Europ. J. Combinatorics (1983) **4**, 169–178

On the Numbers of Bases and Circuits in Simple Binary Matroids

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Quirk and Seymour have shown that a connected simple graph has at least as many spanning trees as circuits. This paper extends and strengthens their result by showing that in a simple binary matroid M the quotient of the number of bases by the number of circuits is at least 2. Moreover, if M has no coloops and rank r, this quotient exceeds $6(r+1)/19$.

1. INTRODUCTION

Welsh [3, pp. 287–288] raised the problem of comparing the numbers $b(M)$ and $c(M)$ of bases and circuits in a matroid M, noting that W. Quirk and P. D. Seymour [3, p. 287] had shown that if M is the cycle matroid of a simple graph, then

$$b(M) \geqslant c(M). \tag{1.1}$$

This paper extends and strengthens Quirk and Seymour's result. Two main results are proved. First, in Section 2, it is shown that if M is a simple binary matroid, then

$$b(M) \geqslant 2c(M), \tag{1.2}$$

with equality being attained only by the direct sum of the Fano matroid and a free matroid.

The second main result, which will be proved in Section 3, shows that if M has rank r and no coloops, then

$$b(M) > \tfrac{6}{19}(r+1)c(M), \tag{1.3}$$

provided again that M is simple and binary. The reason for restricting attention here to simple binary matroids is that expression (1.1) need not hold for arbitrary simple matroids or even for loopless graphic matroids. To see, this, consider, for example, the uniform matroids $U_{2,m}$ for $m \geqslant 6$ and $U_{1,n}$ for $n \geqslant 4$.

We observe here that inequality (1.3) is a sharper bound than expression (1.2) unless M is the direct sum of a free matroid and a matroid of rank less than six. We have included expression (1.2) because it is used in the proof of inequality (1.3). Indeed, without it, one obtains the weaker bound

$$b(M) > \tfrac{3}{10}(r+1)c(M).$$

This raises the question as to how much one may increase the constant $\frac{6}{19}$ in inequality (1.3). The referee conjectures that

$$b(M) \geqslant \tfrac{1}{2}(r+1)c(M) \tag{1.4}$$

observing that

$$\lim_{r\to\infty} \frac{b(PG(r-1,2))}{(r+1)c(PG(r-1,2))} = 1.$$

Notice that equality is attained in expression (1.4) by the Fano matroid.

The matroid terminology used here will, in general, follow Welsh [3]. In particular, if M is a matroid, then $E(M)$ denotes its ground set and rk M its rank. The sets of circuits and bases of M will be denoted by $\mathscr{C}(M)$ and $\mathscr{B}(M)$ respectively and, if $e \in E(M)$, then $\mathscr{C}_e(M)$ and $\mathscr{B}_e(M)$ will denote the sets of circuits and bases of M containing e. A

0195-6698/83/020169+10 $02.00/0

circuit or a cocircuit of M having exactly n elements will be called an *n-circuit* or an *n-cocircuit* respectively. A *series class* of M is a maximal subset X of $E(M)$ such that if x and y are distinct elements of X, then $\{x, y\}$ is a 2-cocircuit. A series class is *nontrivial* if it contains at least two elements. The elements x and y are *in series* if they lie in the same series class.

The matroid obtained from M by deleting all its coloops will be denoted by $\hat{M}$. If $\{x_1, x_2, \ldots, x_m\} \subseteq E(M)$, then $M\backslash x_1, x_2, \ldots, x_m$ and $M/x_1, x_2, \ldots, x_m$ will denote respectively the deletion and contraction of $\{x_1, x_2, \ldots, x_m\}$ from M. Moreover, the numbers of bases and circuits of M which contain $\{x_1, x_2, \ldots, x_m\}$ will be denoted by $b_{x_1,x_2,\ldots,x_m}(M)$ and $c_{x_1,x_2,\ldots,x_m}(M)$ respectively. Hence, if $\{x_1, x_2, \ldots, x_m\}$ is independent in M, then

$$b_{x_1,x_2,\ldots,x_m}(M) = b(M/x_1, x_2, \ldots, x_m).$$

Frequent use will be made here of the well-known fact (see, for example, [3, pp. 267–268]) that $b(M)$ satisfies the following deletion–contraction formula.

If e *is an element of* M, *then*

$$b(M) = b(M\backslash e) + b(M\backslash e) \tag{1.5}$$

unless $\{e\}$ *is a component of* M *in which case*

$$b(M) = b(M\backslash e) = b(M/e).$$

Using this, it is easy to show that if $\{e_1, e_2\}$ is a cocircuit but not a circuit of M, then

$$b(M) = 2b(M\backslash e_1, e_2) + b(M/e_1, e_2). \tag{1.6}$$

This observation is a special case of a general identity for Tutte–Grothendieck invariants which is discussed in detail in [2].

The main results of this paper compare $b(M)$ and $c(M)$ when M is a simple binary matroid. However, the next result applies to all matroids M which are not free. For such a matroid, the average circuit size will be denoted $\gamma(M)$.

1.1. THEOREM. *Let M be a rank-r matroid on a set of n elements and suppose that $r < n$. Then*

$$b(M) \geq \frac{\gamma(M)}{n-r} c(M). \tag{1.7}$$

Moreover, equality holds here if and only if M is isomorphic to $U_{k,k} \oplus U_{r-k,n-k}$ for some k in $\{0, 1, 2, \ldots, r\}$.

PROOF. Consider the set of ordered pairs (B, C) where B is a basis of M and C is a fundamental circuit with respect to B. Every basis has precisely $n-r$ fundamental circuits, so the number of such ordered pairs is $(n-r)b(M)$. On the other hand, if C is a circuit, then for all elements e of C, the set $C\backslash e$ extends to a basis of M having C as a fundamental circuit. Thus the number of ordered pairs of the required type is at least as large as $\sum_{C\in\mathscr{C}(M)} |C|$. Hence

$$\begin{aligned}(n-r)b(M) &\geq \sum_{C\in\mathscr{C}(M)} |C| \\ &= \gamma(M)c(M),\end{aligned}$$

and expression (1.7) follows immediately.

Now suppose that equality holds in expression (1.7) for the matroid M. Then equality also holds for $\hat{M}$, the matroid obtained from M by deleting all coloops. But, by the argument given above, this can only occur if, for every circuit C of $\hat{M}$ and every element e of C, the set $C\backslash e$ is contained in a unique basis of $\hat{M}$. As $\hat{M}$ has no coloops, it follows that $\hat{M}/(C\backslash e)$ has rank zero. Therefore $C\backslash e$ is a basis for $\hat{M}$ and so all circuits of $\hat{M}$ have cardinality equal to $\operatorname{rk}\hat{M}+1$. It follows that $\hat{M}$ is uniform and the proof of Theorem 1.1 is complete.

1.2. COROLLARY. *Let M be a simple matroid which is not free. Then*

$$b(M)\geqslant\frac{3}{n-r}c(M)$$

with equality holding if and only if $M\cong U_{r-2,r-2}\oplus U_{2,n-r+2}$.

A special case of the next result will be used in Section 3 to complete the proof of inequality (1.3).

1.3. THEOREM. *Let $\{e_1, e_2, \ldots, e_m\}$ be an independent set Z in a simple binary matroid M. Then*

$$b_{e_1,e_2,\ldots,e_m}(M)\geqslant c_{e_1,e_2,\ldots,e_m}(M). \tag{1.8}$$

PROOF. Choose a basis B of M containing Z. Now, if C is a circuit of M containing Z, choose an element x_C of C and then extend $C\backslash x_C$ to a basis B_C of M contained in $B\cup C$. The element x_C is chosen to be a member of $(C\cap B)\backslash Z$ provided that this set is nonempty; otherwise we choose x_C in $C\backslash B$. Notice that in both cases B_C will contain Z. We show next that if B' is a basis containing Z, then there are at most two circuits C for which B' can equal B_C. Assume that $C_1, C_2, \ldots, C_k$ are distinct circuits, but that $B_{C_1}, B_{C_2}, \ldots, B_{C_k}$ can all be chosen to equal B'. Let i and j be different elements of $\{1, 2, \ldots, k\}$. If both $(C_i\cap B)\backslash Z$ and $(C_j\cap B)\backslash Z$ are nonempty, then $C_i\backslash B=C_j\backslash B$ and so $C_i\triangle C_j\subseteq B$; a contradiction. If both $(C_i\cap B)\backslash Z$ and $(C_j\cap B)\backslash Z$ are empty, then, as $|(C_i\backslash B)\triangle(C_j\backslash B)|=2$, $|C_i\triangle C_j|=2$, contrary to the fact that M is simple. It follows that $k=2$.

We now show that if $B_{C_1}=B_{C_2}$ where $C_1\neq C_2$, then an alternative choice of bases may be made to avoid this. From above, we may assume that $(C_1\cap B)\backslash Z\neq\varnothing=(C_2\cap B)\backslash Z$. Then $C_1\backslash B=(C_2\backslash x_{C_2})\backslash B$. Since $|C_1\triangle C_2|\geqslant 3$ and $|(C_1\triangle C_2)\backslash B|=1$, we get that $|(C_1\cap B)\backslash Z|\geqslant 2$. Hence there are at least two elements which may be chosen as the element x_{C_1} and so there is a candidate for B_{C_1} which is different from B_{C_2}. Since this alternative choice of B_{C_1} cannot also equal B_C for a circuit $C\notin\{C_1, C_2\}$, expression (1.8) follows.

For small values of m, the preceding result has been strengthened and it has been determined precisely when equality is attained in expression (1.8). However, these results will not be needed here and so they have been omitted.

2. ARBITRARY SIMPLE BINARY MATROIDS

The purpose of this section is to prove the following result. The Fano matroid will be denoted by F_7.

2.1. THEOREM. *Let M be a simple binary matroid. Then*

$$b(M)\geqslant 2c(M). \tag{2.1}$$

Moreover, equality holds here if and only if $\hat{M}\cong F_7$.

To prove this theorem, we shall use a sequence of lemmas, the first four of which are devoted to establishing the theorem for matroids of rank less than five. The number of k-circuits in a matroid M will be denoted by $c^k(M)$.

2.2. LEMMA. *Let M be a simple binary matroid of rank r. Then*

$$c^{r+1}(M) \leq \frac{1}{r+1} b(M).$$

PROOF. Every $(r+1)$-circuit of M contains exactly $r+1$ bases and, as M is simple and binary, no basis is in more than one such circuit.

2.3. LEMMA. *Let M be a rank-3 binary matroid having no loops or coloops and suppose that, for some element e of M, $M \backslash e$ is simple. Then*

$$c^4(M\backslash e) + c^3(M) + c_e^2(M) \leq \tfrac{3}{10}(b_e(M) + 2b(M\backslash e)).$$

Moreover, equality is attained here only if M is isomorphic to the matroid in Figure 1 where e is as shown.

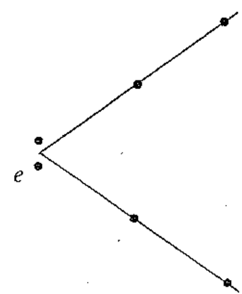

FIGURE 1

PROOF. As $M\backslash e$ is simple, either M is simple, or M has just one 2-circuit, which must contains e. Since M has rank 3 and is binary having no loops or coloops, the simple matroid associated with M is F_7, $M(K_4)$, $U_{3,4}$, or the parallel connection of two three-point lines. It is routine to complete the proof of the lemma simply by checking each of these cases.

2.4. LEMMA. *Let M be a simple binary rank-4 matroid having no coloops. Then*

$$c^4(M) + c^3(M) \leq \tfrac{3}{10} b(M), \tag{2.2}$$

with equality being attained only if M is isomorphic to the matroid G_7 consisting of three 3-point lines all sharing a common point.

PROOF. We argue by induction on $|E(M)|$. If M has elements e and f which are in series, then M/f satisfies the hypotheses of Lemma 2.3. Therefore

$$c^4(M/f\backslash e) + c^3(M/f) + c_e^2(M/f) \leq \tfrac{3}{10}(b_e(M/f) + 2b(M/f\backslash e)). \tag{2.3}$$

But $\{e, f\}$ is a 2-cocircuit of M and so, by proposition (1.6), the right-hand side of expression (2.3) equals $\frac{3}{10}b(M)$. Moreover, it is straightforward to check that the left-hand side of expression (2.3) equals $c^4(M) + c^3(M)$ and so expression (2.2) holds if M has a 2-cocircuit. Furthermore, by Lemma 2.3, equality holds in expression (2.3) only if M/f is isomorphic to the matroid in Figure 1. But e and f are in series in M and so, if equality holds, then $M \cong G_7$.

We may now assume that M has no 2-cocircuits. Then the induction assumption may be applied to $M\backslash e$ for all elements e of M. Evidently, for some element f, $M\backslash f \not\cong G_7$. Hence

$$\sum_{e\in E(M)} (c^4(M\backslash e)+c^3(M\backslash e)) < \tfrac{3}{10} \sum_{e\in E(M)} b(M\backslash e).$$

In $\sum_{e\in E(M)} b(M\backslash e)$, each basis of M is counted once for each element of its complement. Thus

$$\begin{aligned}\sum_{e\in E(M)} b(M\backslash e) &= (|E(M)|-\operatorname{rk} M)b(M)\\ &= (|E(M)|-4)b(M)\end{aligned}$$

and, on arguing similarly for circuits, we obtain that

$$(|E(M)|-4)c^4(M)+(|E(M)|-3)c^3(M) < \tfrac{3}{10}(|E(M)|-4)b(M).$$

The required result follows on dividing this inequality throughout by $|E(M)|-4$.

2.5. LEMMA. *Let M be a simple binary matroid of rank not exceeding 4. Then*

$$b(M) \geq 2c(M), \tag{2.4}$$

with equality being attained only if $M \cong F_7$ or $F_7 \oplus U_{1,1}$.

PROOF. It is easy to check that if $M \cong F_7$ or $F_7 \oplus U_{1,1}$, then $b(M)=2c(M)$. We now show that for all other simple binary matroids of rank less than five the inequality in expression (2.4) is strict. If e is a coloop of M, then $c(M\backslash e)=c(M)$ and $b(M\backslash e)=b(M)$ so we may assume that M has no coloops. Therefore, if $\operatorname{rk} M<4$, then M is isomorphic to $M(K_4)$, $U_{3,4}$, $U_{2,3}$, $U_{0,0}$ or the parallel connection of two three-point lines. In each case, it is easy to check that $b(M)>2c(M)$.

Now suppose that $\operatorname{rk} M=4$. Then, by Lemmas 2.2 and 2.4,

$$\begin{aligned}c(M) &= c^3(M)+c^4(M)+c^5(M)\\ &\leq \tfrac{3}{10}b(M)+\tfrac{1}{5}b(M)=\tfrac{1}{2}b(M).\end{aligned}$$

In fact, the inequality here is strict since equality can only occur if $M\cong G_7$ and, in that case, $c^5(M)=0$.

It was noted above that $b(F_7)=2c(F_7)$. Hence, if $\hat{M}\cong F_7$, then $b(M)=2c(M)$. We shall now prove that if M is simple and binary, then

$$b(M)>2c(M) \quad \text{provided } \hat{M}\not\cong F_7. \tag{2.5}$$

For the remainder of this section, N will denote a minimal counterexample to inequality (2.5). Evidently N has no coloops. Moreover, by Lemma 2.5 $\operatorname{rk} N \geq 5$. We shall show next that $\operatorname{rk} N=5$. Two preliminary lemmas will be required.

2.6. LEMMA. *If $\operatorname{rk} N=r$, then*

$$c^{r+1}(N) > \sum_{k=3}^{r-1} c^k(N).$$

PROOF. For all elements e of N, $N\backslash e$ is not a counterecample to inequality (2.5), so either $b(N\backslash e)>2c(N\backslash e)$, or $\widehat{N\backslash e}\cong F_7$. In both cases

$$2c(N\backslash e) \leq b(N\backslash e),$$

and so

$$2 \sum_{e \in E(N)} c(N\backslash e) < \sum_{e \in E(N)} b(N\backslash e), \tag{2.6}$$

where the inequality here may be taken to be strict since for at least one element f of N, $\widehat{N\backslash f} \not\cong F_7$. From inequality (2.6) we get that

$$2 \sum_{C \in \mathscr{C}(N)} (|E(N)| - |C|) < \sum_{B \in \mathscr{B}(N)} (|E(N)| - |B|).$$

Since N has no coloops, it follows that

$$2 \sum_{k=3}^{r+1} (|E(N)| - k) c^k(N) < (|E(N)| - r) b(N).$$

Hence $\sum_{k=3}^{r+1} (r-k)c^k(N) < (|E(N)| - r)(\frac{1}{2}b(N) - c(N))$. But $b(N) \leq 2c(N)$ and so

$$\sum_{k=3}^{r+1} (r-k)c^k(N) < 0.$$

Therefore

$$c^{r+1}(N) > \sum_{k=3}^{r-1} (r-k)c^k(N) \geq \sum_{k=3}^{r-1} c^k(N).$$

2.7. Lemma. *Let M be a simple binary matroid of rank r. Then*

$$rc^r(M) \leq b(M).$$

Proof. We argue by induction on $|E(M)|$. If M has a coloop e, then we obtain the required result by using Lemma 2.2, for

$$c^r(M) = c^r(M\backslash e) \leq \frac{1}{(r-1)+1} b(M\backslash e) = \frac{1}{r} b(M).$$

If M has no coloops, then the result follows by applying the inductive hypothesis to $M\backslash e$ for all elements e and then adding the resulting inequalities and dividing by $|E(M)| - r$.

2.8. Proposition. $\operatorname{rk} N = 5$.

Proof. Since $\operatorname{rk} N \geq 5$, it suffices to show that $\operatorname{rk} N \leq 5$. Let $\operatorname{rk} N = r$. Then, by Lemma 2.6.

$$\begin{aligned} c(N) &= \sum_{k=3}^{r-1} c^k(N) + c^r(N) + c^{r+1}(N) \\ &< c^r(N) + 2c^{r+1}(N). \end{aligned}$$

Thus, by Lemmas 2.2 and 2.7

$$c(N) < \left(\frac{1}{r} + \frac{2}{r+1}\right) b(N). \tag{2.7}$$

But $\frac{1}{2}b(N) \leq c(N)$, hence

$$\frac{1}{2} < \frac{1}{r} + \frac{2}{r+1}.$$

It follows that $r^2 - 5r - 2 < 0$, and so $r \leq 5$ and the proposition is proved.

We observe here that if it can be shown that

$$c^5(N) \leq \tfrac{1}{6}b(N), \tag{2.8}$$

then inequality (2.7) may be strengthened to give that $c(N) < \frac{1}{2}b(N)$ and this contradicts the fact that N is a counterexample to inequality (2.5). Therefore to complete the proof of Theorem 2.1, it suffices to prove expression (2.8). To do this we shall use the next four lemmas.

2.9. LEMMA. *Let M be a rank-3 binary matroid having no loops or coloops. Suppose that e and f are distinct elements of M such that $M\backslash e, f$ is simple. Then*

$$c^3_{e,f}(M) + c^4_e(M\backslash f) \leq \tfrac{1}{4}(b_{e,f}(M) + 2b_e(M\backslash f)). \tag{2.9}$$

PROOF. We observe first that

$$c^4_e(M\backslash f) \leq \tfrac{1}{3}b_e(M\backslash f), \tag{2.10}$$

since every 4-circuit of $M\backslash f$ containing e contains exactly 3 members of $\mathscr{B}_e(M\backslash f)$, and every member of $\mathscr{B}_e(M\backslash f)$ is in at most one such 4-circuit. Hence to prove the lemma it suffices to show that

$$c^3_{e,f}(M) \leq \tfrac{1}{4}b_{e,f}(M) + \tfrac{1}{6}b_e(M\backslash f). \tag{2.11}$$

Because $M\backslash e, f$ is simple and M is binary, the left-hand side is at most one, and expression (2.11) certainly holds unless it is one. Assume therefore that $\{e, f, g\}$ is a circuit for some element g. As M has no coloops, it has at least two bases containing e and f and at least two bases containing e and g; that is, $b_{e,f}(M) \geq 2$ and $b_{e,g}(M\backslash f) \geq 2$. Thus expression (2.11) holds unless $b_e(M\backslash f) = 2$. But, in that case, by expression (2.10), $c^4_e(M\backslash f) = 0$ and expression (2.9) follows immediately.

2.10. LEMMA. *Let e be an element of a rank-4 loopless binary matroid M such that $M\backslash e$ is simple and M/e has no coloops. Then*

$$c^4_e(M) \leq \tfrac{1}{4}b_e(M).$$

PROOF. We argue by induction on $|E(M)|$. The result is immediate if e is a coloop of M. Now suppose that $E(M) - \{e\}$ contains elements f and g which are in series in M. Then it is not difficult to check that, on applying the preceding lemma to M/g, we obtain the required result. Thus we may assume that $E(M) - \{e\}$ contains no 2-cocircuits of M. Hence the inductive hypothesis may be applied to $M\backslash f$ for all f in $E(M) - \{e\}$. The lemma follows on summing the resulting inequalities over all such f.

2.11. LEMMA. *Let e be an element of a rank-4 binary matroid M such that $M\backslash e$ is simple and M has no loops or coloops. Then*

$$c^4_e(M) + 3c^5(M\backslash e) \leq b(M\backslash e).$$

PROOF. Partition the set of bases of M not containing e into subsets $\mathscr{B}_1$ and $\mathscr{B}_2$ where $\mathscr{B}_1$ consists of those bases B for which the fundamental circuit of e with respect to B has cardinality 4.

Now, as M has no coloops, every 4-circuit of M containing e is the fundamental circuit of e with respect to at least two members of $\mathscr{B}_1$. Hence

$$c^4_e(M) \leq \tfrac{1}{2}|\mathscr{B}_1|. \tag{2.12}$$

If C is a 5-circuit of $M\backslash e$, then there is at most one 4-circuit C' such that $C'-C=\{e\}$ and so at least one of the bases of $M\backslash e$ contained in C is in $\mathscr{B}_2$. Thus

$$c^5(M\backslash e)\leq|\mathscr{B}_2|. \tag{2.13}$$

Moreover, by Lemma 2.3,

$$c^5(M\backslash e)\leq\tfrac{1}{5}b(M\backslash e)=\tfrac{1}{5}(|\mathscr{B}_1|+|\mathscr{B}_2|). \tag{2.14}$$

Hence

$$\begin{aligned}c_e^4(M)+3c^5(M\backslash e)&=c_e^4(M)+\tfrac{1}{2}c^5(M\backslash e)+\tfrac{5}{2}c^5(M\backslash e)\\&\leq\tfrac{1}{2}|\mathscr{B}_1|+\tfrac{1}{2}|\mathscr{B}_2|+\tfrac{1}{2}(|\mathscr{B}_1|+|\mathscr{B}_2|)\end{aligned}$$

on applying expressions (2.12), (2.13) and (2.14). The required result follows immediately.

2.12. LEMMA. *Let M be a simple rank-5 binary matroid having no coloops. Then*

$$c^5(M)\leq\tfrac{1}{6}b(M).$$

PROOF. We argue by induction on $|E(M)|$. If, for every element e of M, $M\backslash e$ has no coloops, then the result follows by applying the inductive hypothesis to $M\backslash e$ for every e and then adding the resulting inequalities.

It follows that we may assume that M has a 2-cocircuit $\{e,f\}$. Then, on applying Lemma 2.11 to M/f, we get that

$$c_{e,f}^5(M)+3c^5(M\backslash e,f)\leq b(M\backslash e,f). \tag{2.15}$$

Furthermore, applying Lemma 2.10 to M/f gives that

$$c_{e,f}^5(M)\leq\tfrac{1}{4}b_{e,f}(M)$$

and so

$$2c_{e,f}^5(M)\leq\tfrac{1}{2}b_{e,f}(M). \tag{2.16}$$

On adding expressions (2.15) and (2.16) and using proposition (1.6) we get that

$$3c^5(M)\leq\tfrac{1}{2}b(M),$$

thereby completing the proof of the lemma.

Since N, a minimal counterexample to inequality (2.5), has rank 5 and no coloops, the last lemma implies that $c^5(N)\leq\frac{1}{6}b(N)$; that is, expression (2.8) holds, and, as noted earlier, Theorem 2.1 follows immediately.

3. SIMPLE BINARY MATROIDS WITHOUT COLOOPS

The purpose of this section is to prove the following result:

3.1. THEOREM. *Let M be a rank-r simple binary matroid having no coloops. Then*

$$c(M)<\frac{19}{6(r+1)}b(M). \tag{3.1}$$

PROOF. We argue by induction on $|E(M)|$. Suppose, first, that M has no 2-cocircuits. Then, by the induction assumption, for all elements e of M,

$$c(M\backslash e)<\frac{19}{6(r+1)}b(M\backslash e).$$

Therefore

$$\sum_{e\in E(M)} c(M\backslash e)<\frac{19}{6(r+1)}\sum_{e\in E(M)} b(M\backslash e),$$

that is,

$$\sum_{C\in\mathscr{C}(M)} (|E(M)|-|C|)<\frac{19}{6(r+1)}(|E(M)|-r)b(M).$$

Arguing now as in the proof of Lemma 2.6, we get that $\sum_{k=3}^{r-1} c^k(M)<c^{r+1}(M)$ and hence, by Lemmas 2.2 and 2.7,

$$c(M)<\left(\frac{2}{r+1}+\frac{1}{r}\right)b(M). \tag{3.2}$$

If $r\geqslant 6$, then

$$\frac{2}{r+1}+\frac{1}{r}\leqslant\frac{19}{6(r+1)}$$

and so (3.2) implies (3.1). If $r\leqslant 5$, then $\frac{1}{2}<[19/6(r+1)]$ and so, by Theorem 2.1, inequality (3.1) holds.

We may now assume that M has a nontrivial series class $\{e_1, e_2, \ldots, e_m\}$. If $m\geqslant 3$, or $m=2$ and $\{e_1, e_2\}$ is not in a 3-circuit, then M/e_1 is simple and $c(M)=c(M/e_1)$. Moreover, by the induction assumption,

$$c(M/e_1)<\frac{19}{6r}b(M/e_1).$$

Therefore we may complete the proof in this case by showing that

$$\frac{19}{6r}b(M/e_1)\leqslant\frac{19}{6(r+1)}b(M). \tag{3.3}$$

But this inequality is equivalent to each of the following:

$$(r+1)b(M/e_1)\leqslant r(b(M/e_1)+b(M\backslash e_1));$$

$$b(M/e_1)\leqslant rb(M\backslash e_1);$$

$$b(M)\leqslant (r+1)b(M\backslash e_1).$$

We now verify the last of these inequalities, thereby establishing expression (3.3). The collection of bases of M may be partitioned into two subsets $\mathscr{B}_1$ and $\mathscr{B}_2$ where a basis B is in $\mathscr{B}_1$ if $|B\cap\{e_1, e_2, \ldots, e_m\}|=m-1$, and is in $\mathscr{B}_2$ if $B\supseteq\{e_1, e_2, \ldots, e_m\}$.

Now let B_1 be an arbitrary basis of $M\backslash e_1$. Then clearly $B_1\supseteq\{e_2, e_3, \ldots, e_m\}$. Replacing $\{e_2, e_3, \ldots, e_m\}$ in B_1 by any $(m-1)$-element subset of $\{e_1, e_2, \ldots, e_m\}$ gives a member of $\mathscr{B}_1$. Moreover, every member of $\mathscr{B}_1$ can be obtained in this way from exactly one basis of $M\backslash e_1$. Therefore

$$|\mathscr{B}_1|=mb(M\backslash e_1). \tag{3.4}$$

The fundamental circuit of e_1 with respect to B_1 in the matroid M contains at most $r-m+1$ elements other than $e_1, e_2, \ldots, e_m$. For each such element y, $(B_1\cup\{e_1\})\backslash\{y\}$ is a member of $\mathscr{B}_2$. Moreover, every member of $\mathscr{B}_2$ arises in this way from some basis of $M\backslash e_1$. Therefore

$$|\mathscr{B}_2|\leqslant(r-m+1)b(M\backslash e_1), \tag{3.5}$$

and expression (3.3) follows immediately on combining expressions (3.4) and (3.5).

It remains to consider the case when M has a series class $\{e_1, e_2\}$ which is contained in a 3-circuit $\{e_1, e_2, p\}$. If this 3-circuit is a component of M, then inequality (3.1) follows without difficulty from the induction assumption. The remaining alternative is that M is the parallel connection of $M_1 = M\backslash e_1, e_2$ and $M_2 = M|\{e_1, e_2, p\}$ with respect to the basepoint p [1] and p is not a coloop in M_1. In that case, by proposition (1.6),

$$b(M) = 2b(M\backslash e_1, e_2) + b(M/e_1, e_2) = 2b(M_1) + b_p(M_1).$$

Moreover,

$$c(M) = c(M_1) + c_p(M_1) + 1 = 2c(M_1) - c(M_1\backslash p) + 1,$$

and

$$\operatorname{rk} M = \operatorname{rk} M_1 + 1.$$

Thus

$$\begin{aligned}
&\tfrac{19}{6}b(M) - (\operatorname{rk} M + 1)c(M) \\
&\quad = \tfrac{19}{6}(2b(M_1) + b_p(M_1)) - (\operatorname{rk} M_1 + 2)(2c(M_1) - c(M_1\backslash p) + 1) \\
&\quad = 2(\tfrac{19}{6}b(M_1) - (\operatorname{rk} M_1 + 1)c(M_1)) \\
&\qquad + \tfrac{19}{6}b_p(M_1) - 2c(M_1) + 2c(M_1\backslash p) + \operatorname{rk}(M_1)c(M_1\backslash p) - \operatorname{rk} M_1 - 2 \\
&\quad \geq \tfrac{19}{6}b_p(M_1) - 2c_p(M_1) + \operatorname{rk}(M_1)c(M_1\backslash p) - \operatorname{rk} M_1 - 2,
\end{aligned}$$

where the last step follows from the induction assumption. Now consider $c(M_1\backslash p)$. If this is zero, then M_1 is an m-circuit for some $m \geq 3$. But we have assumed that M has no series classes with more than two elements. Hence M is the parallel connection of two three-point lines and, for this matroid, inequality (3.1) holds. Thus we may assume that $c(M_1\backslash p) \geq 1$, and therefore

$$\tfrac{19}{6}b(M) - (\operatorname{rk} M + 1)c(M) \geq \tfrac{19}{6}b_p(M_1) - 2c_p(M_1) - 2 \geq 2(b_p(M_1) - c_p(M_1)),$$

since $b_p(M_1) \geq 2$. But, by Theorem 1.3, $b_p(M_1) \geq c_p(M_1)$, so inequality (3.1) holds and the proof of Theorem 3.1 is complete.

Acknowledgement

The author thanks the referee for his assistance. It is his proof of Theorem 2.1 that appears here. An important step in it, Lemma 2.7, is also fundamental to the proof of Theorem 3.1.

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Received 13 July 1981 and in revised form 9 September 1982

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