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Some Local Extremal Connectivity Results for Matroids

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For Paul Erdős on his 80th birthday

Tutte proved that if e is an element of a 3-connected matroid M such that neither $M\backslash e$ nor M/e is 3-connected, then e is in a 3-circuit or a 3-cocircuit. In this paper, we prove a broad generalization of this result. Among the consequences of this generalization are that if X is an $(n-1)$-element subset of an n-connected matroid M such that neither $M\backslash X$ nor M/X is connected, then, provided $|E(M)| \geq 2(n-1) \geq 4$, X is in both an n-element circuit and an n-element cocircuit. When $n = 3$, we describe the structure of M more closely using $\Delta - Y$ exchanges. Several related results are proved and we also show that, for all fields F other than $GF(2)$, the set of excluded minors for F-representability is closed under both $\Delta - Y$ and $Y - \Delta$ exchanges.

1. Introduction

Tutte's celebrated wheels-and-whirls theorem [28] is one of a number of matroid results that make structural assertions for a matroid that satisfies a certain extremal connectivity condition (see, for example, [1, 2, 4, 5, 14, 15, 16, 17, 18, 20, 29]). In particular, Tutte's theorem asserts that if M is a 3-connected matroid for which no single-element deletion or single-element contraction is 3-connected, then M is a wheel or a whirl. The extremal connectivity condition in this result is clearly global. Tutte [28] also proved the following result, in which the extremal connectivity condition is local.

Theorem 1.1. *Let e be an element of a 3-connected matroid M and suppose that neither $M\backslash e$ nor M/e is 3-connected. Then e is in a 3-element circuit or a 3-element cocircuit of M.*

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In this paper, the focus will be on using local extremal connectivity conditions to deduce local structural information for a matroid. In particular, we shall prove a broad generalization of Theorem 1.1.

The matroid terminology used here will follow Oxley [22]. In particular, if M is a matroid and T is a subset of its ground set $E(M)$, then $\mathrm{cl}(T)$ will denote the closure of T, and $r(T)$ will denote the rank of T. The simple matroid associated with M will be denoted by $\widetilde{M}$. The m-element circuits and m-element cocircuits of M will be called simply *m-circuits* and *m-cocircuits* or, when $m = 3$, *triangles* and *triads.*

If k is a positive integer, a partition $\{X, Y\}$ of $E(M)$ is a *k-separation* of a matroid M if $\min\{|X|, |Y|\} \geq k$ and

$$r(X) + r(Y) - r(M) \leq k - 1.$$

If n is a positive integer such that M has no k-separations for any $k < n$, then M is *n-connected* [28]. Thus a matroid is 2-connected if and only if it is connected. A survey of the properties of n-connected matroids can be found in [22, Chapter 8]. One such property that we shall use frequently is the following well-known result (see, for example, [17, Lemma 2.2]).

Lemma 1.2. *Let M be an n-connected matroid having at least $2(n-1)$ elements. Then all circuits and all cocircuits of M have at least n elements.*

If t is a non-negative real number, then $\lfloor t \rfloor$ and $\lceil t \rceil$ will denote, respectively, the greatest integer not exceeding t and the least integer not less than t. The restriction on the cardinality of $E(M)$ imposed in the last lemma is frequently applied when one is considering n-connected matroids. It is a very weak restriction [12, 23] for the only matroids it excludes are those uniform matroids $U_{r,k}$ such that $r \in \{\lfloor k/2 \rfloor, \lceil k/2 \rceil\}$ and $k \in \{0, 1, 2, \ldots, 2n-3\}$.

We shall assume familiarity with the operations of parallel connection and 2-sum of matroids (see, for example, [22, Chapter 7]). A related, but more general, operation is generalized parallel connection. Let M_1 and M_2 be matroids such that $M_1|T = M_2|T$, where $T = E(M_1) \cap E(M_2)$. Let $N = M_1|T$ and suppose that $\widetilde{N}$ is a modular flat of $\widetilde{M_1}$. The *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 across N is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. This construction was introduced by Brylawski [7] when M_1 and M_2 are simple matroids, but it extends easily to the more general case considered above (see, for example, [22, Section 12.4]). Brylawski [7, 8] identified numerous attractive properties of the construction. When $|T| = 1$, $P_N(M_1, M_2)$ is just the parallel connection $P(M_1, M_2)$ [6] of M_1 and M_2.

This operation of generalized parallel connection is closely related to the graph operation of *clique-sum* that has been used by Robertson and Seymour [24] in their important work on graph minors. Let G_1 and G_2 be graphs whose sets of edge labels are disjoint except that each has a triangle whose edges are labelled by e, f, and g. Let G be the graph that is obtained by identifying these two triangles so that edges with the same labels coincide. Then the cycle matroid of G is precisely the matroid $P_N(M(G_1), M(G_2))$, where

$N = M(G_1)|\{e,f,g\} \cong U_{2,3}$. The graph $G\backslash\{e,f,g\}$ that is obtained from G by deleting the identified edges is the 3-*sum* of G_1 and G_2. Seymour [26] extended the definition of 3-sum to the class of binary matroids.

In the last example, N was chosen to be a triangle, such a set always being a modular flat in a simple binary matroid. This special case of the generalized parallel connection will be the one that will be of most importance here. In this case, we shall write $P_\Delta(M_1, M_2)$ for $P_N(M_1, M_2)$. Moreover, we shall be mainly concerned with the case when $M_1 \cong M(K_4)$. When G_1 and G_2 are as in the last paragraph, if $G_1 \cong K_4$, then taking the 3-sum of G_1 and G_2 amounts to replacing the triangle $\{e,f,g\}$ of G_2 by a vertex of degree 3 that is joined to the three vertices of the triangle. For graphs, this operation is called a $\Delta - Y$ *exchange*. If M_1 and M_2 are matroids where ground sets meet in a triangle Δ and $M_1 \cong M(K_4)$, then we shall say that $P_\Delta(M_1, M_2)\backslash\Delta$ has been obtained from M_2 by a $\Delta - Y$ *exchange*. Truemper [27] studied this operation and its properties when M_2 is binary but we shall drop this restriction on M_2. As examples, note that one obtains the matroids $U_{3,5}$, F_7^*, and $(F_7^-)^*$ from $U_{2,5}$, F_7, and F_7^-, respectively, by a single $\Delta - Y$ exchange.

In graphs, one studies not only $\Delta - Y$ exchanges but also $Y - \Delta$ exchanges. For matroids, the latter operation can be defined using duality. If M_2 is a matroid having a triad $\{e,f,g\}$, then $\{e,f,g\}$ is a triangle Δ of M_2^*. Thus $P_\Delta(M(K_4), M_2^*)\backslash\Delta$ is well-defined. Hence so is its dual $[P_\Delta(M(K_4), M_2^*)\backslash\Delta]^*$. The last matroid will be said to be obtained from M_2 by a $Y - \Delta$ *exchange*.

In the next section, we state the main connectivity result of the paper and note some of its consequences. Section 3 contains some preliminary lemmas needed to prove the main result, and Section 4 contains the proof. In Section 5, we look more closely at 3-connected matroids and extend the main result for such matroids. Finally, in Section 6, we prove that, for all fields F other than $GF(2)$, the set of excluded minors for F-representability is closed under both $\Delta - Y$ and $Y - \Delta$ exchanges.

2. The main result and some consequences

We begin this section by stating the main connectivity result of the paper. Because the statement of this result is a little cumbersome, we follow this by extracting some special cases of the theorem that are of most interest.

Theorem 2.1. *Let M be an n-connected matroid having at least $2(n-1)$ elements. Suppose that, for some integer k such that $(2/3)(n-2) < k \leq n-1$, there is a k-element subset Z of $E(M)$ such that both $M\backslash Z$ and M/Z are $(n-k)$-separated. Then*

(i) Z is in an n-circuit of M; or
(ii) Z is in an n-cocircuit of M; or
(iii) k is even and $2n + 2k - 2 \leq |E(M)| \leq 6n - 3k - 8$; or
(iv) k is odd and $2n + 2k - 2 \leq |E(M)| \leq 6n - 3k - 10$.

Moreover,

(v) if $k = n-1$, then both (i) and (ii) hold; and

(vi) *if* $k = n - 2$, *then* (i) *holds; or* (ii) *holds; or* $n = 4$, $|E(M)| = 10$, $r(M) = 5$, *and* $E(M\backslash Z)$ *has a partition into two* 4-*circuits of* M *and another partition into two* 4-*cocircuits of* M.

The proof of this theorem will be given in Section 4. We now examine some consequences of the theorem. The first is a well-known result of Tutte [28].

Corollary 2.2. *Let* e *be an element of a connected matroid* M. *Then* $M\backslash e$ *or* M/e *is connected.*

Proof. Suppose that both $M\backslash e$ and M/e are disconnected. Then, since every matroid with fewer than two elements is connected, $|E(M\backslash e)| \geq 2$, so $|E(M)| \geq 3$. By (v) of the theorem, e is in both a 2-circuit C and a 2-cocircuit C^*. Since $|C \cap C^*|$ cannot be 1, we must have that $C = C^*$. But now $E(M) = C$ since M is connected and any circuit of M meeting C must contain C. Thus $|E(M)| < 3$; a contradiction. □

Next suppose that $n = 3$ and $k = 1$. In that case, part (iv) of the theorem asserts that $6 \leq |E(M)| \leq 5$, so (i) or (ii) holds. Theorem 1.1 now follows from Theorem 2.1 provided $|E(M)| \geq 4$. But if $|E(M)| < 4$, the hypotheses of (1.1) fail.

In the case $k = n - 1$, Theorem 2.1 makes a very strong assertion:

Corollary 2.3. *Let* n *be an integer exceeding two and* M *be an* n-*connected matroid having at least* $2(n-1)$ *elements. If* M *has an* $(n-1)$-*element subset* Z *such that both* $M\backslash Z$ *and* M/Z *are disconnected, then* M *has both an* n-*circuit containing* Z *and an* n-*cocircuit containing* Z.

Next we consider the implications of the last result for graphic matroids. It is well known that the notions of n-connectedness of a graph G and n-connectedness of its cycle matroid $M(G)$ do not, in general, coincide. More precisely, we have the following result [10, 13, 19].

Proposition 2.4. *Let* n *be an integer exceeding one. If* G *is a connected graph with at least three vertices and* $G \ncong K_3$, *then* $M(G)$ *is an* n-*connected matroid if and only if* G *is an* n-*connected graph having no cycles with fewer than* n *edges.*

Using this proposition, we get the next result as a consequence of Theorem 2.1.

Corollary 2.5. *Let* e *and* f *be distinct edges of a* 3-*connected simple graph* G. *Suppose that neither* $G\backslash e, f$ *nor* $G/e, f$ *is both loopless and* 2-*connected. Then* G *has both a triangle containing* $\{e, f\}$ *and a degree*-3 *vertex that is incident with both* e *and* f.

Proof. The last corollary implies that $M(G)$ has both a triangle and a triad containing $\{e, f\}$. There are three possibilities for the subgraph induced by this triad. However, one easily checks that, because G is 3-connected, the only acceptable possibility is that the triad consists of the three edges incident with a degree-3 vertex. $\square$

In Section 5, we shall prove a generalization of the last corollary for 3-connected matroids that are not necessarily graphic. For n exceeding three, there are no graphic matroids satisfying the hypotheses of Corollary 2.3. To see this, suppose that $M(G)$ does satisfy the hypotheses. Then the corollary implies that, in the graph G, the $(n-1)$-element set Z is in both an n-edge cycle and an n-edge cut. But it is straightforward to check that this cannot happen in an n-connected graph that has no cycles with fewer than n edges.

3. Some preliminaries

In this section, we prove two lemmas that will be used in the proof of Theorem 2.1. The first of these is straightforward; the second is more difficult and will play a crucial role in the proof of the theorem.

Lemma 3.1. *Let $\{P,Q\}$ be a partition of the ground set of an n-connected matroid M. Then*

$$r(P)+r(Q)-r(M)\geq \min\{|P|,|Q|,n-1\}.$$

Proof. By semimodularity, the inequality holds trivially if $\min\{|P|,|Q|,n-1\}=0$. Hence assume that $\min\{|P|,|Q|,n-1\}>0$. If $\min\{|P|,|Q|\}\geq n-1$, then, since $\{P,Q\}$ is not an $(n-1)$-separation of M, it follows that

$$r(P)+r(Q)-r(M)\geq n-1=\min\{|P|,|Q|,n-1\}.$$

If $\min\{|P|,|Q|\}=|P|<n-1$, then, since $\{P,Q\}$ is not a $|P|$-separation of M, we deduce that

$$r(P)+r(Q)-r(M)\geq |P|=\min\{|P|,|Q|,n-1\}. \qquad \square$$

Lemma 3.2. *Let M be an n-connected matroid having at least $2(n-1)$ elements. Let Z be a k-element subset of $E(M)$ for some k in $\{1,2,\ldots,n-1\}$ and suppose that $M\backslash Z$ is $(n-k)$-separated. Then*

(i) Z is in an n-cocircuit of M;

or

(ii) for every $(n-k)$-separation $\{U,V\}$ of $M\backslash Z$,

(a) $\min\{|U|,|V|\}\geq n$; and

(b) if $|U|=n$, then U is an n-circuit of M.

Moreover, if $k=n-1$ and Z is in an n-circuit of M, then (i) holds.

Proof. Suppose that $M\backslash Z$ has an $(n-k)$-cocircuit C^*. Then $C^*\cup Z$ contains a cocircuit of M. But, by Lemma 1.2, this cocircuit has at least n elements. Since $|C^*\cup Z|=n$, we deduce that $C^*\cup Z$ is a cocircuit of M; that is, (i) holds.

We may now assume that $M\backslash Z$ has no $(n-k)$-cocircuits. Let $\{U, V\}$ be an $(n-k)$-separation of $M\backslash Z$ and suppose that $r(U) = |U|$. Then, as $r(M\backslash Z) = r(M)$ and

$$r(U) + r(V) - r(M\backslash Z) \le n - k - 1,$$

we deduce that

$$r(M) - r(V) \ge |U| - (n - k - 1),$$

that is,

$$r(M/V) \ge |U| + k - (n - 1). \tag{1}$$

The matroid M/V has $U \cup Z$ as its ground set, so has $|U| + k$ elements. Moreover, since M has no cocircuits of cardinality less than n, every cocircuit of M/V has at least n elements. So every circuit of $(M/V)^*$ has at least n elements. But, by (1), $(M/V)^*$ has rank at most $n-1$. It follows that $(M/V)^*$ is uniform of rank $n-1$, so every n-element subset of $U \cup Z$ is a circuit of $(M/V)^*$, and hence is a cocircuit of M/V. In particular, Z is in an n-cocircuit of M/V and hence is in an n-cocircuit of M. Thus $M\backslash Z$ has an $(n-k)$-cocircuit; a contradiction. We conclude that if $\{U, V\}$ is an $(n-k)$-separation of $M\backslash Z$, then $r(U) \neq |U|$. Since M has no circuits with fewer than n elements, it follows that $\min\{|U|, |V|\} \ge n$. Moreover, if $\min\{|U|, |V|\} = |U| = n$, then U is an n-circuit of M. Hence if (i) does not hold, (ii) does.

It remains to prove that the last sentence in the lemma is true. Hence suppose that $|Z| = n-1$ and that Z is in an n-circuit $Z \cup e$ of M, but that Z is not in an n-cocircuit of M. Let $\{U, V\}$ be a 1-separation of $M\backslash Z$ and suppose, without loss of generality, that $e \in U$. Then

$$r(U) + r(V) - r(M\backslash Z) \le 0 \tag{2}$$

and $r(M\backslash Z) = r(M)$. Moreover, since $e \in U$ and $Z \cup e$ is an n-circuit, semimodularity implies that

$$r(U \cup Z) \le r(U) + r(Z \cup e) - r(e) = r(U) + n - 2.$$

Thus, by (2), $r(U \cup Z) + r(V) - r(M) \le n-2$. Since, by (ii), $\min\{|U \cup Z|, |V|\} \ge n$, we deduce that $\{U \cup Z, V\}$ is an $(n-1)$-separation of the n-connected matroid M; a contradiction. □

4. The proof of the main theorem

In this section, we shall prove Theorem 2.1. In addition, we note a variant of the theorem that guarantees the existence of an n-circuit or an n-cocircuit in M but does not ensure that such a set contains Z.

Proof of Theorem 2.1. Let $\{U, V\}$ and $\{X, Y\}$ be $(n-k)$-separations of $M\backslash Z$ and M/Z, respectively. Then

$$r(U) + r(V) - r(M\backslash Z) \le n - k - 1 \tag{1}$$

and

$$r_{M/Z}(X) + r_{M/Z}(Y) - r(M/Z) \le n - k - 1. \tag{2}$$

By Lemma 1.2, Z is both independent and coindependent in M. Thus $r(M\backslash Z) = r(M)$ and $r(Z) = k$. Substituting into (1) and (2), we deduce that

$$r(U) + r(V) - r(M) \leq n - k - 1 \tag{3}$$

and

$$r(X \cup Z) + r(Y \cup Z) - r(M) \leq n - 1. \tag{4}$$

Adding (3) and (4), we get

$$[r(U) + r(X \cup Z)] + [r(V) + r(Y \cup Z)] - 2r(M) \leq 2n - k - 2. \tag{5}$$

Thus, by semimodularity,

$$[r(U \cap X) + r(U \cup X \cup Z)] + [r(V \cap Y) + r(V \cup Y \cup Z)] - 2r(M) \leq 2n - k - 2.$$

Regrouping terms, we get

$$[r(U \cap X) + r(V \cup Y \cup Z) - r(M)] + [r(V \cap Y) + r(U \cup X \cup Z) - r(M)] \leq 2n - k - 2. \tag{6}$$

Now each of $\{U \cap X, V \cup Y \cup Z\}$ and $\{V \cap Y, U \cup X \cup Z\}$ is a partition of $E(M)$. Thus, by Lemma 3.1,

$$\begin{aligned} r(U \cap X) + r(V \cup Y \cup Z) - r(M) &\geq \min\{|U \cap X|, |V \cup Y \cup Z|, n - 1\} \\ &= \min\{|U \cap X|, n - 1\}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} r(V \cap Y) + r(U \cup X \cup Z) - r(M) &\geq \min\{|V \cap Y|, |U \cup X \cup Z|, n - 1\} \\ &= \min\{|V \cap Y|, n - 1\}. \end{aligned} \tag{8}$$

On combining (6), (7), and (8), we deduce that

$$\min\{|U \cap X|, n - 1\} + \min\{|V \cap Y|, n - 1\} \leq 2n - k - 2. \tag{9}$$

But the roles of U and V are indistinguishable in the above argument, so we may interchange U and V in the last inequality to get

$$\min\{|V \cap X|, n - 1\} + \min\{|U \cap Y|, n - 1\} \leq 2n - k - 2. \tag{10}$$

Lemma 4.1.
(i) $|U \cap X| \leq n - 1 - \lceil k/2 \rceil$ *or* $|V \cap Y| \leq n - 1 - \lceil k/2 \rceil$*; and*
(ii) $|V \cap X| \leq n - 1 - \lceil k/2 \rceil$ *or* $|U \cap Y| \leq n - 1 - \lceil k/2 \rceil$.

Proof. By symmetry, it suffices to prove (i). Suppose first that $|U \cap X| \geq n - 1$. Then, by (9),

$$\min\{|V \cap Y|, n - 1\} \leq n - k - 1,$$

so certainly

$$|V \cap Y| \leq n - 1 - \left\lceil \frac{k}{2} \right\rceil.$$

Thus we may assume that $|U \cap X| < n-1$ and, similarly, that $|V \cap Y| < n-1$. Then, by (9),

$$\min\{|U \cap X|, |V \cap Y|\} \leq \frac{1}{2}(2n-k-2),$$

so

$$\min\{|U \cap X|, |V \cap Y|\} \leq n-1-\left\lceil \frac{k}{2} \right\rceil. \qquad \square$$

Now suppose that M has neither an n-circuit nor an n-cocircuit containing Z. Then, by Lemma 3.2 and its dual,

$$\min\{|U|, |V|\} \geq n \tag{11}$$

and

$$\min\{|X|, |Y|\} \geq n. \tag{12}$$

By Lemma 4.1(i), we may assume, without loss of generality, that

$$|U \cap X| \leq n-1-\left\lceil \frac{k}{2} \right\rceil. \tag{13}$$

Then, by (11),

$$|U \cap Y| \geq \left\lceil \frac{k}{2} \right\rceil + 1 \tag{14}$$

and, by (12),

$$|V \cap X| \geq \left\lceil \frac{k}{2} \right\rceil + 1. \tag{15}$$

But, by Lemma 4.1(ii), $|U \cap Y|$ or $|V \cap X|$ is at most $n-1-\lceil k/2 \rceil$. Thus

$$\left\lceil \frac{k}{2} \right\rceil + 1 \leq n-1-\left\lceil \frac{k}{2} \right\rceil,$$

so

$$2\left\lceil \frac{k}{2} \right\rceil \leq n-2.$$

Hence, if $k = n-1$, then M has either an n-circuit or an n-cocircuit containing Z. Indeed, by Lemma 3.2 and its dual, M has both an n-circuit and an n-cocircuit containing Z.

We may now assume that $k \leq n-2$. Next we note that $\lceil k/2 \rceil + 1 \leq n-1$, so, by (14),

$$\min\{|U \cap Y|, n-1\} \geq \left\lceil \frac{k}{2} \right\rceil + 1. \tag{16}$$

Hence, by (10) and (16),

$$\begin{aligned} \min\{|V \cap X|, n-1\} &\leq 2n-k-2-\min\{|U \cap Y|, n-1\} \\ &\leq 2n-k-2-\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \\ &= 2n-3-\left\lceil \frac{3k}{2} \right\rceil. \end{aligned}$$

Thus

$$|V \cap X| \leq 2n-3-\left\lceil \frac{3k}{2} \right\rceil \tag{17}$$

unless

$$n - 1 \le 2n - 3 - \left\lceil \frac{3k}{2} \right\rceil. \tag{18}$$

But, by assumption, $k > (2/3)(n-2)$, so (18) fails. Therefore (17) holds.

Similarly, by using (10) with (15), we get that

$$|U \cap Y| \le 2n - 3 - \left\lceil \frac{3k}{2} \right\rceil. \tag{19}$$

By Lemma 4.1(ii),

$$|U \cap Y| \le n - 1 - \left\lceil \frac{k}{2} \right\rceil \tag{20}$$

or

$$|V \cap X| \le n - 1 - \left\lceil \frac{k}{2} \right\rceil. \tag{21}$$

If (20) occurs, then, by interchanging X and Y in the argument used to give (17), we deduce that

$$|V \cap Y| \le 2n - 3 - \left\lceil \frac{3k}{2} \right\rceil. \tag{22}$$

If (21) occurs, then, by interchanging U and V in the argument used to give (19), we again deduce that (22) holds.

If (20) holds, then, on combining it with (13), we get that

$$|U| \le 2n - 2 - 2\left\lceil \frac{k}{2} \right\rceil.$$

Moreover, by (17) and (22),

$$|V| \le 4n - 6 - 2\left\lceil \frac{3k}{2} \right\rceil.$$

Thus, as

$$|E(M)| = |U| + |V| + |Z| = |U| + |V| + k,$$

we get that

$$|E(M)| \le 6n - 8 - 2\left\lceil \frac{k}{2} \right\rceil - 2\left\lceil \frac{3k}{2} \right\rceil + k. \tag{23}$$

If (21) holds, then, by (13), $|X| \le 2n - 2 - 2\lceil k/2 \rceil$ and, by (19) and (22), $|Y| \le 4n - 6 - 2\lceil 3k/2 \rceil$ and again (23) holds. The upper bounds on $|E(M)|$ in (iii) and (iv) follow immediately from (23).

To prove the lower bounds on $|E(M)|$ in (iii) and (iv), we note that, by (3),

$$r(M) \ge r(U) + r(V) - (n - k - 1). \tag{24}$$

Since $\min\{|U|, |V|\} \ge n$, it follows that

$$\min\{r(U), r(V)\} \ge n - 1. \tag{25}$$

Hence, by (24),

$$r(M) \ge n - 1 + k. \tag{26}$$

Now, as $\{X, Y\}$ is an $(n-k)$-separation of M/Z, it is an $(n-k)$-separation of $M^*\backslash Z$ (see, for example, [22, Proposition 8.1.5]). Hence we have the following analogue of (24):

$$r^*(M) \geq r^*(X) + r^*(Y) - (n-k-1).$$

Since

$$\min\{r^*(X), r^*(Y)\} \geq n-1, \tag{27}$$

it follows that

$$r^*(M) \geq n-1+k. \tag{28}$$

The lower bound on $|E(M)|$ follows immediately on combining (26) and (28).

To prove (vi), suppose that $k = n-2$ but that neither (i) nor (ii) holds. Then (iv) cannot occur, since it asserts that $4n-6 \leq |E(M)| \leq 3n-4$. Thus (iii) occurs, so

$$4n-6 \leq |E(M)| \leq 3n-2 \tag{29}$$

and n is even. Hence $n = 4$ and $|E(M)| = 3n-2$. Therefore $|X| = |Y| = |U| = |V| = 4$, so, by Lemma 3.2 and its dual, X and Y are both 4-cocircuits of M, and U and V are both 4-circuits of M. Moreover, since equality holds in (29), equality also holds in (26), that is, $r(M) = 5$. □

By making minor modifications to this proof, one can obtain the following result.

Theorem 4.2. *Let M be an n-connected matroid having at least $2(n-1)$ elements. Suppose that, for some positive integer k such that $\frac{2}{3}(n-3) < k \leq n-1$, there is a k-element subset Z of $E(M)$ such that both $M\backslash Z$ and M/Z are $(n-k)$-separated. Then*

(i) M has an n-circuit; or

(ii) M has an n-cocircuit; or

(iii) k is even and $2n+2k+2 \leq |E(M)| \leq 6n-3k-10$; or

(iv) k is odd and $2n+2k+2 \leq |E(M)| \leq 6n-3k-12$.

Moreover,

(v) if $k = n-3$, then (i) or (ii) holds.

On taking $n = 4$ in (v) of this theorem, we obtain the following result of Wong [29, Theorem 4.1].

Corollary 4.3. *Let M be a 4-connected matroid with at least six elements. Suppose that M has an element e such that neither $M\backslash e$ nor M/e is 4-connected. Then M has a 4-circuit or a 4-cocircuit.*

Proof of Theorem 4.2. This follows the proof of Theorem 2.1 up to inequalities (11) and (12). Then, by using Lemma 3.2 and its dual, one can sharpen these inequalities to get

$$\min\{|U|, |V|\} \geq n+1$$

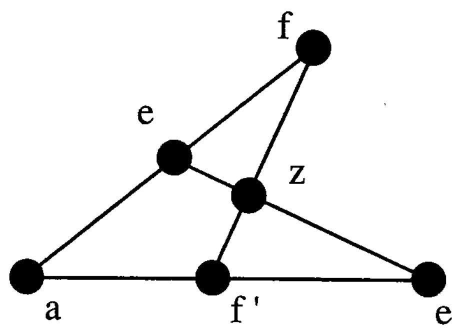

Figure 1

and

$$\min\{|X|, |Y|\} \geq n+1.$$

Moreover, inequalities (25) and (27) can also be sharpened by one. The theorem is obtained by following the implications of these sharpened inequalities throughout the proof and using Theorem 2.1. □

5. An extension of Theorem 2.1 for 3-connected matroids

In Section 2, we noted that when $n = 3$ Theorem 2.1 can be used to give a detailed description of the local structure around the elements e and f when the matroid M is graphic. In this section, we use the operation of generalized parallel connection to extend that theorem to arbitrary matroids. The following theorem establishes that if a 3-connected matroid M has elements e and f such that both $M\backslash e,f$ and $M/e,f$ are disconnected, then M can be obtained from one of its minors by an operation that is very close to a $\Delta - Y$ exchange.

Theorem 5.1. *Let e and f be distinct elements of a 3-connected matroid M. Suppose that*

(i) $\{e,f\}$ is in both a triangle $\{e,f,a\}$ and a triad $\{e,f,z\}$ of M.

Then either

(ii) M is isomorphic to $U_{2,4}$, or

(iii) M is isomorphic to $P_\Delta(M(K_4), M/z)\backslash\{e,f\}$, where $\Delta = \{e,f,a\}$.

In particular, if both $M\backslash e,f$ and $M/e,f$ are disconnected, then (i) holds, hence so does one of (ii) and (iii).

Proof. Corollary 2.3 implies that if both $M\backslash e,f$ and $M/e,f$ are disconnected, then (i) holds. Hence it suffices to prove that if (i) holds, then one of (ii) and (iii) holds. If M has a triangle that is also a triad, then $M \cong U_{2,4}$ (see, for example, [22, Proposition 8.1.7]). Thus we may assume that z is not on the line containing $\{e,f,a\}$, so $a \neq z$. Before proving the rest of the theorem, we shall make some notational changes. The matroid M/z has $\{e,f,a\}$ as a triangle. Let M_2 be the matroid obtained from M/z by relabelling e and f as e' and f', respectively. Let M_1 be $M(K_4)$ labelled as in Figure 1. Let $\Delta = \{e',f',a\}$. We shall complete the proof of the theorem by showing that $M = P_\Delta(M_1, M_2)\backslash\{e', f'\}$. To do this we shall prove the following.

Lemma 5.2. *Every flat of M is a flat of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$.*

This will mean that M is a quotient [9, 11] of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$. Since these matroids have the same rank and the same ground set, it will follow that the two matroids are equal (see, for example, [22, Corollary 7.3.4]).

The proof of Lemma 5.2 will use the following two lemmas.

Lemma 5.3. *Let X be a flat of M that avoids $\{e, f, z\}$. Then*

$$\mathrm{cl}_M(X \cup z) \subseteq X \cup \{e, f, z\}.$$

Proof. Since $E(M) - \{e, f, z\}$ is a hyperplane of M, it follows that X is a flat of $M\backslash e, f, z$. But z is a coloop of $M\backslash e, f$, so $M/z\backslash e, f = M\backslash e, f, z$. Hence X is a flat of $(M/z)\backslash e, f$. Thus $\mathrm{cl}_{M/z}(X) \subseteq X \cup \{e, f\}$, so $\mathrm{cl}_M(X \cup z) \subseteq X \cup \{e, f, z\}$. □

Lemma 5.4. *The set $\{e, f, z\}$ is a cocircuit of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$.*

Proof. Certainly $\{e, f, z\}$ is a cocircuit of $P_\Delta(M_1, M_2)$. Suppose that $\{e, f, z\}$ is not a cocircuit of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$. In that case, $\{e, f, z\}$ is a union of cocircuits of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$. We deduce that $\{e, f\}$ and $\{z\}$ are cocircuits of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$ by considering intersections with the circuit $\{e, f, a\}$ of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$. Hence

$$r(P_\Delta(M_1, M_2)\backslash\{e, f, e', f', z\}) \leq r(P_\Delta(M_1, M_2)) - 2 = r(M) - 2.$$

But

$$P_\Delta(M_1, M_2)\backslash\{e, f, e', f', z\} = M_2\backslash e', f' = M/z\backslash e, f = M\backslash e, f, z.$$

Since $\{e, f, z\}$ is a cocircuit of M, it follows that

$$r(P_\Delta(M_1, M_2)\backslash\{e, f, e', f', z\}) = r(M) - 1.$$

This contradiction completes the proof of Lemma 5.4. □

Proof of Lemma 5.2. Let F be a flat of M. Then one of the following five possibilities must occur:

(i) $z \in F$ and $a \in F$;
(ii) $z \in F$ and $a \notin F$;
(iii) $z \notin F$ and $|\{e, f\} \cap F| = 0$;
(iv) $z \notin F$ and $|\{e, f\} \cap F| = 1$;
(v) $z \notin F$ and $|\{e, f\} \cap F| = 2$.

To prove Lemma 5.2, we shall show that, in each case, F is a flat of $P_\Delta(M_1, M_2)\backslash\{e', f'\}$. Throughout this argument, if X is a subset of $E(M/z)$, then X' will denote the corresponding subset of $E(M_2)$. Hence X' is obtained from X by, if necessary, relabelling e as e' and relabelling f as f'.

(i) $F-z$ is a flat F_1 of M/z. Since $a \in F$ and $\{e,f,a\}$ is a circuit of M, the flat F_1 either contains or avoids $\{e,f\}$. Consider $z \cup F_1 \cup F_1'$. This set meets $E(M_2)$ in F_1', a flat of M_2. Moreover, $z \cup F_1 \cup F_1'$ meets $E(M_1)$ in either $E(M_1)$ or $\{a,z\}$, depending on whether F does or does not contain $\{e,f\}$. Hence $z \cup F_1 \cup F_1'$ meets $E(M_1)$ in a flat of M_1. We conclude that $z \cup F_1 \cup F_1'$ is a flat of $P_\Delta(M_1,M_2)$. Thus $(z \cup F_1 \cup F_1') - \{e',f'\}$ is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$, that is, F is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$.

(ii) $F-z$ is again a flat F_1 of M/z and $z \cup F_1 \cup F_1'$ again meets $E(M_2)$ in F_1', a flat of M_2. But, since $a \notin F$, at most one of e and f is in F, so $z \cup F_1 \cup F_1'$ meets $E(M_1)$ in $\{e,z,e'\}$, $\{f,z,f'\}$, or $\{z\}$. Each of the last three sets is a flat of M_1. Thus $z \cup F_1 \cup F_1'$ is a flat of $P_\Delta(M_1,M_2)$, so, as in case (i), F is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$.

(iii) $\mathrm{cl}_M(F \cup z)$ is a flat of M containing z. Thus, by case (i) or (ii), $\mathrm{cl}_M(F \cup z)$ is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$. But, by Lemma 5.3, $\mathrm{cl}_M(F \cup z) \subseteq F \cup \{e,f,z\}$, and, by Lemma 5.4, $\{e,f,z\}$ is a cocircuit of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$. Thus $\mathrm{cl}_M(F \cup z) - \{e,f,z\}$ is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$; that is, F is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$.

(iv) We may assume, without loss of generality, that $F \cap \{e,f\} = \{e\}$. Since $\{e,f,z\}$ is a cocircuit of M meeting the flat F in $\{e\}$, we deduce that $F-e$ is a flat of M. Then, by Lemma 5.3, $\mathrm{cl}_M((F-e) \cup z) \subseteq (F-e) \cup \{e,f,z\}$. Certainly $e \notin \mathrm{cl}_M(F-e)$ and $z \notin \mathrm{cl}_M((F-e) \cup e)$. Thus, by the Mac Lane-Steinitz exchange property, $e \notin \mathrm{cl}_M((F-e) \cup z)$. Hence $\mathrm{cl}_M((F-e) \cup z) \subseteq (F-e) \cup \{f,z\}$. Thus $\mathrm{cl}_{M/z}(F-e) \subseteq (F-e) \cup f$, so $\mathrm{cl}_{M_2}(F-e)$ is $F-e$ or $(F-e) \cup f'$. Consider $F \cup \mathrm{cl}_{M_2}(F-e)$. This meets $E(M_2)$ in $\mathrm{cl}_{M_2}(F-e)$, a flat of M_2, and meets $E(M_1)$ in $\{e\}$ or $\{e,f'\}$, each of which is a flat of M_1. Thus $F \cup \mathrm{cl}_{M_2}(F-e)$ is a flat of $P_\Delta(M_1,M_2)$, so $[F \cup \mathrm{cl}_{M_2}(F-e)] - \{e',f'\}$ is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$; that is, F is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$.

(v) $F \supseteq \{e,f\}$ so $a \in F$. Moreover, $F-\{e,f\}$ is a flat of M, so, by Lemma 5.3, $\mathrm{cl}_M((F-\{e,f\}) \cup z) \subseteq (F-\{e,f\}) \cup \{e,f,z\}$. Now $e \notin \mathrm{cl}_M(F-\{e,f\})$ and $z \notin \mathrm{cl}_M((F-\{e,f\}) \cup e)$, so, by the Mac Lane-Steinitz exchange property, $e \notin \mathrm{cl}_M((F-\{e,f\}) \cup z)$. Similarly $f \notin \mathrm{cl}_M((F-\{e,f\}) \cup z)$. Hence $\mathrm{cl}_{M/z}(F-\{e,f\}) = F-\{e,f\}$. Thus $F \cap E(M_2)$, which equals $F-\{e,f\}$, is a flat of M_2; and $F \cap E(M_1)$, which equals $\{e,f,a\}$, is a flat of M_1. Therefore F is a flat of $P_\Delta(M_1,M_2)$, so F is a flat of $P_\Delta(M_1,M_2)\backslash\{e',f'\}$.

This completes the proof of Lemma 5.2. □

By the remarks following Lemma 5.2, this finishes the proof of Theorem 5.1. □

Next we note an example showing that one possible strengthening of Theorem 5.1 fails. Let M be the cycle matroid of the 4-spoked wheel and let e and f be diametrically opposite rim edges. Then $M\backslash e,f$ is disconnected and neither M/e nor M/f is 3-connected. However, $\{e,f\}$ is in neither a triangle nor a triad of M.

Seymour [25, Lemma 2.3] showed that $U_{2,4}$ is the only 3-connected matroid M with $|E(M)| \geq 4$ such that $M\backslash Z$ and M/Z are disconnected for all 2-element subsets Z of $E(M)$. We conclude this section by using Theorem 5.1 to prove an extension of this result.

Theorem 5.5. *Let M be a 3-connected matroid with $|E(M)| \geq 4$ and suppose that M has a circuit C such that, for all 2-element subsets Z of C, both $M\backslash Z$ and M/Z are disconnected. Then either*

(i) $M \cong U_{2,4}$; *or*
(ii) $|C| = 3$ *and M has a triad $\{a, b, c\}$ such that* $M^*|(C \cup \{a, b, c\}) \cong M(K_4)$ *and* $M^* = P_\Delta(M^*|(C \cup \{a, b, c\}), M^*\backslash C)$, *where* $\Delta = \{a, b, c\}$.

Proof. We begin by establishing that $|C| = 3$. By Lemma 1.2, it suffices to show that $|C| < 4$. Assume that $|C| \geq 4$ and let v, w, x, and y be distinct elements of C. Then, by Theorem 5.1, M has elements a, b, and c such that $\{v, w, a\}$ and $\{v, x, b\}$ are triads and $\{v, y, c\}$ is a triangle of M. Now $c \notin C$, otherwise the circuit $\{v, y, c\}$ is a proper subset of the circuit C. Since the circuit $\{v, y, c\}$ cannot have exactly one common element with either of the cocircuits $\{v, w, a\}$ or $\{v, x, b\}$, we deduce that $a = b = c$. By cocircuit elimination and the fact that M is 3-connected, it follows that $\{v, w, x\}$ is a cocircuit of M. But this cocircuit meets the circuit $\{v, y, c\}$ in a single element. This contradiction completes the proof that $|C| = 3$.

If $|E(M)| = 4$, then certainly $M \cong U_{2,4}$. Thus suppose that $|E(M)| \geq 5$. Let $C = \{d, e, f\}$. By Theorem 5.1, there are triads $\{d, e, a\}$, $\{d, f, b\}$, and $\{e, f, c\}$ of M. If a, b, and c are not distinct, cocircuit elimination implies that $\{d, e, f\}$ is a triad of M. Since $\{d, e, f\}$ is also a triangle, this contradicts the fact that $|E(M)| \geq 5$ (see, for example, [22, Proposition 8.1.7]). Thus we may assume that a, b, and c are distinct.

Consider $M^*|\{a, b, c, d, e, f\}$. This matroid has $\{d, e, a\}$, $\{d, f, b\}$, and $\{e, f, c\}$ among its circuits, hence it is spanned by $\{d, e, f\}$. But $\{d, e, f\}$ is also a cocircuit of M^* and so is a union of cocircuits of $M^*|\{a, b, c, d, e, f\}$. By considering intersections with the circuits $\{d, e, a\}$, $\{d, f, b\}$, and $\{e, f, c\}$, we deduce that $\{d, e, f\}$ is a cocircuit of $M^*|\{a, b, c, d, e, f\}$. Thus $\{d, e, f\}$ cannot also be a circuit of $M^*|\{a, b, c, d, e, f\}$. Hence $\{d, e, f\}$ is a basis of $M^*|\{a, b, c, d, e, f\}$, and $\{a, b, c\}$ is a hyperplane of this restriction. It now follows without difficulty that $\{a, b, c\}$ is a circuit of this restriction, so $\{a, b, c\}$ is a triad of M, and $M^*|\{a, b, c, d, e, f\} \cong M(K_4)$.

Finally, we note that $M^*/\{a, b, c\}$ has $\{d, e, f\}$ as both a parallel class and a cocircuit, and hence as a separator. Thus, by [7; 22, Proposition 12.4.15],

$$M^* = P_\Delta(M^*|\{a, b, c, d, e, f\}, M^*\backslash C),$$

where $\Delta = \{a, b, c\}$. □

For graphs, Theorem 5.5 asserts the following.

Corollary 5.6. *Let C be a cycle of a 3-connected simple graph G and suppose that, for every pair of distinct edges x and y of C, neither $G\backslash x, y$ nor $G/x, y$ is both loopless and 2-connected. Then C is a 3-cycle each of whose vertices has degree three. Moreover, the three edges of G that have exactly one endpoint in $V(C)$ form an edge cut (see Figure 2).*

We remark that, in Figure 2, the vertices v_1, v_2, and v_3 all have degree three, and the vertices v_4, v_5, and v_6 are distinct unless $G \cong K_4$, in which case all three coincide.

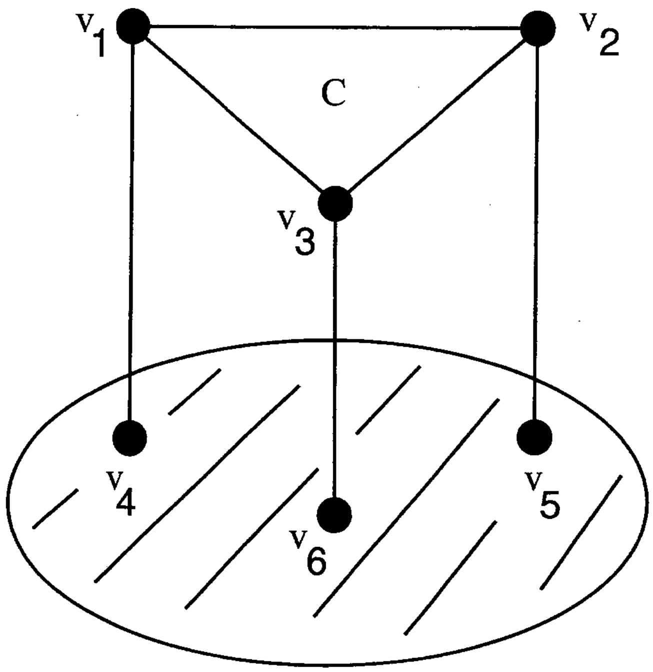

Figure 2 The structure of G in Corollary 5.6.

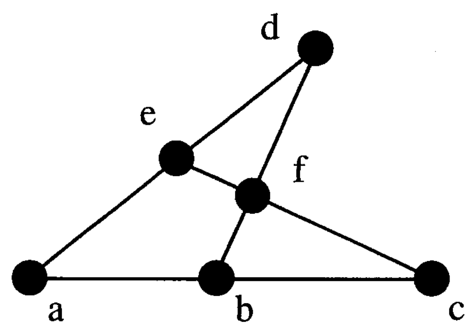

Figure 3

6. The excluded minors for F-representability

The operation of generalized parallel connection across a triangle has featured prominently in the connectivity results of the earlier sections. In this section, we prove a representability result involving this operation, namely that, for all fields F with at least three elements, the set of excluded minors for F-representability is closed under both $\Delta - Y$ and $Y - \Delta$ exchanges.

Theorem 6.1. *Suppose that M is an excluded minor for representability over a field F where $F \ncong GF(2)$. If $\{a,b,c\}$ is a triangle Δ of M, then $P_\Delta(M(K_4),M)\backslash\Delta$ is an excluded minor for representability over F.*

Proof. Let $M(K_4)$ be labelled as in Figure 3 and write P for $P_\Delta(M(K_4),M)$. Since M is an excluded minor for F-representability, M is 3-connected. Also, as $F \ncong GF(2)$, it follows that $M \not\cong U_{2,4}$. Thus the triangle Δ of M is not a triad of M. Hence $r(M\backslash\Delta) = r(M) = r$, say, so

$$r(M) + 1 \leq r(M\backslash\Delta) + 1 \leq r(P\backslash\Delta) \leq r(P) = r(M) + r(M(K_4)) - r(\Delta) = r(M) + 1.$$

Thus equality holds throughout the last line and $r(P\backslash\Delta) = r + 1$.

Now assume that $P\backslash\Delta$ is F-representable. Then, since $P\backslash\Delta$ is clearly simple, we may view it as a restriction of $PG(r, F)$. Since

$$r(\{d, e, f\}) + r(M\backslash\Delta) = 3 + r(M) = r(P\backslash\Delta) + 2,$$

the flats in $PG(r, F)$ that are spanned by $\{d, e, f\}$ and $E(M\backslash\Delta)$ meet in a line, L say. Let a', b', and c' be the points of intersection of L and the lines of $PG(r, F)$ spanned by $\{d, e\}$, $\{d, f\}$, and $\{e, f\}$, respectively. Let

$$P' = PG(r, F)|(E(P\backslash\Delta) \cup \{a', b', c'\})$$

and let

$$M' = P'\backslash\{d, e, f\}.$$

Clearly P' is F-representable, so M' is F-representable. Moreover, $\{d, e, f\}$ is a cocircuit of P'. We shall show that M is isomorphic to M' under the function ψ that fixes every element of $E(M\backslash\Delta)$ and maps a, b, and c to a', b', and c', respectively.

Now $P'/d\backslash c'$ is obtained from $M'\backslash c'$ by adding e in parallel with a' and adding f in parallel with b'. Thus $P'/d\backslash\{a', b', c'\}$ is isomorphic to $M'\backslash c'$ under the function α that fixes every element of $M\backslash\Delta$ and maps e to a' and f to b'.

Similarly, $P/d\backslash c$ is obtained from $M\backslash c$ by adding e in parallel with a and adding f in parallel with b. Thus $P/d\backslash\{a, b, c\}$ is isomorphic to $M\backslash c$ under the function β that fixes every element of $M\backslash\Delta$ and maps e to a and f to b.

Now observe that

$$P'/d\backslash\{a', b', c'\} = P'\backslash\{a', b', c'\}/d = P\backslash\Delta/d = P\backslash\{a, b, c\}/d = P/d\backslash\{a, b, c\}.$$

Hence the function $\alpha\beta^{-1}$ is an isomorphism between $M\backslash c$ and $M'\backslash c'$ that fixes every element of $M\backslash\Delta$ and maps a to a' and b to b'. Clearly $\alpha\beta^{-1}$ is the restriction of ψ to $E(M\backslash c)$. By a similar argument, the restrictions of ψ to each of $E(M\backslash a)$ and $E(M\backslash b)$ are isomorphisms. Since $\{a, b, c\}$ is a circuit of M while its image under ψ, namely $\{a', b', c'\}$, is a circuit of M', we conclude that ψ is indeed an isomorphism between M and M'. But M' is F-representable, so M is F-representable. This contradiction implies that $P\backslash\Delta$ is not F-representable.

To complete the proof that $P\backslash\Delta$ is an excluded minor for F-representability, it suffices to show that every single-element deletion and every single-element contraction of $P\backslash\Delta$ is F-representable. First we note, from above, that $P/d\backslash c$ is obtained from $M\backslash c$ by two parallel extensions. Hence $P/d\backslash c$ is F-representable, so $(P\backslash\Delta)/d$ is F-representable. Similarly, both $(P\backslash\Delta)/e$ and $(P\backslash\Delta)/f$ are F-representable. Moreover, $(P\backslash\Delta)\backslash d$ has $\{e, f\}$ as a 2-cocircuit and, indeed, is equal to the 2-sum, with basepoint c, of $M\backslash a, b$ and a triangle on $\{e, f, c\}$. Thus $(P\backslash\Delta)\backslash d$ is F-representable and, similarly, so are $(P\backslash\Delta)\backslash e$ and $(P\backslash\Delta)\backslash f$.

Now suppose that $x \in E(M\backslash\Delta)$. Then $P_\Delta(M(K_4),M)\backslash x = P_\Delta(M(K_4),M\backslash x)$. Moreover, each of $M(K_4)$ and $M\backslash x$ is F-representable and, since $M(K_4)$ is binary, it is uniquely F-representable (see, for example, [22, Proposition 10.1.3]). So $P_\Delta(M(K_4),M)\backslash x$ is F-representable [8, Proposition 7.6.11]. Hence $(P\backslash\Delta)\backslash x$ is F-representable. Similarly, if $x \in E(M)-\text{cl}_M(\Delta)$, then $P_\Delta(M(K_4),M)/x = P_\Delta(M(K_4),M/x)$. By [8, Proposition 7.6.11] again, $P_\Delta(M(K_4),\widetilde{M/x})$ is F-representable. Thus P/x and, hence $(P\backslash\Delta)/x$, is F-representable. It remains to show that, for $x \in \text{cl}_M(\Delta)-\{a,b,c\}$, the matroid $(P\backslash\Delta)/x$ is F-representable. But P/x is isomorphic to the parallel connection, with basepoint c, of M/x and a 4-point line on $\{d,e,f,c\}$. Hence P/x is F-representable and so too is $(P\backslash\Delta)/x$. □

We now know that if $|F| \geq 3$, the set $\mathscr{M}_F$ of excluded minors for F-representability is closed under $\Delta - Y$ exchanges. On combining this result with the well-known fact that $\mathscr{M}_F$ is closed under duality, we immediately deduce that $\mathscr{M}_F$ is closed under $Y - \Delta$ exchanges:

Corollary 6.2. *Suppose that M is an excluded minor for representability over a field F where $F \not\cong GF(2)$. If $\{a,b,c\}$ is a triad of M, then $\{a,b,c\}$ is a triangle Δ of M^* and $[P_\Delta(M(K_4),M^*)\backslash\Delta]^*$ is an excluded minor for representability over F.*

For $|F| \geq 3$, $\mathscr{M}_F$ has only been completely determined when $|F| = 3$ [3, 25]:

$$\mathscr{M}_{GF(3)} = \{U_{2,5}, U_{3,5}, F_7, F_7^*\}.$$

As noted in the introduction, each of $U_{3,5}$ and F_7^* can be obtained from $U_{2,5}$ and F_7 by a single $\Delta - Y$ exchange. It is well known that $\mathscr{M}_{GF(4)}$ contains $U_{2,6}$, P_6, and $U_{4,6}$, where P_6 is the 6-element rank-3 matroid consisting of a single 3-point line with three other points off that line. Applying a $\Delta - Y$ exchange to $U_{2,6}$ gives P_6, and applying a $\Delta - Y$ exchange to P_6 gives $U_{4,6}$. The non-Fano matroid F_7^- is also in $\mathscr{M}_{GF(4)}$. Its dual $(F_7^-)^*$ is obtained from it by a single $\Delta - Y$ exchange. The only remaining known member of $\mathscr{M}_{GF(4)}$ is an 8-element rank-4 matroid [21] with no triangles or triads, so one cannot perform either a $\Delta - Y$ exchange or a $Y - \Delta$ exchange on it.

References

[1] Akkari, S. (1992) A minimal 3-connectedness result for matroids. *Discrete Math.* **103** 221–232.

[2] Akkari, S. and Oxley, J. (1991) Some extremal connectivity results for matroids. *J. Combin. Theory Ser. B* **52** 301–320.

[3] Bixby, R. E. (1979) On Reid's characterization of the ternary matroids. *J. Combin. Theory Ser. B* **26** 174–204.

[4] Bixby, R. E. (1982) A simple theorem on 3-connectivity. *Linear Algebra Appl.* **45** 123–126.

[5] Bixby, R. E. and Coullard, C. R. (1987) Finding a smallest 3-connected minor maintaining a fixed minor and a fixed element. *Combinatorica* **7** 231–242.

[6] Brylawski, T. H. (1971) A combinatorial model for series-parallel networks. *Trans. Amer. Math. Soc.* **154** 1–22.

[7] Brylawski, T. H. (1975) Modular constructions for combinatorial geometries. *Trans. Amer. Math. Soc.* **203** 1–44.

[8] Brylawski, T. H. (1986) Constructions. In: White, N. (ed.) *Theory of Matroids*, Cambridge University Press, Cambridge 127–223.

[9] Crapo, H. H. (1967) Structure theory for geometric lattices. *Rend. Sem. Mat. Univ. Padova* **38** 14–22.

[10] Cunningham, W. H. (1981) On matroid connectivity. *J. Combin. Theory Ser. B* **30** 94–99.

[11] Higgs, D. A. (1968) Strong maps of geometries. *J. Combin. Theory* **5** 185–191.

[12] Inukai, T. and Weinberg, L. (1978) Theorems on matroid connectivity. *Discrete Math.* **22** 311–312.

[13] Inukai, T. and Weinberg, L. (1981) Whitney connectivity of matroids. *SIAM J. Alg. Disc. Methods* **2** 108–120.

[14] Lemos, M. (1989) On 3-connected matroids. *Discrete Math.* **73** 273–283.

[15] Murty, U. S. R. (1974) Extremal critically connected matroids. *Discrete Math.* **8** 49–58.

[16] Oxley, J. G. (1981) On connectivity in matroids and graphs. *Trans. Amer. Math. Soc.* **265** 47–58.

[17] Oxley, J. G. (1981) On matroid connectivity. *Quart. J. Math. Oxford Ser. (2)* **32** 193–208.

[18] Oxley, J. G. (1981) On 3-connected matroids. *Canad. J. Math.* **33** 20–27.

[19] Oxley, J. G. (1981) On a matroid generalization of graph connectivity. *Math. Proc. Camb. Phil. Soc.* **90** 207–214.

[20] Oxley, J. G. (1984) On minor-minimally-connected matroids. *Discrete Math.* **51** 63–72.

[21] Oxley, J. G. (1986) On the matroids representable over GF(4). *J. Combin. Theory Ser. B* **41** 250–252.

[22] Oxley, J. G. (1992) *Matroid Theory*, Oxford University Press, New York.

[23] Richardson, W. R. H. (1973) Decomposition of chain-groups and binary matroids. *Proc. Fourth Southeastern Conf. on Combinatorics, Graph Theory and Computing*, Utilitas Mathematica, Winnipeg 463–476.

[24] Robertson, N. and Seymour, P. D. (1984) Generalizing Kuratowski's Theorem. *Congressus Numerantium* **45** 129–138.

[25] Seymour, P. D. (1979) Matroid representation over GF(3). *J. Combin. Theory Ser. B* **26** 159–173.

[26] Seymour, P. D. (1980) Decomposition of regular matroids. *J. Combin. Theory Ser. B* **28** 305–359.

[27] Truemper, K. (1992) *Matroid Decomposition*, Academic Press, New York.

[28] Tutte, W. T. (1966) Connectivity in matroids. *Canad. J. Math.* **18** 1301–1324.

[29] Wong, P.-K. (1978) On certain n-connected matroids. *J. Reine Angew. Math.* **299/300** 1–6.