ON BIPARTITE RESTRICTIONS OF BINARY MATROIDS

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ABSTRACT. In a 1965 paper, Erdős remarked that a graph $G$ has a bipartite subgraph that has at least half the number of edges of $G$. The purpose of this note is to prove a matroid analogue of Erdős's original observation. It follows from this matroid result that every loopless binary matroid has a restriction that uses more than half of its elements and has no odd circuits; and, for $2 \leq k \leq 5$, every bridgeless graph $G$ has a subgraph that has a nowhere-zero $k$-flow and has more than $\frac{k-1}{k}|E(G)|$ edges.

1. INTRODUCTION

The matroid terminology used in this note will follow Oxley [9]. The results considered here relate to the critical problem of Crapo and Rota [5]. The reader is referred to the survey paper of Brylawski and Oxley [4] for the theoretical background to these results that is not included here. We shall require only a minimal amount of this theory. Let $M$ be a matroid. Its simplification is denoted by $\text{si}(M)$. When $M$ is $GF(q)$-representable and loopless having rank $r$, we say that $M$ is affine over $GF(q)$ or $q$-affine if $\text{si}(M)$ is isomorphic to a restriction of the affine geometry $AG(r-1, q)$. Equivalently, $M$ is affine over $GF(q)$ if, whenever there is a subset $T$ of the projective geometry $PG(r-1, q)$ such that $\text{si}(M)$ is isomorphic to $PG(r-1, q)/T$, there is a hyperplane $H$ of $PG(r-1, q)$ such that $H$ avoids $T$. It is well known that if $G$ is a graph, then $M(G)$ is affine over $GF(q)$ if and only if $G$ is $q$-colourable. In particular, $G$ is bipartite if and only if $M(G)$ is affine over $GF(2)$. On the other hand, $M^*(G)$ is affine over $GF(q)$ if and only if $G$ has a nowhere-zero $q$-flow. An arbitrary binary matroid is affine if and only if all of its circuits have even cardinality. We follow Welsh [13] in calling such a matroid bipartite.

2. THEOREM AND SOME CONSEQUENCES

Erdős’s observation [7] that every loopless graph has a bipartite subgraph that has at least half the number of edges of $G$ was sharpened by Edwards [6], and Erdős, Gyárfás, and Kohayakawa [8] gave simpler proofs of Edwards’s results. The following theorem generalizes Erdős’s original result to matroids. For a matroid $M$ and for $k$ in $\{0, 1, \ldots, r(M)\}$, let $h_k(M)$ denote the number of flats of $M$ of rank $r(M) - k$.
Theorem 2.1. Let $k$ be a non-negative integer, $P$ be a matroid of rank at least $k + 1$, and $T$ be a subset of $E(P)$ that contains no loops. Let $d = \max_{e \in T} h_k(P/e)$. Then $P$ has a flat $F$ of rank $r(P) - k$ such that

$$\frac{|T - F|}{|T|} \geq 1 - \frac{d}{h_k(P)}. $$

Proof. Let $r(P) = r$. Construct the bipartite graph $G$ with vertex classes $V_1$ and $V_2$ where $V_1$ is the set of rank-$(r - k)$ flats of $P$, and $V_2$ is $T$. An element $e$ of $T$ is adjacent to a rank-$(r - k)$ flat $F$ of $P$ if and only if $e \in F$.

If $e \in T$, then the number of rank-$(r - k)$ flats of $P$ containing $e$ is the same as the number of rank-$(r - 1 - k)$ flats of $P/e$. As $d = \max_{e \in T} h_k(P/e)$, we deduce that the vertex $e$ of $V_2$ has degree at most $d$ in $G$. Hence $|E(G)| \leq d|T|$. Since $|V_1| = h_k(P)$, it follows that $V_1$ contains a vertex of degree at most $\frac{d|T|}{h_k(P)}$. Thus $P$ has a rank-$(r - k)$ flat $F$ that avoids at least $|T| - \frac{d|T|}{h_k(P)}$ elements of $T$. Thus

$$\frac{|T - F|}{|T|} \geq 1 - \frac{d}{h_k(P)}. $$

Corollary 2.2. Let $M$ be a loopless non-empty $GF(q)$-representable matroid. Then $M$ has a $q$-affine restriction $N$ such that

$$|E(N)| > \frac{2q-1}{q}|E(M)|. $$

Proof. Let the size of a largest parallel class of $M$ be $t$ and $r = r(M)$. Replace each element of $PG(r - 1, q)$ by $t$ elements in parallel, letting the resulting matroid be $P$. Then $M$ can be viewed as a restriction of $P$. Let $T = E(M)$. Applying the theorem with $k = 1$, we get that there is a hyperplane $H$ of $P$ such that

$$\frac{|T - H|}{|T|} \geq \frac{h_1(P) - \max_{e \in T} h_1(P/e)}{h_1(P)}. $$

Clearly $h_1(P) = h_1(PG(r - 1, q)) = \frac{q^r - 1}{q - 1}$ while, for all elements $e$ of $P$, by symmetry, $h_1(P/e) = h_1(PG(r - 2, q)) = \frac{q^r - 1}{q - 1}$. Thus

$$\frac{h_1(P) - \max_{e \in T} h_1(P/e)}{h_1(P)} = \frac{q^r - q^{r-1}}{q^r - 1} = \frac{q - 1}{q} \left( 1 - \frac{q^r}{q^r - 1} \right) > \frac{q - 1}{q}. $$

Since $si(M|(T - H))$ is the ground set of a restriction of $AG(r - 1, q)$, we obtain the required result.

The bound in Corollary 2.2 can be restated as

$$|E(N)| \geq \frac{2q-1}{q}|E(M)| + \frac{1}{q}. $$

This bound is sharp with equality being attained when $M \cong PG(r - 1, q)$ since a largest affine restriction of this matroid is isomorphic to $AG(r - 1, q)$.

Corollary 2.2 has some interesting consequences. Taking $q = 2$, we immediately get the following result which implies Erdős’s result. Welsh [13]
showed that a binary matroid is bipartite if and only if its ground set can be written as a disjoint union of cocircuits.

**Corollary 2.3.** Let $M$ be a loopless non-empty binary matroid. Then $M$ has a bipartite restriction that has more than half the elements of $M$.

**Corollary 2.4.** Let $G$ be a loopless non-empty graph and $k$ be a positive integer. Then $G$ has a $k$-colourable subgraph that has more than $\frac{k-1}{k}|E(G)|$ edges.

*Proof.* Suppose that $G$ has $n$ vertices and that its largest parallel class has size $t$. Then we can view $G$ as a subgraph of the graph $K^t_n$ that is obtained from $K_n$ by replacing each edge by $t$ parallel edges. There is an obvious one-to-one correspondence between the flats of $M(K^t_n)$ and those of $M(K_n)$. Moreover, the flats of $M(K_n)$ of rank $n-k$ correspond to the partitions of $V(K_n)$ into exactly $k$ classes where an edge of $K_n$ is in the flat if and only if both ends lie in the same class. The number of such partitions is $S(n,k)$, the Stirling number of the second kind. Moreover, the complement of a flat of rank $n-k$ is a complete $k$-partite graph. As $n-k = r(M(K_n)) - (k-1)$, when we apply Theorem 2.1 substituting $M(K_n)$ for $P$ and $k-1$ for $k$, we find that the right-hand side of the inequality is $\frac{S(n,k)-S(n-1,k)}{S(n,k)}$. But $S(n,k) = S(n,k-1) + kS(n-1,k)$. Thus

$$\frac{S(n,k) - S(n-1,k)}{S(n,k)} = \frac{S(n,k-1) + (k-1)S(n-1,k)}{S(n,k-1) + kS(n-1,k)} > \frac{k-1}{k}.$$

Since the last result is derived from such a general matroid result, it is not surprising that a stronger graph result is known. Andersen, Grant, and Linial [1] showed that, for all positive integers $k$, every loopless graph $G$ has a $k$-colourable subgraph $H$ with

$$|E(H)| \geq \frac{k-1}{k}|E(G)| + \alpha_k(|V(G)| - 1)$$

where $\alpha_k = 1/k$ when $k \geq 3$, while $\alpha_2 = 1/4$.

Next we consider some variants on the results above that use the following special case of a result of Asano, Nishizeki, Saito, and Oxley [2].

**Lemma 2.5.** Let $M$ be a $GF(q)$-representable matroid and $X$ be a subset of $E(M)$. Then $X$ is minimal with the property that $M \setminus X$ is $q$-affine if and only if $X$ is minimal with the property that $M/X$ is $q$-affine.

Applying the last result to the theorem, we immediately obtain the following.

**Corollary 2.6.** Let $M$ be a loopless non-empty $GF(q)$-representable matroid. Then there is a contraction $N$ of $M$ that is $q$-affine and satisfies

$$|E(N)| > \frac{2-1}{q}|E(M)|.$$
The next result is obtained from the last corollary by taking \( q = 2 \) and using duality.

**Corollary 2.7.** Let \( M \) be a non-empty binary matroid without coloops. Then \( M \) has a restriction \( N \) having more than half the elements of \( M \) such that every cocircuit of \( N \) has even cardinality.

Finally, we note another consequence of Corollary 2.6 and duality, this one involving nowhere-zero \( k \)-flows. We state it only for \( k \) at most five because Seymour [11] has proved that every bridgeless graph has a nowhere zero 6-flow. That result is the best partial result towards Tutte’s 5-Flow Conjecture [12], that every bridgeless graph has a nowhere-zero 5-flow.

**Corollary 2.8.** Let \( G \) be a bridgeless graph and suppose \( k \in \{2, 3, 4, 5\} \). Then \( G \) has a subgraph \( H \) that has a nowhere-zero \( k \)-flow and has more than \( \frac{k-1}{k} |E(G)| \) edges.

A bridgeless graph has a nowhere-zero 2-flow if and only if its edge set is a disjoint union of cycles or, equivalently, every vertex has even degree. By the last result, every bridgeless graph \( G \) has a subgraph \( H \) that is a disjoint union of cycles such that \( |E(H)| > \frac{1}{2} |E(G)| \). This bound can be significantly strengthened. Indeed, Bermond, Jackson, and Jaeger [3, Lemma 3.2] have proved that one can always find such a subgraph \( H \) with \( |E(H)| \geq \frac{2}{3} |E(G)| \). The latter bound is sharp since, by Petersen’s Theorem [10], equality holds for every cubic bridgeless graph \( G \).

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**References**


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