CONSTRUCTIVE CHARACTERIZATIONS OF 3-CONNECTED MATROIDS OF PATH WIDTH THREE

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Abstract. A matroid M is sequential or has path width 3 if M is 3-connected and its ground set has a sequential ordering, that is, an ordering $(e_1, e_2, \ldots, e_n)$ such that $(\{e_1, e_2, \ldots, e_k\}, \{e_{k+1}, e_{k+2}, \ldots, e_n\})$ is a 3-separation for all k in $\{3, 4, \ldots, n-3\}$. This paper proves that every sequential matroid is easily constructible from a uniform matroid of rank or corank two by a sequence of moves each of which consists of a slight modification of segment-cosegment or cosegment-segment exchange. It is also proved that if N is an n-element sequential matroid, then N is representable over all fields with at least $n-1$ elements; and there is an attractive family of self-dual sequential 3-connected matroids such that N is a minor of some member of this family.

1. Introduction

The matroid terminology used here will follow Oxley [6] with the following exceptions. The simplification and cosimplification of a matroid M will be denoted by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively. The *full closure* $\mathrm{fcl}(X)$ of a set X in M is the minimal set Y containing X such that Y is closed in both M and M^*. We can obtain $\mathrm{fcl}(X)$ by beginning with X and alternately taking the closure and the coclosure of the current set until no new elements can be added. For a 2-connected matroid M, Cunningham and Edmonds [3] gave a tree decomposition that displays all of its 2-separations. When M is 3-connected, in order to gain control of the 3-separations so that they could be displayed in a corresponding tree, Oxley, Semple, and Whittle [8] defined 3-separations (Y_1, Y_2) and (Z_1, Z_2) to be *equivalent* if $\{\mathrm{fcl}(Y_1), \mathrm{fcl}(Y_2)\} = \{\mathrm{fcl}(Z_1), \mathrm{fcl}(Z_2)\}$. Their tree decomposition was only guaranteed to display one representative from each equivalence class of non-sequential 3-separations, where a 3-separation (X_1, X_2) of M is *sequential* if $E(M) \in \{\mathrm{fcl}(X_1), \mathrm{fcl}(X_2)\}$. One class of 3-connected matroids whose tree decompositions consist of a single vertex are *sequential matroids*, that is, those 3-connected matroids for which the ground set has an ordering $(e_1, e_2, \ldots, e_n)$ such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all

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i with $0 \leq i \leq n$. Such an ordering of the ground set is called *sequential*, and sequential matroids are also said to have *path width three*. Hall, Oxley, and Semple [5] considered the possible sequential orderings of a sequential matroid N and identified the structures in N that permit these different sequential orderings.

In this paper, we give a simple constructive description of all sequential matroids and we use this result to show that every n-element sequential matroid is representable over all fields with at least $n-1$ elements. In addition, we introduce an attractive family of self-dual sequential matroids such that every sequential matroid is a minor of some member of this family. One consequence of the main results of this paper is the following theorem. For each non-negative integer m, let Γ_{2m+1} be the graph that is constructed as follows: take $m+1$ copies of K_3 on disjoint vertex sets $\{u_1, v_1, w_1\}, \{u_2, v_2, w_2\}, \dots,$ and $\{u_{m+1}, v_{m+1}, w_{m+1}\}$; for all i in $\{1, 2, \dots, m\}$, add the edges u_iu_{i+1}, v_iv_{i+1}, and w_iw_{i+1}; adjoin one additional vertex v_0 and add the edges v_0u_1, v_0v_1, and v_0w_1.

Theorem 1.1. *Let M be an n-element binary sequential matroid with $n \geq 5$. Then M is isomorphic to a minor of $M(\Gamma_{2n-9})$.*

2. Overview

In this section, after some preliminary definitions, we state the main results of the paper. The proofs of these results will occupy the rest of the paper. We begin by describing a family of matroids introduced in [7] that will be of particular importance in this paper. For each $k \geq 3$, take a basis $\{y_1, y_2, \dots, y_k\}$ of $PG(k-1, \mathbb{R})$ and a line L that is freely placed relative to this basis. By modularity, for each i, the hyperplane of $PG(k-1, \mathbb{R})$ that is spanned by $\{y_1, y_2, \dots, y_k\} - \{y_i\}$ meets L. Let x_i be the point of intersection. We shall denote by Θ_k the restriction of $PG(k-1, \mathbb{R})$ to $\{y_1, y_2, \dots, y_k, x_1, x_2, \dots, x_k\}$. The reader can easily check that Θ_3 is isomorphic to $M(K_4)$. Alternatively, for all $k \geq 3$, we can define Θ_k to be the matroid with ground set $\{y_1, y_2, \dots, y_k, x_1, x_2, \dots, x_k\}$ whose circuits consist of all 3-element subsets of $\{x_1, x_2, \dots, x_k\}$; all sets of the form $(\{y_1, y_2, \dots, y_k\} - \{y_i\}) \cup \{x_i\}$, where $i \in \{1, 2, \dots, k\}$; and all sets of the form $(\{y_1, y_2, \dots, y_k\} - \{y_j\}) \cup \{x_g, x_h\}$, where j, g, and h are distinct elements of $\{1, 2, \dots, k\}$ [7, Lemma 2.2]. When we want to emphasize its ground set, we shall sometimes write Θ_k as $\Theta_k(X, Y)$ noting that $\Theta_k(X, Y)|X \cong U_{2,k}$. As observed in [7, Lemma 2.1], the matroid Θ_k is isomorphic to its dual under the map that interchanges x_i and y_i for all i. Moreover, by [7, Lemma 2.4], X is a modular flat and Y is a basis in Θ_k. We remark that the sets X and Y here were called A and B in [7]. We shall call the elements x_i and y_i *partners* in Θ_k. Evidently, for every permutation σ of $\{1, 2, \dots, k\}$, the map that takes x_i and y_i to $x_{\sigma(i)}$ and $y_{\sigma(i)}$, respectively, is an automorphism of Θ_k.

Let M_1 and M_2 be matroids such that $M_1|T = M_2|T$, where $T = E(M_1) \cap E(M_2)$. Assume that T is a modular flat of M_1. The *generalized parallel connection* $P_T(M_1, M_2)$ *of* M_1 *and* M_2 *across* T is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets Z of $E(M_1) \cup E(M_2)$ such that $Z \cap E(M_1)$ is a flat of M_1 and $Z \cap E(M_2)$ is a flat of M_2.

Two additional operations, based on generalized parallel connection, will be important here. A *segment* in a matroid N is a subset Z of $E(N)$ such that $N|Z \cong U_{2,k}$ for some $k \geq 3$. A *cosegment* of N is a segment of N^*. Now let X be a coindependent segment $\{x_1, x_2, \dots, x_k\}$ of a matroid M. Since X is a modular line in Θ_k, the generalized parallel connection $P_X(\Theta_k, M)$ of Θ_k and M across X exists. Hence the matroid $P_X(\Theta_k, M)\backslash X$ is certainly defined. We denote this matroid by $\Delta_X(M)$ and call this operation a *segment-cosegment exchange on* X. It is shown in [7, Lemma 2.5] that Y is a cosegment in $\Delta_X(M)$. When $k = 3$, a segment-cosegment exchange on X is a $\Delta - Y$ exchange on X.

To define the dual operation to segment-cosegment exchange, let M be a matroid having a k-element independent cosegment X. Then $M^*|X \cong U_{2,k}$ and we define $\nabla_X(M)$ to be $(\Delta_X(M^*))^*$, that is, $[P_X(\Theta_k, M^*)\backslash X]^*$. We call this operation *cosegment-segment exchange on* X.

It is sometimes convenient, when performing a segment-cosegment exchange or a cosegment-segment exchange, to preserve the ground set of the original matroid M. This is done by relabelling y_i by x_i in $\Delta_X(M)$ or $\nabla_X(M)$, respectively.

The only sequential matroids with at most two elements are $U_{0,0}, U_{0,1}, U_{1,1}$, and $U_{1,2}$. The next theorem shows how, from every sequential matroid M with at least three elements, one can obtain a uniform matroid of rank or corank two. This result enables us to determine precisely the fields over which M is representable.

Theorem 2.1. *Let M be a sequential matroid with $|E(M)| \geq 3$. Then a uniform matroid N that has at least three elements and is of rank or corank two can be obtained from M by using the following algorithm. Moreover, M is representable over a field $\mathbb{F}$ if and only if $|\mathbb{F}| \geq |E(N)| - 1$.*

(i) *Let $N = M$.*
(ii) *Take a sequential ordering $(e_1, e_2, \dots, e_n)$ of N that begins with a maximal rank-2 dependent flat X of N or N^*. Let $k = |X|$.*
(iii) *If $\{e_1, e_2, e_3\}$ is a circuit, go to (iv); if $\{e_1, e_2, e_3\}$ is a cocircuit, go to (v).*

(iv) *If* $X = E(N)$, *then go to (vi); otherwise replace* N *by* $\mathrm{co}(\Delta_X(N))$ *and go to (ii).*
(v) *If* $X = E(N)$, *then go to (vii); otherwise replace* N *by* $\mathrm{si}(\nabla_X(N))$ *and go to (ii).*
(vi) *Output* $N = U_{2,k}$ *and stop.*
(vii) *Output* $N = U_{k-2,k}$ *and stop.*

Essentially by reversing the steps in the last theorem, we can obtain all sequential matroids. A flat F of a matroid M is *proper* if $F \neq E(M)$.

Theorem 2.2. *The class* $\mathcal{M}$ *of sequential matroids with at least three elements coincides with the class of matroids that can be constructed by the following procedure.*

(i) *Let* $\mathcal{M}_0 = \{U_{2,m}, U_{m-2,m} : m \geq 3\}$.
(ii) *Choose* N *in* $\mathcal{M}_0$ *and take a sequential ordering* $(e_1, e_2, \ldots, e_n)$ *of* N *that begins with a maximal rank-2 dependent flat* F *of* N *or* N^*. *Take a subset* X *of* F *with* $|X| = k \geq 3$.
(iii) *If* $\{e_1, e_2, e_3\}$ *is a circuit, go to (iv); if* $\{e_1, e_2, e_3\}$ *is a cocircuit, go to (v).*
(iv) *For a subset* X_1 *of* X *that is a proper flat of* N^*, *add* $P_X(\Theta_k, N)\backslash X_1$ *to* $\mathcal{M}_0$ *and go to (ii).*
(v) *For a subset* X_1 *of* X *that is a proper flat of* N, *add* $(P_X(\Theta_k, N^*))^*/X_1$ *to* $\mathcal{M}_0$ *and go to (ii).*

Next we describe an attractive family of self-dual sequential matroids with the property that every sequential matroid is a minor of some member of the family. The matroids we construct here are based on the matroid Θ_n. To begin, we take $M' = \Theta_n(B, A')$ and $M'' = \Theta_n(B, A'')$ where $A' \cap A'' = \emptyset$. By [7, Lemma 2.4], B is a modular flat of M' and M''. Hence $(P_B(M', M''))^*$ is well-defined having ground set $A' \cup B \cup A''$ and rank $n+2$. We shall denote this matroid by Θ_n^2. Note that Θ_n^2 is well-defined. To see this, observe that, since both M' and M'' are isomorphic to Θ_n, each element b of B has a partner a' in A' and a partner a'' in A''. It follows that once M' has been labelled, the labelling of M'' is determined. We shall call a' and a'' the *partners* of b in Θ_n^2. Observe that

$$\Theta_n^2/A'' = \Theta_n(A', B). \tag{2.1}$$

To see this, note that $\Theta_n^2/A'' = (P_B(M', M''))^*/A'' = (P_B(M', M'')\backslash A'')^* = (P_B(M', M''\backslash A''))^* = (M')^* = \Theta_n(A', B)$.

The dual $(K_5 - e)^*$ of the graph that is obtained by deleting an edge from K_5 is the triangular prism graph. Since $\Theta_3 \cong M(K_4)$, the reader can easily check that

$$\Theta_3^2 \cong M((K_5 - e)^*) = M^*(K_5 - e). \tag{2.2}$$

We shall show in Lemma 5.1 that A' and A'' are modular lines of Θ_n^2. We now inductively define the matroid Θ_n^{2m+1} for all integers m and n with $m \geq 0$ and $n \geq 3$. The ground set of this matroid is the disjoint union of $2m+2$ sets $B_1, A_1, B_2, A_2, \dots, B_{m+1}, A_{m+1}$, each of which has exactly n elements. Let $M_0 = \Theta_n(A_1, B_1)$. For all i in $\{1, 2, \dots, m\}$, let M_i be a copy of Θ_n^2 with ground set $A_i \cup B_{i+1} \cup A_{i+1}$ where $M_i|A_i \cong U_{2,n} \cong M_i|A_{i+1}$. Define $\Theta_n^1 = M_0$, and, for all $m \geq 1$, let $\Theta_n^{2m+1} = P_{A_m}(M_m, \Theta_n^{2m-1})$. A straightforward induction argument using partners establishes that Θ_n^{2m+1} is well-defined. As we shall show in Lemmas 5.5 and 5.9, Θ_n^{2m+1} is isomorphic to its dual and $\Theta_3^{2m+1} \cong M(\Gamma_{2m+1})$.

The matroid Θ_n^{2n-9} is a universal sequential matroid in a sense that is made precise by the next result.

Theorem 2.3. *If M is an n-element matroid for $n \geq 5$, then M is sequential if and only if M is isomorphic to a 3-connected minor of Θ_n^{2n-9}.*

A sequential matroid with at least two elements can equivalently be described as a 3-connected matroid M having a 2-element subset X such that $\mathrm{fcl}(X) = E(M)$. In view of this, it is natural to consider those 2-connected matroids M that have an element x such that $\mathrm{fcl}(\{x\}) = E(M)$ or, equivalently, that have an ordering $(e_1, e_2, \dots, e_n)$ of $E(M)$ in which $\{e_1, e_2, \dots, e_i\}$ is 2-separating for all i with $0 \leq i \leq n$. The next result is an analogue of Theorem 2.3 for such matroids. For $m \geq 1$, let Φ_m be the graph that is formed as follows. Begin with a path $v_1, b_1, v_2, b_2, \dots, b_{m-1}, v_m$ with edges $b_1, b_2, \dots, b_{m-1}$; add a new vertex v_0 and, for all i in $\{1, 2, \dots, m\}$, add an edge a_i joining v_0 to v_i; finally, add an edge b_m parallel to a_m. Evidently, Φ_m is a planar graph that is isomorphic to its dual.

Theorem 2.4. *Let M be a 2-connected matroid with $|E(M)| = n \geq 2$. Then M has an element x such that $\mathrm{fcl}(\{x\}) = E(M)$ if and only if M is a 2-connected minor of $M(\Phi_{n-1})$.*

The rest of the paper is structured as follows. The next section contains some definitions together with some basic connectivity results that will be needed to prove the main theorems. That section also contains a proof of Theorem 2.4. In Section 4, we prove Theorems 2.1 and 2.2 and describe another family of sequential matroids of which every sequential matroid is a minor. This family is used in the proof of Theorem 2.3, which is given in Section 5.

3. Preliminaries

This section contains some definitions and lemmas needed to prove the theorems stated in the last section. We shall also prove Theorem 2.4 here.

For a matroid M on a set E, the *connectivity function* λ_M of M is defined, for all subsets Z of E, by $\lambda_M(Z) = r(Z) + r(E-Z) - r(M)$. We shall often abbreviate λ_M as λ. The set Z or the partition $(Z, E-Z)$ is *k-separating* if $\lambda(Z) < k$. The partition $(Z, E-Z)$ is a *k-separation* if it is k-separating and $|Z|, |E-Z| \geq k$; and M is *n-connected* if, for all j with $1 \leq j \leq n-1$, there are no $(n-j)$-separations in M. A k-separating set Z, or a k-separating partition $(Z, E-Z)$, or a k-separation $(Z, E-Z)$ is *exact* if $\lambda(Z) = k-1$ and is *minimal* if $\min\{|Z|, |E-Z|\} = k$. A k-separation $(Z, E-Z)$ is *vertical* if $r(Z), r(E-Z) \geq k$.

The connectivity function of a matroid M has a number of attractive properties. For example, $\lambda_M(Z) = \lambda_M(E-Z)$. Moreover, the connectivity functions of M and its dual M^* are equal. To see this, it suffices to note the easily verified fact that

$$\lambda_M(Z) = r(Z) + r^*(Z) - |Z|.$$

Now suppose that M is a 3-connected matroid. Following [5], we use the term 3-*sequence* for an ordered partition $(E_1, E_2, \dots, E_n)$ of $E(M)$ into non-empty sets such that if $i \in \{1, 2, \dots, n-1\}$ and both $|\bigcup_{j=1}^{i} E_j|$ and $|\bigcup_{j=i+1}^{n} E_j|$ exceed one, then $\bigcup_{j=1}^{i} E_j$ is exactly 3-separating. If, for some m in $\{1, 2, \dots, n\}$, there is an ordering $\overrightarrow{E_m}$ of E_m, say $\overrightarrow{E_m} = (x_1, x_2, \dots, x_k)$, such that $(E_1, E_2, \dots, E_{m-1}, \{x_1\}, \{x_2\}, \dots, \{x_k\}, E_{m+1}, \dots, E_n)$ is a 3-sequence, then we also write this 3-sequence as $(E_1, E_2, \dots, E_{m-1}, x_1, x_2, \dots, x_k, E_{m+1}, \dots, E_n)$ or $(E_1, E_2, \dots, E_{m-1}, \overrightarrow{E_m}, E_{m+1}, \dots, E_n)$. A 3-sequence of the form $(A, x_1, x_2, \dots, x_m, B)$ such that $|A|, |B| \geq 2$ will be called an (A, B) 3-*sequence*. This terminology agrees with [5]. Note, however, that in [4] a '3-sequence' is what we have called here an '(A, B) 3-sequence'.

Evidently if M is a 3-connected matroid and $E(M) = \{e_1, e_2, \dots, e_t\}$, a sequence $(e_1, e_2, \dots, e_t)$ is a 3-sequence if and only it is a sequential ordering of $E(M)$. When we refer to a sequential ordering of M of the form $(E_1, E_2, \dots, E_n)$, we mean that there are orderings $\overrightarrow{E_1}, \overrightarrow{E_2}, \dots, \overrightarrow{E_n}$ of $E_1, E_2, \dots, E_n$ such that $(\overrightarrow{E_1}, \overrightarrow{E_2}, \dots, \overrightarrow{E_n})$ is a sequential ordering of M. A sequential matroid M is also said to have path width three because there is a path P with vertex set $E(M)$ such that the partition of $E(M)$ induced by each edge of P is a 3-separating partition of $E(M)$.

The next result is elementary. Its proof appears, for example, in [4, Lemma 4.1].

Lemma 3.1. *Let $(A, e_1, e_2, \dots, e_n, B)$ be an (A, B) 3-sequence of a 3-connected matroid. Then, for each i, either*

(i) $e_i \in \operatorname{cl}(A \cup \{e_1, \dots, e_{i-1}\}) \cap \operatorname{cl}(\{e_{i+1}, \dots, e_n\} \cup B)$, *or*

(ii) $e_i \in \text{cl}^*(A \cup \{e_1, \dots, e_{i-1}\}) \cap \text{cl}^*(\{e_{i+1}, \dots, e_n\} \cup B)$,

but not both.

We call e_i a *guts* or *coguts* element of $(A, e_1, e_2, \dots, e_n, B)$ depending on whether (i) or (ii) of the last lemma holds or, equivalently, on whether e_i is in the closure or coclosure of $A \cup \{e_1, e_2, \dots, e_{i-1}\}$. It was shown in [4, Lemma 4.6] that this labelling is robust in that if e_i is a guts element of some (A, B) 3-sequence, then it is a guts element of all (A, B) 3-sequences.

The class of sequential matroids is well-behaved. For example, it is clear that the dual of a sequential matroid is sequential. Moreover, we have the following attractive property.

Lemma 3.2. *Every 3-connected minor N of a sequential matroid M is sequential. In particular, if $\overrightarrow{E}$ is a sequential ordering of M, then the induced ordering on $E(N)$ is sequential.*

The last lemma follows immediately from the next lemma, which, in turn, is a consequence of the well-known and easily verified fact that the connectivity function is monotone under taking minors.

Lemma 3.3. *Let $(e_1, e_2, \dots, e_n)$ be an ordering of the ground set of a matroid M. If $\lambda_M(\{e_1, e_2, \dots, e_i\}) \leq k$ for all i and N is a minor of M, then $\lambda_N(\{e_1, e_2, \dots, e_i\} \cap E(N)) \leq k$ for all i.*

Next we insert the proof of Theorem 2.4. The argument here uses properties of the operation of parallel connection. Later, in the proof of Theorem 2.3, we shall use similar properties of the generalized parallel connection. In a matroid M, if $e \in E(M)$ and $Z \subseteq E(M)$, we write $e \in \text{cl}^{(*)}(Z)$ to indicate that $e \in \text{cl}(Z)$ or $e \in \text{cl}^*(Z)$.

Proof of Theorem 2.4. Evidently $\text{fcl}(\{a_1\}) = E(M(\Phi_{n-1})) = \text{fcl}(\{b_{n-1}\})$ and, by Lemma 3.3, it follows that every 2-connected minor N of $M(\Phi_{n-1})$ has an element whose full closure is $E(N)$.

Now suppose that M has an element x such that $\text{fcl}(\{x\}) = E(M)$. We shall argue by induction on n that

2.4.1. *M is isomorphic to a minor of $M(\Phi_{n-1})$ in which x is mapped to a_1 or b_{n-1}.*

If $n = 2$, then $M \cong U_{1,2} \cong M(\Phi_1)$ so (2.4.1) holds. Assume it holds for $n < k$ and let $n = k \geq 3$. Since $\text{fcl}(\{x\}) = E(M)$, there is an ordering $(e_1, e_2, \dots, e_n)$ of $E(M)$ such that $x = e_1$ and, for all $i \geq 2$, the element

$e_i \in \mathrm{cl}^{(*)}(\{e_1, e_2, \dots, e_{i-1}\})$. We shall show that (2.4.1) holds when $e_3 \in \mathrm{cl}(\{e_1, e_2\})$. If, instead, $e_3 \in \mathrm{cl}^*(\{e_1, e_2\})$, then $e_3 \in \mathrm{cl}_{M^*}(\{e_1, e_2\})$ so we can apply the same argument to get that M^* is isomorphic to a minor of $M(\Phi_{n-1})$ in which e_1 is mapped to a_1 or b_{n-1}. Because there is an isomorphism between $M^*(\Phi_{n-1})$ and $M(\Phi_{n-1})$ that interchanges a_1 and b_{n-1}, the required result will follow.

Since $e_2 \in \mathrm{cl}^{(*)}(\{e_1\})$ and $e_3 \in \mathrm{cl}(\{e_1, e_2\})$, the matroid $M|\{e_1, e_2, e_3\}$ is isomorphic to $U_{1,3}$ or $U_{2,3}$. Each of the last two matroids is isomorphic to a minor of $M(\Phi_2)$ in which e_1 is mapped to a_1 or b_2, so the required result holds for $|E(M)| = 3$. We may now assume that $|E(M)| \geq 4$. Then one easily checks that $\lambda_{M/e_3}(\{e_1, e_2\}) = \lambda_M(\{e_1, e_2\}) - 1 = 0$, so M/e_3 is not 2-connected. Thus, by a result of Brylawski [2] (see also [6, 7.1.16 and 7.1.17]), M is the parallel connection, with basepoint e_3, of $M|\{e_1, e_2, e_3\}$ and $M\backslash e_1, e_2$, and each of the last two matroids is 2-connected. It follows, by Lemma 3.3, that the sequence $(e_3, e_4, \dots, e_n)$ has the property that $\{e_3, e_4, \dots, e_i\}$ is 2-separating in $M\backslash e_1, e_2$ for all i in $\{3, 4, \dots, n-1\}$. Thus, by the induction assumption, $M\backslash e_1, e_2$ is isomorphic to a minor of $M(\Phi_{n-3})$ in which e_3 is mapped to either a_1 or b_{n-3}.

After combining the two possibilities for $M\backslash e_1, e_2$ with the two possibilities for $M|\{e_1, e_2, e_3\}$, we get a total of four cases. But, because each of $M\backslash e_1, e_2$ and $M|\{e_1, e_2, e_3\}$ is graphic, it is straightforward to check that, in each case, M is isomorphic to a minor of $M(\Phi_{n-1})$ in which e_1 is mapped to a_1 or b_{n-1}. The theorem follows by induction. □

The next result [4, Lemma 3.2] contains two more elementary properties of sequential matroids.

Lemma 3.4. *Let $(e_1, e_2, \dots, e_n)$ be a sequential ordering of a 3-connected matroid M, and let $i < j$.*

(i) *If $e_j \in \mathrm{cl}^{(*)}(\{e_1, e_2, \dots, e_i\})$, then*

$$(e_1, e_2, \dots, e_i, e_j, e_{i+1}, \dots, e_{j-1}, e_{j+1}, e_n)$$

is also a sequential ordering of M.

(ii) *If $r(\{e_1, e_2, \dots, e_k\}) = 2$ and $(z_1, z_2, \dots, z_k)$ is an arbitrary permutation of $\{e_1, e_2, \dots, e_k\}$, then $(z_1, z_2, \dots, z_k, e_{k+1}, \dots, e_n)$ is also a sequential ordering of M.*

One of the most useful features of the connectivity function λ of M is that it is submodular, that is, for all $J, K \subseteq E(M)$,

$$\lambda(J) + \lambda(K) \geq \lambda(J \cap K) + \lambda(J \cup K).$$

This means that if J and K are k-separating, and one of $J \cap K$ or $J \cup K$ is not $(k-1)$-separating, then the other must be k-separating. The next lemma specializes this fact.

Lemma 3.5. *Let M be a 3-connected matroid, and let J and K be 3-separating subsets of $E(M)$.*

(i) *If $|J \cap K| \geq 2$, then $J \cup K$ is 3-separating.*
(ii) *If $|E(M) - (J \cup K)| \geq 2$, then $J \cap K$ is 3-separating.*

Another consequence of the submodularity of λ is the following very useful result for 3-connected matroids, which has come to be known as Bixby's Lemma [1] (see also [6, Proposition 8.4.6]).

Lemma 3.6. *Let M be a 3-connected matroid and e be an element of M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case, $\mathrm{co}(M\backslash e)$ is 3-connected, while, in the second case, $\mathrm{si}(M/e)$ is 3-connected.*

Lemma 3.7. *For all $k \geq 3$, the matroid Θ_k is sequential. Moreover, if $\overrightarrow{X}$ and $\overrightarrow{Y}$ are arbitrary permutations of X and Y, respectively, then $(\overrightarrow{X}, \overrightarrow{Y})$ is a sequential ordering of Θ_k.*

Proof. It was noted in [8] that Θ_k is 3-connected. Now let $\overrightarrow{Z}$ be an initial subsequence of $(\overrightarrow{X}, \overrightarrow{Y})$. Then either $Z \subseteq X$ or $E(\Theta_k) - Z \subseteq Y$. Hence $r(Z) = 2$ or $r^*(E(\Theta_k) - Z) = 2$, and we deduce that $\lambda(Z) = 2$. □

The next result follows, for example, by [9].

Lemma 3.8. *Let M_1 and M_2 be 3-connected matroids and $E(M_1) \cap E(M_2) = T$. Assume that $M_1|T = M_2|T$ and that T is a rank-2 modular flat of M_1. Then $P_T(M_1, M_2)$ is 3-connected.*

Corollary 3.9. *Let N be a 3-connected matroid having a rank-2 subset X such that $|X| = k \geq 3$. Then $P_X(\Theta_k, N)$ is 3-connected.*

Lemma 3.10. *Let $(e_1, e_2, \ldots, e_n)$ be a sequential ordering of a sequential matroid M. Let $A = \{e_1, e_2\}$ and $B = \{e_{n-1}, e_n\}$. For $3 \leq i \leq n-2$, if e_i is a guts element of the (A, B) 3-sequence $(A, e_3, e_4, \ldots, e_{n-2}, B)$ but $e_i \notin \mathrm{cl}(A) \cup \mathrm{cl}(B)$, then $\mathrm{si}(M/e_i)$ is not 3-connected, so $\mathrm{co}(M\backslash e_i)$ is sequential.*

Proof. The partition $(\{e_1, e_2, \ldots, e_{i-1}\}, \{e_{i+1}, e_{i+2}, \ldots, e_n\})$ is a vertical 2-separation of M/e_i, so $\mathrm{si}(M/e_i)$ is not 3-connected. By Lemma 3.6, $\mathrm{co}(M\backslash e_i)$ is 3-connected and the lemma follows by Lemma 3.2. □

Lemma 3.11. *Let M_1 and M_2 be sequential matroids. Let $T = E(M_1) \cap E(M_2)$ and assume that $M_1|T = M_2|T$ and that T is a rank-2 modular flat of M_1. Let M_1 and M_2 have sequential orderings $(\overrightarrow{U_1}, \overrightarrow{T_1})$ and $(\overrightarrow{T_2}, \overrightarrow{U_2})$ where $T_1 = T = T_2$. Then $P_T(M_1, M_2)$ is a sequential matroid having $(\overrightarrow{U_1}, \overrightarrow{T_1}, \overrightarrow{U_2})$ as a sequential ordering.*

Proof. For each i, let $E_i = E(M_i)$. Then $E_i = U_i \cup T_i$. Let $M = P_T(M_1, M_2)$ and abbreviate $E(M)$ as E. By Lemma 3.8, M is 3-connected. If $r(M_1) = 2$, then $T = E_1$, so $M = M_2$ and the result is immediate. Hence, we may assume that $r(M_1), r(M_2) \geq 3$. Thus $E(M_1) - T$ and $E(M_2) - T$ span T in M_1 and M_2, respectively. Now, since T has rank 2 in M_2, the ordering $(\overrightarrow{T_1}, \overrightarrow{U_2})$ of E_2 is sequential in M_2.

To complete the proof, we shall show that $(\overrightarrow{U_1}, \overrightarrow{T_1}, \overrightarrow{U_2})$ is a sequential ordering of M. Let $\overrightarrow{Z}$ be an initial subsequence of $(\overrightarrow{U_1}, \overrightarrow{T_1}, \overrightarrow{U_2})$ with $|Z|, |E - Z| \geq 3$. By symmetry, we may assume that $Z \subseteq U_1 \cup T$. Now U_1 and U_2 span M_1 and M_2, respectively. Thus, if $Z \supseteq U_1$, then, as $E - Z \supseteq U_2$, we have

$$r(Z) + r(E - Z) - r(M) = r(M_1) + r(M_2) - r(M) = 2.$$

Now suppose $Z \subseteq U_1$. Then, by submodularity,

$$\begin{aligned} r(Z) + r(E - Z) - r(M) &= r(Z) + r((E_1 - Z) \cup E_2) - r(M) \\ &\leq r(Z) + [r(E_1 - Z) \\ &\quad + r(E_2) - r(\mathrm{cl}(E_1 - Z) \cap \mathrm{cl}(E_2))] - r(M) \\ &\leq r(Z) + r(E_1 - Z) + r(E_2) \\ &\quad - r(T) - [r(M_1) + r(M_2) - 2] \\ &= r(Z) + r(E_1 - Z) - r(M_1) \\ &\leq 2. \end{aligned}$$

We conclude that if $Z \subseteq U_1 \cup T$, then $(Z, E - Z)$ is 3-separating and the lemma follows. □

Lemma 3.12. *Let M be a 3-connected matroid, (J, K) be a 3-separation of M, and $Z \subseteq \mathrm{cl}(J) \cap \mathrm{cl}(K)$. If both $|J - Z|$ and $|K - Z|$ exceed two, then*

(i) *$M \backslash Z$ is connected;*
(ii) *$\mathrm{co}(M \backslash Z)$ is 3-connected; and*
(iii) *every non-trivial series class of $M \backslash Z$ has exactly two elements, one in $J - Z$ and the other in $K - Z$.*

Proof. We have

$$\begin{aligned} 2 &= r(J) + r(K) - r(M) \\ &\geq r(J - Z) + r(K \cup Z) - r(M) \\ &\geq 2. \end{aligned}$$

Thus equality holds throughout and so $\mathrm{cl}(J-Z) \supseteq J$. Hence $\mathrm{cl}(J-Z) \supseteq Z$. Likewise, $\mathrm{cl}(K-Z) \supseteq Z$.

We shall prove all three parts simultaneously. Suppose that, for some k in $\{1, 2\}$, the matroid $M\backslash Z$ has an exact k-separation (R, G). Since $Z \subseteq \mathrm{cl}(J-Z) \cap \mathrm{cl}(K-Z)$, both R and G meet both $J-Z$ and $K-Z$, otherwise $(R \cup Z, G)$ or $(R, G \cup Z)$ is an exact k-separation of M; a contradiction.

Since $|J-Z| \geq 3$, we may assume that $|G \cap (J-Z)| \geq 2$. Now $\lambda_{M\backslash Z}(G) = k-1$ and $\lambda_{M\backslash Z}(J-Z) = 2$. Hence, by the submodularity of λ, we get

$$\lambda_{M\backslash Z}(G \cap (J-Z)) + \lambda_{M\backslash Z}(G \cup (J-Z)) \leq 2 + k - 1.$$

Since $(E-Z) - (G \cup (J-Z)) = R \cap (K-Z)$, we have

$$\lambda_{M\backslash Z}(G \cap (J-Z)) + \lambda_{M\backslash Z}(R \cap (K-Z)) \leq 2 + k - 1.$$

As $Z \subseteq \mathrm{cl}(R \cup (K-Z)) \cap \mathrm{cl}(G \cup (J-Z))$, it follows that

$$\lambda_M(G \cap (J-Z)) + \lambda_M(R \cap (K-Z)) \leq 2 + k - 1.$$

Since $|G \cap (J-Z)| \geq 2$, we have $\lambda_M(G \cap (J-Z)) \geq 2$ and so $\lambda_M(R \cap (K-Z)) \leq k-1$. As $R \cap (K-Z)$ is non-empty, it follows that $k \neq 1$. Hence $k = 2$ and $|R \cap (K-Z)| = 1$. Therefore $M\backslash Z$ is connected. Moreover, as $|K-Z| \geq 3$, we have $|G \cap (K-Z)| \geq 2$. Thus we can interchange J and K in the argument in this paragraph to get that $|R \cap (J-Z)| = 1$. Hence $|R| = 2$. Therefore R is a cocircuit of $M\backslash Z$ and every non-trivial series class of $M\backslash Z$ has exactly one element in $J-Z$ and exactly one element in $K-Z$. Furthermore, $M\backslash Z$ has no 2-separation (R, G) in which both R and G are dependent, so $\mathrm{co}(M\backslash Z)$ is 3-connected (see, for example, [10, (5.1)]). □

Let n be an integer with $n \geq 3$. Let V_n be the rank-3 matroid with a $(2n)$-point ground set $R \cup S \cup f$ consisting of two distinct n-point lines, R and S, with $R = \{t, r_1, r_2, \ldots, r_{n-1}\}$ and $S = \{t, s_1, s_2, \ldots, s_{n-1}\}$, and a point f placed so that $\{f, r_i, s_i\}$ is a line for all i in $\{1, 2, \ldots, n-1\}$. Evidently, $V_3 \cong M(K_4)$. We call R and S the *distinguished lines* of V_n. The points t and f are called the *tip* and the *focus* of V_n. When $n \geq 4$, the distinguished lines, the tip, and the focus of V_n are uniquely determined. When $n = 3$, we designate two 3-point lines of V_n as the distinguished lines and this determines the tip and the focus.

In V_n, both R and S are modular lines. We now describe an important family Π_n^m of matroids. For some $m \geq 1$, let $M_1, M_2, \ldots, M_m$ be copies of

V_n. Let R_i and S_i be the distinguished lines of M_i, and t_i and f_i be its tip and its focus. For each i in $\{1, 2, \ldots, m-1\}$, let φ_i be a bijection from S_i to R_{i+1}. Identify each element of S_i with its image under φ_i and let N_m be the matroid that is constructed as follows: $N_1 = M_1$; $N_2 = P_{R_2}(M_2, N_1)$; $\ldots$; $N_m = P_{R_m}(M_m, N_{m-1})$. Evidently, N_m depends on the bijections φ_i. We denote by Π_n^m the collection of all possible such matroids N_m.

Lemma 3.13. *The matroid N_m is sequential.*

Proof. We establish the lemma by proving the following by induction on m:

3.13.1. *N_m is 3-connected having a sequential ordering of the form (Z_m, S_m).*

If $m = 1$, then $(\overrightarrow{R_1 - t_1}, f_1, \overrightarrow{S_1})$ is a sequential ordering of N_m where $\overrightarrow{R_1 - t_1}$ and $\overrightarrow{S_1}$ are arbitrary orderings of $R_1 - t_1$ and S_1, respectively. Assume the assertion is true for $m = k$ and let $m = k + 1$. Then $N_{k+1} = P_{R_{k+1}}(M_{k+1}, N_k)$. By the induction assumption, N_k is 3-connected having a sequential ordering of the form (Z_k, S_k). Moreover, M_{k+1} is 3-connected having a sequential ordering of the form $(R_{k+1}, f_{k+1}, S_{k+1} - t_{k+1})$. Since each element of S_k is identified with its image in R_{k+1} under the bijection φ_{k+1}, it follows by Lemma 3.11 that N_{k+1} is 3-connected having a sequential ordering of the form $(Z_k, S_k, f_{k+1}, S_{k+1} - t_{k+1})$. As $|S_{k+1} - t_{k+1}| \geq 2$, it follows that N_{k+1} has a sequential ordering of the form (Z_{k+1}, S_{k+1}). Thus (3.13.1) holds and hence so does the lemma. □

4. Constructing all sequential matroids

In this section, we prove Theorems 2.1 and 2.2. In addition, we establish a crucial step in the proof of Theorem 2.3.

Proof of Theorem 2.1. We shall first establish that if we begin at step (ii) of the algorithm with a sequential matroid N having a sequential ordering $(e_1, e_2, \ldots, e_n)$ and pass through the loop in the algorithm once to return to (ii), then the resulting matroid N' has the following properties:

(a) N' is sequential;
(b) N' is representable over a field F if and only if N is representable over F; and
(c) either $|E(N')| < |E(N)|$, or N' has a sequential ordering that begins with a maximal rank-2 flat X' of N' or $(N')^*$ such that $|X'| > |X|$.

To establish this, we observe that N' is either $\text{co}(\Delta_X(N))$ or $\text{si}(\nabla_X(N))$. By duality, it suffices to treat the first case. Then $r(X) = 2$. Note that, since $X \neq E(N)$ and N is sequential, we have $|E(N) - X| \geq 3$ and $r(N) \geq 3$.

Since $2 = \lambda_N(X) = r(X) + r^*(X) - |X|$, it follows that X is coindependent in N. Hence $\Delta_X(N)$ is well-defined. Now $\Delta_X(N) = P_X(\Theta_k, N)\backslash X$. By [7, Corollary 3.7], $\Delta_X(N)$ and N are representable over exactly the same fields. Hence (b) holds.

Now recall that $E(\Theta_k) = X \cup Y$. Let $E(N) - X = Z$. We know that Θ_k and N are sequential having sequential orderings of the form (Y, X) and (X, Z), respectively. Since $r(X) = 2$, Lemma 3.11 implies that $P_X(\Theta_k, N)$ is sequential. Now $(Y \cup X, E(N) - X)$ is a 3-separation of $P_X(\Theta_k, N)$. Since $X = \text{cl}(Y \cup X) \cap \text{cl}(E(N) - X)$, it follows by Lemma 3.12 that $\text{co}(P_X(\Theta_k, N)\backslash X)$ is 3-connected. Thus, by Lemma 3.2, the last matroid is sequential, that is, $\text{co}(\Delta_X(N))$ is sequential. Finally, we observe that either $|E(\text{co}(\Delta_X(N))| < |E(N)|$, or $\text{co}(\Delta_X(N)) = \Delta_X(N)$. In the former case, $|E(N')| < |E(N)|$ and (c) holds. In the latter case, $\Delta_X(N)$ has a sequential ordering of the form $(Y, e_{k+1}, e_{k+2}, \ldots, e_n)$. Now Y is a union of triads in $\Delta_X(N)$ and e_{k+1} is a coloop of $N\backslash X$. By [7, Lemma 2.8], $N\backslash X = \Delta_X(N)\backslash Y$. Hence $Y \cup e_{k+1}$ spans a rank-2 flat of $(N')^*$, and again (c) holds. We conclude that (a)-(c) hold.

Because of (c), the algorithm terminates in a finite number of steps yielding the required uniform matroid N of rank or corank 2. Moreover, the original matroid M is representable over exactly the same fields as N and so is $\mathbb{F}$-representable if and only if $|\mathbb{F}| \geq |E(N)| - 1$. □

The following is an immediate consequence of Theorem 2.1.

Corollary 4.1. *If a sequential matroid M is representable over a field with n elements, then M is representable over all fields with at least n elements.*

Next we show how, by reversing the steps in Theorem 2.1, we can build all sequential matroids from uniform matroids of rank or corank two.

Lemma 4.2. *Let N be a sequential matroid. Let $(e_1, e_2, \ldots, e_n)$ be a sequential ordering of $E(N)$ in which $\{e_1, e_2, \ldots, e_m\}$ is a rank-2 flat for some $m \geq 3$. Let $X = \{e_1, e_2, \ldots, e_k\}$ where $3 \leq k \leq m$. If $X_1 \subseteq X$ and X_1 is a proper flat of N^*, then $P_X(\Theta_k, N)\backslash X_1$ is sequential.*

Proof. Let $M_1 = P_X(\Theta_k, N)$. Then M_1 is certainly sequential. Let $X_1 = \{e_1, e_2, \ldots, e_s\}$ and suppose that X_1 is a proper flat of N^*. We need to show that $M_1\backslash X_1$ is sequential. By Lemma 3.2, it suffices to show that $M_1\backslash X_1$ is 3-connected. Evidently $\text{cl}_N(X) = \{e_1, e_2, \ldots, e_m\}$. Suppose first that $r(N) = 2$. Then $P_X(\Theta_k, N)\backslash X_1$ is 3-connected unless $|\text{cl}_N(X) - X_1| \leq 2$. In the exceptional case, $E(N) - X_1$ is independent in N so X_1 spans N^*. Hence X_1 is not a proper flat of N^*.

We may now suppose that $r(N) \geq 3$. By Lemma 3.12, $\mathrm{co}(M_1 \backslash X_1)$ is 3-connected. The lemma will follow if we can show that $M_1 \backslash X_1$ has no non-trivial series classes. Assume the contrary, letting S be such a series class. Then, by Lemma 3.12 again, S has exactly one element in Y and exactly one other element, say z, which is in $E(N) - X_1$. Thus M_1 has a cocircuit of the form $S \cup X'$ where $X' \subseteq X_1$. Since N is a restriction of M_1, we deduce that N has a cocircuit that contains z and some subset of X_1. Hence X_1 is not a flat of N^*; a contradiction. $\square$

Note that, up to an obvious relabelling, the matroid $P_X(\Theta_k, N) \backslash X_1$ in the last result can alternatively be obtained from N by first adding a single element in parallel to each element of $X - X_1$ to give the matroid N' and then finding $\Delta_X(N')$.

Lemma 4.3. *Let N be a 3-connected matroid and X be a k-element segment in N for some $k \geq 3$. Assume that X_1 is a subset of X such that $P_X(\Theta_k, N) \backslash X_1$ is 3-connected but is not isomorphic to $U_{k,k+2}$. Then either*

(i) *X_1 is a proper flat of N^*; or*
(ii) *$X_1 = X$ and $Y \cup f$ is a cosegment of $P_X(\Theta_k, N) \backslash X_1$ for some f in $E(N) - X$.*

Proof. Suppose that X_1 is not a proper flat of N^*. Since $P_X(\Theta_k, N) \backslash X_1$ is 3-connected, $X_1 \neq E(N^*)$. Then, for some subset X_2 of X_1, there is an element f of $E(N) - X_1$ such that $X_2 \cup f$ is a cocircuit of N. Let H_1 be the hyperplane $E(N) - (X_2 \cup f)$ of N.

Suppose first that $f \in \mathrm{cl}_N(X)$. Then

$$\lambda_N(X_2 \cup f) = r(X_2 \cup f) + (r^*(X_2 \cup f) - |X_2 \cup f|) = 2 + (-1) = 1,$$

so we have a contradiction to the fact that N is 3-connected unless $|H_1| \leq 1$. Consider the exceptional case. We must have $r(N) = 2$. Thus $P_X(\Theta_k, N) \backslash X_1$ is 3-connected having rank k, so $|E(N) - X_1| \geq 2$. But $|E(N) - X_1| \leq |H_1| + 1 \leq 2$, so $|E(N) - X_1| = 2$. Thus $P_X(\Theta_k, N) \backslash X_1$ has $k + 2$ elements and rank k, so $P_X(\Theta_k, N) \backslash X_1 \cong U_{k,k+2}$; a contradiction.

We may now assume that $f \notin \mathrm{cl}_N(X)$. Then $r(N) \geq 3$. Since $X_2 \cup f$ is a cocircuit of N, by orthogonality with the triangles contained in $\mathrm{cl}_N(X)$, we deduce that $|\mathrm{cl}_N(X) - X_2| \leq 1$. We have $X_2 \subseteq X_1 \subseteq X \subseteq \mathrm{cl}_N(X)$. Suppose that $X = X_2$. Take $\{y_1, y_2\} \subseteq Y$. Then $Y - \{y_1, y_2\}$ is a flat of Θ_k of rank $k - 2$. Thus $(Y - \{y_1, y_2\}) \cup H_1$ is a flat of $P_X(\Theta_k, N)$. But $r((Y - \{y_1, y_2\}) \cup H_1) = r(H_1) + |Y - \{y_1, y_2\}|$ since each 3-element subset of Y is a triad of $P_X(\Theta_k, N)$. Thus $(Y - \{y_1, y_2\}) \cup H_1$ is a hyperplane of $P_X(\Theta_k, N)$. The complementary cocircuit is $\{y_1, y_2, f\} \cup X_2$, so $\{y_1, y_2, f\}$ is a cocircuit of $P_X(\Theta_k, N) \backslash X_1$ and (ii) holds.

It remains to consider the case when $X_2 \neq X$. Then $X = \mathrm{cl}_N(X)$ and $\mathrm{cl}_N(X) - X_2 = \{g\}$ for some element g. Thus Θ_k has a hyperplane H_2 that meets X in $\{g\}$. Moreover, $H_2 = (Y - g') \cup g$ for some g' in Y. Thus $H_1 \cup H_2$ is a hyperplane of $P_X(\Theta_k, N)$ and $(X - g) \cup g' \cup f$ is the complementary cocircuit. Hence $P_X(\Theta_k, N)\backslash X_1$ has $\{g', f\}$ as a cocircuit. This contradicts the fact that $P_X(\Theta_k, N)\backslash X_1$ is 3-connected. □

We are now ready to prove our construction yields the class of sequential matroids.

Proof of Theorem 2.2. It follows by Lemma 4.2 that every matroid produced by the prescribed procedure is sequential. Now let M be an arbitrary sequential matroid. Then, by using the algorithm in Theorem 2.1, we obtain a uniform matroid of rank or corank two. Up to duality, a typical step in this algorithm consists of replacing N by $\mathrm{si}(\nabla_X(N))$ for some maximal cosegment X of N. Now consider how we can recover N from $\mathrm{si}(\nabla_X(N))$. We know by Lemma 3.12 and duality that each non-trivial parallel class of $\nabla_X(N)$ contains exactly one element of Y. Now $\nabla_X(N) = (P_X(\Theta_k, N^*)\backslash X)^*$. Hence $\mathrm{si}(\nabla_X(N)) = (P_X(\Theta_k, N^*)\backslash X)^*\backslash W$ where, for some $Y_0 \subseteq Y$ with $|Y_0| = |W|$, each element of Y_0 is parallel to a unique element of W, and $W \cap Y = \emptyset$.

Recall that $\Theta_k(X, Y)$ denotes the copy of Θ_k in which X is a segment and Y is a cosegment. By [7, Corollary 2.12], when we maintain the same ground set after each segment-cosegment and cosegment-segment exchange, we have that $N = \Delta_X(\nabla_X(N))$. This means, in the notation we are using, that

$$N = P_Y(\Theta_k(Y, X), (P_X(\Theta_k(X, Y), N^*)\backslash X)^*)\backslash Y.$$

In $(P_X(\Theta_k(X, Y), N^*)\backslash X)^*$, each element of Y_0 is parallel to an element of W. Thus $N \cong P_Y(\Theta_k(Y, X), (P_X(\Theta_k(X, Y), N^*)\backslash X)^*\backslash W)\backslash Y_1$ where $Y_1 = Y - Y_0$, that is, $N \cong P_Y(\Theta_k(Y, X), \mathrm{si}(\nabla_X(N)))\backslash Y_1$ where this isomorphism relabels each member of Y_0 by the parallel element from W. Now we may assume that N is not uniform of corank 2 otherwise N is already in $\mathcal{M}_0$. Since N is 3-connected, by Lemma 4.3, either Y_1 is a proper flat of $(\mathrm{si}(\nabla_X(N)))^*$, or X is not a maximal cosegment of $P_Y(\Theta_k(Y, X), \mathrm{si}(\nabla_X(N)))\backslash Y_1$, that is, of N. This contradiction means that the steps described in the theorem essentially correspond to reversing the steps in the algorithm in Theorem 2.1 and this completes the proof. □

The next result will be important in our proof of Theorem 2.3, which appears in the next section. It provides an alternative to the latter result by giving another family of sequential matroids of which every sequential matroid is a minor. In the next theorem, we consider a field $GF(q)$ over which an n-element sequential matroid M is representable. By Theorem 2.1, we can find such a field simply by choosing $q \geq n-1$. But M may also be representable over smaller fields and we want our result to cover these fields too.

Theorem 4.4. *Let M be an n-element sequential matroid and suppose that M is representable over $GF(q)$. For $\mu \in \{n, q+1\}$, if $n \geq 6$, then M is a minor of a member of Π_μ^{n-5} unless $M \cong U_{n-2,n}$, in which case, M is a minor of a member of Π_μ^{n-4}. If $n \in \{3,4,5\}$, then M is a minor of a member of Π_μ^1.*

Proof. Let $r(M) = r$. As M is representable over $GF(q)$, we may view M as a restriction of $PG(r-1,q)$. We shall denote the closure of a set X in $PG(r-1,q)$ by $\langle X \rangle$.

Let $(e_1, e_2, \ldots, e_n)$ be a sequential ordering of M. Let $A = \{e_1, e_2\}$ and $B = \{e_{n-1}, e_n\}$ and consider the (A,B) 3-sequence $(A, e_3, e_4, \ldots, e_{n-2}, B) = (A, \overrightarrow{X}, B)$. Break $(A, e_3, e_4, \ldots, e_{n-2}, B)$ up according to the presence of coguts elements as $(G_0, c_1, G_1, c_2, \ldots, c_s, G_s)$, where $G_0 = \mathrm{cl}(\{e_1, e_2\})$ and $G_s = \mathrm{cl}(\{e_{n-1}, e_n\})$, the elements $c_1, c_2, \ldots, c_s$ are the coguts elements of $\overrightarrow{X}$, and, for all i in $\{1, 2, \ldots, s-1\}$, the set of guts elements lying between c_i and c_{i+1}, which may be empty, is G_i.

If $s = 0$, then $M \cong U_{2,n}$, so $q \geq n-1$ and M is a minor of V_n and of V_{q+1}. Hence M is a minor of a member of Π_μ^1. Now suppose that $s \geq 1$. We shall show that M is a minor of a member of Π_μ^s.

First we show the following.

4.4.1. *For each i in $\{1, 2, \ldots, s-1\}$, the set G_i is a subset of the line G_i' of $PG(r-1,q)$ that is the intersection of $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup c_i \rangle$ and $\langle c_{i+1} \cup G_{i+1} \cup \cdots \cup G_s \rangle$.*

We have

$$r + 2 = r(G_0 \cup c_1 \cup G_1 \cup \cdots \cup c_i \cup G_i) + r(c_{i+1} \cup G_{i+1} \cup \cdots \cup G_s).$$

But $G_i \subseteq \mathrm{cl}_M(G_0 \cup c_1 \cup G_1 \cup \cdots \cup c_i) \cap \mathrm{cl}_M(c_{i+1} \cup G_{i+1} \cup \cdots \cup G_s)$. Thus

$$\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup c_i \rangle \cap \langle c_{i+1} \cup G_{i+1} \cup \cdots \cup G_s \rangle$$

is a rank-2 flat G_i' of $PG(r-1,q)$ containing G_i. Hence (4.4.1) holds.

We shall let G_0' and G_s' be the lines of $PG(r-1,q)$ spanned by G_0 and G_s, respectively.

4.4.2. *For each i in $\{0, 1, \ldots, s-1\}$, there is a unique point t_i of $PG(r-1,q)$ in $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i \rangle \cap \langle G_{i+1} \cup c_{i+2} \cup \cdots \cup G_s \rangle$ and $G_i' \cap G_{i+1}' = \{t_i\}$.*

We have $r(G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i \cup c_{i+1}) = r(G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i) + 1$. Thus

$$r(G_0 \cup c_1 \cup G_1 \cup \cdots \cup c_i \cup G_i) + r(G_{i+1} \cup c_{i+2} \cup \cdots \cup G_s) = r + 1.$$

Hence $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i \rangle \cap \langle G_{i+1} \cup c_{i+2} \cup \cdots \cup G_s \rangle$ is a point t_i. Now $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i \rangle \supseteq G'_i$ and $\langle G_{i+1} \cup c_{i+2} \cup \cdots \cup G_s \rangle \supseteq G'_{i+1}$. Hence $G'_i \cap G'_{i+1} \subseteq \{t_i\}$. But $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_{i+1} \rangle$ has rank one more than $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i \rangle$, so

$$\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_{i+1} \rangle \cap \langle G_{i+1} \cup c_{i+2} \cup \cdots \cup G_s \rangle$$

has rank 2 and contains and so equals G'_{i+1}. This intersection also contains t_i, so $t_i \in G'_{i+1}$. Similarly, $t_i \in G'_i$. Thus (4.4.2) holds.

Next observe that

4.4.3. *G'_i meets G'_{i+j} for some $j \geq 2$ if and only if $t_i = t_{i+1} = \cdots = t_{i+j-1}$.*

This follows because t_i is the unique point of intersection of $\langle G_0 \cup c_1 \cup G_1 \cup \cdots \cup G_i \rangle$ and $\langle G_{i+1} \cup c_{i+2} \cup \cdots \cup G_s \rangle$.

We now define a sequence of sets of distinguished points beginning with $D_0 = E(M) \cup t_0$. Assume that $D_0, D_1, \ldots, D_{i-1}$ have been defined. To define D_i, we introduce a function α_i. We let $\alpha_i(c_i) = t_{i-1}$; for each element x of $D_{i-1} \cap (G'_{i-1} - t_{i-1})$, let $\alpha_i(x)$ be the element of G'_i that also lies on the line spanned by $\{x, c_i\}$. Define

$$D_i = (D_{i-1} - c_i - G'_{i-1}) \cup t_{i-1} \cup \{\alpha_i(x) : x \in D_{i-1} \cap (G'_{i-1} - t_{i-1})\} \cup t_i$$

where, if $i = s$, we take $t_s = t_{s-1}$.

The next two assertions are straightforward consequences of this definition.

4.4.4. $D_0 \cap G'_i \subseteq D_1 \cap G'_i \subseteq \cdots \subseteq D_i \cap G'_i$.

4.4.5. $D_s \cap G'_i \subseteq D_{s-1} \cap G'_i \subseteq \cdots \subseteq D_{i+1} \cap G'_i = \{t_i\}$.

We show next that

4.4.6.

$$|D_i \cap G'_i| \leq \begin{cases} |E(M) - (G_{i+1} \cup G_{i+2} \cup \cdots \cup G_s)| - (s-i) + 1 & \text{if } i < s \\ |E(M)| & \text{if } i = s. \end{cases}$$

We argue by induction on i. If $i = 0$, then $D_i \cap G'_i$ is $(E(M) \cap G'_0) \cup t_0$, which equals $G_0 \cup t_0$, so the result follows. Assume that (4.4.6) holds for $i < j$ and let $i = j$.

Suppose first that $j < s$. Then $D_j \cap G'_j$ is contained in the union of $\{t_{j-1}, t_j\} \cup G_j$ and the set $\{\alpha_j(x) : x \in D_{j-1} \cap (G'_{j-1} - t_{j-1})\}$. Thus

$$\begin{aligned} |D_j \cap G'_j| &\leq 2 + |G_j| + |D_{j-1} \cap G'_{j-1}| - 1 \\ &= |G_j| + |D_{j-1} \cap G'_{j-1}| + 1. \end{aligned}$$

But, by the induction assumption,

$$|D_{j-1} \cap G'_{j-1}| \leq |E(M) - (G_j \cup G_{j+1} \cup \cdots \cup G_s)| - (s - (j-1)) + 1.$$

Hence

$$\begin{aligned} |D_j \cap G'_j| &\leq |G_j| + |E(M) - (G_j \cup \cdots \cup G_s)| - (s-j) + 1 \\ &\leq |E(M) - (G_{j+1} \cup \cdots \cup G_s)| - (s-j) + 1, \end{aligned}$$

where the last step follows since $G_j, G_{j+1}, \ldots, G_s$ are disjoint.

Finally, observe that if $j = s$, then $t_j = t_{j-1}$, so we can decrease the bound on $|D_j \cap G'_j|$ by one to get $|D_s \cap G'_s| \leq |E(M)|$. Hence (4.4.6) holds.

Now let $\widehat{D_s} = D_0 \cup D_1 \cup \cdots \cup D_s$. Then t_{s-1} is certainly in $\widehat{D_s}$. Moreover,

4.4.7. $|\widehat{D_s} \cap G'_i| \leq \mu$ *for all* i *in* $\{0, 1, \ldots, s\}$.

This is immediate if $\mu = q+1$. If $\mu = n$, it follows from (4.4.6) because $\widehat{D_s} \cap G'_i = (D_0 \cup D_1 \cup \cdots \cup D_s) \cap G'_i = D_i \cap G'_i$, where the last equality holds by (4.4.4) and (4.4.5).

Now suppose that $|\widehat{D_s} \cap G'_s| = m$. Take $\mu - m$ points of $G'_s - \widehat{D_s}$ and adjoin these elements to $\widehat{D_s}$ continuing to call the resulting set $\widehat{D_s}$. We now have $|\widehat{D_s} \cap G'_s| = \mu$. For each element z of $(\widehat{D_s} \cap G'_s) - t_{s-1}$, there is a unique point $\beta_s(z)$ of G'_{s-1} on the line through z and c_s. Some of these elements are already in $\widehat{D_s}$. Adjoin the other such points to $\widehat{D_s}$ letting the resulting set be $\widehat{D_{s-1}}$. Evidently

$$|\widehat{D_{s-1}} \cap G'_s| = \mu.$$

We assert that

4.4.8. $|\widehat{D_{s-1}} \cap G'_{s-1}| = \mu$.

By construction, it is clear that $|\widehat{D_{s-1}} \cap G'_{s-1}| \geq \mu$. Assume this inequality is strict. Then there is a point y of $\widehat{D_{s-1}} \cap G'_{s-1}$ that does not lie on a line through c_s and some element of $(\widehat{D_s} \cap G'_s) - t_{s-1}$. Then $y \in [(D_0 \cup D_1 \cup \cdots \cup D_s) \cap G'_{s-1}] - t_{s-1}$ so, by (4.4.4) and (4.4.5), $y \in D_{s-1} \cap (G'_{s-1} - t_{s-1})$. Thus the construction of D_s produces a point $\alpha_s(y)$ of G'_s that lies on the line through y and c_s. Hence $\alpha_s(y) \in (D_s \cap G'_s) - t_{s-1}$ and we have a contradiction that establishes (4.4.8).

Now using $\widehat{D_{s-1}} \cap G'_{s-1}$ in place of $\widehat{D_s} \cap G'_s$, we can construct a new set $\widehat{D_{s-2}}$ by adjoining points of G'_{s-2} to $\widehat{D_{s-1}}$. The same argument used above guarantees that

$$\mu = |\widehat{D_{s-2}} \cap G'_{s-2}| = |\widehat{D_{s-1}} \cap G'_{s-1}| = |\widehat{D_s} \cap G'_s|.$$

Repeating this process, we eventually obtain a set $\widehat{D_0}$ such that

$$|\widehat{D_0} \cap G'_i| = \mu \text{ for all } i \text{ in } \{0, 1, \ldots, s\}.$$

Finally, we consider the matroid $\widehat{M}$ that equals $PG(r-1, q)|\widehat{D_0}$. Note that $\widehat{M}$ consists of $s+1$ lines, $L_0, L_1, \ldots, L_s$, each containing exactly μ points, along with s additional points, $c_1, c_2, \ldots, c_s$. Moreover, for all i in $\{0, 1, \ldots, s\}$, we have $L_{i-1} \cap L_i = \{t_{i-1}\}$ and, for each point e of $L_{i-1} - t_{i-1}$, there is a unique point e' of L_i such that $\{e, c_i, e'\}$ is a line of $\widehat{M}$.

Now clearly $\widehat{M}|(L_{s-1} \cup c_s \cup L_s) \cong V_\mu$ and L_{s-1} is a modular line of this restriction. It follows by a result of Brylawski [2] (see also [6, Proposition 12.4.15]) that $\widehat{M}$ is the generalized parallel connection across L_{s-1} of $\widehat{M}|(L_{s-1} \cup c_s \cup L_s)$ and $\widehat{M} \backslash [(L_s - t_{s-1}) \cup c_s]$. A routine induction argument establishes that $\widehat{M}$ is in Π^s_μ. We conclude that M is a minor of a member of Π^s_μ.

We have $1 \le s \le n - |G_0 \cup G_s| \le n-4$. If $s = n-4$, then $M \cong U_{n-2,n}$, so $q \ge n-1$ and $\mu \ge n$. Thus if $M \not\cong U_{n-2,n}$, we may assume that $s \le n-5$. The theorem follows without difficulty. □

5. The Universal Sequential Matroid

In this section, we shall prove an extension of Theorem 2.3. We begin with a result needed in the definition of Θ_n^{2m+1}. Recall that $\Theta_n^2 = (P_B(M', M''))^*$ where $M' = \Theta_n(B, A')$ and $M'' = \Theta_n(B, A'')$.

Lemma 5.1. *The sets A' and A'' are modular lines of Θ_n^2.*

Proof. Clearly A' is a rank-2 flat of Θ_n^2. By [2, Theorem 3.3] (see also [6, Proposition 6.9.2(iii)]), A' is modular provided that $r(A') + r(F) = r(\Theta_n^2)$ for all flats F of Θ_n^2 avoiding A' such that $F \cup A'$ spans Θ_n^2. Now, for all such flats F, we must have $r(\Theta_n^2) > r(F) \ge r(\Theta_n^2) - 2$. Since $r(A') = 2$, it suffices to show that $r(F) \ne r(\Theta_n^2) - 1$. Assume the contrary. Then $E(\Theta_n^2) - F$ is a cocircuit of Θ_n^2, so $E(\Theta_n^2) - F$ is a circuit C of $P_B(M', M'')$ containing A'. But A' is independent in $P_B(M', M'')$, so C properly contains A'. Now, for all b in B, if a' is the partner of b in A', then $(A' - a') \cup b$ is a circuit of $P_B(M', M'')$. Hence $b \notin C$ so $C \cap B = \emptyset$. Since $C \supsetneq A'$, we deduce that $C \cap A'' \neq \emptyset$. As A'' is a cosegment of $P_B(M', M'')$, by orthogonality, $|C \cap A''| \geq |A''| - 1$. But, if $a'' \in A''$, then $(A'' - a'') \cup A'$ properly contains the circuit $(A'' - a'') \cup (A' - a')$ of $P_B(M', M'')$, where a' and a'' are the partners of some element b of B. We conclude that the circuit C does not exist, so A' is indeed a modular flat of Θ_n^2. □

We defined Θ_n^{2m+1} in Section 2. The latter has B_1 as an n-element cosegment. We now define Θ_n^{2m} to be $\Theta_n^{2m+1} \backslash B_1$. Thus $\Theta_n^0 \cong U_{2,n}$. Using the notation in the definition of Θ_n^{2m+1}, this implies that $\Theta_n^2 = \Theta_n^3 \backslash B_1 = M_1$. But M_1 is a copy of Θ_n^2 with ground set $A_1 \cup B_2 \cup A_2$, so the notation is consistent. In general, for all $m \geq 2$,

$$\Theta_n^{2m} = P_{A_m}(M_m, \Theta_n^{2m-1}) \backslash B_1 = P_{A_m}(M_m, \Theta_n^{2m-1} \backslash B_1) = P_{A_m}(M_m, \Theta_n^{2m-2}).$$

The matroid Θ_n^{2m+1} has a number of attractive properties, many of which are summarized in the next result.

Lemma 5.2. (i) *$r(\Theta_n^{2m+1}) = (m+1)n$ and $|E(\Theta_n^{2m+1})| = 2(m+1)n$;*
(ii) *$\Theta_n^{2m+1} \backslash (B_{k+2} \cup A_{k+2} \cup \cdots \cup A_{m+1}) = \Theta_n^{2k+1}$;*
(iii) *Θ_n^{2m+1} has $B_1, B_2, \ldots, B_{m+1}$ as cosegments and $A_1, A_2, \ldots, A_{m+1}$ as segments;*
(iv) *$\Theta_n^3 / A_2 \cong (\Theta_n^2)^*$;*
(v) *Θ_n^{2m+1} has $B_1 \cup B_2 \cup \cdots \cup B_{m+1}$ as a basis;*
(vi) *A_i is a modular flat of Θ_n^{2m+1} for all i in $\{1, 2, \ldots, m+1\}$;*
(vii) *$\Theta_n^{2m+1} \backslash B_1 / A_1 \cong \Theta_n^{2m-1}$.*

Proof. Parts (i) and (ii) follow easily by induction and by the definition, respectively. Part (iii) follows from the fact that M_0 has B_1 as a cosegment and A_1 as a segment, while, for all $i \geq 1$, the matroid M_i has A_i and A_{i+1} as segments and B_{i+1} as a cosegment.

For (iv), we have

$$\Theta_n^3 / A_2 = P_{A_1}(M_1, \Theta_n^1) / A_2 = P_{A_1}(M_1 / A_2, \Theta_n^1).$$

But, by (2.1), M_1 / A_2 has A_1 as a segment and B_1 as a cosegment and is isomorphic to Θ_n. Thus

$$\Theta_n^3 / A_2 = P_{A_1}(\Theta_n, \Theta_n^1) \cong (\Theta_n^2)^*.$$

We prove (v) by induction. Clearly B_1 is a basis of Θ_n^1. Assume that $B_1 \cup B_2 \cup \cdots \cup B_m$ is a basis of Θ_n^{2m-1}. Hence Θ_n^{2m+1} has a basis containing $B_1 \cup B_2 \cup \cdots \cup B_m$ and contained in $B_1 \cup B_2 \cup \cdots \cup B_m \cup A_m \cup B_{m+1} \cup A_{m+1}$. As $|B_1 \cup B_2 \cup \cdots \cup B_{m+1}| = r(\Theta_n^{2m+1})$ and $\mathrm{cl}(B_1 \cup B_2 \cup \cdots \cup B_m) \supseteq A_m$, while $\mathrm{cl}(B_1 \cup B_2 \cup \cdots \cup B_m \cup A_m \cup B_{m+1}) \supseteq A_{m+1}$, we deduce that $B_1 \cup B_2 \cup \cdots \cup B_{m+1}$ is a basis of Θ_n^{2m+1}.

To prove (vi), we argue by induction on m. We know that A_1 is a modular flat of M_0. Assume that $A_1, A_2, \dots, A_m$ are modular flats of Θ_n^{2m-1}. Then, by [2] (see [6, Proposition 12.4.14(ii)]), $P_{A_m}(M_m, \Theta_n^{2m-1}) = P_{A_m}(\Theta_n^{2m-1}, M_m)$. Thus, by [2] (see [6, Proposition 12.4.14(iii)]), $E(M_m)$ and $E(\Theta_n^{2m-1})$ are modular flats of Θ_n^{2m+1}. Thus, by [2] (see [6, Proposition 6.9.7]), since $A_1, A_2, \dots, A_m$ are modular flats of Θ_n^{2m-1}, and A_{m+1} is a modular flat of M_m, we deduce that $A_1, A_2, \dots, A_m$, and A_{m+1} are modular flats of Θ_n^{2m+1}.

To prove (vii), we argue by induction on m that $\Theta_n^{2m+1}\backslash B_1/A_1 \cong \Theta_n^{2m-1}$ with the cosegments and segments of the former being $B_2, B_3, \dots, B_{m+1}$ and $A_2, A_3, \dots, A_{m+1}$, respectively. The result holds for $m = 1$ by (2.1). Assume it holds for $m < k$ and let $m = k \geq 2$. Then

$$\begin{aligned}
\Theta_n^{2k+1}\backslash B_1/A_1 &= P_{A_k}(M_k, \Theta_n^{2k-1})\backslash B_1/A_1 \\
&= P_{A_k}(M_k, \Theta_n^{2k-1}\backslash B_1/A_1) \\
&\cong P_{A_k}(M_k, \Theta_n^{2k-3}) \ \text{ by the induction assumption;} \\
&\cong \Theta_n^{2k-1}.
\end{aligned}$$

□

The following lemma provides a useful link between Θ_n^{2m} and Θ_n^{2m-1}. In the notation above, for k in $\{2m, 2m+1\}$, let $A_{k+1} = \{a'_1, a'_2, \dots, a'_n\}$ and $B_{k+1} = \{b'_1, b'_2, \dots, b'_n\}$, where a'_i is a partner of b'_i.

Lemma 5.3. *If $k \geq 1$, then $\Theta_n^k\backslash\{a'_2, a'_3, \dots, a'_{n-1}\}$ has a'_1 and a'_n in series with b'_n and b'_1, respectively. Moreover,*

$$\Theta_n^k\backslash\{a'_2, a'_3, \dots, a'_{n-1}\}/\{a'_1, a'_n\} \cong \begin{cases} \Theta_n^{k-1} & \text{if } k = 2m, \\ (\Theta_n^{k-1})^* & \text{if } k = 2m+1. \end{cases}$$

Proof. By the definition of Θ_n^k, the result follows easily by induction once we have shown it for k in $\{1, 2\}$, and it is not difficult to check for $k = 1$. Now let $k = 2$. We have $\Theta_n^2 = (P_{B_2}(\Theta_n(B_2, A_1), \Theta_n(B_2, A_2)))^*$. So

$$\Theta_n^2\backslash\{a'_2, a'_3, \dots, a'_{n-1}\} = (P_{B_2}(\Theta_n(B_2, A_1), \Theta_n(B_2, A_2)/\{a'_2, a'_3, \dots, a'_{n-1}\}))^*.$$

But $\Theta_n(B_2, A_2)/\{a'_2, a'_3, \dots, a'_{n-1}\}$ is obtained from $\Theta_n|B_2$ by adding a'_1 and a'_n in parallel to b'_n and b'_1, respectively. Thus

$$\Theta_n(B_2, A_2)/\{a'_2, a'_3, \dots, a'_{n-1}\}\backslash\{a'_1, a'_n\} \cong U_{2,n}.$$

Hence

$$\Theta_n^2\backslash\{a'_2, a'_3, \dots, a'_{n-1}\}/\{a'_1, a'_n\} = (\Theta_n(B_2, A_1))^* = \Theta_n(A_1, B_2)$$

and the lemma follows. □

The next result is an immediate consequence of Lemma 3.11.

Lemma 5.4. *The matroid* Θ_n^{2m+1} *is sequential. Moreover, every ordering of its ground set of the form* $(B_1, A_1, B_2, A_2, \ldots, B_{m+1}, A_{m+1})$ *is sequential.*

We know that Θ_n is isomorphic to its dual. We show next that the same is true for Θ_n^{2m+1}

Lemma 5.5. *The matroid* Θ_n^{2m+1} *is isomorphic to its dual.*

Proof. We argue by induction on m. We know that the assertion is true for $m = 0$. Assume it true for $m < k$ and consider $(\Theta_n^{2k+1})^*$. From Lemma 5.4, this matroid is sequential having $(A_{k+1}, B_{k+1}, A_k, \ldots, A_1, B_1)$ as a sequential ordering where $A_{k+1}, A_k, \ldots, A_1$ are cosegments and $B_{k+1}, B_k, \ldots, B_1$ are segments. Thus $(\Theta_n^{2k+1})^*/B_2$ has $B_1 \cup A_1$ and $A_{k+1} \cup B_{k+1} \cup \cdots \cup A_2$ as separators. Now

$$(\Theta_n^{2k+1})^*|(B_2 \cup A_2 \cup \cdots \cup A_{k+1}) = (\Theta_n^{2k+1}/(A_1 \cup B_1))^* = (\Theta_n^{2k+1}/A_1 \backslash B_1)^*$$

where the last step follows from the fact that B_1 is a separator of Θ_n^{2k+1}/A_1. By Lemma 5.2(vii) and the induction assumption, $(\Theta_n^{2k+1}/A_1\backslash B_1)^* \cong \Theta_n^{2k-1}$, and $(\Theta_n^{2k+1}/A_1\backslash B_1)^*$ has $A_2, A_3, \ldots, A_{k+1}$ and $B_2, B_3, \ldots, B_{k+1}$ as cosegments and segments, respectively. Thus, by Lemma 5.2(vi), B_2 is a modular flat of $(\Theta_n^{2k+1})^*|(B_2 \cup A_2 \cup \cdots \cup A_{k+1})$. Also

$$\begin{aligned}(\Theta_n^{2k+1})^*|(B_1 \cup A_1 \cup B_2) &= (\Theta_n^{2k+1}/(A_2 \cup B_3 \cup \cdots \cup A_{k+1}))^* \\ &= (\Theta_n^{2k+1}\backslash(B_3 \cup A_3 \cup \cdots A_{k+1})/A_2)^* \\ &= (\Theta_n^3/A_2)^* \\ &\cong \Theta_n^2\end{aligned}$$

where the last step follows by Lemma 5.2(iv). Note that $(\Theta_n^3/A_2)^*$ has B_1 and B_2 as segments and A_1 as a cosegment.

By [2] (see [6, Proposition 12.4.15], $(\Theta_n^{2k+1})^*$ is the generalized parallel connection across B_2 of the restrictions of $(\Theta_n^{2k+1})^*$ to $B_2 \cup A_2 \cup \cdots \cup A_{k+1}$ and $B_1 \cup A_1 \cup B_2$. Thus $(\Theta_n^{2k+1})^* = P_{B_2}(\Theta_n^{2k-1}, \Theta_n^2)$ where B_2 is a segment of both Θ_n^{2k-1} and Θ_n^2. Hence $(\Theta_n^{2k+1})^* \cong \Theta_n^{2k+1}$. □

Using Lemmas 5.4 and 3.10, we get the following result.

Lemma 5.6. *If the 3-connected matroid M is obtained from* Θ_n^{2m+1} *by contracting elements of* $B_1 \cup B_2 \cup \cdots \cup B_{m+1}$ *and simplifying, then M is sequential.*

The main result of this section is the following converse to the last lemma.

Theorem 5.7. *Let M be an n-element sequential matroid and suppose that M is representable over $GF(q)$. For $\mu \in \{n, q+1\}$, if $n \geq 5$, then M is isomorphic to a minor of* Θ_μ^{2n-9}*, while if $n \in \{3, 4\}$, then M is isomorphic to* *a minor of* Θ^1_μ. *More specifically, a minor isomorphic to* M *can be obtained from* Θ^{2n-9}_μ *or* Θ^1_μ *by contracting elements of* $\cup_i B_i$ *and simplifying, and deleting elements of* $\cup_i A_i$ *and cosimplifying.*

The proof of this theorem will use the following result.

Lemma 5.8. *Every matroid in* Π^m_μ *is a minor of* Θ^{2m+1}_μ *obtained by contracting elements of* $B_1 \cup B_2 \cup \cdots \cup B_{m+1}$ *and simplifying.*

Proof. The unique member of Π^1_μ is V_μ and this matroid is easily seen to be obtained from Θ^3_μ by simplifying the matroid we get by contracting all but two elements of B_1 and all but one element of B_2. In particular, we may assume that A_2 is one of the distinguished n-point lines of V_μ.

Now, as our induction assumption, we suppose that every matroid M in Π^m_μ is isomorphic to a minor of Θ^{2m+1}_μ that is obtained by contracting elements of $B_1 \cup B_2 \cup \cdots \cup B_{m+1}$ and simplifying. In addition, suppose that, under this isomorphism, S_m is mapped to A_{m+1} where, in constructing M, the last copy of V_μ that is adjoined has R_m and S_m as its distinguished lines.

Next we assume that $M \in \Pi^{m+1}_\mu$. Then $M = P_{R_{m+1}}(V_\mu, N)$ where $N \in \Pi^m_\mu$ and the copy of V_μ has R_{m+1} and S_{m+1} as its distinguished lines. By construction, R_{m+1} coincides with S_m. Moreover, by the induction assumption, N is isomorphic to a minor of Θ^{2m+1}_μ under which S_m is mapped to A_{m+1}. Hence we can relabel R_{m+1} as A_{m+1}. Let t and f be the tip and the focus of the distinguished copy of V_μ. Now take a copy of Θ^2_μ on $A_{m+1} \cup B_{m+1} \cup A_{m+2}$ letting f be an element of B_{m+1} and letting t_{m+1} and t_{m+2} be the partners of f in A_{m+1} and A_{m+2}, respectively. By contracting $B_{m+1} - f$ from this copy of Θ^2_μ, we obtain a copy of V_μ with an element added in parallel to the tip. This parallel pair is $\{t_{m+1}, t_{m+2}\}$. By deleting t_{m+1}, we ensure that A_{m+2} labels one of the distinguished lines of this copy of V_μ, and the lemma follows by induction. □

Proof of Theorem 5.7. If $n \in \{3, 4, 5\}$, then clearly M is isomorphic to a minor of Θ^1_μ. Now suppose that $n \geq 6$. If $M \cong U_{n-2,n}$, then $q + 1 \geq n$, so $\mu \geq n$. Thus M is isomorphic to a minor of Θ^1_μ and hence to a minor of Θ^{2n-9}_μ. If $M \ncong U_{n-2,n}$, then, by Theorem 4.4, M is isomorphic to a minor of Π^{n-5}_μ. But, by Lemma 5.8, every member of Π^{n-5}_μ is a minor of $\Theta^{2(n-5)+1}_\mu$, that is, of Θ^{2n-9}_μ. We conclude that the second sentence of the theorem holds. It remains to establish that the assertion in the last sentence of the theorem is true. If $n \in \{3, 4\}$, then M is uniform of rank 1 or 2 and the asserted result holds.

Assume that $n \geq 5$. Then $M \cong \Theta_\mu^{2n-9}\backslash X/Y$. We shall show that if $e \in A_i \cap Y$, then we can remove e in a cosimplification step. This will suffice to prove the required result because there is an isomorphism between Θ_μ^{2n-9} and its dual that interchanges $\cup_{i=1}^{n-4} A_i$ and $\cup_{i=1}^{n-4} B_i$. Let $A_i = \{a'_1, a'_2, \dots, a'_\mu\}$ and $e = a'_\mu$. Then $\{a'_1, a'_2, \dots, a'_{\mu-1}\}$ is a parallel class of Θ_μ^{2n-9}/a'_μ so M is isomorphic to a minor of $\Theta_\mu^{2n-9}/a'_\mu\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}$. One possibility here is that $i = n-4$. Suppose $i < n-4$. Then the last matroid is the parallel connection, with basepoint a'_1, of $\Theta_\mu^{2i-1}/a'_\mu\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}$ and $[\Theta_\mu^{2n-9}|(A_i \cup B_{i+1} \cup \dots \cup A_{n-4})]/a'_\mu\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}$. In this case, because M is 3-connected, it is isomorphic to a minor of one of the two matroids involved in this parallel connection.

First let M be isomorphic to a minor of $\Theta_\mu^{2i-1}/a'_\mu\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}$ where we allow $i = n-4$. By Lemma 5.3, $\Theta_\mu^{2i-1}\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}$ has a'_1 and a'_μ in non-trivial series classes. Hence, when we remove a'_μ, we can do so as part of a cosimplification.

Now suppose that M is isomorphic to a minor of

$$[\Theta_\mu^{2n-9}|(A_i \cup B_{i+1} \cup \dots \cup A_{n-4})]/a'_\mu\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}.$$

Observe that $[\Theta_\mu^{2n-9}|(A_i \cup B_{i+1} \cup \dots \cup A_{n-4})] \cong \Theta_\mu^{2(n-4-i)}$. Thus, by Lemma 5.3,

$$[\Theta_\mu^{2n-9}|(A_i \cup B_{i+1} \cup \dots \cup A_{n-4})]\backslash\{a'_2, a'_3, \dots, a'_{\mu-1}\}$$

has a'_1 and a'_μ in non-trivial series classes and, after these elements are contracted, we obtain a matroid isomorphic to $\Theta_\mu^{2(n-i)-9}$. Again, when we remove a'_μ, we can do so as part of a cosimplification. □

Theorem 2.3 follows by combining Lemma 5.6 and Theorem 5.7. The graph Γ_{2m+1} in the next lemma was defined in Section 1.

Lemma 5.9. $\Theta_3^{2m+1} \cong M(\Gamma_{2m+1})$.

Proof. Evidently $\Gamma_1 \cong K_4$ so $\Theta_3^1 \cong M(\Gamma_1)$. By (2.2), $\Theta_3^2 \cong M((K_5 - e)^*)$ and the lemma follows without difficulty from this. □

In Θ_3^{2m+1}, the set B_1 corresponds to v_0u_1, v_0v_1, v_0w_1; each A_i corresponds to $\{u_iv_i, v_iw_i, w_iu_i\}$; and, for all i in $\{1, 2, \dots, m\}$, the set B_{i+1} corresponds to $\{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}\}$.

By combining Theorem 5.7 and Lemma 5.9, we immediately obtain the next result, which implies Theorem 1.1.

Corollary 5.10. *Let M be an n-element binary sequential matroid. Then M is graphic being a minor of Θ_3^{2n-9} when $n \geq 5$; and a minor of Θ_3 when $n \in \{3, 4\}$.*

Corollary 5.11. *Let M be an n-element graphic sequential matroid with $n \geq 4$. Then $M \cong M(G)$ where G is a minor of Γ_{2n-9}.*

We remark that we have made no attempt to find the minimum value of m such that every n-element graphic sequential matroid is a minor of Γ_{2m+1}. In this regard, observe that, from considering vertex degrees, one can show that the wheel with $2k$ spokes is a minor of Γ_{2k-3} but is not a minor of Γ_{2k-5}.

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