ON SERIES-PARALLEL EXTENSIONS OF UNIFORM MATROIDS

BRAHIM CHAOURAR AND JAMES OXLEY

Abstract. This paper gives an excluded-minor characterization of the class of matroids that are series-parallel extensions of uniform matroids.

1. Introduction

The terminology used here will follow Oxley [6]. A matroid M is a *series-parallel extension* of a matroid N if M can be obtained from N by a sequence of operations each consisting of a series or parallel extension, where the last two operations involve the addition of an element in series or in parallel, respectively, to an existing element. Let $\mathcal{M}$ be the class of all series-parallel extensions of uniform matroids together with all minors of such matroids. Clearly $\mathcal{M}$ is closed under the taking of duals. It is not difficult to show that, to obtain all the members of $\mathcal{M}$ from uniform matroids, one must allow, along with the operations of series and parallel extension, the addition of loops or coloops. The purpose of this note is to characterize $\mathcal{M}$ by excluded minors. There are exactly five 3-connected matroids of rank 3 on a 6-element set. These matroids can be obtained from $M(K_4)$ by relaxing zero, one, two, three, or four circuit-hyperplanes. The matroids are, respectively, $M(K_4)$, the rank-3 whirl $\mathcal{W}^3$, Q_6, P_6, and the uniform matroid $U_{3,6}$ (see [6, p. 295]). All but the last of these five matroids is an excluded minor for $\mathcal{M}$. There are two further excluded minors, $U_{2,4} \oplus_2 U_{2,4}$, which consists of two disjoint 3-point lines in the plane, and $U_{2,4} \oplus U_{2,4}$.

Theorem 1.1. *A matroid M is a minor of a series-parallel extension of a uniform matroid if and only if M has no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, P_6, Q_6, $U_{2,4} \oplus_2 U_{2,4}$, or $U_{2,4} \oplus U_{2,4}$.*

2. The Proof

The proof of Theorem 1.1 will use the following three lemmas. The first is a well-known extension (see, for example, [8, Theorem 14.2.2] or [6, Corollary 11.2.15]) of a graph result of Dirac [3], Ádám [1], and Duffin [4]; the second is a result of Bixby [2]; and the third was proved by Walton [7] (see also [5]).

Lemma 2.1. *A connected matroid with at least one element is a series-parallel network if and only if it has no minor isomorphic to $U_{2,4}$ or $M(K_4)$.*

Lemma 2.2. *Let M be a connected non-binary matroid. If $e \in E(M)$, then M has a $U_{2,4}$-minor using e.*

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Lemma 2.3. *Let M be a 3-connected matroid having no minor isomorphic to any of $M(K_4)$, $\mathcal{W}^3$, P_6, or Q_6. Then M is uniform.*

Proof of Theorem 1.1. It is straightforward to check that each of $M(K_4)$, $\mathcal{W}^3$, P_6, Q_6, $U_{2,4} \oplus_2 U_{2,4}$, and $U_{2,4} \oplus U_{2,4}$ is an excluded minor for $\mathcal{M}$. Now let N be an excluded minor that is not in this list. If N is disconnected, then each component of N is in $\mathcal{M}$. No component of N can be a series-parallel network so, by Lemma 2.1, each component has a minor isomorphic to $M(K_4)$ or $U_{2,4}$. As N has no $M(K_4)$-minor, it follows that N has $U_{2,4} \oplus U_{2,4}$ as a minor; a contradiction. We conclude that N is connected. If N is 3-connected, then, by Lemma 2.3, N is uniform; a contradiction. We deduce that N is not 3-connected. Thus N is a 2-sum with basepoints p_1 and p_2 of two connected matroids N_1 and N_2 each of which has at least three elements. Both N_1 and N_2 are minors of N so neither has $M(K_4)$ as a minor. If N_i has no $U_{2,4}$-minor, then it a series-parallel network and so N is a series-parallel extension of a member of $\mathcal{M}$; a contradiction. Therefore both N_1 and N_2 have minors isomorphic to $U_{2,4}$. Thus, by Lemma 2.2, each N_i has a $U_{2,4}$-minor using p_i. Hence N has $U_{2,4} \oplus_2 U_{2,4}$ as a minor; a contradiction. □

Since $\mathcal{M}$ can be obtained from the class of uniform matroids by a sequence of series extensions, parallel extensions, or direct sums with loops or coloops, it is natural to consider the class of matroids that can be derived from the class of uniform matroids by series extensions, parallel extensions, and direct sums. This class is easily seen to be minor-closed and all its excluded minors are connected. The next result is obtained by making the obvious modifications to the last proof.

Corollary 2.4. *The excluded minors for the class of matroids that can be constructed from uniform matroids by a sequence of series extensions, parallel extensions, or direct sums are $M(K_4)$, $\mathcal{W}^3$, P_6, Q_6, and $U_{2,4} \oplus_2 U_{2,4}$.*

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Riyadh College of Technology, P.O. Box 42826, Riyadh 11551, Saudi Arabia
E-mail address: bchaourar@hotmail.com

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803–4918, USA
E-mail address: oxley@math.lsu.edu