ON BIPARTITE RESTRICTIONS OF BINARY MATROIDS

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Abstract. In a 1965 paper, Erdős remarked that a graph G has a bipartite subgraph that has at least half the number of edges of G. The purpose of this note is to prove a matroid analogue of Erdős's original observation. It follows from this matroid result that every loopless binary matroid has a restriction that uses more than half of its elements and has no odd circuits; and, for $2 \leq k \leq 5$, every bridgeless graph G has a subgraph that has a nowhere-zero k-flow and has more than $\frac{k-1}{k}|E(G)|$ edges.

1. Introduction

The matroid terminology used in this note will follow Oxley [9]. The results considered here relate to the critical problem of Crapo and Rota [5]. The reader is referred to the survey paper of Brylawski and Oxley [4] for the theoretical background to these results that is not included here. We shall require only a minimal amount of this theory. Let M be a matroid. Its simplification is denoted by $\mathrm{si}(M)$. When M is $GF(q)$-representable and loopless having rank r, we say that M is *affine over* $GF(q)$ or q-*affine* if $\mathrm{si}(M)$ is isomorphic to a restriction of the affine geometry $AG(r-1,q)$. Equivalently, M is affine over $GF(q)$ if, whenever there is a subset T of the projective geometry $PG(r-1,q)$ such that $\mathrm{si}(M)$ is isomorphic to $PG(r-1,q)|T$, there is a hyperplane H of $PG(r-1,q)$ such that H avoids T. It is well known that if G is a graph, then $M(G)$ is affine over $GF(q)$ if and only if G is q-colourable. In particular, G is bipartite if and only if $M(G)$ is affine over $GF(2)$. On the other hand, $M^*(G)$ is affine over $GF(q)$ if and only if G has a nowhere-zero q-flow. An arbitrary binary matroid is affine if and only if all of its circuits have even cardinality. We follow Welsh [13] in calling such a matroid *bipartite*.

2. The Theorem and Some Consequences

Erdős's observation [7] that every loopless graph has a bipartite subgraph that has at least half the number of edges of G was sharpened by Edwards [6], and Erdős, Gyárfás, and Kohayakawa [8] gave simpler proofs of Edwards's results. The following theorem generalizes Erdős's original result to matroids. For a matroid M and for k in $\{0,1,\dots,r(M)\}$, let $h_k(M)$ denote the number of flats of M of rank $r(M)-k$.

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Theorem 2.1. *Let k be a non-negative integer, P be a matroid of rank at least $k+1$, and T be a subset of $E(P)$ that contains no loops. Let $d = \max_{e \in T} h_k(P/e)$. Then P has a flat F of rank $r(P)-k$ such that*

$$\frac{|T-F|}{|T|} \geq 1 - \frac{d}{h_k(P)}.$$

Proof. Let $r(P) = r$. Construct the bipartite graph G with vertex classes V_1 and V_2 where V_1 is the set of rank-$(r-k)$ flats of P, and V_2 is T. An element e of T is adjacent to a rank-$(r-k)$ flat F of P if and only if $e \in F$.

If $e \in T$, then the number of rank-$(r-k)$ flats of P containing e is the same as the number of rank-$(r-1-k)$ flats of P/e. As $d = \max_{e \in T} h_k(P/e)$, we deduce that the vertex e of V_2 has degree at most d in G. Hence $|E(G)| \leq d|T|$. Since $|V_1| = h_k(P)$, it follows that V_1 contains a vertex of degree at most $\frac{d|T|}{h_k(P)}$. Thus P has a rank-$(r-k)$ flat F that avoids at least $|T| - \frac{d|T|}{h_k(P)}$ elements of T. Thus

$$\frac{|T-F|}{|T|} \geq 1 - \frac{d}{h_k(P)}.$$

□

Corollary 2.2. *Let M be a loopless non-empty $GF(q)$-representable matroid. Then M has a q-affine restriction N such that*

$$|E(N)| > \tfrac{q-1}{q}|E(M)|.$$

Proof. Let the size of a largest parallel class of M be t and $r = r(M)$. Replace each element of $PG(r-1,q)$ by t elements in parallel, letting the resulting matroid be P. Then M can be viewed as a restriction of P. Let $T = E(M)$. Applying the theorem with $k = 1$, we get that there is a hyperplane H of P such that

$$\frac{|T-H|}{|T|} \geq \frac{h_1(P) - \max_{e \in T} h_1(P/e)}{h_1(P)}.$$

Clearly $h_1(P) = h_1(PG(r-1,q)) = \frac{q^r-1}{q-1}$ while, for all elements e of P, by symmetry, $h_1(P/e) = h_1(PG(r-2,q)) = \frac{q^{r-1}-1}{q-1}$. Thus

$$\frac{h_1(P) - \max_{e \in T} h_1(P/e)}{h_1(P)} = \frac{q^r - q^{r-1}}{q^r - 1} = \left(\frac{q-1}{q}\right)\left(\frac{q^r}{q^r-1}\right) > \frac{q-1}{q}.$$

Since $\mathrm{si}(M|(T-H))$ is the ground set of a restriction of $AG(r-1,q)$, we obtain the required result. □

The bound in Corollary 2.2 can be restated as

$$|E(N)| \geq \tfrac{q-1}{q}|E(M)| + \tfrac{1}{q}.$$

This bound is sharp with equality being attained when $M \cong PG(r-1,q)$ since a largest affine restriction of this matroid is isomorphic to $AG(r-1,q)$.

Corollary 2.2 has some interesting consequences. Taking $q = 2$, we immediately get the following result which implies Erdős's result. Welsh [13] showed that a binary matroid is bipartite if and only if its ground set can be written as a disjoint union of cocircuits.

Corollary 2.3. *Let M be a loopless non-empty binary matroid. Then M has a bipartite restriction that has more than half the elements of M.*

Corollary 2.4. *Let G be a loopless non-empty graph and k be a positive integer. Then G has a k-colourable subgraph that has more than $\frac{k-1}{k}|E(G)|$ edges.*

Proof. Suppose that G has n vertices and that its largest parallel class has size t. Then we can view G as a subgraph of the graph K_n^t that is obtained from K_n by replacing each edge by t parallel edges. There is an obvious one-to-one correspondence between the flats of $M(K_n^t)$ and those of $M(K_n)$. Moreover, the flats of $M(K_n)$ of rank $n-k$ correspond to the partitions of $V(K_n)$ into exactly k classes where an edge of K_n is in the flat if and only if both ends lie in the same class. The number of such partitions is $S(n,k)$, the Stirling number of the second kind. Moreover, the complement of a flat of rank $n-k$ is a complete k-partite graph. As $n-k = r(M(K_n)) - (k-1)$, when we apply Theorem 2.1 substituting $M(K_n)$ for P and $k-1$ for k, we find that the right-hand side of the inequality is $\frac{S(n,k)-S(n-1,k)}{S(n,k)}$. But $S(n,k) = S(n,k-1) + kS(n-1,k)$. Thus

$$\frac{S(n,k)-S(n-1,k)}{S(n,k)} = \frac{S(n,k-1)+(k-1)S(n-1,k)}{S(n,k-1)+kS(n-1,k)} > \frac{k-1}{k}.$$

□

Since the last result is derived from such a general matroid result, it is not surprising that a stronger graph result is known. Andersen, Grant, and Linial [1] showed that, for all positive integers k, every loopless graph G has a k-colourable subgraph H with

$$|E(H)| \geq \tfrac{k-1}{k}|E(G)| + \alpha_k(|V(G)|-1)$$

where $\alpha_k = 1/k$ when $k \geq 3$, while $\alpha_2 = 1/4$.

Next we consider some variants on the results above that use the following special case of a result of Asano, Nishizeki, Saito, and Oxley [2].

Lemma 2.5. *Let M be a $GF(q)$-representable matroid and X be a subset of $E(M)$. Then X is minimal with the property that $M\backslash X$ is q-affine if and only if X is minimal with the property that M/X is q-affine.*

Applying the last result to the theorem, we immediately obtain the following.

Corollary 2.6. *Let M be a loopless non-empty $GF(q)$-representable matroid. Then there is a contraction N of M that is q-affine and satisfies*

$$|E(N)| > \tfrac{q-1}{q}|E(M)|.$$

The next result is obtained from the last corollary by taking $q = 2$ and using duality.

Corollary 2.7. *Let M be a non-empty binary matroid without coloops. Then M has a restriction N having more than half the elements of M such that every cocircuit of N has even cardinality.*

Finally, we note another consequence of Corollary 2.6 and duality, this one involving nowhere-zero k-flows. We state it only for k at most five because Seymour [11] has proved that every bridgeless graph has a nowhere zero 6-flow. That result is the best partial result towards Tutte's 5-Flow Conjecture [12], that every bridgeless graph has a nowhere-zero 5-flow.

Corollary 2.8. *Let G be a bridgeless graph and suppose $k \in \{2, 3, 4, 5\}$. Then G has a subgraph H that has a nowhere-zero k-flow and has more than $\frac{k-1}{k}|E(G)|$ edges.*

A bridgeless graph has a nowhere-zero 2-flow if and only if its edge set is a disjoint union of cycles or, equivalently, every vertex has even degree. By the last result, every bridgeless graph G has a subgraph H that is a disjoint union of cycles such that $|E(H)| > \frac{1}{2}|E(G)|$. This bound can be significantly strengthened. Indeed, Bermond, Jackson, and Jaeger [3, Lemma 3.2] have proved that one can always find such a subgraph H with $|E(H)| \geq \frac{2}{3}|E(G)|$. The latter bound is sharp since, by Petersen's Theorem [10], equality holds for every cubic bridgeless graph G.

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