1, D_2\}$-Type-3. First assume that D_4 is $\{D_1, D_2\}$-Type-3. Then, without loss of generality, either $D_4 = \{x_3, x_4, x_5, x_9\}$ or $D_4 = \{x_3, x_4, x_6, x_9\}$. If $D_4 = \{x_3, x_4, x_5, x_9\}$, then $|D_3 \cap D_4| = 2$ and $|D_2 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 3.8. Similarly, if $D_4 = \{x_3, x_4, x_6, x_9\}$, then $|D_1 \cap D_4| = 2$ and $|D_3 \cap (D_1 \cup D_4)| < 2$, again contradicting Lemma 3.8. Thus D_4 is not $\{D_1, D_2\}$-Type-3.

Now assume that D_4 is $\{D_1, D_2\}$-Type-2. Then $|D_4 \cap \{x_3, x_5\}| \leq 1$; otherwise, $\{x_3, x_5\} \subseteq D_4$ and $|D_1 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 3.8. If $|D_4 \cap \{x_3, x_5\}| = 1$, then, without loss of generality, $x_3 \in D_4$. Since D_4 is $\{D_1, D_2\}$-Type-2, we have $x_6 \in D_4$. Furthermore, either x_7 or x_8 is in D_4; otherwise, $|D_1 \cap D_3 \cap D_4| = 1$ and $|D_i \cap D_j| = 1$ for all distinct $i, j \in \{1, 3, 4\}$, and so we contradict Lemma 3.6 as $|E(M)| \geq 11$. Hence, we may assume, $D_4 = \{x_3, x_6, x_7, x_9\}$. But then $|D_3 \cap D_4| = 2$ and $|D_1 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 3.8.

It now follows that D_4 avoids $\{x_3, x_5\}$, and so $x_4, x_6 \in D_4$. Furthermore, as M has no disjoint 4-cocircuits, we may assume $x_7 \in D_4$. Thus $D_4 = \{x_4, x_6, x_7, x_9\}$. By (P2), M has a 4-cocircuit D_5 containing x_{10}. By applying the argument that showed D_4 is not $\{D_1, D_2\}$-Type-3 to D_5, we have that D_5 is not $\{D_1, D_2\}$-Type-3. If D_5 is $\{D_1, D_2\}$-Type-2, then, by the analysis of the previous paragraph, $\{x_4, x_6\} \subseteq D_5$ and $\{x_7, x_8\} \cap D_5 \neq \emptyset$. If $D_5 = \{x_4, x_6, x_7, x_{10}\}$, then $|D_4 \cap D_5| = 3$, and so, by Lemma 3.1, M is isomorphic to $U_{3,6}$; a contradiction. If $D_5 = \{x_4, x_6, x_8, x_{10}\}$, then

$|D_4 \cap D_5| = 2$ and $|D_1 \cap (D_4 \cup D_5)| < 2$, contradicting Lemma 3.8. Therefore D_5 is $\{D_1, D_2\}$-Type-1. It is easily checked that, by symmetry, we may assume that D_5 is (D_1, D_2)-Type-1.

By symmetry, we may assume $\{x_1, x_3\} \subseteq D_5$. Furthermore, D_5 contains either x_7 or x_9; otherwise, $D_4 \cap D_5 = \emptyset$. But, if $D_5 = \{x_1, x_3, x_7, x_{10}\}$, then $|D_3 \cap D_5| = 2$ and $|D_4 \cap (D_3 \cup D_5)| < 2$, contradicting Lemma 3.8. Similarly, if $D_5 = \{x_1, x_3, x_9, x_{10}\}$, then $|D_1 \cap D_5| = 2$ and $|D_3 \cap (D_1 \cup D_5)| < 2$, again contradicting Lemma 3.8. This completes the proof of the lemma. □

Lemma 3.13. *Let $|E(M)| \geq 11$, and suppose M has no two disjoint 4-cocircuits. Let D_1, D_2 and D_3 be distinct 4-cocircuits of M such that $|D_1 \cap D_2| = 2$ and $D_3 \not\subseteq D_1 \cup D_2$. Then D_3 is not $\{D_1, D_2\}$-Type-2.*

Proof. Suppose D_3 is $\{D_1, D_2\}$-Type-2. Then, without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_3, x_5, x_7, x_8\}$. By (P2), M has a 4-cocircuit D_4 containing x_9. By symmetry and Lemma 3.12, we may assume that D_4 is (D_1, D_2)-Type-1, in which case, D_4 meets $\{x_3, x_4\}$ but avoids $\{x_5, x_6\}$. Since $|D_1 \cap D_4| = 2$, it follows by Lemma 3.8 that $|D_3 \cap (D_1 \cup D_4)| \geq 2$, so $D_4 \cap \{x_7, x_8\} \neq \emptyset$. Hence, without loss of generality, either $D_4 = \{x_1, x_3, x_7, x_9\}$ or $D_4 = \{x_1, x_4, x_7, x_9\}$. First assume that $D_4 = \{x_1, x_3, x_7, x_9\}$.

3.13.1. *Let D be a 4-cocircuit of M such that $D \not\subseteq D_1 \cup D_2 \cup D_3 \cup D_4$. Then $|\{x_1, x_3\} \cap D| = 1$.*

By Lemma 3.12, D is $\{D_1, D_2\}$-Type-1. Furthermore, as $|D_3 \cap D_4| = 2$ and D_2 is (D_3, D_4)-Type-2, it follows by Lemma 3.12 that D is $\{D_3, D_4\}$-Type-1. Also, as $D_1 \cap D_4 = \{x_1, x_3\}$, Lemma 3.9 implies that $\{x_1, x_3\} \not\subseteq D$.

If $\{x_1, x_3\} \cap D = \emptyset$, then, as D is $\{D_1, D_2\}$-Type-1 and $\{D_3, D_4\}$-Type-1, we have $\{x_2, x_7\} \subseteq D$ as well as $x_5 \in D$. Hence $D \cap (D_1 \cup D_2 \cup D_3 \cup D_4) = \{x_2, x_5, x_7\}$. Now $|D_2 \cap D| = 2$ and $|D_3 \cap D| = 2$. Furthermore, D_4 is $\{D_2, D\}$-Type-2, D_1 is $\{D_3, D\}$-Type-2, and D is $\{D_1, D_4\}$-Type-2. Therefore, by Lemma 3.12, if D' is a 4-cocircuit of M such that $D' \not\subseteq (D_1 \cup D_2 \cup D_3 \cup D_4 \cup D)$, then D' is $\{D_2, D\}$-Type-1, $\{D_3, D\}$-Type-1, and $\{D_1, D_4\}$-Type-1. As $|E(M)| \geq 11$, M has such a cocircuit D'. By Lemma 3.9, if $x_1 \in D'$, then $x_2 \notin D'$ and $x_3 \notin D'$. Since D' is a $\{D_2, D\}$-Type-1 and $\{D_3, D\}$-Type-1, we have $x_5, x_7 \in D'$; a contradiction to Lemma 3.9 as $\{x_5, x_7\} = D_3 \cap D$. Thus $x_1 \notin D'$. Similarly, $x_3 \notin D'$. But then D' is not $\{D_1, D_4\}$-Type-1; a contradiction. Hence 3.13.1 holds.

Now let D_5 be a 4-cocircuit of M that contains x_{10}. By symmetry and 3.13.1, we may assume $x_1 \in D_5$ and $x_3 \notin D_5$. Since D_2 is (D_3, D_4)-Type-2, it follows by Lemma 3.12 that D_5 is $\{D_3, D_4\}$-Type-1, and so

$x_7 \in D_5$ but $x_5 \notin D_5$. Therefore, as D_5 is $\{D_1, D_2\}$-Type-1, it follows by Lemma 3.8 that D_5 contains one of x_4 and x_6.

If $x_4 \in D_5$, then $|D_1 \cap D_4 \cap D_5| = 1$ and

$$|D_1 \cap D_4| = |D_1 \cap D_5| = |D_4 \cap D_5| = 2,$$

contradicting Lemma 3.10. Thus $x_6 \in D_5$ and so $D_5 = \{x_1, x_6, x_7, x_{10}\}$. By (P2), M has a 4-cocircuit D_6 containing x_{11}. By 3.13.1, either $x_1 \in D_6$ or $x_3 \in D_6$. If $x_1 \in D_6$, then, by the previous argument concerning D_5 and now applied to D_6, we get $D_6 = \{x_1, x_6, x_7, x_{11}\}$. But then $|D_5 \cap D_6| = 3$ and so, by Lemma 3.1, $M \cong U_{3,6}$; a contradiction. Therefore $x_1 \notin D_6$ and so $x_3 \in D_6$. Observe that $|D_2 \cap D_5| = 2$ and D_3 is $\{D_2, D_5\}$-Type-2 and so, by Lemma 3.12, D_6 is $\{D_2, D_5\}$-Type-1. But D_6 is also $\{D_1, D_2\}$-Type-1 and $\{D_3, D_4\}$-Type 1. Therefore D_6 contains an element from each of the sets $\{1, 6\}$, $\{1, 2\}$, and $\{1, 5, 8, 9\}$. This is impossible as D_6 has exactly four elements and $x_1 \notin D_6$. We conclude that $D_4 \neq \{x_1, x_3, x_7, x_9\}$.

We may now assume that $D_4 = \{x_1, x_4, x_7, x_9\}$. Now $|D_1 \cap D_4| = 2$ and D_3 is $\{D_1, D_4\}$-Type-2. Therefore, by Lemma 3.12,

3.13.2. *if D is a 4-cocircuit of M such that $D \not\subseteq D_1 \cup D_3 \cup D_4$, then D is $\{D_1, D_4\}$-Type-1.*

We next show that

3.13.3. *M has a 4-cocircuit containing x_1 and an element not in $\{x_1, x_2, \ldots, x_9\}$.*

Let D_5 be a cocircuit containing x_{10}. If $x_1 \notin D_5$, then, as D_5 is $\{D_1, D_2\}$-Type-1 and, by 3.13.2, $\{D_1, D_4\}$-Type-1, it follows that $\{x_2, x_4\} \subseteq D_5$ and $\{x_3, x_5, x_6, x_7, x_9\} \cap D_5 = \emptyset$. Further, as M has no disjoint 4-cocircuits, $D_5 \cap D_3 \neq \emptyset$. Therefore, $D_5 = \{x_2, x_4, x_8, x_{10}\}$. As $|E(M)| \geq 11$, M has a 4-cocircuit D_6 containing x_{11}. By the same reasoning, $\{x_2, x_4, x_8\} \subseteq D_6$, so $|D_5 \cap D_6| = 3$; a contradiction. Thus 3.13.3 holds.

By 3.13.3, we may assume that M has a 4-cocircuit D_5 containing x_1 and x_{10}. We show that

3.13.4. *$x_3 \notin D_5$.*

If $x_3 \in D_5$, then, as D_5 is $\{D_1, D_2\}$-Type-1 and $\{D_1, D_4\}$-Type-1, we have $\{x_2, x_4, x_5, x_6, x_7, x_9\} \cap D_5 = \emptyset$. Furthermore, $|D_1 \cap D_5| = 2$ and so, by Lemma 3.8, $x_8 \in D_5$. Therefore $D_5 = \{x_1, x_3, x_8, x_{10}\}$. By (P2), M has a 4-cocircuit D_6 containing x_{11}. Since $|D_3 \cap D_5| = 2$ and D_4 is $\{D_3, D_5\}$-Type-2, it follows by Lemma 3.12 that D_6 is $\{D_3, D_5\}$-Type-1. As D_6 is also $\{D_1, D_2\}$-Type-1 and, by 3.13.2, $\{D_1, D_4\}$-Type-1, it is easily checked that

$x_1 \in D_6$, in which case $\{x_2, x_4, x_5, x_7, x_{10}\} \cap D_6 = \emptyset$. This implies that each of $D_6 \cap \{3, 8\}$, $D_6 \cap \{3, 6\}$, and $D_6 \cap \{3, 9\}$ is non-empty, and so $x_3 \in D_6$. But then $D_1 \cap D_5 \cap D_6 = \{1, 3\}$, contradicting Lemma 3.9. Thus $x_3 \notin D_5$, thereby establishing 3.13.4.

Since D_5 is $\{D_1, D_2\}$-Type-1 and $\{D_1, D_4\}$-Type-1, but does not contain x_3, it follows that $|\{x_5, x_6\} \cap D_5| = 1$ and $|\{x_7, x_9\} \cap D_5| = 1$. In turn this implies D_5 contains either x_5 or x_7; otherwise, it is disjoint from D_3. If $x_6 \in D_5$, then $x_7 \in D_5$, in which case, $|D_4 \cap D_5| = 2$. But then $|D_3 \cap (D_4 \cup D_5)| = 1$, contradicting Lemma 3.8. Therefore $x_6 \notin D_5$, so $x_5 \in D_5$. Then $|D_2 \cap D_5| = 2$, and so, by Lemma 3.8, $|D_3 \cap (D_2 \cup D_5)| \geq 2$, which implies $x_7 \in D_5$. By (P2), M has a 4-cocircuit D_6 containing x_{11}. If $x_1 \in D_6$, then, by an argument analogous to that which determined D_5, we have $D_6 = \{x_1, x_5, x_7, x_{11}\}$ and so $|D_5 \cap D_6| = 3$; a contradiction. Thus $x_1 \notin D_6$. Since D_6 is $\{D_1, D_2\}$-Type-1 and $\{D_1, D_4\}$-Type 1, it is easily checked that $D_6 = \{x_2, x_3, x_4, x_{11}\}$. But then $|D_1 \cap D_6| = 3$; a contradiction to Lemma 3.1. This completes the proof of Lemma 3.13. $\square$

Lemma 3.14. *Let $|E(M)| \geq 11$, and suppose M has no two disjoint 4-cocircuits. Let D_1, D_2, and D_3 be distinct 4-cocircuits of M such that $|D_1 \cap D_2| = 2$ and $D_3 \not\subseteq D_1 \cup D_2$. Then D_3 is not $\{D_1, D_2\}$-Type-3.*

Proof. Suppose that D_3 is $\{D_1, D_2\}$-Type-3. Then, without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_3, x_4, x_5, x_7\}$. Note that D_2 is (D_1, D_3)-Type-3.

3.14.1. *Let D be a 4-cocircuit of M such that $D \not\subseteq D_1 \cup D_2 \cup D_3$. Then D is neither $\{D_1, D_2\}$-Type-3 nor $\{D_1, D_3\}$-Type-3.*

Without loss of generality, we may assume $x_8 \in D$. First suppose that D is $\{D_1, D_2\}$-Type-3. If D is (D_1, D_2)-Type-3, then $D_1 \cap D_3 \cap D = \{x_3, x_4\}$, contradicting Lemma 3.9. Therefore assume that D is (D_2, D_1)-Type-3. By symmetry, we may assume, $D = \{x_3, x_5, x_6, x_8\}$.

By (P2), M has a 4-cocircuit D' containing x_9. Since D' is not $\{D_1, D_2\}$-Type-2, it follows by Lemmas 3.9 and 3.13 that D' is neither $\{D_1, D_2\}$-Type-3 nor $\{D_1, D_2\}$-Type-2. Therefore, by Lemma 3.11, D' is $\{D_1, D_2\}$-Type-1. By considering the way in which D_1, D_2, D_3, and D relate to each other, we may assume, by symmetry, that $x_1 \in D'$ and that $|D' \cap \{x_3, x_4\}| = 1$ and $|D' \cap \{x_5, x_6\}| = 0$. If $x_4 \in D'$, then $x_8 \in D'$; otherwise, $D \cap D' = \emptyset$. But then, D' is $\{D_3, D\}$-Type-2, contradicting Lemma 3.13 as $|D_3 \cap D| = 2$.

Therefore $x_3 \in D'$. Now $x_8 \in D'$; otherwise D' is $\{D_2, D\}$-Type-2, contradicting Lemma 3.13 as $|D_2 \cap D| = 2$. Hence $D' = \{x_1, x_3, x_8, x_9\}$. By

(P2), M has a 4-cocircuit D'' containing x_{10}. As above, D'' is $\{D_1, D_2\}$-Type-1, so $|\{x_1, x_2\} \cap D''| = 1$. By Lemma 3.13, D'' is not $\{D_3, D_4\}$-Type-2. Furthermore, as either $\{x_1, x_{10}\} \subseteq D''$ or $\{x_2, x_{10}\} \subseteq D''$, it follows that D'' is not $\{D_3, D\}$-Type-2. Thus, by Lemma 3.11, D'' is $\{D_3, D\}$-Type-1, and so $|\{x_3, x_5\} \cap D''| = 1$. Say $x_1 \in D''$. Then, by Lemma 3.9, $x_3 \notin D''$, so $x_5 \in D''$. But then D'' is $\{D_1, D_3\}$-Type-3 and $\{D, D'\}$-Type-3, and so $|\{2, 7\} \cap D''| = 1$ and $|\{6, 9\} \cap D''| = 1$; a contradiction. Thus $x_1 \notin D''$ and so $x_2 \in D''$. If $x_3 \in D''$, then D'' is $\{D_2, D\}$-Type-3, and so $D'' = \{x_2, x_3, x_8, x_{10}\}$. But then $D \cap D' \cap D'' = \{x_3, x_8\}$, contradicting Lemma 3.9. Therefore $x_3 \notin D''$ and $x_5 \in D''$. So D'' is $\{D_1, D'\}$-Type-3, in which case $|\{x_4, x_8, x_9\} \cap D''| = 2$; a contradiction. Hence D is not $\{D_1, D_2\}$-Type-3. Since D_2 is (D_1, D_3)-Type-3, it follows by symmetry that D is not $\{D_1, D_3\}$-Type-3. Thus 3.14.1 holds.

By Lemmas 3.13 and 3.14.1, every 4-cocircuit D of M such that $D \not\subseteq D_1 \cup D_2 \cup D_3$ is both $\{D_1, D_2\}$-Type-1 and $\{D_1, D_3\}$-Type-1. In fact, we have

3.14.2. *D is both (D_1, D_2)-Type-1 and (D_1, D_3)-Type-1.*

Without loss of generality, we may assume that $x_8 \in D$. Note that D is (D_1, D_2)-Type-1 if and only if it is (D_1, D_3)-Type-1. Suppose D is neither (D_1, D_2)-Type-1 nor (D_1, D_3)-Type-1. Then D is (D_2, D_1)-Type-1 and (D_3, D_1)-Type-1. If $x_5 \in D$, then, without loss of generality, $D = \{x_1, x_3, x_5, x_8\}$, in which case, $|D_1 \cap D_2 \cap D| = 1$ and

$$|D_1 \cap D_2| = |D_1 \cap D| = |D_2 \cap D| = 2,$$

contradicting Lemma 3.10. Therefore $x_5 \notin D$, and so $\{x_6, x_7\} \subseteq D$. But then either $|\{x_1, x_2\} \cap D| = 0$ or $|\{x_3, x_4\} \cap D| = 0$, in which case, D is not (D_2, D_1)-Type-1 or not (D_3, D_1)-Type-1, respectively. Thus 3.14.2 holds.

By (P2), M has a 4-cocircuit D_4 that contains x_8. By 3.14.2, we may assume $D_4 = \{x_1, x_3, x_8, x_9\}$. Furthermore, M has a 4-cocircuit D_5 containing x_{10}. By 3.14.2, D_5 is (D_1, D_2)-Type-1 and (D_1, D_3)-Type-1. This implies $|\{x_1, x_2\} \cap D_5| = 1$ and $|\{x_3, x_4\} \cap D_5| = 1$. By Lemma 3.9, $\{x_1, x_3\} \not\subseteq D_5$ and so D_5 contains one of $\{x_1, x_4\}$, $\{x_2, x_3\}$, and $\{x_2, x_4\}$.

Say $\{x_1, x_4\} \subseteq D_5$. Then $\{x_8, x_9\} \cap D_5 \neq \emptyset$; otherwise, $|D_2 \cap D_4 \cap D_5| = 1$ and $|D_2 \cap D_4| = |D_2 \cap D_5| = |D_4 \cap D_5| = 1$, and so, by Lemma 3.6, $|E(M)| = 10$. Therefore, we may assume $D_5 = \{x_1, x_4, x_8, x_{10}\}$. But then $|D_1 \cap D_4 \cap D_5| = 1$ and $|D_1 \cap D_4| = |D_1 \cap D_5| = |D_4 \cap D_5| = 2$, contradicting Lemma 3.10. Similarly $\{x_2, x_3\} \not\subseteq D_5$, and therefore $\{x_2, x_4\} \subseteq D_5$. Now, $D_5 \cap \{x_8, x_9\} \neq \emptyset$; otherwise, D_4 and D_5 are disjoint. Hence, without loss of generality, $D_5 = \{x_2, x_4, x_8, x_{10}\}$. By (P2), M has a 4-cocircuit D_6 containing x_{11}. As the restrictions on D_5 also apply to D_6, we have

$\{x_2, x_4\} \subseteq D_6$, which contradicts Lemma 3.9 as $D_1 \cap D_5 = \{2, 4\}$. This completes the proof of Lemma 3.14. □

At last we show that M has two disjoint 4-cocircuits if $|E(M)| \geq 11$.

Lemma 3.15. *Let $|E(M)| \geq 11$. Then M has two disjoint 4-cocircuits.*

Proof. Suppose that M has no two disjoint 4-cocircuits. By Lemma 3.7, M has 4-cocircuits D_1 and D_2 with $|D_1 \cap D_2| = 2$. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$. By (P2), M has a 4-cocircuit D_3 containing x_7. Lemmas 3.13 and 3.14 together with Lemma 3.11 imply that D_3 is $\{D_1, D_2\}$-Type-1. Therefore, without loss of generality, $D_3 = \{x_1, x_3, x_7, x_8\}$. Let D be a 4-cocircuit of M such that $D \not\subseteq D_1 \cup D_2 \cup D_3$. Since $|D_1 \cap D_3| = 2$, it again follows by Lemmas 3.11, 3.13, and 3.14 that D is $\{D_1, D_2\}$-Type-1 as well as $\{D_1, D_3\}$-Type-1. We next show

3.15.1. *D is not both (D_2, D_1)-Type-1 and (D_3, D_1)-Type-1.*

If D is both (D_2, D_1)-Type-1 and (D_3, D_1)-Type-1, then, without loss of generality, $\{x_5, x_7\} \subseteq D$. In turn, this implies $x_1 \in D$, so we may assume $D = \{x_1, x_5, x_7, x_9\}$. By (P2), M has a 4-cocircuit D'' containing x_{10}. As $|D_2 \cap D| = 2$ and $|D_3 \cap D| = 2$, it follows by Lemmas 3.11, 3.13, and 3.14 that D' is $\{D_1, D_2\}$-Type-1, $\{D_1, D_3\}$-Type-1, $\{D_2, D\}$-Type-1, and $\{D_3, D\}$-Type-1. This implies that $x_1 \in D'$, and it is easily checked that either $\{x_4, x_9\} \subseteq D'$ or $\{x_6, x_8\} \subseteq D'$. If $\{x_4, x_9\} \subseteq D'$, then $D' = \{x_1, x_4, x_9, x_{10}\}$. But then $|D_2 \cap D_3 \cap D'| = 1$ and $|D_2 \cap D_3| = |D_2 \cap D'| = |D_3 \cap D'| = 1$, and so, by Lemma 3.6, $|E(M)| = 10$; a contradiction. Similarly, if $\{x_6, x_8\} \subseteq D'$, then $|D_1 \cap D \cap D'| = 1$ and $|D_1 \cap D| = |D_1 \cap D'| = |D \cap D'| = 1$ and we contradict Lemma 3.6. This proves 3.15.1.

In addition to 3.15.1, we also have

3.15.2. $\{x_2, x_3\} \subseteq D$.

By 3.15.1, D is at least one of (D_1, D_2)-Type-1 and (D_1, D_3)-Type-1. If D is (D_1, D_2)-Type-1, then, since D is $\{D_1, D_3\}$-Type-1, we have $|\{x_1, x_3\} \cap D| = 1$. If $x_1 \in D$, then $x_4 \in D$ and $D \cap (D_1 \cup D_2 \cup D_3) = \{x_1, x_4\}$, and so $|D_2 \cap D_3 \cap D| = 1$ and $|D_2 \cap D_3| = |D_2 \cap D| = |D_3 \cap D| = 1$. But then, by Lemma 3.6, $|E(M)| = 10$; a contradiction. Therefore $x_1 \notin D$, and so $\{x_2, x_3\} \subseteq D$. Similarly, if D is (D_1, D_3)-Type-1, we have $\{x_2, x_3\} \subseteq D$. Thus 3.15.2 holds.

By (P2), M has a 4-cocircuit D_4 containing x_9. By 3.15.2, $\{x_2, x_3\} \subseteq D_4$. Therefore, as $|E(M)| \geq 11$, we deduce that M has a 4-cocircuit D_5 such that $D_5 \not\subseteq D_1 \cup D_2 \cup D_3 \cup D_4$. But then, by 3.15.2, we have $\{x_2, x_3\} \subseteq D_5$.

Therefore $D_1 \cap D_4 \cap D_5 = \{x_2, x_3\}$, contradicting Lemma 3.9. This last contradiction completes the proof of Lemma 3.15. □

Having established that M has two disjoint 4-cocircuits if $|E(M)| \geq 11$, the last step before proving the necessary direction of Theorem 1.4 for $|E(M)| \geq 16$ is to show that $E(M)$ can be partitioned into 4-cocircuits if $|E(M)| \geq 16$. Before showing this, we prove two preliminary results.

Lemma 3.16. *Let $X \subseteq E(M)$ such that $M|X \cong M(K_{2,4})$, and let D be a 4-cocircuit of M meeting X. Then either D contains exactly one element from each of the four series pairs of $M|X$, or $D \cap X$ is a series pair of $M|X$.*

Proof. Suppose the lemma does not hold. For all $i \in \{1,2,3,4\}$, let $\{x_i, y_i\}$ denote the series pairs of $M|X$. Since M is 4-connected, $D \cap X \neq \{x_i, y_i, x_j, y_j\}$ for distinct $i, j \in \{1,2,3,4\}$. Therefore, for some i and j, we have $|D \cap \{x_i, y_i\}| = 1$ and $|D \cap \{x_j, y_j\}| = 0$. But $\{x_i, x_j, y_i, y_j\}$ is a circuit; a contradiction. Thus the lemma holds. □

Lemma 3.17. *If $|E(M)| \geq 13$, then M has three pairwise-disjoint 4-cocircuits.*

Proof. Suppose that $|E(M)| \geq 13$ and M has no three pairwise-disjoint 4-cocircuits. By Lemma 3.15, M has two disjoint 4-cocircuits, D_1 and D_2 say. Moreover, by Lemma 3.5, $M|(D_1 \cup D_2) \cong M(K_{2,4})$. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_5, x_6, x_7, x_8\}$, and let $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the series pairs in $M|(D_1 \cup D_2)$. By (P2), M has a 4-cocircuit D_3 containing x_9. Since $D_3 \cap (D_1 \cup D_2)$ is nonempty, it follows by Lemma 3.16 that $D_3 \cap (D_1 \cup D_2)$ is a series pair of $M|(D_1 \cup D_2)$. Thus, without loss of generality, $D_3 = \{x_1, x_5, x_9, x_{10}\}$. Let D be a 4-cocircuit of M such that $D \not\subseteq D_1 \cup D_2 \cup D_3$. We next show that

3.17.1. *$D_3 \cap D \neq \emptyset$.*

If $D_3 \cap D = \emptyset$, then, as $D \cap (D_1 \cup D_2)$ is nonempty, we may assume by Lemma 3.16 that $D = \{x_2, x_6, x_{11}, x_{12}\}$. By Lemma 3.5, $M|(D_3 \cup D) \cong M(K_{2,4})$ and so, by orthogonality, $\{x_1, x_2\}$ and $\{x_5, x_6\}$ are series pairs in $M|(D_3 \cup D)$. Thus, without loss of generality, we may assume $\{x_9, x_{11}\}$ and $\{x_{10}, x_{12}\}$ are also series pairs in $M|(D_3 \cup D)$.

Now M has a 4-cocircuit D' containing x_{13}. Furthermore, $D' \cap (D_1 \cup D_2)$ and $D' \cap (D_3 \cup D)$ are both nonempty. By Lemma 3.16,

$$D' \cap (D_1 \cup D_2) \in \{\{x_1, x_5\}, \{x_2, x_6\}, \{x_3, x_7\}, \{x_4, x_8\}\}$$

and

$$D' \cap (D_3 \cup D) \in \{\{x_1, x_2\}, \{x_5, x_6\}, \{x_9, x_{11}\}, \{x_{10}, x_{12}\}\}.$$

As $|D'| = 4$, the intersections $D' \cap (D_1 \cup D_2)$ and $D' \cap (D_3 \cup D)$ are not disjoint. But then D' meets a circuit of $M|(D_1 \cup D_2)$ in exactly one element; a contradiction. Thus 3.17.1 holds.

We also have

3.17.2. $\{x_1, x_5\} \cap D = \emptyset$.

If $x_1 \in D$, then, by orthogonality, $x_5 \in D$, and so we may assume that $D = \{x_1, x_5, x_{11}, x_{12}\}$. Now $(D_3 \cup D) - x_1$ contains a cocircuit and, by orthogonality, this cocircuit avoids x_5. Hence, as M is 4-connected, $\{x_9, x_{10}, x_{11}, x_{12}\}$ is a 4-cocircuit of M disjoint from D_1 and D_2; a contradiction. Thus $x_1 \notin D$ and, similarly, $x_5 \notin D$, and 3.17.2 holds.

By (P2), M has a 4-cocircuit D_4 containing x_{11}. By 3.17.1 and 3.17.2, we may assume $x_9 \in D_4$. Furthermore, as D_4 meets $D_1 \cup D_2$, we may assume that by Lemma 3.16 that $D_4 = \{x_2, x_6, x_9, x_{11}\}$. Now M has a 4-cocircuit D_5 containing x_{12}. By 3.17.1 and 3.17.2, $D_3 \cap D_5 \neq \emptyset$ and $\{x_1, x_5\} \cap D_5 = \emptyset$. Moreover, replacing D_3 with D_4 in the above argument shows that $D_4 \cap D_5 \neq \emptyset$ and $\{x_2, x_6\} \cap D_5 = \emptyset$. Therefore we may assume that $D_5 = \{x_3, x_7, x_9, x_{12}\}$. Now M has a 4-circuit C containing $\{x_4, x_9\}$. By orthogonality, C meets each of $\{x_1, x_5, x_{10}\}$, $\{x_2, x_6, x_{11}\}$, and $\{x_3, x_7, x_{12}\}$. But then $|C| \geq 5$; a contradiction. This completes the proof of Lemma 3.17. □

The next lemma extends Lemmas 3.15 and 3.17.

Lemma 3.18. *If $|E(M)| \geq 16$, then $E(M)$ can be partitioned into 4-element blocks, where each block is a 4-cocircuit.*

Proof. Suppose that $|E(M)| \geq 16$. We first show that M has four pairwise-disjoint 4-cocircuits. By Lemma 3.17, M has three pairwise-disjoint 4-cocircuits, D_1, D_2, and D_3 say. Moreover, by Lemma 3.5, we have $M|(D_i \cup D_j) \cong M(K_{2,4})$ for all distinct $i, j \in \{1, 2, 3\}$. Let z_1, z_2, z_3, and z_4 be distinct elements of $E(M) - (D_1 \cup D_2 \cup D_3)$. By (P2), each of these elements is in a 4-cocircuit, Z_1, Z_2, Z_3, and Z_4 say, of M. If $Z_i \cap (D_1 \cup D_2 \cup D_3) = \emptyset$ for some i, then M has four pairwise-disjoint 4-cocircuits. Therefore assume $Z_i \cap (D_1 \cup D_2 \cup D_3)$ is nonempty for all i. Then, by Lemma 3.16, we have $|Z_i \cap (D_1 \cup D_2 \cup D_3)| = 3$ and $|Z_i \cap D_1| = |Z_i \cap D_2| = |Z_i \cap D_3| = 1$ for all i. If, for distinct i and j, we have $Z_i \cap Z_j \neq \emptyset$, then it is easily checked that $|Z_i \cap Z_j| = 3$, contradicting Lemma 3.1. It now follows that Z_1, Z_2, Z_3, and Z_4 are four pairwise-disjoint 4-cocircuits of M.

Now suppose that $E(M)$ cannot be partitioned into 4-cocircuits. Let $\{D_1, D_2, \ldots, D_n\}$ be a maximum-sized set of pairwise-disjoint 4-cocircuits of M. Then, by above, $n \geq 4$. Let x be an element of $E(M) - (D_1 \cup D_2 \cup \cdots \cup D_n)$. By (P2), M has a 4-cocircuit D containing x. Furthermore, $D \cap (D_1 \cup D_2 \cup \cdots \cup D_n) \neq \emptyset$. Without loss of generality, we may assume that $D \cap D_1 \neq \emptyset$, and so $D \cap (D_1 \cup D_2 \cup D_3 \cup D_4) \neq \emptyset$. But, for all distinct $i, j \in \{1, 2, 3, 4\}$, we have $M|(D_i \cup D_j) \cong M(K_{2,4})$ and so, by Lemma 3.16, $|D \cap (D_1 \cup D_2 \cup D_3 \cup D_4| \geq 4$; a contradiction. The lemma now follows. □

We are now ready to prove the necessary direction of Theorem 1.4 when $|E(M)| \geq 16$.

Proof of Theorem 1.4 for $|E(M)| \geq 16$. Suppose $|E(M)| \geq 16$. Then, by Lemma 3.18, there is a partition of $E(M)$ into 4-cocircuits $D_1, D_2, \ldots, D_n$, where $D_i = \{w_i, x_i, y_i, z_i\}$ for all i. By Lemma 3.5, $M|(D_1 \cup D_i) \cong M(K_{2,4})$ for all $i \in \{2, 3, \ldots, n\}$, so we may assume M has 4-circuits $\{w_1, x_1, w_i, x_i\}$, $\{w_1, y_1, w_i, y_i\}$, $\{w_1, z_1, w_i, z_i\}$, $\{x_1, y_1, x_i, y_i\}$, $\{x_1, z_1, x_i, z_i\}$, and $\{y_1, z_1, y_i, z_i\}$ for all such i. Consider the 4-circuits $\{w_1, x_1, w_i, x_i\}$ and $\{w_1, x_1, w_j, x_j\}$. By circuit elimination and orthogonality, $\{w_i, x_i, w_j, x_j\}$ is a 4-circuit of M. Similarly, for all distinct $i, j \in \{2, 3, \ldots, n\}$, we have $\{w_i, y_i, w_j, y_j\}$, $\{w_i, z_i, w_j, z_j\}$, $\{x_i, y_i, x_j, y_j\}$, $\{x_i, z_i, x_j, y_j\}$, and $\{y_i, z_i, y_j, z_j\}$ are 4-circuits of M. In turn, as $M|(D_i \cup D_j) \cong M(K_{2,4})$ for all distinct i and j, we have

3.18.1. *$\{w_i, w_j\}$, $\{x_i, x_j\}$, $\{y_i, y_j\}$, and $\{z_i, z_j\}$ are the series pairs in $M|(D_i \cup D_j)$ for all distinct i and j.*

Now consider $K_{4,n}$, where $n \geq 4$. Label the edge set of $K_{4,n}$ so that

$$\big\{\{w_1, x_1, y_1, z_1\}, \{w_2, x_2, y_2, z_2\}, \ldots, \{w_n, x_n, y_n, z_n\}\big\}$$

is a partition of $E(K_{4,n})$, where each block is a bond of $K_{4,n}$, and $\{w_i, x_i, w_j, x_j\}$, $\{w_i, y_i, w_j, y_j\}$, $\{w_i, z_i, w_j, z_j\}$, $\{x_i, y_i, x_j, y_j\}$, $\{x_i, z_i, x_j, z_j\}$, and $\{y_i, z_i, y_j, z_j\}$ are 4-cycles of $K_{4,n}$ for distinct $i, j \in \{1, 2, \ldots, n\}$. We next show that the identity map φ from $E(M(K_{4,n}))$ to $E(M)$ is a weak map from $M(K_{4,n})$ to M. Let C be a circuit of $M(K_{4,n})$. Then $|C| \in \{4, 6, 8\}$. If C is a 4-circuit, then, by above, $\varphi(C)$ is a 4-circuit of M. Now assume that $|C| = 6$. Then, without loss of generality, we may assume

$$C = \{w_i, x_i, x_j, y_j, y_k, w_k\},$$

where i, j, and k are distinct elements in $\{1, 2, \ldots, n\}$. Using circuit elimination on the 4-circuits $\{w_i, x_i, w_j, x_j\}$ and $\{w_j, y_j, w_k, y_k\}$ of M, it follows that $\{w_i, x_i, x_j, y_j, y_k, w_k\}$ contains a circuit of M. By orthogonality and 3.18.1, it is easily checked that

$$\{w_i, x_i, x_j, y_j, y_k, w_k\}$$

is a 6-circuit of M. Thus if C is a 6-circuit of $M(K_{4,n})$, then $\varphi(C)$ is a 6-circuit of M. Lastly, assume that $|C| = 8$. Then, without loss of generality, we may assume

$$C = \{w_i, w_j, x_j, x_k, y_k, y_l, z_l, z_i\},$$

where i, j, k, and l are distinct elements in $\{1, 2, \dots, n\}$. Now $\{w_i, w_j, x_j, x_k, y_k, y_i\}$ and $\{y_i, y_l, z_l, z_i\}$ are circuits of M. By circuit elimination, $\{w_i, w_j, x_j, x_k, y_k, y_l, z_l, z_i\}$ contains a circuit of M. If this last set is not a circuit, then, by orthogonality and 3.18.1, it contains a 6-circuit of M. Without loss of generality, we may assume that this 6-circuit is $\{w_i, w_j, x_j, x_k, y_k, z_i\}$. But then, as $\{x_j, y_j, x_k, y_k\}$ is a 4-circuit of M, it follows by circuit elimination that

$$X = \{w_i, w_j, x_j, x_k, z_i, y_j\}$$

contains a circuit of M. By orthogonality and 3.18.1, X contains no circuit of M. Thus C is an 8-circuit of M. It now follows that if C is a circuit of $M(K_{4,n})$, then $\varphi(C)$ is a circuit of M. Hence M is a weak-map image of $M(K_{4,n})$ under φ.

We next show that

3.18.2. *$M|(D_i \cup D_j \cup D_k \cup D_l) \cong M(K_{4,4})$ for all distinct $i, j, k, l \in \{1, 2, \dots, n\}$.*

By above, $M|(D_i \cup D_j \cup D_k \cup D_l)$ is a weak-map image of $M(K_{4,4})$. Furthermore, as $M|(D_i \cup D_j \cup D_k \cup D_l)$ has an 8-circuit, it follows that $r(M|(D_i \cup D_j \cup D_k \cup D_l)) \geq 7$. Since $M|(D_i \cup D_j) \cong M(K_{2,4})$, we have $r(M|(D_i \cup D_j)) = 5$ and so, by the 4-circuits of M established above,

$$r(M|(D_i \cup D_j \cup D_k) \leq 6.$$

In turn, as D_l is a cocircuit of $M|(D_i \cup D_j \cup D_k \cup D_l)$, we deduce that $r(M|(D_i \cup D_j \cup D_k \cup D_l)) \leq 7$. Thus $r(M|(D_i \cup D_j \cup D_k \cup D_l)) = 7$, that is, $r(M|(D_i \cup D_j \cup D_k \cup D_l)) = r(M(K_{4,4}))$. Since $M|(D_i \cup D_j \cup D_k \cup D_l)$ is connected, it follows by Theorem 1.5 that $M|(D_i \cup D_j \cup D_k \cup_l) \cong M(K_{4,4})$. Thus 3.18.2 holds.

We next prove by induction on n that $r(M) = r(M(K_{4,n}))$ for all $n \geq 4$. If $n = 4$, then, by 3.18.2, $r(M) = r(M(K_{4,4}))$. Therefore suppose that $n \geq 5$ and that, for all matroids M' satisfying (P2) and whose ground set can be partitioned into m 4-cocircuits, where $4 \leq m \leq n-1$, we have $r(M') = r(M(K_{4,m}))$. Let M' denote the matroid $M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1})$. We first show that M' satisfies (P2). Evdiently, every element of M' is in a 4-cocircuit. Let x and y be distinct elements of M'. If x and y are in distinct 4-cocircuits D_i and D_j of M', then, by orthogonality and M satisfying (P2), M' has a 4-circuit containing $\{x, y\}$. Thus assume x and y are in the same 4-cocircuit D_i of M'. By considering D_i with D_1 if $i \neq 1$ or D_2 if $i = 1$, it follows that M' has a 4-circuit containing $\{x, y\}$. Lastly, if M' is not 4-connected, then it has a 2- or 3-separation (A, B). Since $n \geq 5$, it is easily checked that, for distinct i, j, k, and l, there are four 4-cocircuits

D_i, D_j, D_k, and D_l of M' such that $|A \cap (D_i \cup D_j \cup D_k \cup D_l)| \geq 3$ and $|B \cap (D_i \cup D_j \cup D_k \cup D_l) \geq 3$. Now, by [3, Lemma 8.2.3],

$$\begin{aligned} 2 &\geq r(A) + r(B) - r(M') \\ &\geq r(A \cap (D_i \cup D_j \cup D_k \cup D_l)) + r(B \cap (D_i \cup D_j \cup D_k \cup D_l)) \\ &\qquad - r(D_i \cup D_j \cup D_k \cup D_l). \end{aligned}$$

But then $M|(D_i \cup D_j \cup D_k \cup D_l)$ is not 4-connected, contradicting 3.18.2. It follows that M' is 4-connected, and so M' satisfies (P2). By induction,

$$r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1}) = r(M(K_{4,n-1}))$$

and so, as D_n is a cocircuit of M,

$$r(M) = r(M|(D_1 \cup D_2 \cup \cdots \cup D_{n-1}) + 1 = r(M(K_{4,n})).$$

Finally, as M is connected, it now follows by Theorem 1.5 that $M \cong M(K_{4,n})$, thereby completing the proof of Theorem 1.4. □

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Appendix

Let M be a 4-connected matroid satisfying (P2) with $|E(M)| \leq 15$. Then M is one of thirty-five matroids. These thirty-five matroids comprise of $U_{3,6}$, twenty-one 8-element paving matroids, ten 9-element paving matroids, R_{10}, a 12-element matroid, and a 14-element matroid. The matroid R_{10} is the unique splitter for the class of regular matroids and for which

$$\left[\begin{array}{c|ccccc} & -1 & 1 & 0 & 0 & 1 \\ & 1 & -1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & -1 & 1 & 0 \\ & 0 & 0 & 1 & -1 & 1 \\ & 1 & 0 & 0 & 1 & -1 \end{array}\right]$$

is a representation of it over all fields. Precise descriptions of the 8-, 9-, 12-, and 14-element matroids are given below. For ease of reference, the notation is in keeping with the notation in [4], 8-**Element Matroids.** If $|E(M)| = 8$, then M is one of twenty-one rank-4 paving matroids. Let $E(M) = \{1, 2, \ldots, 8\}$. Up to isomorphism, to describe M, it is sufficient, to list the 4-circuits of M. The first table consists of those matroids M having the property that, for every 4-circuit C, there is another 4-circuit of M meeting C in exactly one element.

M	4-Circuits of M
$M_{8,1}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{2,4,5,6\}$, $\{1,4,7,8\}$, $\{1,3,6,8\}$
$M_{8,2}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{2,4,5,6\}$, $\{1,3,6,8\}$, $\{4,6,7,8\}$
$M_{8,3}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{1,4,6,8\}$, $\{2,4,7,8\}$
$M_{8,3^+}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{1,4,6,8\}$, $\{2,4,7,8\}$, $\{1,3,7,8\}$
$M_{8,4}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{4,6,7,8\}$
$M_{8,4^+}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{4,6,7,8\}$, $\{1,3,6,8\}$
$M_{8,5}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{2,3,6,8\}$, $\{3,4,5,6\}$, $\{1,4,7,8\}$, $\{3,5,7,8\}$, $\{2,4,6,8\}$
$M_{8,6}$	$\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,2,5,8\}$, $\{2,3,6,7\}$, $\{3,4,5,6\}$, $\{2,4,7,8\}$, $\{1,3,6,8\}$

The second table consists of those matroids M having a 4-circuit C such that every other 4-circuit of M meets C in exactly two elements. Note that, in the table, F_7^+ denotes the free coextension of F_7.

M	4-Circuits of M
F_7^+	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$ $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{1,4,6,7\}$
$M_{8,7}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{2,4,6,7\}$
$M_{8,7^+}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, $\{2,4,6,7\}$, $\{3,4,6,7\}$
$M_{8,8a}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{2,4,5,8\}$, $\{3,4,6,7\}$
$M_{8,8b}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,3,6,8\}$, $\{2,4,5,8\}$, $\{2,4,6,7\}$
$M_{8,9a}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,6,8\}$, $\{2,4,6,7\}$
$M_{8,9b}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,6,8\}$, $\{3,4,6,7\}$
$M_{8,9b^+}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,6,8\}$, $\{3,4,6,7\}$, $\{2,4,5,7\}$
$M_{8,10}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{3,4,6,7\}$
$M_{8,10^+}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{3,4,6,7\}$, $\{2,3,6,8\}$
$M_{8,10^{++}}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{3,4,6,7\}$, $\{2,3,6,8\}$, $\{2,4,5,7\}$
$M_{8,11}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,2,7,8\}$, $\{1,3,5,7\}$, $\{1,4,6,8\}$, $\{3,4,5,8\}$, $\{2,3,6,7\}$, $\{2,4,5,7\}$
$M_{8,12}$	$\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,3,5,7\}$, $\{1,4,5,8\}$, $\{2,3,7,8\}$, $\{2,4,6,7\}$, $\{3,4,6,8\}$

9-**Element Matroids.** If $|E(M)| = 9$, then M is one of ten rank-4 paving matroids. Let $E(M) = \{1, 2, \ldots, 9\}$. Again, to describe M, it suffices, up to isomorphism, to list the 4-circuits of M. Here, if M is such a matroid, then its set of 4-circuits contains every 4-element subset of each the sets $\{1,2,3,4,5\}$, $\{4,5,7,8,9\}$, and $\{2,3,6,8,9\}$. The remaining 4-circuits of M are given in the next table.

M	Remaining 4-Circuits of M				
$M_{9,1}$	$\{1,2,6,7\}$,	$\{1,3,7,8\}$,	$\{1,4,6,9\}$,	$\{1,5,6,8\}$	
$M_{9,1a}$	$\{1,2,6,7\}$,	$\{1,3,7,8\}$,	$\{1,4,6,9\}$,	$\{1,5,6,8\}$,	$\{3,4,6,7\}$
$M_{9,1b}$	$\{1,2,6,7\}$,	$\{1,3,7,8\}$,	$\{1,4,6,9\}$,	$\{1,5,6,8\}$,	$\{3,5,6,7\}$
$M_{9,2}$	$\{1,2,6,7\}$,	$\{1,3,7,8\}$,	$\{1,4,6,9\}$	$\{3,5,6,7\}$	
$M_{9,3}$	$\{1,4,6,8\}$,	$\{1,2,7,8\}$,	$\{1,5,6,9\}$,	$\{1,3,7,9\}$,	$\{2,4,6,7\}$
$M_{9,3^+}$	$\{1,4,6,8\}$,	$\{1,2,7,8\}$,	$\{1,5,6,9\}$,	$\{1,3,7,9\}$,	$\{2,4,6,7\}$,
	$\{3,5,6,7\}$				
$M_{9,4}$	$\{1,4,6,8\}$,	$\{1,2,7,8\}$,	$\{1,5,6,9\}$,	$\{1,3,7,9\}$,	$\{2,5,6,7\}$
$M_{9,4^+}$	$\{1,4,6,8\}$,	$\{1,2,7,8\}$,	$\{1,5,6,9\}$,	$\{1,3,7,9\}$,	$\{2,5,6,7\}$,
	$\{3,4,6,7\}$				
$M_{9,5}$	$\{1,4,6,8\}$,	$\{1,2,7,8\}$,	$\{1,5,6,9\}$,	$\{3,4,6,7\}$	
$M_{9,6}$	$\{1,4,6,8\}$,	$\{1,2,7,9\}$,	$\{3,5,6,7\}$		

12- and 14-Element Matroids. The unique 4-connected 12-element matroid satisfying (P2) and the unique 4-connected 14-element matroid satisfying (P2) have $GF(4)$-representations

$$\left[\begin{array}{c|ccccccc} & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ I_5 & 0 & 1 & 0 & 1 & \alpha & 1 & \alpha \\ & 0 & 0 & 1 & 1 & \alpha^2 & 1 & \alpha^2 \\ & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{c|cccccccc} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ I_6 & 0 & 1 & 0 & 0 & 1 & 0 & \alpha & \alpha \\ & 0 & 0 & 1 & 0 & 0 & 1 & \alpha^2 & \alpha^2 \\ & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right],$$

respectively, where $\alpha^2 + \alpha + 1 = 0$.

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