Conference (Universität Ulm), pp. 257–59.
Strietz, H. (1977). Über Erzeugendenmengen endlicher Partitionenverbände, *Studia Scient. Math. Hung.* **12**, 1–17.
Tůma, J. (1980). A structure theorem for lattices of generalized partitions, in *Lattice Theory* (Szeged), Colloq. Math. Soc. János Bolyai **33**, pp. 737–57. North-Holland, Amsterdam.
Tůma, J. (1985). A simple geometric proof of a theorem on M_n, *Comment. Math. Carolin.* **26**, 233–9.
Werner, H. (1976). Which partition lattices are congruence lattices?, in *Lattice Theory* (Szeged, 1974), Colloq. Math. Soc. János Bolyai 14, pp. 433–53. North-Holland, Amsterdam.
White, N. (ed.) (1986). *Theory of Matroids*, Cambridge University Press.
Whitman, P. M. (1946). Lattices, equivalence relations, and subgroups, *Bull. Amer. Math. Soc.* **52**, 507–22.
Wille, R. (1967). Verbandstheoretische Kennzeichnung *n*-stufiger Geometrien, *Arch. Math.* **18**, 465–8.
Wille, R. (1976). A note on simple lattices, in *Lattice Theory* (Szeged, 1974), Colloq. Math. Soc. János Bolyai 14, pp. 455–62. North-Holland, Amsterdam.

6

The Tutte Polynomial and Its Applications

THOMAS BRYLAWSKI† and JAMES OXLEY‡

6.1. Introduction

The theory of numerical invariants for matroids is one of the many aspects of matroid theory having its origins within graph theory. Indeed, most of the fundamental ideas in matroid invariant theory were developed for graphs by Veblen (1912), Birkhoff (1912–13), Whitney (1933c), and Tutte (1947; 1954) when considering colorings and flows in graphs. The applications of matroid invariant theory now extend well beyond graphs, reaching into such fields as coding theory, percolation theory, electrical network theory, and statistical mechanics. In addition, many new graph-theoretic applications of the theory have been found. The purpose of this chapter is to review the many diverse applications of matroid invariant theory. In White (1987), Chapters 7 and 8 deal with several fundamental examples of matroid invariants, and we shall make frequent reference to these chapters here, particularly the former (Zaslavsky, 1987).

A matroid *isomorphism invariant* is a function f on the class of all matroids such that

$$f(M) = f(N) \quad \text{whenever } M \cong N. \tag{6.1}$$

The starting point for our chapter is the observation that several numbers associated with a matroid $M(E)$, such as its number of bases, its number of independent sets, and its number of spanning sets, are isomorphism invariants satisfying the following two basic recursions. Recall that if $T \subseteq E$, then $M(T)$ denotes the submatroid of M on T.

† Supported in part by ONR grant N00014-86-K-0449.

‡ Supported in part by NSF grant No. DMS-8500494 and by a grant from the Louisiana Education Quality Support Fund through the Board of Regents.

For every element e of M,

$$f(M) = f(M-e) + f(M/e) \text{ if } e \text{ is neither a loop nor an isthmus,} \quad (6.2)$$

$$f(M) = f(M(e))f(M-e) \text{ otherwise.} \quad (6.3)$$

If $\mathscr{K}$ is a class of matroids closed under isomorphism and the taking of minors, and f is a function on $\mathscr{K}$ satisfying (6.1), (6.2), and (6.3), then f is called a *Tutte–Grothendieck* or *T–G invariant.*
Theorem 7.2.4 of White (1987) notes that the characteristic polynomial $p(M; \lambda)$ of a matroid M satisfies the deletion–contraction formula

$$p(M; \lambda) = p(M-e; \lambda) - p(M/e; \lambda)$$

for all elements e that are neither loops nor isthmuses of M. This prompts consideration of matroid isomorphism invariants satisfying (6.3) and the following generalization of (6.2).

For some fixed non-zero numbers σ and τ,

$$f(M) = \sigma f(M-e) + \tau f(M/e) \quad (6.4)$$

provided e is neither a loop nor an isthmus.

Such invariants will be called *generalized T–G invariants.*

The fundamental result of this theory is that every T–G invariant f is an evaluation of a certain two-variable polynomial $t(x, y)$ where, for an isthmus I and a loop L,

$$f(I) = x \quad \text{and} \quad f(L) = y. \quad (6.5)$$

From this result, it is straightforward to deduce a characterization of all generalized T–G invariants. The precise statements of these results will be given in the next section, which contains a review of the basic results in matroid invariant theory. A more detailed development of this theory appears in Brylawski (1982). The primary focus of this chapter is on the applications of the theory. In particular, we concentrate mainly on those applications that are related to graphs and coding theory, preferring to treat a few applications in detail rather than to give a superficial treatment of all the applications. We hope that the extensive list of references will compensate in part for our failure to provide encyclopedic treatment of every application.

6.2. The Tutte Polynomial

In this section we state the fundamental results characterizing all T–G and generalized T–G invariants. We also investigate several closely related but coarser matroid invariants.

For an arbitrary matroid $M(E)$ having rank and nullity functions r and n respectively, the *rank generating polynomial* $S(M; x, y)$ of M is defined by

$$S(M; x, y) = \sum_{X \subseteq E} x^{r(E)-r(X)} y^{n(X)} = \sum_{X \subseteq E} x^{r(E)-r(X)} y^{|X|-r(X)}. \quad (6.6)$$

Thus

$$S(M; x, y) = \sum_i \sum_j a_{ij} x^i y^j \quad (6.7)$$

where a_{ij} is the number of submatroids of M of rank $r(M)-i$ and nullity j. Brylawski (1982) calls $S(M; x, y)$ the *corank-nullity polynomial* of M, the *corank* of a set X in M being $r(E)-r(X)$. Note that this usage differs from that in Welsh (1976) where 'corank' means rank in the dual matroid. Clearly, $S(M; x, y)$ is an isomorphism invariant for the class of all matroids. Moreover, one can easily check that if I is an isthmus and L is a loop, then

$$S(I; x, y) = x+1 \quad \text{and} \quad S(L; x, y) = y+1. \quad (6.8)$$

6.2.1. Lemma. *$S(M; x, y)$ is a T–G invariant for the class of all matroids.*

Proof. Let e be an element of the matroid $M(E)$. Clearly

$$S(M; x, y) = \sum_{\substack{X \subseteq E \\ e \notin X}} x^{r(E)-r(X)} y^{n(X)} + \sum_{\substack{X \subseteq E \\ e \in X}} x^{r(E)-r(X)} y^{n(X)}. \quad (6.9)$$

Consider the first term on the right-hand side. Clearly this equals $\sum_{X \subseteq E-e} x^{r(E)-r(X)} y^{n(X)}$. Moreover,

$$r(E) = \begin{cases} r(E-e)+1 & \text{if } e \text{ is an isthmus,} \\ r(E-e) & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{\substack{X \subseteq E \\ e \notin X}} x^{r(E)-r(X)} y^{n(X)} = \begin{cases} x \sum_{X \subseteq E-e} x^{r(E-e)-r(X)} y^{n(X)} & \text{if } e \text{ is an isthmus,} \\ \sum_{X \subseteq E-e} x^{r(E-e)-r(X)} y^{n(X)} & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{\substack{X \subseteq E \\ e \notin X}} x^{r(E)-r(X)} y^{n(X)} = \begin{cases} xS(M-e; x, y) & \text{if } e \text{ is an isthmus,} \\ S(M-e; x, y) & \text{otherwise.} \end{cases} \quad (6.10)$$

Now consider the second term on the right-hand side of (6.9). This equals $\sum_{Y \subseteq E-e} x^{r((E-e)\cup e)-r(Y\cup e)} y^{n(Y \cup e)}$. Let r' and n' denote the rank and nullity functions of M/e. Then, for all $Y \subseteq E-e$,

$$r'(Y) = \begin{cases} r(Y \cup e) & \text{if } e \text{ is a loop,} \\ r(Y \cup e) - 1 & \text{otherwise;} \end{cases}$$

and

$$n'(Y) = \begin{cases} n(Y \cup e) - 1 & \text{if } e \text{ is a loop,} \\ n(Y \cup e) & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{\substack{X\subseteq E\\ e\in X}} x^{r(E)-r(X)}y^{n(X)} = \begin{cases} y\sum_{Y\subseteq E-e} x^{r'(E-e)-r'(Y)}y^{n'(Y)} & \text{if } e \text{ is a loop,}\\ \sum_{Y\subseteq E-e} x^{r'(E-e)-r'(Y)}y^{n'(Y)} & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{\substack{X\subseteq E\\ e\in X}} x^{r(E)-r(X)}y^{n(X)} = \begin{cases} yS(M/e;\ x,\ y) & \text{if } e \text{ is a loop,}\\ S(M/e;\ x,\ y) & \text{otherwise.}\end{cases} \tag{6.11}$$

On substituting from (6.10) and (6.11) into (6.9), we obtain

$$S(M;\ x,\ y) = \begin{cases} S(M-e;\ x,\ y) + S(M/e;\ x,\ y) & \text{if } e \text{ is neither an isthmus nor a loop,}\\ (x+1)S(M-e;\ x,\ y) & \text{if } e \text{ is an isthmus,}\\ (y+1)S(M/e;\ x,\ y) & \text{if } e \text{ is a loop.}\end{cases}$$

But $S(I;\ x,\ y) = x+1$ and $S(L;\ x,\ y) = y+1$. Moreover, if e is a loop, then $M/e = M-e$. The lemma now follows easily. □

The next theorem, the main result of this section, extends the preceding lemma by showing that not only is $S(M;\ x,\ y)$ a T–G invariant but, more importantly, it is essentially the universal T–G invariant. The sets of isomorphism classes of matroids and non-empty matroids will be denoted by $\mathscr{M}$ and $\mathscr{M}'$, respectively. Note that (ii) and (iii) in this theorem are no more than restatements of the fundamental recursions (6.2) and (6.3).

6.2.2. Theorem. (*Brylawski, 1972b*) *There is a unique function t from* $\mathscr{M}$ *into the polynomial ring* $\mathbb{Z}[x,\ y]$ *having the following properties*:

(i) $t(I;\ x,\ y) = x$ *and* $t(L;\ x,\ y) = y$.
(ii) (*Deletion–contraction*) *If e is an element of the matroid M and e is neither a loop nor an isthmus, then*

$$t(M;\ x,\ y) = t(M-e;\ x,\ y) + t(M/e;\ x,\ y).$$

(iii) *If e is a loop or an isthmus of the matroid* $M(E)$, *then*

$$t(M;\ x,\ y) = t(M(e);\ x,\ y)t(M-e;\ x,\ y).$$

Furthermore, let R be a commutative ring and suppose that f is any function from $\mathscr{M}'$ *into R. If f satisfies* (6.2) *and* (6.3) *whenever* $|E| \geqslant 2$, *then, for all matroids M,*

$$f(M) = t(M;\ f(I),\ f(L)).$$

Proof. By Lemma 6.2.1, if $t(M;\ x,\ y) = S(M;\ x-1,\ y-1)$, then (i)–(iii) hold. Now the only non-empty matroids that cannot be decomposed using (ii) or (iii) are I and L, and, for these matroids, $t(M;\ x,\ y)$ is fixed by (i). Hence an easy induction argument establishes that t is unique. Finally, the last part of the theorem can be proved using another straightforward induction argument, the details of which are left to the reader. □

We shall call the function $t(M;\ x,\ y)$ the *Tutte polynomial* of M. Evidently $t(M;\ x,\ y)$ can be written as $\sum_i\sum_j b_{ij}x^iy^j$ where $b_{ij} \geqslant 0$ for all i and j. We shall usually abbreviate this double summation to $\sum b_{ij}x^iy^j$. It follows immediately from the last proof that

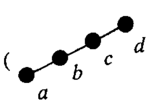
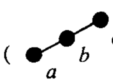
$$t(M;\ x,\ y) = S(M;\ x-1,\ y-1). \tag{6.12}$$

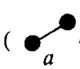
Hence

$$t(M;\ x,\ y) = \sum_{X\subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{n(X)}. \tag{6.13}$$

The Tutte polynomial can be calculated directly from this summation, or alternatively, it can be determined by using the recursions 6.2.2(ii) and (iii). The second of these techniques is illustrated as follows.

6.2.3. Example. Let M be $U_{2,4}$, the rank 2 uniform matroid on a set of four elements, that is, the 4-point line. In the calculation below, we shall abbreviate $t(N;\ x,\ y)$ throughout as (N), and represent each matroid affinely, where a loop e is written as **e**. By repeated application of 6.2.2(i), (ii), and (iii) we have:

() = () + (●abc)

= () + (●ab) + (●ab) + ({**a**, **b**})

= x(●a) + 2(●ab) + y(**a**)

= x^2 + 2(●a) + 2(**a**) + y^2

= $x^2 + 2x + 2y + y^2$.

Thus

$$t(U_{2,4};\ x,\ y) = x^2 + 2x + 2y + y^2.$$

Evidently $t(U_{2,4};\ x,\ y)$ is symmetric with respect to x and y. Since $U_{2,4}$ is a self-dual matroid, this observation is a special case of the next result, the proof of which follows easily from (6.13) by using the fact that if $X \subseteq E$, then its rank in $M^*(E)$ is $|X| - r(E) + r(E-X)$, where r is the rank function of $M(E)$.

6.2.4. Proposition. *For all matroids M,*

$$t(M^*;\ x,\ y) = t(M;\ y,\ x).$$

An easy induction argument beginning with 6.2.2(iii) establishes the next result.

6.2.5. Proposition. *For matroids $M_1(E_1)$ and $M_2(E_2)$ where E_1 and E_2 are disjoint,*

$$t(M_1 \oplus M_2; x, y) = t(M_1; x, y)t(M_2; x, y).$$

Evidently 6.2.2(iii) is a special case of 6.2.5. A consequence of this observation and the preceding result is that one can use the direct sum formula 6.2.5 in place of 6.2.2(iii) in defining the Tutte polynomial.

The following characterization of generalized T–G invariants is a straightforward extension of Theorem 6.2.2. Its proof is left to the reader.

6.2.6. Corollary. (*Oxley & Welsh, 1979b*) *Let σ and τ be non-zero elements of a field F. Then there is a unique function t' from $\mathscr{M}$ into the polynomial ring $F[x, y]$ having the following properties:*

(i) *$t'(I; x, y) = x$ and $t'(L; x, y) = y$.*
(ii) *If e is an element of the matroid M and e is neither a loop nor an isthmus, then*
$$t'(M; x, y) = \sigma t'(M-e; x, y) + \tau t'(M/e; x, y).$$
(iii) *If e is a loop or an isthmus of the matroid M, then*
$$t'(M; x, y) = t'(M(e); x, y)t'(M-e; x, y).$$

Furthermore, this function t' is given by
$$t'(M; x, y) = \sigma^{|E|-r(E)}\tau^{r(E)}t(M; x/\tau, y/\sigma).$$

We defer to the exercises consideration of a still more general invariant which admits a multiplicative constant on the right-hand side of 6.2.6(iii).

6.2.7. Example. Suppose that every element of a matroid $M(E)$ has, independently of all other elements, a probability $1-p$ of being deleted from M and assume that $0<p<1$. We call the resulting restriction minor $\omega(M)$ of M a *random submatroid* of M, corresponding in the obvious way to a random graph on m vertices when M is the cycle matroid of the complete graph K_m. If we let $\Pr(M)$ denote the probability that $\omega(M)$ has the same rank as M, then, evidently, $\Pr(I) = p$ and $\Pr(L) = 1$. Moreover,

$$\Pr(M) = \begin{cases} (1-p)\Pr(M-e) + p\Pr(M/e) & \text{if } e \text{ is neither a loop nor an isthmus,} \\ \Pr(M(e))\Pr(M-e) & \text{otherwise.} \end{cases}$$

It follows by Corollary 6.2.6 that
$$\Pr(M) = (1-p)^{|E|-r(E)}p^{r(E)}t(M; 1, 1/(1-p)).$$

Section 7.3 of White (1987) is devoted to the beta invariant for matroids. This invariant is a member of the class of matroid isomorphism invariants that satisfy the additive recursion (6.2) but not necessarily the multiplicative

recursion (6.3). Such matroid isomorphism invariants will be called (*T–G*) *group invariants*. The next result shows that the theory already developed can be used to characterize invariants of this type.

6.2.8. Proposition. *If A is an Abelian group, then there is a unique function g from $\mathscr{M}'$ into A such that*

(i) *$g(M) = g(M-e) + g(M/e)$ provided e is neither a loop nor an isthmus of the matroid M; and*
(ii) *$g(U_{i,i} \oplus U_{0,j}) = \alpha_{ij}$ for all i and j such that $i+j>0$.*

Moreover, if $t(M; x, y) = \sum_i \sum_j b_{ij}x^iy^j$, then $g(M) = \sum_i \sum_j b_{ij}\alpha_{ij}$.

Proof. Evidently, if we define $g(M) = \sum_i \sum_j b_{ij}\alpha_{ij}$ for all matroids M, then g satisfies (i). Moreover, as $t(U_{i,i} \oplus U_{0,j}; x, y) = x^iy^j$, g satisfies (ii). Thus there is at least one function satisfying the required conditions. To obtain that such a function is unique, we argue by induction noting that the only non-empty matroids that cannot be decomposed using (i) are those consisting entirely of loops and isthmuses. As the value of the function is fixed on such matroids by (ii), the required result follows. □

We shall now illustrate the use of the last result. If M is a matroid, then $i_{r-j}(M)$ will denote the number of independent sets of M having $r(M)-j$ elements.

6.2.9. Proposition. *For a non-negative integer k, the isomorphism invariant $i_{r-k}(M)$ is a group invariant and, if $t(M; x, y) = \sum_i \sum_j b_{ij}x^iy^i$, then*

$$i_{r-k}(M) = \sum_i \sum_j b_{ij}\binom{i}{k}.$$

Proof. Let e be an element of M and suppose that e is neither a loop nor an isthmus. Partition the set $\mathscr{I}_{r(M)-k}$ of independent sets of M having $r(M)-k$ elements into subsets $\mathscr{I}'_{r(M)-k}$ and $\mathscr{I}''_{r(M)-k}$ consisting, respectively, of those members of $\mathscr{I}_{r(M)-k}$ that contain e and those that do not. Evidently $\mathscr{I}''_{r(M)-k}$ is in one-to-one correspondence with the set of independent sets of $M-e$ having $r(M-e)-k$ elements. Moreover, $\mathscr{I}'_{r(M)-k}$ is in one-to-one correspondence with the set of independent sets of M/e having $r(M/e)-k$ elements. It follows that $i_{r-k}(M) = i_{r-k}(M-e) + i_{r-k}(M/e)$. Thus $i_{r-k}(M)$ is a group invariant. Now clearly $i_{r-k}(U_{i,i} \oplus U_{0,j}) = \binom{i}{i-k} = \binom{i}{k}$ and so, by Proposition 6.2.8, we conclude that for all matroids M,

$$i_{r-k}(M) = \sum_i \sum_j b_{ij}\binom{i}{k}.$$ □

By (6.6), $i_{r-k}(M)$ is the coefficient of x^k in $S(M; x, y)$. Indeed, the preceding proposition is just a special case of the result (see Exercise 6.4) that every coefficient of $S(M; x, y)$ is a group invariant.

Theorem 6.2.2 and Corollary 6.2.6 give characterizations of T–G invariants and group invariants that are neatly expressible in terms of the Tutte polynomial. In order to address the problem of precisely which matroid isomorphism invariants can be determined from the Tutte polynomial, a function f from $\mathcal{M}$ into a set Ω will be called a *Tutte invariant* if it has the property that $f(M)=f(N)$ whenever M and N have the same Tutte polynomial. Thus all generalized T–G and group invariants are examples of Tutte invariants. There are many other examples that are not of one of these types. For instance, since

$$t(M; x, y)=S(M; x-1, y-1)=\sum_{X\subseteq E}(x-1)^{r(E)-r(X)}(y-1)^{n(X)},$$

$$r(M) \text{ is the highest power of } x \text{ in } t(M; x, y), \tag{6.14}$$

while

$$n(M) \text{ is the highest power of } y \text{ in } t(M; x, y). \tag{6.15}$$

Thus rank and nullity are Tutte invariants. Clearly, since $|E|=r(M)+n(M)$, cardinality of the ground set is also a Tutte invariant. The next result characterizes all Tutte invariants, although it is essentially just a restatement of the definition. We leave the straightforward proof as an exercise.

6.2.10. Proposition. *Let Ω be a set and f be a function from $\mathcal{M}$ into Ω such that $f(M)=f(N)$ whenever $t(M; x, y)=t(N; x, y)$. Then $f(M)$ is a function of the coefficients b_{ij} of $t(M; x, y)$.*

Notice that this proposition does not assert that we can find an explicit formula for a particular Tutte invariant in terms of the Tutte polynomial coefficients. Nevertheless, most of the examples of Tutte invariants that we shall consider will have such an explicit formula. For example, by (6.14) and (6.15), we have

$$r(M)=\max\{i: b_{ij}>0 \text{ for some } j\}, \text{ and} \tag{6.16}$$

$$n(M)=\max\{j: b_{ij}>0 \text{ for some } i\}. \tag{6.17}$$

Having stated the characterizations of T–G, generalized T–G, group, and Tutte invariants, we now give a number of the more basic applications of these results. For a matroid M, we denote by $b(M)$, $i(M)$, and $s(M)$, the numbers of bases, independent sets, and spanning sets, respectively, of M. It was asserted in section 6.1 that these three numbers are T–G invariants. We now prove this.

6.2.11. Proposition.

(i) $b(M)=t(M; 1, 1)=S(M; 0, 0)$;
(ii) $i(M)=t(M; 2, 1)=S(M; 1, 0)$;
(iii) $s(M)=t(M; 1, 2)=S(M; 0, 1)$; *and*
(iv) $2^{|E|}=t(M; 2, 2)=S(M; 1, 1)$.

Proof. We begin by proving (i). Let e be an element of the matroid M and suppose that e is neither a loop nor an isthmus. Partition the set of bases of M into subsets $\mathcal{B}'$ and $\mathcal{B}''$ consisting, respectively, of those bases containing e and those bases not containing e. Now $\mathcal{B}'$ is equal to the set of bases of $M-e$, while the set of bases of M/e is $\{B-e: B\in\mathcal{B}''\}$. Thus $|\mathcal{B}'|=b(M-e)$ and $|\mathcal{B}''|=b(M/e)$. Therefore if e is neither a loop nor an isthmus of M, then $b(M)=b(M-e)+b(M/e)$. On the other hand, if e is a loop or an isthmus, then it is clear that $b(M)=b(M(e))b(M-e)$. Thus $b(M)$ satisfies (6.2) and (6.3). But clearly $b(I)=b(L)=1$. Therefore, by Theorem 6.2.2, $b(M)=t(M; 1, 1)$. Moreover, by (6.12), $t(M; 1, 1)=S(M; 0, 0)$. Thus (i) holds.

The proof technique used above can also be used to prove (ii) and we leave this to the reader. To prove (iii), note that $s(M)=i(M^*)$. Thus, by (ii), $s(M)=t(M^*; 2, 1)$. But by Proposition 6.2.4, $t(M^*; 2, 1)=t(M; 1, 2)$, and (iii) follows.

Finally, we have by (6.12) and (6.8) that $t(M; 2, 2)=S(M; 1, 1)=\sum_{X\subseteq E}1^{r(E)-r(X)}1^{n(X)}=2^{|E|}$. □

Since $b(M)=t(M; 1, 1)$ and $t(M; x, y)=\sum_i\sum_j b_{ij}x^iy^j$, it follows that $\sum_i\sum_j b_{ij}=b(M)$. In section 6.6A, we shall describe how, by ordering the ground set of M, we can interpret each of the coefficients b_{ij} as counting a particular set of bases of M.

The characteristic polynomial, the Möbius function, and the beta invariant were all considered in detail in Chapter 7 of White (1987). We now relate each of these functions to the Tutte polynomial. Recall that for a matroid M having a lattice of flats L, the characteristic polynomial $p(M; \lambda)$ of M is defined by the equation

$$p(M; \lambda)=\sum_{F\in L}\mu_M(\varnothing, F)\lambda^{r(M)-r(F)} \tag{6.18}$$

and satisfies the identity

$$p(M; \lambda)=\sum_{X\subseteq S}(-1)^{|X|}\lambda^{r(M)-r(X)}. \tag{6.19}$$

We noted in section 6.1 that $p(M; \lambda)$ is a generalized T–G invariant. Using Corollary 6.2.6, the characterization of such invariants, we obtain

$$p(M; \lambda)=(-1)^{r(M)}t(M; 1-\lambda, 0)=(-1)^{r(M)}S(M; -\lambda, -1). \tag{6.20}$$

By (6.18), $\mu(M)=\mu_M(\varnothing, E)=p(M;0)$. Hence the Möbius function $\mu(M)$ is a generalized T–G invariant and

$$\mu(M)=(-1)^{r(M)}t(M;1,0)=(-1)^{r(M)}S(M;0,-1). \tag{6.21}$$

6.2.12. Proposition. *$\beta(M)$ is a group invariant whose value is b_{10}. Moreover, if $|E|\geqslant 2$, then $b_{10}=b_{01}$.*

Proof. As noted earlier, $\beta(M)$ satisfies the additive recursion 6.2.5 and hence is a group invariant. Now, by Theorems 7.3.2(b) and 7.3.4 of White (1987),

$$\beta(U_{i,i}\oplus U_{0,j})=\begin{cases}1 & \text{if } i=1 \text{ and } j=0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence by Proposition 6.2.8, $\beta(M)=b_{10}$. The fact that $b_{10}=b_{01}$ for $|E|\geqslant 2$ follows by a straightforward induction argument using the additive recursion 6.2.8(i). □

The identity $b_{10}=b_{01}$ noted in the preceding proposition is one of a number of identities that hold for the coefficients b_{ij} of the Tutte polynomial. The next result (Brylawski, 1972b; 1982) completely characterizes all identities of the form $\sum_i \sum_j \alpha_{ij}b_{ij}=\gamma$ where γ and all the α_{ij} are constants.

6.2.13. Theorem. *The following identities form a basis for the affine linear relations that hold among the coefficients b_{ij} in the Tutte polynomial*

$$t(M;x,y)=\sum_{i\geqslant 0}\sum_{j\geqslant 0} b_{ij}x^iy^j$$

where M is a rank r geometry having m elements none of which is an isthmus.

(i) $b_{ij}=0$ *for all* $i>r$ *and all* $j\geqslant 0$;
(ii) $b_{r0}=1$; $b_{rj}=0$ *for all* $j>0$;
(iii) $b_{r-1,0}=m-r$; $b_{r-1,j}=0$ *for all* $j>0$;
(iv) $b_{ij}=0$ *for all i and j such that* $1\leqslant i\leqslant r-2$ *and* $j\geqslant m-r$;
(v) $b_{0,m-r}=1$; $b_{0j}=0$ *for all* $j>m-r$;
(vi) $\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}b_{st}=0$ for all k such that $0\leqslant k\leqslant m-3$.

Moreover, (vi) *holds for all matroids M(E) such that $|E|>k$. Hence,*

(vii) $b_{00}=0$ *if* $|E|\geqslant 1$;
(viii) $b_{10}=b_{01}$ *if* $|E|\geqslant 2$;
(ix) $b_{20}-b_{11}+b_{02}=b_{10}$ *if* $|E|\geqslant 3$; *and*
(x) $b_{30}-b_{21}+b_{12}-b_{03}=b_{11}-2b_{02}+b_{10}$ *if* $|E|\geqslant 4$.

Next we consider what sorts of matroid properties are Tutte invariants. First we note four elementary examples of such invariants. These assertions are easily proved by induction.

6.2.14. Example. If $b_{r(M),t}>0$, then t is the number of loops of M and $b_{r(M),t}=1$.

6.2.15. Example. If $b_{s,n(M)}>0$, then s is the number of isthmuses of M and $b_{s,n(M)}=1$.

6.2.16. Example. The number of rank 1 flats of M is $r(M)+b_{r(M)-1,t}$ where t is the number of loops of M.

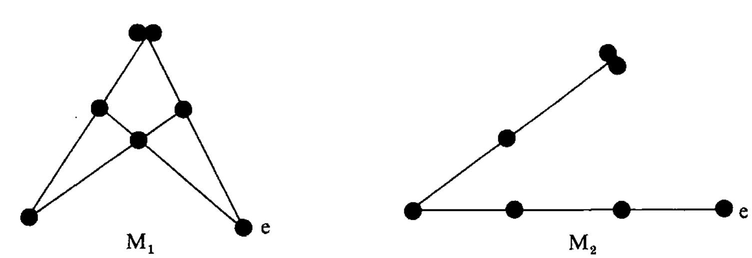
6.2.17. Example. Provided $r(M)$ and $n(M)$ are both positive, the number of connected components of M is $\min\{j: b_{0j}>0\}=\min\{i: b_{i0}>0\}$. If exactly one of $r(M)$ and $n(M)$ is zero, then M has $|E|$ components.

To see the sorts of properties that are not Tutte invariants, we look at two non-isomorphic matroids having the same Tutte polynomial.

6.2.18. Example. Let M_1 and M_2 be the matroids for which affine representations are shown in Figure 6.1. Their Tutte polynomials are equal because $M_1-e\cong M_2-e$ and $M_1/e\cong M_2/e$. We now list various properties that M_1 and M_2 do not share. Each such property is an example of a matroid isomorphism invariant that is not a Tutte invariant, and hence cannot be determined from the Tutte polynomial.

(i) $M_1\cong M_\Gamma$ for a planar graph Γ, so M_1 is both graphic and cographic and hence is unimodular and binary. On the other hand, M_2 has a 4-point line as a restriction so it is not even binary.
(ii) M_2 is transversal, but M_1 is not even a gammoid.
(iii) Although M_1 and M_2 have the same number of flats of rank 1 and the same number of 4- and 3-element lines, M_1 has two 2-element lines

Figure 6.1.



whereas M_2 has three. Hence M_1 and M_2 have different numbers of flats and different numbers of hyperplanes.

(iv) M_1 and M_2 each have one 2-element circuit and six 3-element circuits, but M_1 has five 4-element circuits and M_2 has six such circuits. Thus M_1 and M_2 have different numbers of circuits.

By Proposition 6.2.4, M_1^* and M_2^* have the same Tutte polynomial. Moreover, unlike M_1 and M_2, each of M_1^* and M_2^* is a geometry. We defer to the exercises (Exercise 6.9) consideration of an elementary counting argument which has as a consequence that for any number N there are at least N geometries all of which have the same Tutte polynomial.

Returning to the basic Tutte recursion (6.2), namely $f(M) = f(M-e) + f(M/e)$, we note that there are several slightly different techniques which are used to show that, when $f(M)$ enumerates the family $\mathscr{F}(M)$,

$$|\mathscr{F}(M)| = |\mathscr{F}(M-e)| + |\mathscr{F}(M/e)|.$$

(1) There is a bijection b: $\mathscr{F}(M) \to \mathscr{F}(M-e) \dot\cup \mathscr{F}(M/e)$. Thus the members of $\mathscr{F}(M)$ are partitioned into two classes corresponding to the analogous families in the deletion and the contraction. This technique was used in the proof of 6.2.11(i).

(2) There is an injection i: $\mathscr{F}(M/e) \to \mathscr{F}(M-e)$ such that $|\mathscr{F}(M)| = \sum_{x \in \mathscr{F}(M-e)} m(x)$, where $m(x)$ is 2 when x is in the image of i and is 1 otherwise. This idea is used, for example, to prove that the number of acyclic orientations of a graph is a T–G invariant (6.3.17), where, in this case, $m(x)$ counts the number of ways to orient the edge e while maintaining the property of being acyclic.

(3) There are two surjections π_1: $\mathscr{F}(M) \to \mathscr{F}(M/e)$ and π_2: $\mathscr{F}(M-e) \to \mathscr{F}(M/e)$, such that, for all x in $\mathscr{F}(M/e)$, $|\pi_1^{-1}(x)| = |\pi_2^{-1}(x)| + 1$. Here we think of partitioning both $\mathscr{F}(M)$ and $\mathscr{F}(M-e)$ into $f(M/e)$ blocks such that corresponding blocks have one more member in M than in $M-e$. An example occurs in the calculation of the number of different score vectors that arise from the orientations of a graph (see Proposition 6.3.19).

Many T–G invariants are evaluations of the Tutte polynomial when y, or dually x, is 0, 1, or 2. We now list some salient features of such invariants f.

(1) $y=0$ if and only if $f(M)=0$ whenever M has a loop, or, equivalently, whenever parallel elements can be ignored. In this case, we obtain the recursion $f(G) = f(G-e) + f(\overline{G/e})$ for any geometry G and any non-isthmus e. Here $\overline{G/e}$ denotes the simplification of the matroid G/e.

(2) $y=1$ if and only if loops can be ignored so that $f(M) = f(\tilde{M})$, where $\tilde{M}$ is obtained from M by deleting its loops. Here, if A is a set of parallel elements of M that is not a component, then

$$f(M) = f(M-A) + |A| f(M/A).$$

(3) $y=2$ if and only if the recursion $f(M) = f(M-e) + f(M/e)$ holds not only when e is a non-loop or a non-isthmus, but also if e is a loop.

In some of the applications of T-G techniques, it is convenient to work with a four-variable version of the Tutte polynomial. This is defined on the class $\mathscr{M}_P$ of *pointed matroids*, that is, matroids M_d having a distinguished point d. It is not difficult to modify the proof of Theorem 6.2.2 to establish the next result and we leave the details to the reader. Evidently, if e is an element of M_d other than d, then $M_d - e$ and M_d/e are members of $\mathscr{M}_P$, the distinguished point of each being d.

6.2.19. Proposition. *There is a unique function t_P from $\mathscr{M}_P$ into the polynomial ring $\mathbb{Z}[x', x, y', y]$ having the following properties:*

(i) $t_P(M_d(d)) = x'$ *if $M_d(d)$ is an isthmus, and*

$t_P(M_d(d)) = y'$ *if $M_d(d)$ is a loop.*

(ii) *If e is an element of a member M_d of $\mathscr{M}_P$* and $e \neq d$, *then*

$t_P(M_d) = t_P(M_d - e) + t_P(M_d/e)$.

(iii) *If e is a loop or an isthmus of a member M_d of $\mathscr{M}_P$ and $e \neq d$, then*

$t_P(M_d) = t_P(M_d - e)t(M_d(e))$. *In particular,* $t_P(M_d(e)) = t(M_d(e))$.

The polynomial $t_P(M_d; x', x, y', y)$ is called the *pointed Tutte polynomial.* The following proposition, which summarizes some of the basic properties of this polynomial, is not difficult to prove and is left to the reader as an exercise.

6.2.20. Proposition. *Suppose that $M_d \in \mathscr{M}_P$. Then*

(i) *for some f and g in $\mathbb{Z}[x, y]$,*

$$t_P(M_d; x', x, y', y) = x'f(x, y) + y'g(x, y).$$

Moreover, for this f and g,

(ii) $t(M_d; x, y) = xf(x, y) + yg(x, y)$;

(iii) $t_P(M_d^*; x', x, y', y) = t_P(M_d; y', y, x', x) = x'g(y, x) + y'f(y, x)$;

(iv) *if d is neither a loop nor an isthmus of M_d, then*

$$t(M_d - d; x, y) = (x-1)f(x, y) + g(x, y) \text{ and}$$

$$t(M_d/d; x, y) = f(x, y) + (y-1)g(x, y); \text{ and}$$

(v) *if d is a loop or an isthmus of M_d, then*

$$t(M_d - d;\, x, y) = t(M_d/d;\, x, y) = \begin{cases} f(x, y) & \text{if } d \text{ is an isthmus,} \\ g(x, y) & \text{if } d \text{ is a loop.} \end{cases}$$

We have seen that the rank generating polynomial is fundamental in the class of T–G invariants. Another related polynomial that arises in applications of T–G techniques is the *cardinality-corank polynomial* $S_{KC}(M; x, y)$. This is defined by

$$S_{KC}(M; x, y) = \sum_{X \subseteq E} x^{|X|} y^{r(E) - r(X)}. \tag{6.22}$$

We leave the reader to prove that this polynomial is a generalized T–G invariant, as stated in the following.

6.2.21. Proposition. $S_{KC}(M; x, y) = x^r t\left(M; \dfrac{x+y}{x}, x+1\right)$.

6.3. T–G Invariants in Graphs

In this section we shall review the occurrence of T–G invariants in graphs. Although the most important such invariants occur in the context of colorings and flows, a number of others arise, for example, in connection with acyclic and totally cyclic orientations, score vectors, and network reliability. Here, most of our attention will be devoted to colorings and flows. We begin with the former.

6.3.A. Colorings

Iet Γ be a graph and λ be a positive integer. A *proper vertex coloring of* Γ *with* λ *colors* or a *proper* λ*-coloring* of Γ is a function f from $V(\Gamma)$ into $\{1, 2, \ldots, \lambda\}$ such that if $uv \in E(\Gamma)$, then $f(u) \neq f(v)$. The number of such colorings will be denoted $\chi_\Gamma(\lambda)$. It was noted in Chapter 7 of White (1987) that $\chi_\Gamma(\lambda)$ is a polynomial in λ. This polynomial is called the *chromatic polynomial* of Γ. The next result relates $\chi_\Gamma(\lambda)$ to the Tutte polynomial of M_Γ.

6.3.1. Proposition. *For a graph* Γ *having* $k(\Gamma)$ *connected components,*

$$\chi_\Gamma(\lambda) = \lambda^{k(\Gamma)} p(M_\Gamma; \lambda) = \lambda^{k(\Gamma)} (-1)^{|V(\Gamma)| - k(\Gamma)} t(M_\Gamma; 1 - \lambda, 0).$$

Proof. Let $f(\Gamma; \lambda) = \lambda^{-k(\Gamma)} \chi_\Gamma(\lambda)$. Ostensibly f depends on the particular graph Γ. However, we shall show that in fact f depends only on M_Γ and that if we let $f(M_\Gamma; \lambda) = f(\Gamma; \lambda)$, then this matroid function is a well defined generalized T–G invariant for which $\sigma = 1$ and $\tau = -1$. If M_Γ is I or L, then Γ is formed from a single isthmus or a single loop by adjoining isolated vertices. It follows easily that $f(I; \lambda) = \lambda - 1$ and $f(L; \lambda) = 0$ so that f is well defined if $|E(\Gamma)| = 1$. Assume that f is a well defined matroid function when $|E(\Gamma)| < n$ and let $|E(\Gamma)| = n$.

Now suppose that e is an edge of Γ having endpoints u and v. Assume that e is neither a loop nor an isthmus of Γ. Then we can partition the set of proper λ-colorings of $\Gamma - e$ into those in which u and v are colored alike and those in which they are colored differently. But the first subset is one-to-one correspondence with the set of proper λ-colorings of Γ/e, and the second subset is in one-to-one correspondence with the set of proper λ-colorings of Γ. Evidently Γ and Γ/e have the same number of components. Moreover, as e is not an isthmus, Γ and $\Gamma - e$ have the same number of components. Thus $f(\Gamma - e; \lambda) = f(\Gamma; \lambda) + f(\Gamma/e; \lambda)$, and so, by the induction assumption,

$$f(\Gamma; \lambda) = f(M_{\Gamma - e}; \lambda) - f(M_{\Gamma/e}; \lambda).$$

If e is a loop of Γ, then $f(\Gamma; \lambda) = f(L; \lambda) = 0$, and so

$$f(\Gamma; \lambda) = f(L; \lambda) f(M_{\Gamma - e}; \lambda).$$

Finally, if e is an isthmus of Γ, then the number of ways to properly λ-color Γ equals the number of ways to properly λ-color $\Gamma - e$ so that u and v are colored differently. But in a proper λ-coloring of $\Gamma - e$, once a color is assigned to u, there are λ possible colors that can be assigned to v. Of these, $\lambda - 1$ are different from the color assigned to u. Thus the number of ways to properly λ-color $\Gamma - e$ so that u and v are colored differently is $\dfrac{\lambda - 1}{\lambda} \chi_{\Gamma - e}(\lambda)$. Hence $\chi_\Gamma(\lambda) = \dfrac{\lambda - 1}{\lambda} \chi_{\Gamma - e}(\lambda)$. Since $f(I; \lambda) = \lambda - 1$ and $k(\Gamma - e) = k(\Gamma) + 1$, we conclude that $f(\Gamma; \lambda) = f(I; \lambda) f(\Gamma - e; \lambda)$, and so, by the induction assumption,

$$f(\Gamma; \lambda) = f(I; \lambda) f(M_{\Gamma - e}; \lambda).$$

On comparing the equations for $f(\Gamma; \lambda)$ when e is a loop, an isthmus, and neither a loop nor an isthmus, we conclude by induction that f is well defined as a matroid function. Moreover, the same equations imply that f is a generalized T–G invariant for which $\sigma = 1$ and $\tau = -1$. Since $f(I; \lambda) = \lambda - 1$ and $f(L; \lambda) = 0$, it follows by Corollary 6.2.6 that

$$f(M_\Gamma; \lambda) = (-1)^{|V(\Gamma)| - k(\Gamma)} t(M_\Gamma; 1 - \lambda, 0).$$

Thus

$$\chi_\Gamma(\lambda) = \lambda^{k(\Gamma)} (-1)^{|V(\Gamma)| - k(\Gamma)} t(M_\Gamma; 1 - \lambda, 0).$$

Since $p(M; \lambda) = (-1)^{r(M)} t(M; 1 - \lambda, 0)$ for all matroids M, the rest of the proposition follows easily. □

The chromatic polynomial for graphs was introduced by Birkhoff (1912–13) as a tool for attacking the persistent Four Color Map problem. This problem remained unsolved for another sixty years after Birkhoff's paper, and the eventual resolution of the problem did not use chromatic polynomials. Appel & Haken (1976) were able to reduce the Four Color problem to one that involved checking a very large, but finite, number of cases. Using a computer to do the case checking, they were then able to prove the following result, which is known as the Four Color theorem.

6.3.2. Theorem. *Let Γ be a loopless planar graph. Then Γ has a proper 4-coloring.*

Accounts of the history of the Four Color problem and of the methods used to prove the last theorem can be found in Biggs, Lloyd & Wilson (1976) and in Woodall & Wilson (1978). In terms of chromatic polynomials, the assertion of the theorem is that, for a loopless planar graph Γ, $\chi_\Gamma(4) > 0$.

In general, if Γ is a planar graph, we can construct a geometric dual Γ^* of it (see Chapter 6 of White, 1986). Consider now the number of proper λ-colorings of Γ^*. In section 6.3.B, we shall show how to interpret this number in the graph Γ.

6.3.B. Flows

Let θ be some fixed orientation of an arbitrary graph Γ and let Γ_θ denote the associated directed graph. Let H be an additively written Abelian group with identity element 0 and order $|H|$. An *H-flow* on Γ_θ is an assignment of weights from H to the directed edges of Γ_θ so that, at each vertex v of Γ_θ, the sum in H of the weights of the edges directed into v equals the sum of the weights of the edges directed out from v. If none of the edges receives zero weight, the H-flow is called *nowhere zero*. We denote by $\chi^*_{\Gamma_\theta}(H)$ the number of nowhere-zero H-flows on Γ_θ.

6.3.3. Lemma. *$\chi^*_{\Gamma_\theta}(H)$ does not depend on the orientation θ of Γ.*

Proof. Suppose that the orientation θ' is obtained from θ by reversing the direction of a single edge e. Then, by replacing the weight of e by its additive inverse in H, we determine a bijection between the sets of nowhere-zero H-flows on Γ_θ and $\Gamma_{\theta'}$. Thus $\chi^*_{\Gamma_\theta}(H) = \chi^*_{\Gamma_{\theta'}}(H)$ and an obvious extension of this yields the required result. □

In view of the preceding lemma, we shall abbreviate $\chi^*_{\Gamma_\theta}(H)$ as simply $\chi^*_\Gamma(H)$. The next result should be compared with Proposition 6.3.1.

6.3.4. Proposition. *For a graph Γ having $k(\Gamma)$ connected components,*

$$\chi^*_\Gamma(H) = p(M^*_\Gamma; |H|) = (-1)^{|E(\Gamma)|-|V(\Gamma)|+k(\Gamma)} t(M_\Gamma; 0, 1-|H|).$$

Proof. Let $g(M_\Gamma; H) = \chi^*_\Gamma(H)$. We shall prove that g is well defined and that it is a generalized T–G invariant for which $\sigma = -1$ and $\tau = 1$. Evidently $g(I; H) = 0$ and $g(L; H) = |H| - 1$. So g is well defined if $|E(\Gamma)| = 1$. Assume it is well defined for $|E(\Gamma)| < n$ and let $|E(\Gamma)| = n$.

Now suppose that e is an edge of Γ having endpoints u and v. Assume that e is not a loop or an isthmus of Γ. Partition the set W of nowhere-zero H-flows on Γ/e into subsets W' and W'' where W' consists of those members of W that are also nowhere-zero H-flows on $\Gamma - e$. Thus $|W'| = \chi^*_{\Gamma-e}(H)$. Moreover, W'' is in one-to-one correspondence with the set of nowhere-zero H-flows on Γ. To see this, we note that a member of W'' fails as a nowhere-zero H-flow on $\Gamma - e$ precisely because, at each of the vertices u and v in $\Gamma - e$, the sum of the weights of the edges directed into the vertex does not equal the sum of the weights of the edges directed out. We may assume, without loss of generality, that the resultant flow into u is n. Then the resultant flow out of v is also n and, by directing e from u to v and assigning it the weight n, we obtain a nowhere-zero H-flow on Γ. Since every nowhere-zero H-flow on Γ is uniquely obtainable in this way, it follows that $|W''| = \chi^*_\Gamma(H)$ and so $\chi^*_\Gamma(H) = \chi^*_{\Gamma/e}(H) - \chi^*_{\Gamma-e}(H)$. Thus, by the induction assumption,

$$\chi^*_\Gamma(H) = g(M_{\Gamma/e}; H) - g(M_{\Gamma-e}; H).$$

If e is a loop in Γ, then, corresponding to every nowhere-zero H-flow on $\Gamma - e$, we may take the weight of e to be any one of the non-zero elements of H. This gives a nowhere-zero H-flow on Γ, and every such flow on Γ arises in this way. Thus if e is a loop in M_Γ, then $\chi^*_\Gamma(H) = g(L; H)\chi^*_{\Gamma-e}(H)$ and so, by the induction assumption,

$$\chi^*_\Gamma(H) = g(L; H)g(M_{\Gamma-e}; H).$$

If e is an isthmus of Γ, then $\chi^*_\Gamma(H) = 0 = g(I; H)$ and so

$$\chi^*_\Gamma(H) = g(I; H)\chi^*_{\Gamma-e}(H).$$

On comparing the equations for $\chi^*_\Gamma(H)$ when e is a loop, an isthmus, and neither a loop nor an isthmus, we conclude that g is well defined. The same equations imply that g is a generalized T–G invariant for which $\sigma = -1$ and $\tau = 1$. The proposition now follows immediately from Corollary 6.2.6 and 6.20. □

A consequence of the last result is that $\chi^*_\Gamma(H)$ does not depend on the particular Abelian group H but only on its order. Thus, if $|H| = n$, we shall denote $\chi^*_\Gamma(H)$ by $\chi^*_\Gamma(n)$. In particular, $\chi^*_\Gamma(\mathbb{Z}_n) = \chi^*_\Gamma(n)$. The last result implies that $\chi^*_\Gamma(\lambda)$ is a polynomial in λ. We call this the *flow polynomial* of Γ.

Let Γ^* be a geometric dual of the planar graph Γ. On combining Propositions 6.3.1 and 6.3.4, we deduce the following result.

6.3.5. Corollary. *The number of proper λ-colorings of Γ equals the product of $\lambda^{k(\Gamma)}$ and the number of nowhere-zero $\mathbb{Z}_\lambda$-flows on Γ^*.*

The next result comes from combining this corollary with the Four Color theorem (6.3.2).

6.3.6. Corollary. *Let Γ be a planar graph having no isthmuses. Then Γ has a nowhere-zero $\mathbb{Z}_4$-flow.*

An immediate consequence of Proposition 6.3.4 is that if H is an Abelian group of order k, then Γ has a nowhere-zero H-flow if and only if Γ has a nowhere-zero $\mathbb{Z}_k$-flow. The next result shows that the existence of the latter corresponds to the existence of a nowhere-zero $\mathbb{Z}$-flow for which the weights lie in $[-(k-1), k-1]$. This result holds in the more general context of flows in unimodular matroids (Tutte, 1965), but we prove it here only for graphs.

6.3.7. Proposition. *Let k be an integer exceeding one, Γ be a graph, and θ be a fixed orientation of Γ. Then the following statements are equivalent.*
(i) *Γ_θ has a nowhere-zero $\mathbb{Z}_k$-flow.*
(ii) *Γ_θ has a nowhere-zero $\mathbb{Z}$-flow with weights in $[-(k-1), k-1]$.*

Proof. In this proof, the symbols $0, 1, 2, \ldots, k-1$ will be used to denote integers *and* to denote the members of $\mathbb{Z}_k$, and it will be convenient to switch between these. Suppose Γ_θ has a nowhere-zero $\mathbb{Z}$-flow with weights in $[-(k-1), k-1]$. Then, by regarding these weights as elements of $\mathbb{Z}_k$, we obtain a nowhere-zero $\mathbb{Z}_k$-flow in Γ_θ. Thus (ii) implies (i).

Now suppose that ψ is a nowhere-zero $\mathbb{Z}_k$-flow on Γ_θ. Define a function ϕ on $E(\Gamma_\theta)$ by, for each e in $E(\Gamma_\theta)$, taking $\phi(e)$ to be an integer in $[-(k-1), k-1]$ such that, regarded as a member of $\mathbb{Z}_k$, $\phi(e)$ equals $\psi(e)$. This does not uniquely determine ϕ since there are two choices for $\phi(e)$ for each edge e. We call e positive if $\phi(e)$ is positive, and negative otherwise. Evidently $\phi(e)$ is non-zero.

Now, for each vertex v of Γ, define the weight $w(v)$ of v by

$$w(v) = \sum_{e \in N^+(v)} \phi(e) - \sum_{e \in N^-(v)} \phi(e)$$

where $N^+(v)$ is the set of edges directed into v, and $N^-(v)$ is the set of edges directed out from v. We call v positive, negative, or zero according to whether its weight is positive, negative, or zero. Clearly $w(v) \equiv 0 \pmod k$ for all vertices v. We choose the function ϕ so that $\sum_{v \in V(\Gamma)} |w(v)|$ is a minimum.

A path P in Γ from a vertex u to a vertex v will be called positive provided that the positive edges of P are precisely those whose orientation in Γ_θ agrees with the direction of traversal of P. We show next that there is no positive path in Γ_θ with a negative initial vertex and a positive final vertex. If there is such a path, P', then let ϕ' be defined as follows for all edges e of Γ:

$$\phi'(e) = \begin{cases} \phi(e) - k & \text{if } e \text{ is a positive edge of } P', \\ \phi(e) + k & \text{if } e \text{ is a negative edge of } P', \\ \phi(e) & \text{otherwise.} \end{cases}$$

Evidently, regarded as members of $\mathbb{Z}_k$, $\phi'(e)$ and $\phi(e)$ are equal for all edges e. Moreover, using the fact that $w(v) \equiv 0 \pmod k$ for all vertices v, it is easy to show that if w' is the weight function associated with ϕ', then

$$\sum_{v \in V} |w'(v)| < \sum_{v \in V} |w(v)|.$$

This contradiction to the choice of ϕ establishes that the path P' does not exist.

Let V_P and V_N be the sets of positive and negative vertices, respectively, of $V(\Gamma)$. Let V_1 be V_N together with all the vertices u for which there is a positive path from a member of V_N to u. Then $V - V_1 \supseteq V_P$ and

$$\sum_{v \in V - V_1} w(v) = \sum_{v \in V_P} w(v). \tag{6.23}$$

But

$$\sum_{v \in V - V_1} w(v) = \sum_{v \in V - V_1} \left(\sum_{e \in N^+(v)} \phi(e) - \sum_{e \in N^-(v)} \phi(e) \right). \tag{6.24}$$

If both endpoints of e are in $V - V_1$, the net contribution of e to the right-hand side of (6.24) is zero. Moreover, by the definition of V_1, no positive edge has its tail in V_1 and its head in $V - V_1$, and no negative edge has its tail in $V - V_1$ and its head in V_1. Thus the right-hand side of (6.24) is non-positive. Hence, by (6.23), $\sum_{v \in V_P} w(v) \leqslant 0$. But, as every vertex in V_P is positive, $\sum_{v \in V_P} w(v) \geqslant 0$ with equality only if $V_P = \varnothing$. We conclude that Γ has no positive vertices. A similar argument shows that Γ has no negative vertices. Thus ϕ is a $\mathbb{Z}$-flow on Γ having all its weights in $[-(k-1), k-1]$ and Proposition 6.3.7 is proved. □

If Γ is a graph, a $\mathbb{Z}$-flow with weights in $[-(k-1), k-1]$ is called a *k-flow*.

6.3.C. Tutte's 5-flow Conjecture

In sections 6.3.A and 6.3.B, we noted that certain fundamental T–G invariants occur in the study of both colorings and flows in graphs. In each of these areas, the focal point of much of the research has been one very difficult problem. For colorings this problem was, for many years, the Four Color problem. Now that this has been solved, attention has turned to the more general conjecture of Hadwiger, which we shall discuss later in the chapter (section 6.4). For flows, the outstanding problem has been to prove or disprove the following conjecture of Tutte (1954).

6.3.8. Conjecture. *Every graph without isthmuses has a nowhere-zero 5-flow.*

Appearing with this conjecture was the following weakening of it.

6.3.9. Conjecture. *There is some integer k such that every graph without isthmuses has a nowhere-zero k-flow.*

Both conjectures remained unresolved for over twenty years until Jaeger (1976b) proved that every graph without isthmuses has a nowhere-zero 8-flow. This result was sharpened by Seymour (1981) and, in this section, we shall prove his result, which is still the best partial result toward Conjecture 6.3.8.

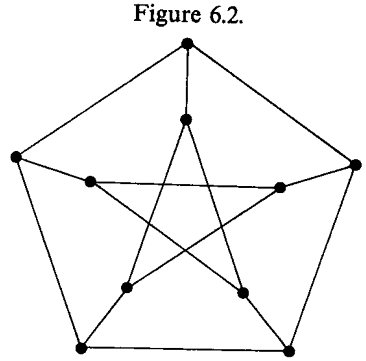
6.3.10. Theorem. *Every graph without isthmuses has a nowhere-zero 6-flow.*

As an example of a graph having no nowhere-zero 4-flow, Tutte (1954) cited the Petersen graph P_{10} (see Figure 6.2). The reader can check this by showing that the flow polynomial of P_{10} is

$$\chi^*_{P_{10}}(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^2-5\lambda+10).$$

In 1966, Tutte (1966a) advanced a variant of 6.3.8, namely that P_{10} is the

Figure 6.2.



unique minimal obstruction to the existence of a nowhere-zero 4-flow. More precisely, he proposed the following.

6.3.11. Conjecture. *If a graph without isthmuses has no nowhere-zero 4-flow, then it has a subgraph contractible to P_{10}.*

Let Γ be a graph without isthmuses. To prove Theorem 6.3.10, we shall show that Γ has a nowhere-zero $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-flow. We begin with the following simple observation, the proof of which is left to the reader.

6.3.12. Lemma. *The following statements are equivalent for a graph Γ.*

(i) *Γ has a nowhere-zero 2-flow.*
(ii) *Every vertex of Γ has even degree.*
(iii) *$E(\Gamma)$ is a disjoint union of circuits.*

We show next that if a minimal counterexample to Theorem 6.3.10 exists, then it is simple and 3-connected.

6.3.13. Lemma. *Suppose that k is an integer exceeding 2. Let Γ be a graph that, among all graphs Δ with no isthmuses and no nowhere-zero k-flows, has $|V(\Delta)| + |E(\Delta)|$ minimum. Then Γ is simple and 3-connected.*

Proof. Evidently Γ is loopless and 2-connected. Moreover, if $\{e_1, e_2\}$ is a circuit of Γ, then $E(\Gamma) \neq \{e_1, e_2\}$, so $\Gamma - e_1$ has no isthmuses. The choice of Γ now implies that $\Gamma - e_1$ has a nowhere-zero k-flow, and, since $k \geqslant 3$, this k-flow can easily be modified to give a nowhere-zero k-flow on Γ. We conclude that Γ is simple.

Now suppose that Γ is not 3-connected. Then, as Γ has at least four vertices, it follows by Lemma 6.3.3 of White (1986) that Γ has a representation as a generalized circuit, each part of which is a block. This means that, for some $m \geqslant 2$, Γ has subgraphs $\Gamma_1, \Gamma_2, .., \Gamma_m$ so that the following conditions hold:

(1) Each Γ_i is connected, loopless, has no cut vertices, and has a non-empty edge set; and, if $m = 2$, both Γ_1 and Γ_2 have at least three vertices.
(2) The edge sets of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ partition the edge set of Γ, and each Γ_i shares exactly two vertices, its *contact vertices*, with $\bigcup_{j \neq i} \Gamma_j$.
(3) If each Γ_i is replaced by an edge joining its contact vertices, the resulting graph is a circuit.

If none of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ consists of a single edge, then each has no isthmuses and therefore, by the choice of Γ, each has a nowhere-zero k-flow. It follows that Γ has such a flow; a contradiction. Thus, we may suppose that Γ_1 consists of a single edge. Hence $m \geq 3$. Therefore Γ is the series connection of two graphs Δ_1 and Δ_2, neither of which has an isthmus. Thus $M(\Gamma) = S(M(\Delta_1), M(\Delta_2)) = [P(M^*(\Delta_1), M^*(\Delta_2))]^*$ (White, 1986, p. 180), so $M^*(\Gamma) = P(M^*(\Delta_1), M^*(\Delta_2))$. Now, by Theorem 7.2.9 of White (1987),

$$p(M^*(\Gamma); \lambda) = p(M^*(\Delta_1); \lambda)p(M^*(\Delta_2); \lambda),$$

that is,

$$\chi_\Gamma^*(\lambda) = \chi_{\Delta_1}^*(\lambda)\chi_{\Delta_2}^*(\lambda).$$

Since each of Δ_1 and Δ_2 has a nowhere-zero k-flow, so does Γ; a contradiction. □

The proof of Theorem 6.3.10 will use the following function defined on the set of subsets of $E(\Gamma)$. For $X \subseteq E(\Gamma)$, we take the *S-closure* $\langle X \rangle$ of X to be the smallest subset Y of $E(\Gamma)$ with the following properties:

(1) $X \subseteq Y$; and
(2) there is no circuit C of Γ such that $0 < |C - Y| \leq 2$.

Evidently if Y_1 and Y_2 both satisfy (1) and (2), then so does $Y_1 \cap Y_2$. Hence $\langle X \rangle$ is well defined. It is not difficult to check that $\langle X \rangle$ can be obtained constructively as follows. If C is a circuit of Γ with $0 < |C - X| \leq 2$, then let $X' = X \cup C$. Repeat this procedure with X' replacing X and continue in this manner until no further elements can be added. The resulting set is $\langle X \rangle$.

We leave it to the reader (Exercise 6.32) to check that S-closure is a closure operator, that is, $X \subseteq \langle X \rangle$; $\langle\langle X \rangle\rangle = \langle X \rangle$; and if $X_1 \subseteq X_2$, then $\langle X_1 \rangle \subseteq \langle X_2 \rangle$.

If f is a $\mathbb{Z}_n$-flow on Γ, then the *support* $S(f)$ of f is the set $\{e \in E(\Gamma): f(e) \neq 0\}$.

6.3.14. Lemma. *Let Γ be a graph and X be a subset of $E(\Gamma)$ such that $\langle X \rangle = E(\Gamma)$. Then there is a $\mathbb{Z}_3$-flow f on Γ with $E(\Gamma) - X \subseteq S(f)$.*

Proof. We argue by induction on $|E(\Gamma) - X|$. If this is zero, the result is immediate so assume that $|E(\Gamma) - X| > 0$. Then $X \neq \langle X \rangle$, so there is a circuit C with $0 < |C - X| \leq 2$. Certainly $\langle X \cup C \rangle = E(\Gamma)$ and so, by the induction assumption, there is a $\mathbb{Z}_3$-flow g on Γ such that $E(\Gamma) - (C \cup X) \subseteq S(g)$. Evidently there is a $\mathbb{Z}_3$-flow h on Γ so that $S(h) = C$. As $|C - X| \leq 2$, we can choose n from $\mathbb{Z}_3$ so that, for all e in $C - X$, $n \neq -g(e)/h(e)$. Let $f = g + nh$. Then, for e in $E(\Gamma) - (X \cup C)$, $f(e) = g(e) \neq 0$. Moreover, for e in $C - X$, $f(e) = g(e) + nh(e)$. This sum is non-zero by the choice of n, and we conclude that, for all e in $E(\Gamma) - X$, $f(e)$ is non-zero. □

By Lemma 6.3.13, we know that we may assume that Γ is simple and 3-connected. The next lemma focusses on such graphs. The proof of Theorem 6.3.10 will be obtained by combining this lemma with Lemma 6.3.14.

6.3.15. Lemma. *A simple 3-connected graph Γ has a collection $C_1, C_2, \ldots, C_m$ of disjoint circuits such that $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle = E(\Gamma)$.*

To prove this lemma, we shall use the following technical but elementary result.

6.3.16. Lemma. *Let Δ be a non-null simple graph in which each vertex has degree at least two. Then Δ has a block Δ' with at least three vertices so that at most one vertex of Δ' is adjacent in Δ to some vertex of Δ not in Δ'.*

Proof. Let $bc(\Delta)$ be the graph having as its vertices the blocks and cut vertices of Δ; the edges of $bc(\Delta)$ join a cut vertex to a block if the block contains the cut vertex. Evidently $bc(\Delta)$ is a forest. Let v be a pendant vertex of $bc(\Delta)$. Then v corresponds to a block of Δ. Since Δ is simple and has no vertices of degree less than 2, this block has at least three vertices. We take this block to be Δ'. □

Proof of Lemma 6.3.15. A subset X of $E(\Gamma)$ will be called *connected* if the subgraph of Γ consisting of X and all incident vertices is connected. Now certainly Γ has a circuit C. Moreover, as Γ is simple, $\langle C \rangle$ is connected. Thus we can choose a maximum positive integer m so that there are disjoint circuits $C_1, C_2, \ldots, C_m$ with $\langle C_1 \cup C_2 \cup \ldots C_m \rangle$ connected.

Let U be the set of vertices of Γ incident with $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle$ and let Δ be the subgraph of Γ obtained by deleting U. If Δ is the null graph, then $U = V(\Gamma)$ and so, as $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle$ is connected, $\langle\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle\rangle = E(\Gamma)$. Thus $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle = E(\Gamma)$ and the lemma holds.

Now suppose that Δ is non-null. No vertex v of Δ is adjacent in Γ to two distinct vertices, say u_1 and u_2, of U; otherwise, since $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle$ is connected, there would be a path joining u_1 and u_2 using only edges of $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle$. This path together with the edges vu_1 and vu_2 forms a circuit contradicting the definition of $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle$. Thus, as every vertex of the simple 3-connected graph Γ has degree at least 3, every vertex of Δ has degree at least 2. Therefore, by Lemma 6.3.16, Δ has a block Δ' with at least three vertices and with at most one vertex adjacent in Δ to some vertex of Δ not in Δ'. Since Γ is 3-connected and $|V(\Delta')| \geq 3$, there are at least 3 vertices of Δ' that are adjacent in Γ to vertices not in Δ'. Hence there are distinct vertices b_1 and b_2 of Δ' both of which are adjacent in Γ to vertices in U. As Δ' is a block with at least three vertices, it has a circuit, say C_{m+1}, using both b_1 and b_2. Let e_1 and e_2 be edges of Γ joining b_1 and b_2 respectively to U. Then $\{e_1, e_2\} \subseteq \langle C_1 \cup C_2 \cup \ldots \cup C_{m+1} \rangle$ and so $\langle C_1 \cup C_2 \cup \ldots C_{m+1} \rangle$ is connected. This contradicts the maximality of m and completes the proof of the lemma. □

We are now ready to prove Theorem 6.3.10.

Proof of Theorem 6.3.10. Let Γ be a graph that, among all graphs Δ with no isthmuses and no nowhere-zero 6-flows, has $|V(\Delta)| + |E(\Delta)|$ minimum. By Lemma 6.3.13, we may assume that Γ is simple and 3-connected. Then, by Lemma 6.3.15, Γ has a set $\{C_1, C_2, \ldots, C_m\}$ of disjoint circuits so that $\langle C_1 \cup C_2 \cup \ldots \cup C_m \rangle = E(\Gamma)$. By Lemma 6.3.14, there is a $\mathbb{Z}_3$-flow f_1 on Γ with $E(\Gamma) - (C_1 \cup C_2 \cup \ldots \cup C_m) \subseteq S(f_1)$. By Lemma 6.3.12, there is a $\mathbb{Z}_2$-flow f_2 on Γ with $S(f_2) = C_1 \cup C_2 \cup \ldots \cup C_m$. Then the $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-flow f defined, for all e in $E(\Gamma)$, by $f(e) = (f_1(e), f_2(e))$ is nowhere zero. We conclude, by Propositions 6.3.4 and 6.3.7, that Γ has a nowhere-zero 6-flow. □

6.3.D. Orientations of Graphs

As we have seen, T–G invariants are important in the study of colorings and flows in graphs. Another area of graph theory in which there have been numerous applications of T–G techniques is in the consideration of certain special types of orientations of graphs. Several of these are considered below and some further examples are considered in section 6.3.G and the exercises. An *acyclic orientation* of a graph Γ is an orientation of Γ in which there are no directed cycles. Let $a(\Gamma)$ denote the number of such orientations of Γ.

6.3.17. Proposition. (*Stanley, 1973b*) $a(\Gamma) = (-1)^{|V(\Gamma)|}\chi_\Gamma(-1) = t(M_\Gamma; 2, 0)$.

Proof. Suppose that the edge e of Γ is neither a loop nor an isthmus and let u and v be the endpoints of e. Partition the set $\mathscr{A}$ of acyclic orientations of $\Gamma - e$ into subsets $\mathscr{A}'$ and $\mathscr{A}''$ where $\mathscr{A}'$ consists of those members θ of $\mathscr{A}$ for which $(\Gamma - e)_\theta$ contains a directed path from u to v or from v to u. It is straightforward to show that if $\theta \in \mathscr{A}'$, then $(\Gamma - e)_\theta$ cannot contain both a directed path from u to v and a directed path from v to u, as otherwise $(\Gamma - e)_\theta$ certainly contains a directed cycle. Therefore, for each orientation θ in $\mathscr{A}'$, there is precisely one orientation of e that will extend θ to an acyclic orientation of Γ. On the other hand, if $\theta \in \mathscr{A}''$, then each of the two orientations of e extends θ to an acyclic orientation of Γ. Since every acyclic orientation of Γ can be uniquely obtained from a member of $\mathscr{A}$ by assigning an orientation to e, it follows that

$$a(\Gamma) = |\mathscr{A}'| + 2|\mathscr{A}''|.$$

But

$$a(\Gamma - e) = |\mathscr{A}'| + |\mathscr{A}''|.$$

Moreover, it is easy to see that

$$a(\Gamma/e) = |\mathscr{A}''|.$$

We conclude that if e is neither a loop nor an isthmus of Γ, then

$$a(\Gamma) = a(\Gamma - e) + a(\Gamma/e).$$

Now suppose that e is an isthmus of Γ. Then $a(\Gamma) = 2a(\Gamma - e)$ unless consists of a single edge, in which case $a(\Gamma) = 2$. Finally, if Γ has a loop, the Γ has no acyclic orientations, that is, $a(\Gamma) = 0$.

We may now apply Theorem 6.2.2 to obtain that

$$a(\Gamma) = t(M_\Gamma; 2, 0).$$

Thus, by Proposition 6.3.1,

$$a(\Gamma) = (-1)^{|V(\Gamma)|}\chi_\Gamma(-1),$$

as required. □

The preceding proposition showed that the number of acyclic orientatio of a graph is a T–G invariant. Certain proper subsets of the set of acycli orientations can also be associated with T–G invariants. The next result give one such example and another example is given in Exercise 6.35. A vertex in a directed graph is a *source* if no edge is directed toward v, and a *sink* no edge is directed away from v. We shall denote by $N_v(\Gamma)$ the number acyclic orientations of Γ in which v is the unique source.

6.3.18. Proposition. (*Greene & Zaslavsky, 1983*) *$N_v(\Gamma)$ is $(-1)^{r(M_\Gamma)}\mu(M_\Gamma)$ if is connected, and is 0 otherwise. Thus $N_v(\Gamma)$ does not depend on the choice the vertex v.*

Proof. The proof of this result differs slightly from the usual pattern in that instead of establishing the deletion–contraction formula for an arbitrary edg e, we show it only for certain special choices of e. In particular, we assum that the edge e has v as an endpoint. Let v' be the other endpoint of e. If has e as its only edge, it is clear that

$$N_v(\Gamma) = \begin{cases} 0, & \text{if } e \text{ is a loop,} \\ 1, & \text{if } e \text{ is an isthmus.} \end{cases} \qquad (6.25$$

Now suppose that Γ has at least two edges and that e is still a loop or a isthmus. Then

$$N_v(\Gamma) = \begin{cases} 0, & \text{if } e \text{ is a loop,} \\ N_v(\Gamma/e), & \text{if } e \text{ is an isthmus.} \end{cases} \qquad (6.26$$

Next assume that e is neither a loop nor an isthmus. Then we can partition the set $\mathscr{S}$ of acyclic orientations of Γ in which v is the unique source into subsets $\mathscr{S}'$ and $\mathscr{S}''$, where $\theta \in \mathscr{S}'$ provided that the only edge of Γ_θ directed into v' is e. Evidently

$$|\mathscr{S}'| = N_v(\Gamma/e) \quad \text{and} \quad |\mathscr{S}''| = N_v(\Gamma - e),$$

hence

$$N_v(\Gamma) = N_v(\Gamma - e) + N_v(\Gamma/e). \qquad (6.27)$$

Now although (6.27) has not been established for an arbitrary edge of Γ, it is clear that by repeated application of (6.25), (6.26), and (6.27), one can determine the value of $N_v(\Gamma)$ for any graph Γ having a distinguished vertex v. Moreover, as (6.26) and (6.27) are Tutte–Grothendieck recursions, a straightforward induction argument establishes that $N_v(\Gamma) = t(M_\Gamma; 1, 0)$. It follows, by (6.21), that $N_v(\Gamma) = (-1)^{r(M_\Gamma)}\mu(M_\Gamma)$, as required. □

Our last result for oriented graphs concerns score vectors. If the graph Γ has vertex set $\{v_1, v_2, \ldots, v_n\}$ and θ is an orientation of Γ, then the *score vector* of Γ_θ is the ordered n-tuple $(s_1, s_2, \ldots, s_n)$ where s_i, the *score* of v_i, is the number of edges of Γ_θ that are directed away from v_i. We shall denote the number of distinct score vectors of Γ by $s(\Gamma)$.

6.3.19. Proposition. (*Stanley, 1980*) $s(\Gamma) = t(M_\Gamma; 2, 1) = i(M_\Gamma)$.

The proof of this proposition will use the following result.

6.3.20. Lemma. *Let e be an edge of Γ joining v_1 and v_2. Suppose that $(s_2, s_2, s_3, s_4, \ldots, s_n)$ and $(s'_1, s'_2, s_3, s_4, \ldots, s_n)$ are score vectors of Γ with $s_2 < s'_2$. Then $(s_1 - 1, s_2 + 1, s_3, s_4, \ldots, s_n)$ is a score vector for Γ.*

Proof. Let θ and θ' be orientations of Γ having $(s_1, s_2, s_3, s_4, \ldots, s_n)$ and $(s'_1, s'_2, s_3, s_4, \ldots, s_n)$, respectively, as their score vectors. If, in Γ_θ, the edge e is directed from v_1 to v_2, then reversing the orientation of e gives an orientation of Γ having $(s_1 - 1, s_2 + 1, s_3, s_4, \ldots, s_n)$ as its score vector. Therefore we may assume that e is directed from v_2 to v_1 in Γ_θ. Now consider the set V_1 of vertices v such that there is a directed path in Γ_θ from v_1 to v. We distinguish two cases:

$$(1)\ v_2 \in V_1, \quad \text{and} \quad (2)\ v_2 \notin V_1.$$

In case (1), on taking a directed path from v_1 to v_2 and adding the edge e, we obtain a directed cycle containing the edge e. Reversing the directions of all the edges in this cycle except e gives an orientation having $(s_1 - 1, s_2 + 1, s_3, s_4, \ldots, s_n)$ as its score vector.

In case (2), $v_2 \in V(\Gamma) - V_1$. Now, by definition, every edge in Γ_θ joining a vertex in V_1 to a vertex in $V(\Gamma) - V_1$ must be directed from the vertex in $V(\Gamma) - V_1$ to the vertex in V_1. Therefore, for any orientation of Γ and, in particular, for $\Gamma_{\theta'}$, the sum of the scores of the vertices in $V(\Gamma) - V_1$ cannot exceed the sum of the scores of these vertices in Γ_θ. But since $s'_2 > s_2$, this is a contradiction and the proof of the lemma is complete. □

Proof of Proposition 6.3.19. The equality of $t(M_\Gamma; 2, 1)$ and $i(M_\Gamma)$ follows immediately from Proposition 6.2.11. We now show that $t(M_\Gamma; 2, 1) = s(\Gamma)$. Let e be an edge of Γ which is neither a loop nor an isthmus and assume that e joins the vertices v_1 and v_2. We shall show that

$$s(\Gamma) = s(\Gamma - e) + s(\Gamma/e). \qquad (6.28)$$

Suppose that $(s'_1, s'_2, s_3, s_4, \ldots, s_n)$ is a score vector for $\Gamma - e$ and that $\mathscr{S}$ is the set of score vectors of $\Gamma - e$ having $(s_3, s_4, \ldots, s_n)$ as the last $n - 2$ entries. Then, by Lemma 6.3.20, there are integers s_1, s_2, and k such that

$$\mathscr{S} = \{(s_1 - j, s_2 + j, s_3, s_4, \ldots, s_n): 0 \leqslant j \leqslant k\}.$$

Now, given an orientation of $\Gamma - e$, we can orient the edge e in two different ways to obtain orientations of Γ. Thus if $\mathscr{S}'$ is the set of score vectors for Γ having $(s_3, s_4, \ldots, s_n)$ as the last $n - 2$ entries, then

$$\mathscr{S}' = \{(s_1 + 1 - j, s_2 + j, s_3, s_4, \ldots, s_n): 0 \leqslant j \leqslant k + 1\}.$$

Hence $|\mathscr{S}'| = |\mathscr{S}| + 1$. Since the only score vector for Γ/e having $(s_3, s_4, \ldots, s_n)$ as the last $n - 2$ entries is $(s_1 + s_2, s_3, s_4, \ldots, s_n)$, we conclude that (6.28) holds.

To complete the proof, it only remains to notice that if Γ has a loop, then $s(\Gamma) = 0$, while if e is an isthmus of Γ, then $s(\Gamma) = 2s(\Gamma - e)$. Since the value of s on an isthmus is 2, the proposition follows immediately on applying Theorem 6.2.2. □

6.3.E. Reliability and Percolation

Classical percolation theory was introduced by Broadbent & Hammersley (1957) to model the flow of liquid through a random medium. As such, the classical theory is a branch of random graph theory. Another closely related branch of random graph theory is the study of the reliability of a network. Here one is interested in determining the probability that, in a random subgraph of the network, two distinguished vertices are joined by a path. In this section we show how the Tutte polynomial is useful in the study of the matroid generalizations of these graph problems.

In matroid reliability and percolation problems, every element e_i of a matroid $M(E)$ has, independently of all other elements, a probability $1 - p_i$ of being deleted from M where, except when otherwise stated, $0 < p_i < 1$. Then, writing q_i for $1 - p_i$, the probability $\Pr(A)$ that a subset A of E consists of precisely those elements that are retained is given by

$$\Pr(A) = \prod_{e_i \in A} p_i \prod_{e_i \notin A} q_i. \qquad (6.29)$$

The standard problem in this area is to find ways to efficiently compute the probability $\Pr(\mathscr{F})$ that the set of retained elements is in some family $\mathscr{F}$. Evidently $\Pr(\mathscr{F}) = \sum_{A \in \mathscr{F}} \Pr(A)$. Usually the family $\mathscr{F}$ is ascending, that is, if $A \in \mathscr{F}$ and $B \supseteq A$, then $B \in \mathscr{F}$. For example, when $\mathscr{F}$ is the family $\mathscr{S}$ of spanning sets of M and all the retention probabilities p_i equal some constant p, then $\Pr(\mathscr{F})$ is the probability $\Pr(M)$ that a random submatroid of $M(E)$ has the same rank as M. We saw in Example 6.2.7 that this probability is

$$q^{|E|-r(M)} p^{r(M)} t(M; 1, 1/q). \tag{6.30}$$

Toward the end of this section, we shall present a procedure for modifying the matroid M so as to adapt the last formula to the case when the retention probabilities p_i are different.

Next we consider how to put the problem of computing network reliability into this framework. Given two distinguished vertices s and t in a graph Γ, we are interested in determining the probability that a random subgraph of Γ contains a path between s and t. To do this, we first form a new graph $\bar{\Gamma}$ from Γ by adjoining a basepoint edge d between s and t. Let $\mathscr{D}$ be the family of subsets A of $E(\Gamma)$ for which $A \cup d$ contains a cycle of $\bar{\Gamma}$ containing d. Equivalently, $A \in \mathscr{D}$ if and only if d is not an isthmus in the subgraph of $\bar{\Gamma}$ induced by $A \cup d$. We shall develop a formula for $\Pr(\mathscr{D})$ for an arbitrary pointed matroid $M_d(E \dot\cup d)$, where, in this more general context, $A \in \mathscr{D}$ if and only if d is not an isthmus of $M_d(A \cup d)$. Our formula for $\Pr(\mathscr{D})$ will involve the pointed Tutte polynomial $t_P(M_d)$ that was introduced in Proposition 6.2.19. We shall first determine $\Pr(\mathscr{D})$ in the case when, for all elements e_i of E, the retention probability p_i equals a constant p. The number of elements in a matroid M will be denoted by $|M|$.

6.3.21. Proposition. *Let the matroid M be $M_d(E \dot\cup d)$ and assume that every element of E has, independently of all other elements, probability $1-p$ of being deleted from M, while the element d has probability 0 of being deleted. Then the probability that, in a random submatroid $\omega(M)$ of M, the element d is not an isthmus is given by the formulas*

(i) $\Pr(\mathscr{D}) = p^{r(M)} q^{|M|-r(M)-1} g(1/p, 1/q)$ *and*
(ii) $\Pr(\mathscr{D}) = 1 - p^{r(M)-1} q^{|M|-r(M)} f(1/p, 1/q)$

where $x'f(x, y) + y'g(x, y) = t_P(M_d(E \dot\cup d); x', x, y', y)$.

Proof. We first show that $\Pr(\mathscr{D})$ obeys the weighted recursion

$$\Pr(\mathscr{D}(M)) = q \Pr(\mathscr{D}(M-e)) + p \Pr(\mathscr{D}(M/e)) \tag{6.31}$$

where e is a point of $M-d$ that is not a loop or an isthmus. By (6.29), we have

$$\Pr(\mathscr{D}(M)) = \sum_{A \in \mathscr{D}} p^{|A|} q^{|E-A|} = \sum_{\substack{A \in \mathscr{D} \\ e \notin A}} p^{|A|} q^{|E-A|} + \sum_{\substack{A \in \mathscr{D} \\ e \in A}} p^{|A|} q^{|E-A|}.$$

Thus

$$\Pr(\mathscr{D}(M)) = q \sum_{\substack{A \in \mathscr{D} \\ e \notin A}} p^{|A|} q^{|(E-e)-A|} + p \sum_{\substack{A' \subseteq E-e \\ A' \cup e \in \mathscr{D}}} p^{|A'|} q^{|(E-e)-A'|}. \tag{6.32}$$

The first summation in (6.32) is clearly over those subsets A of $E-e$ for which d is not an isthmus of $A \cup d$ in $M-e$. Thus this summation is over those members A of $\mathscr{D}(M-e)$. On the other hand, since d is an isthmus of $M_d(A' \cup d \cup e)$ if and only if it is an isthmus of $M_d(A' \cup d \cup e)/e$, the second summation in (6.32) is over those members A' of $\mathscr{D}(M/e)$. Thus (6.31) holds. It follows that if

$$h(M) = (1/q)^{|M|-r(M)} (1/p)^{r(M)} \Pr(\mathscr{D}(M)),$$

then

$$h(M) = h(M-e) + h(M/e) \tag{6.33}$$

for all elements e of $M-d$ that are not loops or isthmuses of M. Moreover, it is routine to check that if e is an element of $M-d$, then

$$h(M) = \begin{cases} (1/q)h(M-e) & \text{if } e \text{ is a loop,} \\ (1/p)h(M-e) & \text{if } e \text{ is an isthmus.} \end{cases} \tag{6.34}$$

Finally, one easily checks that

$$h(M(d)) = \begin{cases} 1/q, & \text{if } M(d) \text{ is a loop,} \\ 0, & \text{if } M(d) \text{ is an isthmus.} \end{cases} \tag{6.35}$$

We conclude, by Proposition 6.2.19, that $h(M_d) = t_P(M_d; x', x, y', y)$ where $x' = 0$, $y' = 1/q$, $x = 1/p$, and $y = 1/q$. Therefore, as $t_P(M_d; x', x, y', y) = x'f(x, y) + y'g(x, y)$, $h(M_d) = (1/q)g(1/p, 1/q)$ and so $\Pr(\mathscr{D}(M)) = q^{|M|-r(M)-1} p^{r(M)} g(1/p, 1/q)$. This establishes (i). A similar argument applied to $\Pr(2^E - \mathscr{D})$ gives (ii). □

With M still equal to $M_d(E \dot\cup d)$, we note that, by (6.29),

$$\Pr(\mathscr{D}(M)) = \sum_{i=0}^{|E|} a_i p^i q^{|E|-i}. \tag{6.36}$$

Thus $\Pr(\mathscr{D}(M))$ is a polynomial in p and q of constant total degree. The coefficient a_i here equals the number of i-element subsets A of E for which d is not an isthmus of $M_d(A \cup d)$. Thus, provided d is not an isthmus or a loop of M, $a_i \geqslant 0$ for all i and $a_{|E|} = 1$. Now suppose that M_d is the polygon matroid of the graph $\bar{\Gamma}$ and let the basepoint edge d join the distinguished vertices s and t. Let k be the least j for which a_j is non-zero. Then k equals the length of the shortest (s, t)-path in $\bar{\Gamma} - d$, and the number of such shortest paths is a_k.

On substituting $1 - p$ for q in (6.36), we obtain $\Pr(\mathscr{D})$ as a polynomial in p alone. We leave as an exercise the problem of determining the coefficients of this polynomial (Exercise 6.37).

Before attacking the reliability problem in the case of unequal retention probabilities, we note a remarkable fact about the evaluation $t(M; 1/p, 1/q)$. Recall that $t_P(M_d; x', x, y', y) = x'f(x, y) + y'g(x, y)$. If M_d is viewed as simply a matroid M rather than as a pointed matroid, then d is no longer distinguished, so $x' = x$, $y' = y$, and $t(M; x, y) = xf(x, y) + yg(x, y)$. In Proposition 6.2.20(iv), it was noted that, if d is neither a loop nor an isthmus of M, then

$$\begin{cases} t(M - d) = (x - 1)f(x, y) + g(x, y) \quad \text{and} \\ t(M/d) = f(x, y) + (y - 1)g(x, y). \end{cases} \tag{6.37}$$

Clearly these formulas can be inverted in the Tutte–Grothendieck ring to give expressions for $f(x, y)$ and $g(x, y)$ in terms of $t(M - d)$ and $t(M/d)$. However, the determinant of this system equals $xy - x - y$, which is zero when $(1/x) + (1/y) = 1$. Since $p + q = 1$, $\Pr(\mathscr{D})$ is computable from the evaluations of $t(M - d)$ and $t(M/d)$ at $x = 1/p$ and $y = 1/q$ only in the most formal sense, that is, when the identity $p + q = 1$ is never invoked. On the other hand, we note that, by (6.37), $t(M/d; 1/p, 1/q) = (p/q)t(M - d; 1/p, 1/q)$. Therefore, for a given matroid M', $t(M''; 1/p, 1/q)$ is the same for any strong map image M'' of M'. To see this, we note that if M'' and M' are so related, then, for some matroid M and element d which is neither a loop nor an isthmus, $M - d = M'$ and $M/d = M''$.

The above remarks are summarized in the following proposition, the first part of which generalizes the following identity, a trivial consequence of 6.2.11(iv):

$$t(M - d; 2, 2) = t(M/d; 2, 2) = 2^{|M| - 1}.$$

6.3.22. Proposition. *Let $q(M) = t(M; 1/p, 1/q)$ where $p + q = 1$ and let d be an element of M that is neither a loop nor an isthmus. Then*

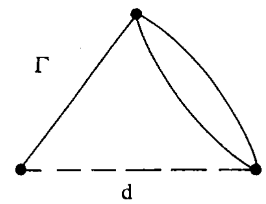
(i) $q(M/d) = (p/q)q(M - d) = f(M; 1/p, 1/q) + (p/q)g(M; 1/p, 1/q)$;
(ii) *Both $q(M - d)$ and $q(M/d)$ are independent of the modular cut of $M - d$ determined by d in M.*

These ideas are illustrated in the following.

6.3.23. Example. Let M be the polygon matroid of the graph Γ in Figure 6.3. Then, it is straightforward to check that

$$t_P(M_d; x, y) = x'(x + y + 1) + y'(y + 1). \tag{6.38}$$

Figure 6.3.



Thus, by 6.3.21(i),

$$\begin{aligned} \Pr(\mathscr{D}(M)) &= p^2 q\left(\left(\frac{1}{q}\right) + 1\right) \\ &= 2p^2 q + p^3 \\ &= 2p^2 - p^3. \end{aligned}$$

Now, by (6.37) and (6.38),

$$t(M - d) = x^2 + xy$$

and

$$t(M/d) = y^2 + y + x.$$

Thus

$$\begin{aligned} q(M/d) &= \frac{1}{q^2} + \frac{1}{q} + \frac{1}{p} \\ &= \frac{1}{pq^2} \\ &= \frac{p}{q}\left(\frac{1}{p^2} + \frac{1}{pq}\right) \\ &= \frac{p}{q} q(M - d). \end{aligned}$$

By 6.3.21(i), if M_1 and M_2 are the polygon matroids of the graphs Γ_1 and Γ_2 shown in Figure 6.4, then $q(M/d) = q(M_1/d_1) = q(M_2/d_2)$. However, $\Pr(\mathscr{D}(M_1)) = p$ while $\Pr(\mathscr{D}(M_2)) = 2p - p^2$.

Proposition 6.3.21 gives two formulas for $\Pr(\mathscr{D})$ in the case when all the retention probabilities are equal. We now turn to the general problem of determining $\Pr(\mathscr{D})$ when M_d is the polygon matroid of a graph $\bar{\Gamma}$ and the retention probabilities p_i can vary from edge to edge. In particular, we shall describe how, if p_i is equal to a k-place binary decimal, we can replace the corresponding edge e_i by an appropriate series–parallel network in which each edge has retention probability equal to 1/2. Since $\Pr(\mathscr{D})$ will be unaffected by the presence of loops, we assume that $\bar{\Gamma}$ is loopless.

Figure 6.4.

Let $0.d_{i1}d_{i2} \ldots d_{ik}$ be the binary decimal for p_i where $d_{ik} = 1$. We form the $(k+1)$-edge series–parallel network N_i as follows. Start with the graph consisting of the edge e_i and its endpoints. Then, since $d_{ik} = 1$, add an edge e_{ik} in parallel with e_i. Assuming the edges $e_{ik}, e_{i(k-1)}, \ldots, e_{i(j+1)}$ have been added, add e_{ij} in parallel with e_i if $d_{ij} = 1$, and in series with e_i otherwise. After the edge e_{i1} has been added, form the 2-sum of this network N_i and $\bar{\Gamma}$ along the basepoint e_i. As an example, if $p_i = 0.0111001$, then N_i is as shown in Figure 6.5.

We repeat the above procedure for every edge of $\bar{\Gamma}$ other than d to obtain a new graph $\tilde{\Gamma}$ with polygon matroid $\tilde{M}_d$. Clearly $\tilde{\Gamma}$ retains the distinguished edge d. In $\tilde{\Gamma}$, we assign to each edge e_{ij} the retention probability 1/2. Then it is straightforward to check that

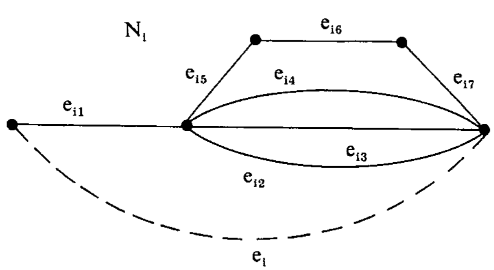
$$\Pr(\mathscr{D}(M_d)) = \Pr(\mathscr{D}(\tilde{M}_d)).$$

But, since $\tilde{M}_d$ has constant retention probability, we get, by 6.3.21(i), that

$$\Pr(\mathscr{D}(M_d)) = (\tfrac{1}{2})^{|\tilde{M}_d|-1} g(\tilde{M}_d; 2, 2). \tag{6.39}$$

The techniques just described when M_d is a polygon matroid can be equally well applied to find $\Pr(\mathscr{D}(M_d))$ for an arbitrary pointed matroid. This technique can also be used in other situations where the retention probabilities can vary. For instance, by replacing each element of an arbitrary matroid M by an appropriate series–parallel network and using (6.30), we can obtain a formula for the probability that a random submatroid of M has the same rank as M.

Figure 6.5.



To conclude this section, we shall extend Proposition 6.3.19 on the number of score vectors that can arise when orienting a graph. The following lemma will be used in the proof of this extension.

6.3.24. Lemma. *Let k be a positive integer and M be a matroid. Let $M^{(k)}$ be the matroid that is obtained from M by replacing each non-loop element by k parallel elements and replacing each loop by k loops. Then*

$$t(M^{(k)}; x, y) = (y^{k-1} + y^{k-2} + \ldots + y + 1)^{r(M)} t\left(M; \frac{y^{k-1} + y^{k-2} + \ldots + y + x}{y^{k-1} + y^{k-2} + \ldots + y + 1}, y^k\right).$$

Proof. Let $f(M) = t(M^{(k)}; x, y)$. Then it is straightforward to show that f is a generalized T–G invariant. The lemma then follows easily from Corollary 6.2.6. We leave it to the reader to complete the details of this argument. □

In the North American National Hockey League (NHL), teams are awarded 2 points for a victory, 1 point for a tie, and 0 points for a loss. To compute the number of possible score vectors at the end of an NHL season, it suffices to compute the number of score vectors for $\Gamma^{(2)}$ where Γ is the graph corresponding to the NHL schedule and $\Gamma^{(k)}$ is obtained from Γ by replacing each edge by k edges in parallel. A special case of the next result is that this number of score vectors can be determined directly from $t(M_\Gamma)$.

6.3.25. Proposition. *Suppose that the vertices of a loopless graph Γ correspond to teams in some league with each edge corresponding to a game that must be played between the two endpoints. Let k be a fixed positive integer so that, for each game a team plays, it may score any number of points from the set $\{0, 1, \ldots, k\}$ provided that the two teams in any game score a total of k points from that game. Then, when all the games have been played, the number of possible score vectors is*

$$t(M_{\Gamma^{(k)}}; 2, 1) = k^{r(M_\Gamma)} t\left(M_\Gamma; \frac{k+1}{k}, 1\right).$$

Proof. The fact that the number of possible score vectors is $t(M_{\Gamma^{(k)}}; 2, 1)$ follows easily from Proposition 6.3.19. The equality of this and $k^{r(M_\Gamma)} t\left(M_\Gamma; \frac{k+1}{k}, 1\right)$ is a consequence of Lemma 6.3.24. □

We note that this result can also be used to treat the case when, from each game, a team may score any number of points from the set $\{-k, -(k-1), \ldots, k-1, k\}$ provided that from any one game a total of 0 points are scored. A one-to-one correspondence between the possible score vectors $(s(v_1), s(v_2), \ldots, s(v_m))$ of Γ in this case and the possible score vectors obtained from orienting the edges of $\Gamma^{(2k)}$ is given by

$$(s(v_1), s(v_2), \ldots, s(v_m)) \to (s(v_1) + k\deg(v_1), s(v_2) + k\deg(v_2), \ldots, s(v_m) + k\deg(v_m)).$$

Hence there are $t(M_{\Gamma^{(2k)}}; 2, 1)$ possible score vectors in this case. We know of no easy formula for determiming the number of score vectors when the sum of the points scored in each contest is allowed to vary.

6.3.F. Two-variable Coloring

In this section we generalize the relationship (6.3.1) between the chromatic polynomial of a graph Γ and the Tutte polynomial of its polygon matroid. So far we have considered only proper colorings of Γ, that is, assignments of colors to the vertices of Γ so that two adjacent vertices receive different colors. We now consider arbitrary colorings where adjacent vertices are no longer required to be colored differently. In such a coloring, an edge is called *monochromatic* if its two endpoints receive the same color. Let Γ be connected and let $c_i(\lambda, \Gamma)$, or briefly $c_i(\lambda)$, denote the number of ways to color Γ with λ colors so that exactly i edges are monochromatic. Then $c_i(\lambda)$ is a polynomial in λ. If Γ has n edges, define

$$\bar{\chi}(\Gamma; \lambda, v) = \frac{1}{\lambda} \sum_{i=0}^{n} c_i(\lambda) v^i. \tag{6.40}$$

Then $\bar{\chi}(\Gamma; \lambda, v)$ is easily determined from the Tutte polynomial of M_Γ.

6.3.26. Proposition. *If M_Γ has rank r, then*

$$\bar{\chi}(\Gamma; \lambda, v) = (v-1)^r t\left(M_\Gamma; \frac{v+\lambda-1}{v-1}, v\right).$$

Proof. We shall sketch three different but suggestive proofs of this identity, leaving the reader the exercise of filling in the details.

(1) We can generalize the recursion for proper colorings to obtain:

$$\bar{\chi}(\Gamma) = \begin{cases} (v+\lambda-1)\bar{\chi}(\Gamma/e) & \text{if } e \text{ is an isthmus,} \\ v\bar{\chi}(\Gamma - e) & \text{if } e \text{ is a loop,} \\ \bar{\chi}(\Gamma - e) + (v-1)\bar{\chi}(\Gamma/e) & \text{otherwise.} \end{cases} \tag{6.41}$$

To see the third part of this, suppose e is neither a loop nor an isthmus of Γ. Then, for any fixed λ and any $i \geqslant 1$, we can partition the λ-colorings of Γ with i monochromatic edges into those in which the two endpoints v and v' of e are colored the same and those in which they are colored differently. Evidently there are $c_{i-1}(\lambda, \Gamma/e)$ members of the first class. Moreover, the number of members of the second class equals the number of λ-colorings of $\Gamma - e$ with i monochromatic edges in which v and v' are colored differently. In turn, this number is the difference between $c_i(\lambda, \Gamma - e)$ and the number of λ-colorings of $\Gamma - e$ with i monochromatic edges in which v and v' are colored the same. Since the last quantity clearly equals $c_i(\lambda, \Gamma/e)$, we have

$$c_i(\lambda, \Gamma) = c_{i-1}(\lambda, \Gamma/e) + c_i(\lambda, \Gamma - e) - c_i(\lambda, \Gamma/e).$$

Using this, the third part of (6.41) is not difficult to deduce.

(2) Following Crapo (1969), define the *coboundary polynomial* of an arbitrary matroid M having lattice of flats $L(M)$ by

$$\bar{\chi}(M; \lambda, v) = \sum_{X \in L(M)} v^{|X|} p(M/X; \lambda)$$
$$= \sum_{\substack{X, Y \in L(M) \\ X \subseteq Y}} v^{|X|} \lambda^{r(M) - r(Y)} \mu(X, Y).$$

We leave it to the reader to check that

$$\bar{\chi}(M; \lambda, v) = (v-1)^r t\left(M; \frac{v+\lambda-1}{v-1}, v\right).$$

To show that $\bar{\chi}(\Gamma; \lambda, v) = \bar{\chi}(M_\Gamma; \lambda, v)$ we note that, in any coloring of Γ, the set X of monochromatic edges forms a flat in M_Γ. Hence the coloring induced on Γ/X by contracting all the edges in X is proper.

(3) This proof is based on the pervasive combinatorial idea of 'counting in two different ways'. It is quite similar, for example, to the calculations involving permutations with restricted position found in Stanley (1986, section 2.3). Let M_{KC} be the $(n+1) \times (r+1)$ matrix with rows indexed by $0, 1, 2, \ldots, n$ and columns indexed by $0, 1, 2, \ldots, r$ whose (i, j)-entry equals the number of subsets A of $E(\Gamma)$ of size i such that A has rank $r - j$ in M_Γ, or equivalently, the subgraph $\Gamma'[A]$ of Γ having edge set A and vertex set $V(\Gamma)$ has $j+1$ connected components. Then

$$\sum_{j=0}^{r} M_{KC}(i, j) \lambda^{j+1} = \sum_{j=0}^{n} \binom{j}{i} c_j(\lambda), \tag{6.42}$$

since each side counts the pairs (A, c) in which A is an i-element subset of $E(\Gamma)$ and c is a λ-coloring of Γ for which each edge in A is monochromatic. Indeed, the left-hand side sums first over all such A and then, for each such subset, counts the number of λ-colorings that are monochromatic on each component of $\Gamma'[A]$. The right-hand side sums over all λ-colorings c according to their number j of monochromatic edges and then picks i of these j edges.

In matrix form, we have, from (6.42), that

$$M_{KC} \cdot \boldsymbol{\lambda} = T \cdot \mathbf{c}_\lambda \tag{6.43}$$

where

$$\boldsymbol{\lambda} = (\lambda, \lambda^2, \ldots, \lambda^{r+1})^t,$$
$$\mathbf{c}_\lambda = (c_0(\lambda), c_1(\lambda), \ldots, c_n(\lambda))^t,$$

and T is the $(n+1) \times (n+1)$ matrix for which $T(i, j) = \binom{j}{i}$, the row and column indices, i and j, ranging over the set $\{0, 1, 2, \ldots, n\}$. It is well known that the inverse T^{-1} of T is given by $T^{-1}(i, j) = (-1)^{i+j}\binom{j}{i}$. By (6.43),

$$T^{-1} \cdot M_{KC} \cdot \boldsymbol{\lambda} = \mathbf{c}_\lambda.$$

Now recall that, for a matroid $M(E)$, the cardinality-corank polynomial $S_{KC}(M; v, \lambda)$ is equal to

$$\sum_{A \subseteq E} v^{|A|}\lambda^{r(E)-r(A)}.$$

Thus

$$S_{KC}(M_\Gamma; v, \lambda) = \sum M_{KC}(i, j)v^i\lambda^j$$

and so

$$S_{KC}(M_\Gamma; v, \lambda) = \left(\frac{1}{\lambda}\right)\mathbf{v}^t \cdot M_{KC} \cdot \boldsymbol{\lambda}$$

where

$$\mathbf{v}^t = (1, v, v^2, \ldots, v^n).$$

Thus, by (6.43),

$$S_{KC}(M_\Gamma; v, \lambda) = \left(\frac{1}{\lambda}\right)\mathbf{v}^t \cdot T \cdot \mathbf{c}_\lambda.$$

It is now not difficult to check that

$$\lambda S_{KC}(M_\Gamma; v, \lambda) = \lambda\bar{\chi}(\Gamma; v+1, \lambda). \tag{6.44}$$

But, by Proposition 6.2.21,

$$S_{KC}(M; v, \lambda) = v^r t\left(M; \frac{v+\lambda}{v}, v+1\right).$$

Substituting this into (6.44), we immediately get 6.3.26. □

The formula in Proposition 6.3.26 is invertible. Hence $t(M_\Gamma)$ can be computed from knowing the distribution of monochromatic edges among all λ-colorings of Γ for at least $r(M_\Gamma)$ values of λ. For the coboundary polynomial in general, it is not difficult to show that

$$t(M; x, y) = \frac{1}{(y-1)^r}\bar{\chi}(M; (x-1)(y-1), y). \tag{6.45}$$

The *dual coboundary polynomial* $\bar{\chi}^*(M; \lambda, v)$ is defined by

$$\bar{\chi}^*(M; \lambda, v) = \bar{\chi}(M^*; v, \lambda). \tag{6.46}$$

Moreover, for a graph Γ, we let $\bar{\chi}^*(\Gamma; \lambda, v) = \bar{\chi}^*(M_\Gamma; \lambda, v)$. Then one can show that the coefficient of v^j in $\bar{\chi}^*(\Gamma; \lambda, v)$ is the number of λ-flows that are zero on precisely j edges of Γ. Using the duality of the Tutte polynomial, we then get the following link between two-variable colorings and two-variable flows in graphs.

6.3.27. Proposition. *If M_Γ has n elements and rank r, then*

$$\bar{\chi}^*(\Gamma; v, \lambda) = \frac{(v-1)^n}{\lambda^r}\bar{\chi}\left(\Gamma; \lambda, \frac{v+\lambda-1}{v-1}\right).$$

Proof.
$$\begin{aligned}\bar{\chi}^*(\Gamma; v, \lambda) &= \bar{\chi}^*(M_\Gamma; v, \lambda)\\ &= \bar{\chi}(M_\Gamma^*; \lambda, v)\\ &= (v-1)^{n-r}t\left(M_\Gamma^*; \frac{v+\lambda-1}{v-1}, v\right)\\ &= (v-1)^{n-r}t\left(M_\Gamma; v, \frac{v+\lambda-1}{v-1}\right)\\ &= (v-1)^{n-r}\frac{(v-1)^r}{\lambda^r}\bar{\chi}\left(\Gamma; (v-1)\left(\frac{\lambda}{v-1}\right), \frac{v+\lambda-1}{v-1}\right)\end{aligned}$$

where the last step follows by (6.45). □

Clearly the last result also holds if we replace Γ by an arbitrary matroid M although, in this more general context, we no longer have the link to colorings and flows.

6.3.G. Other Graph-theoretic Tutte Invariants

In this section, we briefly list some other Tutte invariants for graphs that have appeared in the literature. Most of the results here can be proved by verifying the fundamental recursion 6.2.2(ii).

Dual to acyclic orientations we have *totally cyclic orientations*: those in which every edge of the graph Γ is contained in some directed cycle. To avoid distracting complications, the statements of these results will assume, unless otherwise stated, that Γ has no isthmuses. Then, following the results of Greene & Zaslavsky (1983) or Las Vergnas (1977) we have the following.

6.3.28. Example. The number $a^*(\Gamma)$ of totally cyclic orientations of Γ is given by

$$a^*(\Gamma) = t(M_\Gamma; 0, 2) = |\chi_\Gamma^*(-1)| = |p(M_\Gamma^*; -1)|.$$

Hence if Γ is a planar graph and Γ^* is a geometric dual of Γ, then $a^*(\Gamma) = a(\Gamma^*)$.

6.3.29. Example. Let e be a fixed edge of Γ and $a_e^*(\Gamma)$ be the number of totally cyclic orientations of Γ such that reversing the orientation of e makes the orientation acyclic. Equivalently, $a_e^*(\Gamma)$ equals the number of totally cyclic orientations of Γ such that every directed cycle uses e, that is, such that $\Gamma - e$ is acyclic. Then, provided $|E(\Gamma)| > 1$,

$$a_e^*(\Gamma) = 2\beta(M_\Gamma^*) = 2b_{10} = 2\beta(M_\Gamma).$$

Thus $a_e^*(\Gamma)$ does not depend on the edge e.

Some further results of this type can be found in the exercises for section 6.3.D. The next two examples link totally cyclic orientations and the Möbius function. They are both special cases of a more general result of Greene & Zaslavsky (1983, Theorem 8.3).

6.3.30. Example. Let Γ be a directed graph having a fixed ordering on its edges. Then the number of totally cyclic reorientations τ of Γ such that in each cycle of τ the lowest edge is not reoriented is $|\mu(M_\Gamma^*)|$.

6.3.31. Example. Let Γ be a plane graph. The number of totally cyclic orientations of Γ in which there is no clockwise cycle equals $|\mu(M_\Gamma^*)|$ if Γ has no isthmuses, and equals 0 otherwise.

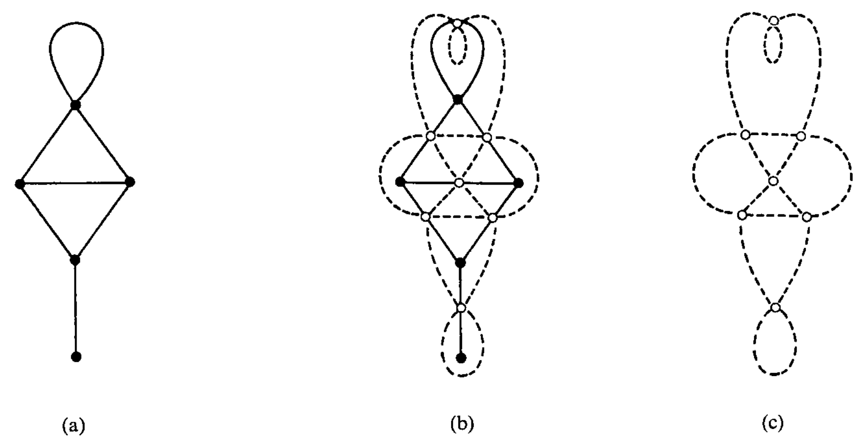
When Stanley (1973b) proved his famous result (6.3.17) for acyclic orientations, he actually obtained the following stronger result in order to interpret evaluations of the chromatic polynomial at negative integers.

6.3.32. Example. Let m be a positive integer and $w(\Gamma; m)$ denote the number of pairs (θ, σ) such that θ is an acyclic orientation of Γ and σ is a function from $V(\Gamma)$ into $\{1, 2, \ldots, m\}$ with the property that if θ directs the edge uv of Γ from u to v then $\sigma(u) \geqslant \sigma(v)$. Suppose that Γ has k connected components and no isolated vertices. Then

$$w(\Gamma; m) = (-1)^{|V(\Gamma)|}\chi_\Gamma(-m) = m^k t(M_\Gamma; m+1, 0).$$

Let Γ be a plane graph and, for each e of Γ, let $v(e)$ be a point in the interior of e. The *medial graph* Γ_m of Γ will appear in the next result and we now describe its construction. If Γ is disconnected, its medial graph is the union of the medial graphs of its components. Now suppose that Γ is connected. The construction of Γ_m in this case is illustrated in Figure 6.6. In general, Γ_m is a plane graph with vertex set $\{v(e): e \in E(\Gamma)\}$. As one can see from Figure 6.6, two such vertices $v(e)$ and $v(f)$ are joined by one edge for every face in which e and f occur successively on the boundary. More

Figure 6.6. (a) Γ; (b) Γ_m superimposed on Γ; (c) Γ_m.



precisely, the edge set of Γ_m depends upon the set of ordered pairs (F, u) where F is a face of Γ and u is a vertex on the boundary of F. For every such pair, we add an edge $v(e)v(f)$ to Γ_m whenever e and f are edges of Γ incident with u that occur successively when one traverses the boundary of F just inside its interior.

It is not difficult to check that Γ_m is 4-regular for all plane graphs Γ. Moreover, if Γ^* is the geometric dual of Γ, then $\Gamma_m = (\Gamma^*)_m$. If Δ is an arbitrary connected, 4-regular plane graph, we can form a graph Γ for which $\Gamma_m = \Delta$ as follows. Two-color the faces of Δ so that two faces sharing an edge are colored differently. This is possible because Δ is Eulerian and hence Δ^* is bipartite; this 2-coloring of the faces of Δ is just a proper 2-coloring of the vertices of Δ^*. We construct Γ by letting its vertices consist of the faces in one of the color classes; two vertices of Γ are joined by an edge for every vertex shared by the corresponding faces of Δ. Note that if we choose the faces of the other color class to be the vertices of the graph, then this construction will produce Γ^*.

An *Eulerian partition* of a graph is a set of closed trails partitioning the edge set of the graph. Clearly every 4-regular plane graph Δ has such a partition P. Suppose that, at a vertex v of Δ, the edges, in cyclic order, are e_1, e_2, e_3 and e_4, where a loop, if it occurs, is listed once for each of its ends. We say that P has a *crossing* at v if one of the closed trails in P uses both e_1 and e_3. It is not difficult to check that Δ has a unique Eulerian partition P_0 in which there is a crossing at every vertex. Note that, since Δ is a plane graph, there are two ways of travelling along each loop.

For an arbitrary plane graph Γ, results in Jaeger (1988a), Las Vergnas (1979, 1981), Martin (1977, 1978), and Rosenstiehl & Read (1978) show that

(i) $t(M_\Gamma; x, x) = \sum_P (x-1)^{\gamma(P)-1}$, where the sum is over all Eulerian partitions P of Γ_m such that each P has no crossings and $\gamma(P)$ is the number of closed trails in the partition P.

In addition, we have

(ii) $t(M_\Gamma; -1, -1) = (-1)^{|E(\Gamma)|}(-2)^{\gamma(P_0)-1}$.

It will follow using Proposition 6.5.4 that both sides of (ii) equal the number of vectors in $C \cap C^*$ where C is the binary code associated with M_Γ.

6.4. The Critical Problem

6.4.A. Definitions and Elementary Results

The critical problem for matroids was introduced by Crapo & Rota (1970) to provide a unified setting for a number of problems in extremal combinatorial theory including such fundamental graph-theoretic problems as Tutte's 5-flow conjecture (6.3.8) and the following celebrated conjecture of Hadwiger (1943).

6.4.1. Conjecture. *Let Γ be a loopless graph and k be a positive integer. If Γ has no proper k-coloring, then some subgraph of Γ is contractible to K_{k+1}.*

The cases $k=1$ and $k=2$ of this conjecture are straightforward. The conjecture was proved by Dirac (1952) when $k=3$. For $k=4$, Wagner (1964) proved that the conjecture is equivalent to what was then the Four Color conjecture. Now that the latter has become the Four Color theorem (6.3.2), we know that Hadwiger's conjecture is true for $k \leqslant 4$.

Several other important instances of the critical problem were noted in section 7.5 of White (1987). In this section, we shall give a more detailed discussion of the critical problem focussing attention on more recent results. Inevitably there will be some overlap between this section and Chapter 7 of White (1987). We begin here by recalling some basic definitions. A *linear functional* on $V(n, q)$, the n-dimensional vector space over $GF(q)$, is a linear map from $V(n, q)$ into $GF(q)$. If $A \subseteq V(n, q)$, then a k-tuple $(f_1, f_2, \ldots, f_k)$ of linear functionals on $V(n, q)$ is said to *distinguish* A if A is disjoint from $\{e: f_i(e) = 0$ for all i such that $1 \leqslant i \leqslant k\}$. Let $M(E)$ be a rank r matroid that is coordinatizable over $GF(q)$. The following fundamental result of Crapo & Rota (1970) was proved in White (1987, Theorem 7.4.1).

6.4.2. Theorem. *If $k \in \mathbb{Z}^+$ and ϕ is a coordinatization of M in $V(n, q)$, then the number of k-tuples of linear functionals on $V(n, q)$ that distinguish $\phi(E)$ equals $q^{k(n-r)}p(M; q^k)$.*

It follows from this result that, for a matroid M coordinatizable over $GF(q)$,

$$p(M; q^k) \geqslant 0 \text{ for all } k \text{ in } \mathbb{Z}^+. \quad (6.47)$$

The *critical exponent* $c(M; q)$ of M is defined by

$$c(M; q) = \begin{cases} \infty & \text{if } M \text{ has a loop,} \\ \min\{j \in \mathbb{N}: p(M; q^j) > 0\} & \text{otherwise.} \end{cases} \quad (6.48)$$

It follows from Theorem 6.4.2 that

$$c(M; q) = \min\{j \in \mathbb{N}: p(M; q^k) > 0 \text{ for all integers } k \geqslant j\}. \quad (6.49)$$

Since the kernel of a linear functional is a hyperplane, the following result follows easily from Theorem 6.4.2.

6.4.3. Corollary. *Let M be a rank r loopless matroid and ϕ be a coordinatization of M in $V(n, q)$. Then*

$$c(M; q) = \min\left\{j \in \mathbb{N}: V(n, q) \text{ has hyperplanes } H_1, H_2, \ldots, H_j \text{ such that } \left(\bigcap_{i=1}^{j} H_i\right) \cap \phi(E) = \varnothing\right\}$$

$$= \min\{j \in \mathbb{N}: V(n, q) \text{ has a subspace of dimension } n-j \text{ having empty intersection with } \phi(E)\}.$$

A noteworthy and somewhat surprising aspect of Theorem 6.4.2 and Corollary 6.4.3 is that the value of $c(M; q)$ does not depend upon the particular coordinatization ϕ. From (6.48) and Propositions 6.3.1 and 6.3.4, we deduce that when M is isomorphic to the polygon matroid of a graph Γ, $c(M; q)$ is the least integer c such that the chromatic number of Γ does not exceed q^c; when M is isomorphic to the bond matroid of Γ, $c(M; q)$ is the least integer c for which Γ has a nowhere-zero q^c-flow.

For a matroid M that is coordinatizable over $GF(q)$, the critical problem is the problem of determining the critical exponent, $c(M; q)$. This is theoretically possible for any matroid M, simply by calculating $p(M; \lambda)$. In general, however, this will require exponentially many steps. In particular then, the critical problem becomes one of efficiently determining $c(M; q)$ by, for example, recognizing M as a member of a class of matroids whose critical exponents are bounded above.

We now note some basic properties of the critical exponent. In each of these, we shall assume that M is a matroid coordinatizable over $GF(q)$.

6.4.4. Proposition. *If M is loopless and T is a subset of $E(M)$, then*

$$c(M(T); q) \leqslant c(M; q) \leqslant c(M(T); q) + c(M(E-T); q).$$

Proof. This is an immediate consequence of Corollary 6.4.3. □

6.4.5. Proposition. (*Asano, Nishizeki, Saito & Oxley, 1984*) *Suppose that $S \subseteq E(M)$ and $k \in \mathbb{N}$. Then the following are equivalent.*

(i) *S is minimal with the property that $c(M-S; q) \leqslant k$.*
(ii) *S is minimal with the property that $c(M/S; q) \leqslant k$.*
(iii) *S is minimal with the property that M has a minor with ground set $E(M)-S$ and critical exponent not exceeding k.*

This result is proved in Asano, Nishizeki, Saito & Oxley (1984) by using Tutte's theory of chain-groups (1965). We prove it here using a deletion–contraction argument on the characteristic polynomial. We shall require several preliminaries.

6.4.6. Lemma. *Let S be a subset of $E(M)$ for which $c(M-S; q) \leqslant k$. Then S has a subset T such that $c(M/T; q) \leqslant k$.*

Proof. We argue by induction on $|S|$, noting that the result is immediate if this is zero. Assume the result to be true for $|S| = n-1$ and let $|S| = n$. Choose an element e of S. Then $p((M-e)-(S-e); q^k) = p(M-S; q^k) > 0$. Therefore, by the induction assumption, $S-e$ has a subset T such that $p((M-e)/T; q^k) > 0$. The required result holds if $p(M/(T\cup e); q^k) > 0$. Therefore we may assume that $p(M/(T\cup e); q^k) = 0$. Now $(M-e)/T = (M/T)-e$ and therefore $p((M/T)-e; q^k) > 0$. Hence e is not a loop or an isthmus of M/T. Thus

$$p(M/T; q^k) = p((M/T)-e; q^k) - p(M/(T\cup e); q^k) > 0.$$

We conclude that $c(M/T; q) \leqslant k$ and, since $T \subseteq S-e$, the lemma follows. □

6.4.7. Proposition. *Let N be a matroid and λ be a real number such that $p(N'; \lambda) \geqslant 0$ for all minors N' of N. Suppose that T is a subset of $E(N)$ for which $p(N/T; \lambda) > 0$. Then $p(N-T; \lambda) > 0$.*

Proof. This follows by a similar induction argument to that given in the last proof and is left as an exercise for the reader. □

6.4.8. Corollary. (*Oxley, 1978a*) *If T is a subset of $E(M)$, then*

$$c(M-T; q) \leqslant c(M/T; q).$$

Proof. This follows by taking λ equal to each of $q, q^2, q^3, \ldots$ in 6.4.7. □

We are now ready to prove Proposition 6.4.5.

Proof of Proposition 6.4.5. We shall show the equivalence of 6.4.5(i), 6.4.5(ii), and the following statement, which is easily seen to be equivalent to 6.4.5(iii).

(iii′) S is minimal with the property that, for some subset T of S, $c((M-T)/(S-T); q) \leqslant k$.

We begin by showing that (i) implies (iii′). Let S be a minimal set for which $c(M-S; q) \leqslant k$. Then $c((M-S)/\varnothing; q) \leqslant k$. Suppose that S' and T are sets for which $T \subseteq S' \subsetneqq S$ and $c((M-T)/(S'-T); q) \leqslant k$. Then, by Corollary 6.4.8, $c((M-T)-(S'-T); q) \leqslant k$, that is, $c(M-S'; q) \leqslant k$. This contradicts the choice of S.

To show that (iii′) implies (ii), suppose that S satisfies (iii′). Then, as $(M-T)/(S-T) = (M/(S-T))-T$, the latter has critical exponent not exceeding k. Hence, by Lemma 6.4.6, there is a subset T' of T such that $c((M/(S-T))/(T-T'); q) \leqslant k$, that is, $c(M/(S-T'); q) \leqslant k$, or equivalently, $c((M-\varnothing)/(S-T'); q) \leqslant k$. By the choice of S, it follows that $T' = \varnothing$. Hence $c(M/S; q) \leqslant k$. Moreover, if $S' \subsetneqq S$ and $c(M/S'; q) \leqslant k$, then $c((M-\varnothing)/S'; q) \leqslant k$, contrary to the choice of S.

A similar argument shows that (ii) implies (i) and this completes the proof of Proposition 6.4.5. □

A matroid M coordinatizable over $GF(q)$ is called *affine* if $c(M; q) = 1$. To justify this terminology, note that, from Corollary 6.4.3, M is affine if and only if the simplification of M is a subgeometry of the affine space $AG(r, q)$ for some r. Recall here that $AG(r, q)$ is obtained from the projective space $PG(r, q)$ by deleting the points of a hyperplane.

The next two observations come from combining Proposition 6.4.5 with Corollary 6.4.3 and Lemma 6.4.6.

$$c(M; q) = \min\left\{n \in \mathbb{N}:\ E(M) = \bigcup_{i=1}^{n} S_i \text{ and } M(S_i) \text{ is affine for all } i\right\}. \qquad (6.50)$$

$$c(M; q) = \min\left\{n \in \mathbb{N}:\ E(M) = \bigcup_{i=1}^{n} S_i \text{ and } M/(E-S_i) \text{ is affine for all } i\right\}. \qquad (6.51)$$

For a loopless graph Γ, the chromatic number $\chi(\Gamma)$ satisfies

$$\chi(\Gamma) = \min\{n \in \mathbb{Z}^+ : p(M_\Gamma; n) > 0\}. \qquad (6.52)$$

It turns out to be quite fruitful to exploit the similarity between this and the definition of the critical exponent (6.48). Many bounds on the chromatic number of a graph are expressed in terms of vertex degrees. By analogy with this, the next result bounds $c(M; q)$ in terms of the sizes of its bonds. For a matroid N, we denote the set of simple submatroids of N by $\mathbb{R}(N)$. The set of bonds of N will be denoted by $\mathscr{C}^*(N)$.

6.4.9. Proposition. *If M is a loopless matroid coordinatizable over $GF(q)$, then*

$$c(M;q) \leqslant \left\lceil \log_q\left(1 + \max_{N \in \mathcal{R}(M)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)\right)\right\rceil.$$

The proof of this result depends upon the following useful lemma for the characteristic polynomial.

6.4.10. Lemma. (*Oxley, 1978a*) *Let $\{e_1, e_2, \ldots, e_k\}$ be a bond C^* of a matroid M. Then*

$$p(M;\lambda) = (\lambda - k)p(M - C^*;\lambda) + \sum_{j=2}^{k}\sum_{i=1}^{j-1} p(M - \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{j-1}\}/\{e_i, e_j\};\lambda).$$

Proof. We argue by induction on k. If $k = 1$, then e_1 is an isthmus of M and the result is immediate. Assume the result holds for $k < n$ and let $k = n \geqslant 2$. Then

$$p(M;\lambda) = p(M - e_1;\lambda) - p(M/e_1;\lambda). \tag{6.53}$$

If e_2 is not a loop of M/e_1, then

$$p(M/e_1;\lambda) = p(M/e_1 - e_2;\lambda) - p(M/e_1/e_2;\lambda). \tag{6.54}$$

But, if e_2 is a loop of M/e_1, then $p(M/e_1;\lambda) = 0$ and $M/e_1 - e_2 \cong M/\{e_1, e_2\}$, hence, (6.54) also holds in this case. On substituting (6.54) into (6.53), we obtain

$$p(M;\lambda) = p(M - e_1;\lambda) - p(M/e_1 - e_2;\lambda) + p(M/\{e_1, e_2\};\lambda). \tag{6.55}$$

If $k = 2$, then $M - e_1$ and $M - e_2$ have e_2 and e_1 respectively as isthmuses. Thus $p(M - e_1;\lambda) = p(M - e_2;\lambda) = (\lambda - 1)p(M - C^*;\lambda)$, and $M/e_1 - e_2 \cong M - e_2/e_1 \cong M - e_2 - e_1 \cong M - C^*$. On combining these observations with (6.55), we deduce that the required result holds for $k = 2$. We may therefore suppose that $k > 2$. Then, as $M/e_1 - e_2 = M - e_2/e_1$, and e_1 is neither a loop nor an isthmus of $M - e_2$, we have

$$p(M/e_1 - e_2;\lambda) = p(M - e_2 - e_1;\lambda) - p(M - e_2;\lambda). \tag{6.56}$$

On substituting (6.56) into (6.55), we get that

$$p(M;\lambda) = p(M - e_1;\lambda) + p(M - e_2;\lambda) - p(M - \{e_1, e_2\};\lambda) + p(M/\{e_1, e_2\};\lambda).$$

As $C^* - A$ is a bond of $M - A$ for every proper subset A of C^*, we may now apply the induction assumption to each of the matroids $M - e_1$, $M - e_2$, and $M - \{e_1, e_2\}$ to get the required result. The straightforward details are omitted here. □

Proof of Proposition 6.4.9. We argue by induction on $|E(M)|$. The result is true for $|E(M)| = 1$. Assume it to be true for all matroids on sets with fewer than n elements and suppose that $|E(M)| = n$. If M has an element e that is in a 2-element circuit, then $\mathcal{R}(M - e) = \mathcal{R}(M)$, $c(M - e;q) = c(M;q)$ and we can deduce the result by applying the induction assumption to $M - e$. We may now suppose that M is simple. Then, for a bond D^* of M of minimum size, we have, by (6.47) and Lemma 6.4.10, that $c(M;q) \leqslant \max\{\lceil \log_q(|D^*| + 1)\rceil, c(M - D^*;q)\}$. But

$$|D^*| = \min_{C^* \in \mathcal{C}^*(M)} |C^*| \leqslant \max_{N \in \mathcal{R}(M)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right).$$

Moreover, by the induction assumption,

$$c(M - D^*;q) \leqslant \left\lceil \log_q\left(1 + \max_{N \in \mathcal{R}(M - D^*)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)\right)\right\rceil \leqslant \left\lceil \log_q\left(1 + \max_{N \in \mathcal{R}(M)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)\right)\right\rceil.$$

The required result now follows by induction. □

The proof of the following consequence of Proposition 6.4.9 is left as an exercise for the reader.

6.4.11. Corollary. *Suppose that M is coordinatizable over $GF(q)$. If there is a covering of $E(M)$ with bonds each having fewer than q^k elements, then $c(M;q) \leqslant k$.*

If $E(M)$ can be covered by *disjoint* bonds, then we have the following:

6.4.12. Proposition. (*Oxley, 1978a*) *Suppose that M is coordinatizable over $GF(q)$ and $E(M)$ is a disjoint union of bonds. Then M is affine, that is, $c(M;q) = 1$.*

For $q = 2$, the converse of the last proposition holds (Brylawski, 1972b) (see Exercise 6.50). To see that the converse does not hold for $q > 2$, consider the affine plane $AG(2, q)$.

6.4.B. Minimal and Tangential Blocks

If M is a loopless matroid coordinatizable over $GF(q)$, then M and its simplification have the same characteristic polynomial and therefore have the same critical exponent. Thus, for the moment, we shall suppose that M is simple. Then M can be embedded as a submatroid of $PG(n-1, q)$ for some n. In general, several different embeddings are possible. However, using the fact that $PG(n-1, q)$ is isomorphic to the simple matroid associated with $V(n, q)$, we deduce from Corollary 6.4.3 that the value of $c(M;q)$ does not depend on the embedding.

6.4.13. Proposition. *If M is isomorphic to the restriction of $PG(n-1, q)$ to the set E, then*

$$c(M;q) = \min\left\{j \in \mathbb{N}: PG(n-1,q) \text{ has hyperplanes } H_1, H_2, \ldots, H_j \text{ such that } \left(\bigcap_{i=1}^{j} H_i\right) \cap E = \varnothing\right\}$$
$$= \min\left\{j \in \mathbb{N}: PG(n-1,q) \text{ has a flat of rank } n-j \text{ having empty intersection with } E\right\}.$$

For any positive integer k, we shall call a simple matroid M a *k-block over $GF(q)$* if $c(M;q) > k$. M is a *minimal k-block over $GF(q)$* if M is a k-block over $GF(q)$ but no proper submatroid of M is. It follows easily from Corollary 6.4.3 that M is a minimal k-block over $GF(q)$ if and only if $c(M;q) = k+1$ and, for all proper submatroids N of M, $c(N;q) \leqslant k$.

An elementary geometric argument shows that $PG(k,q)$ is a minimal k-block. Moreover, one can easily show using the characteristic polynomial that if Γ is a graph that is edge-minimal with the property of being properly (q^k+1)-colorable, then its polygon matroid $M(\Gamma)$ is a minimal k-block. One important such graph is $M(K_{q^k+1})$. An infinite family of minimal k-blocks can be constructed from these examples by using the fact that if M and N are these minimal k-blocks over $GF(q)$, so is their series connection, $S(M,N)$ (Oxley, 1980) (Exercise 6.60). In view of this observation, it seems natural to consider a strengthened notion of minimality for k-blocks. A subclass of the class of minimal k-blocks that has received considerable attention is the class of *tangential* k-blocks. A simple matroid M that is coordinatizable over $GF(q)$ is a *tangential k-block over $GF(q)$* if M is a k-block over $GF(q)$ but no simple proper minor of M is. It is not difficult to check that both $PG(k,q)$ and $M(K_{q^k+1})$ are tangential k-blocks. Moreover, since M and N are both minors of $S(M,N)$ (Brylawski, 1971), one cannot create new tangential k-blocks simply by taking series connections of these blocks.

The straightforward proof of the next result is left to the reader (Exercise 6.61).

6.4.14. Proposition. *The following statements are equivalent for a simple matroid M that is coordinatizable over $GF(q)$.*

(i) *M is a tangential k-block over $GF(q)$;*
(ii) *$c(M;q) > k$ and $c(N;q) \leqslant k$ for all loopless proper minors N of M;*
(iii) *$c(M;q) = k+1$ and $c(M/F;q) \leqslant k$ for all non-empty flats F of M.*

Tangential blocks were studied originally by Tutte (1966a). He concentrated on tangential 1- and 2-blocks over $GF(2)$ and began by showing that there is only one tangential 1-block over $GF(2)$. Recall that a matroid M coordinatizable over $GF(q)$ is affine if $c(M;q) = 1$.

6.4.15. Proposition. (*Tutte, 1966a*) *The unique tangential 1-block over $GF(2)$ is $M(K_3)$.*

Proof. This follows immediately from the fact that a binary matroid is affine if and only if it has no odd circuits. The proof of this is left to the reader (Exercise 6.50). □

We have already noted that $M(K_5)$ and $PG(2,2)$ are tangential 2-blocks over $GF(2)$. Moreover, as the Petersen graph P_{10} has no 4-flow, $M^*(P_{10})$ is a 2-block over $GF(2)$. Indeed, it is not difficult to check that this 2-block is tangential.

The next theorem is the main result of Tutte's paper (1966a). F_7 denotes the Fano matroid, $PG(2,2)$.

6.4.16. Theorem. *The only tangential 2-blocks over $GF(2)$ of rank at most 6 are F_7, $M(K_5)$, and $M^*(P_{10})$.*

Tutte also conjectured that the restriction on the rank in this theorem could be dropped:

6.4.17. Conjecture. *The only tangential 2-blocks over $GF(2)$ are F_7, $M(K_5)$, and $M^*(P_{10})$.*

Using geometric methods, Datta (1976b; 1981) proved that there are no tangential 2-blocks over $GF(2)$ of rank 7 or 8. Conjecture 6.4.17 remains one of the most important unsolved problems in this area of combinatorics. The most significant advance toward its solution was made by Seymour (1981b) who proved the following result.

6.4.18. Theorem. *Let M be a tangential 2-block over $GF(2)$ and suppose that M is not isomorphic to F_7 or $M(K_5)$. Then M is cographic.*

The proof of this theorem uses a number of very powerful results including the Four Color theorem (6.3.2) and Seymour's decomposition theorem for regular matroids (1980). We omit the details and refer the reader to Welsh (1982) for an outline of the proof.

An interesting consequence of Theorem 6.4.18 is that Conjecture 6.4.17 is equivalent to Tutte's 4-flow conjecture (6.3.11). The proof of this equivalence is straightforward and is based on the observation that a graph Γ without isthmuses has a nowhere-zero 4-flow if and only if $M^*(\Gamma)$ is a 2-block over $GF(2)$.

There are a number of results for tangential blocks over fields other than $GF(2)$. These results indicate that the binary case is certainly the nicest. By arguing in terms of the characteristic polynomial it is straightforward to prove the following result.

6.4.19. Proposition. *Suppose that M is coordinatizable over $GF(q)$, and j and k are positive integers such that j divides k. Then M is a tangential k-block over $GF(q)$ if and only if M is a tangential j-block over $GF(q^{k/j})$.*

We observe here that the assumption that M is coordinatizable over $GF(q)$ is redundant above if M is a tangential k-block over $GF(q)$, but may be needed if M is a tangential j-block over $GF(q^{k/j})$, since such a matroid need not be coordinatizable over $GF(q)$. The next result was proved by Walton & Welsh (1982).

6.4.20. Proposition. *The only tangential 1-blocks over $GF(3)$ are $M(K_4)$ and $U_{2,4}$.*

The proof of this will use the following result of Brylawski (1971).

6.4.21. Proposition. *Let Γ be a loopless series–parallel network. Then Γ has a proper 3-coloring.*

Proof. We argue by induction on $|E(\Gamma)|$ to show that $p(M(\Gamma); 3) > 0$. If Γ is a forest having m edges, then $p(M(\Gamma); \lambda) = (\lambda - 1)^m$, hence $p(M(\Gamma); 3) > 0$. Thus the proposition is true in this case. Assume it to be true for $|E(\Gamma)| < n$ and let $|E(\Gamma)| = n$. We may suppose that Γ is not a forest. Then $E(\Gamma)$ has a subset $\{e_1, e_2\}$ that is either a circuit or a bond of $M(\Gamma)$. In the first case, $p(M(\Gamma); \lambda) = p(M(\Gamma - e_2); \lambda)$ and the result follows by the induction assumption. In the second case, by Lemma 6.4.10,

$$p(M(\Gamma); \lambda) = (\lambda - 2)p(M(\Gamma) - \{e_1, e_2\}; \lambda) + p(M(\Gamma)/\{e_1, e_2\}; \lambda).$$

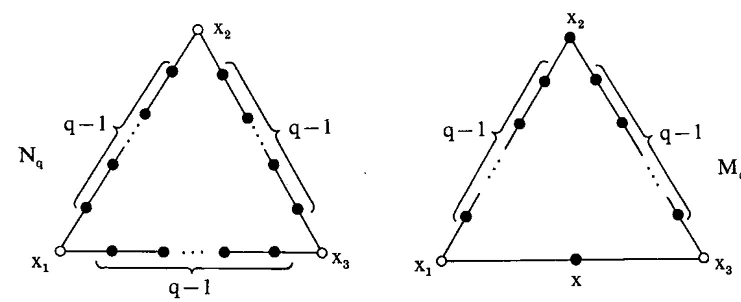
By the induction assumption, when $\lambda = 3$, the first term on the right-hand side is positive. Since the second term is non-negative, the result follows. □

Proof of Proposition 6.4.20. $M(K_4) = M(K_{3^1+1})$ and $U_{2,4} \cong PG(1, 3)$, hence both $M(K_4)$ and $U_{2,4}$ are tangential 1-blocks over $GF(3)$. If M is a tangential 1-block having no minor isomorphic to $M(K_4)$ or $U_{2,4}$, then, by Table 7.1 (p. 146) of White (1986), $M \cong M(\Gamma)$ where Γ is a series–parallel network. By Proposition 6.4.21, $p(M(\Gamma); 3) > 0$. Hence $c(M; 3) = 1$; a contradiction. □

Proposition 6.4.20 and Conjecture 6.4.17 suggest that tangential blocks are relatively scarce. Indeed, Welsh (1980) made several conjectures to this effect. Subsequently, he and Seymour (Walton, 1981) and, independently, Whittle (1987) gave a number of examples to disprove these conjectures, thereby showing that there are many more tangential blocks than had previously been thought.

6.4.22. Example. (Seymour & Welsh, in Walton, 1981) Let q be a prime power exceeding two and let x_1, x_2, and x_3 be three non-collinear points of $PG(2, q)$. Let S_1 be the set of points of $PG(2, q)$ that lie on one of the lines spanned by $\{x_1, x_2\}$, $\{x_2, x_3\}$, and $\{x_3, x_1\}$. Let x be an arbitrary point that is on the line spanned by $\{x_3, x_1\}$ but different from x_1 and x_3. Let S_2 be the set of points of $PG(2, q)$ that lie on one of the lines spanned by $\{x_1, x_2\}$ and $\{x_2, x_3\}$, together with the point x. The geometries N_q and M_q are obtained from $PG\{2, q)$ by restricting to the sets $S_1 - \{x_1, x_2, x_3\}$ and $S_2 - \{x_1, x_3\}$, respectively. Affine representations for these geometries are shown in Figure 6.7, where we note that a number of lines have been left out to avoid cluttering the diagrams. We remark, without proof, that $N_4 \cong AG(2, 3)$ (Exercise 6.62).

Figure 6.7.



6.4.23. Proposition. *M_q and N_q are tangential 1-blocks over $GF(q)$.*

Proof. In view of Proposition 6.4.14, to establish that a member M of $\{M_q, N_q\}$ is a minimal 1-block over $GF(q)$, we need to check that the following hold. We leave these checks to the reader.

(1) The ground set E of M intersects every line of $PG(2, q)$.
(2) For every point p of M, there is a line of $PG(2, q)$ that meets E in p only.

Evidently, for any point e of M, the simplification of M/e is not $PG(1, q)$. Thus no minor of M of lesser rank is also a 1-block. It follows from this that M is indeed a tangential 1-block. □

6.4.24. Example. (Whittle, 1989a) Let A be a subgroup of order m of $GF(q)^*$, the multiplicative group of $GF(q)$. Let $\{v_1, v_2, \ldots, v_r\}$ be a basis B for $V(r, q)$ and let D be $\{v_i + (-1)^{i-j+1}av_j\colon 1 \leqslant i < j \leqslant r$ and $a \in A\}$. The matroids $Q_r(A)$ and $Q'_r(A)$ are obtained by restricting $V(r, q)$ to the sets $B \cup D$ and D, respectively. Thus, for example, $Q'_2(GF(q)^*)$ is precisely the matroid N_q in Figure 6.7, while $Q_2(GF(q)^*)$ is obtained from N_q by adjoining the points x_1,

x_2, and x_3 to N_q. If A is the trivial group, then it is straightforward to show that $Q_r(A) \cong M(K_{r+1})$. More generally, it can be shown that $Q_r(A)$ depends only on r and the group A and not on the prime power q.

The matroid $Q_r(A)$ was introduced by Dowling (1973a, b) and is now known as the *rank r Dowling geometry based on the group A*. In fact, Dowling defined such matroids when A is an arbitrary finite group. Our main interest here will be in the special case defined above, although we remark that Whittle (1989a) has described an interesting extension of the critical problem to Dowling geometries in general.

Since $Q'_2(GF(q)^*) \cong N_q$, Proposition 6.4.23 implies that $Q'_2(GF(q)^*)$ is a tangential 1-block over $GF(q)$. Moreover, when A is the trivial group and r is q^k, $Q_r(A)$ is the tangential k-block $M(K_{q^k+1})$ over $GF(q)$. These observations are special cases of the following result of Whittle (1989a).

6.4.25. Proposition. *Let A be a subgroup of GF(q)*. Then*

(i) *for* $r = \dfrac{q^k-1}{|A|} + 1$, $Q_r(A)$ *is a tangential k-block over GF(q); and*

(ii) *for* $r = \dfrac{q^k-1}{|A|} + 2$, $Q'_r(A)$ *is a tangential k-block over GF(q).*

In an important sequence of papers Whittle (1987, 1988, 1989b) noted several general constructions for using known tangential blocks to find others. Next we describe the simplest of these constructions, which, curiously, was the last to be found. Suppose that M is a rank r geometry that is coordinatizable over $GF(q)$. Let E be a subset of $PG(r, q)$ such that the restriction of $PG(r, q)$ to this set is isomorphic to M. Clearly $\mathrm{cl}_P(E)$ is a hyperplane of $PG(r, q)$ where cl_P denotes the closure operator of $PG(r, q)$. Now take a point p of $PG(r, q)$ that is not in $\mathrm{cl}_P(E)$. Let E' be the set of all points of $PG(r, q)$ lying on some line that contains p and some point of E, that is, $E' = \bigcup_{x \in E} \mathrm{cl}_P(\{x, p\})$. We call the restriction of $PG(r, q)$ to E' a *q-lift* of M. Thus, for example, $PG(2, 2)$ is a 2-lift of the 3-point line $U_{2,3}$, while a 3-lift of $U_{2,3}$ is the complement in $PG(2, 3)$ of $U_{2,3}$.

6.4.26. Proposition. (*Whittle, 1989b*). *If M is a tangential k-block over GF(q) and M′ is a q-lift of M, then M′ is a tangential (k + 1)-block over GF(q).*

The last construction produces tangential $(k+1)$-blocks from tangential k-blocks. Next we shall describe two special cases of a quotient construction of Whittle (1988) that produces tangential k-blocks from tangential k-blocks.

A description of the general quotient construction can be found in Exercise 6.67. A rank r matroid M is *supersolvable* if there is a set $\{F_0, F_1, F_2, \ldots, F_r\}$ of modular flats of M with $r(F_i) = i$ for $0 \leqslant i \leqslant r$ and $F_i \supseteq F_{i-1}$ for $1 \leqslant i \leqslant r$. We call the set $\{F_0, F_1, F_2, \ldots, F_r\}$ a *maximal chain of modular flats.*

Now suppose that M is a supersolvable rank r tangential k-block over $GF(q)$ with $r > k+1$. Then we can embed M in $PG(r-1, q)$. Let $\{F_0, F_1, F_2, \ldots, F_r\}$ be a maximal chain of modular flats in M. Then $F_0 = \emptyset$ and $F_r = E(M)$. Because $r > k+1$, $M \not\cong PG(k, q)$. Let m be the least element of the set $\{i\colon 2 \leqslant i \leqslant r \text{ and } M(F_i) \not\cong PG(i-1, q)\}$. Since M does not have $PG(k, q)$ as a minor, $m \leqslant k+1$. But $r > k+1$ and so F_m is a proper flat of M. As $M(F_m) \not\cong PG(m-1, q)$, the closure of F_m in $PG(r-1, q)$ contains an element x that is not in F_m. Let M'' be the elementary quotient of M by the element x, that is, M'' is formed by first extending M by adding x and then contracting x.

6.4.27. Theorem. (*Whittle, 1987*) *The simplification of M″ is a rank (r − 1) tangential k-block over GF(q).*

We shall not prove this result here but instead we note the following important consequence of it.

6.4.28. Corollary. (*Whittle, 1987*) *For all r such that* $k+1 \leqslant r \leqslant q^k$, *there is a rank r tangential k-block over GF(q).*

Proof. $M(K_{q^k+1})$ is a supersolvable tangential k-block over $GF(q)$. By repeatedly applying the above construction, the corollary follows. □

The second special case of Whittle's quotient construction that we shall consider involves the complete principal truncation $\bar{T}_F(M)$ of the matroid M with respect to the flat F (see White, 1986, p. 149). If $r(F) = j > 0$, we recall that $\bar{T}_F(M)$ is formed from M by putting a set P of $j-1$ independent points freely on F and then contracting P. The bases of $\bar{T}_F(M)$ are the subsets of $E(M)$ of the form B or $B' \cup x$ where x is a non-loop element of F, and B and B' are subsets of $E - F$ that are independent in M such that $|B| = r - j + 1$, $|B'| = r - j$, and $r(B \cup F) = r(B' \cup F) = r$.

6.4.29. Theorem. (*Whittle, 1987*) *Let M be a tangential k-block over GF(q) and F be a proper non-empty modular flat of M. If* $\bar{T}_F(M)$ *is coordinatizable over GF(q), then the simple matroid associated with* $\bar{T}_F(M)$ *is a tangential k-block over GF(q).*

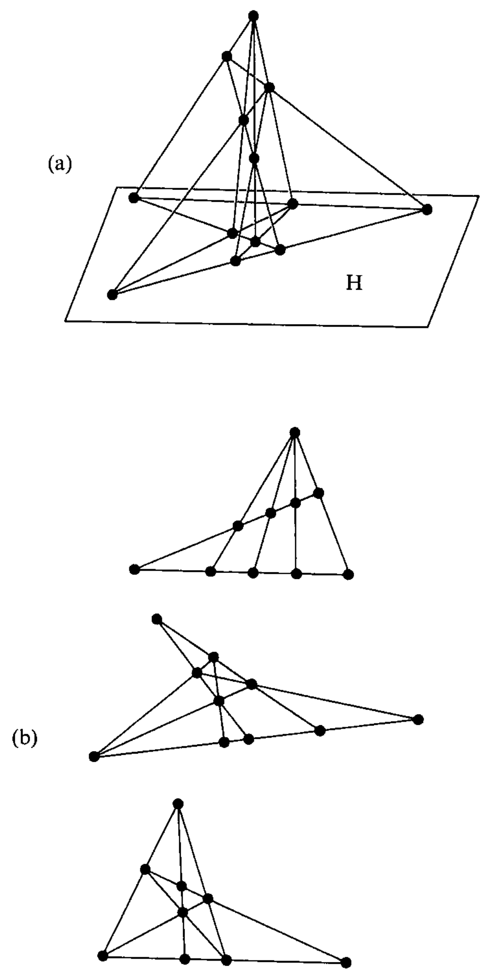
From this, Whittle deduced the following.

6.4.30. Corollary. *For all m with $2 \leqslant m \leqslant q$, the simplification of $\overline{T}_{M(K_m)}(M(K_{q^k+1}))$ is a tangential k-block over GF(q) with rank $q^k - m + 2$ and with $\frac{1}{2}(q^k+1)q^k - \frac{1}{2}m(m+1)+1$ elements.*

It is not difficult to check that, for $n = q^k$, the simplification of $\overline{T}_{M(K_{n-1})}(M(K_{n+1}))$ is isomorphic to the matroid M_n in Example 6.4.22.

We conclude this discussion of tangential blocks with some diagrams of such matroids from Whittle (1985) and with some unsolved problems. The matroid shown in Figure 6.8a is the simplification of $\overline{T}_{M(K_3)}(M(K_6))$ and is a tangential 1-block over $GF(5)$. The restriction of this matroid to the hyperplane H is the simplification of $\overline{T}_{M(K_3)}(M(K_5))$ and is a tangential 1-block over $GF(4)$.

Figure 6.8.



The three matroids shown in Figure 6.8b are all examples of tangential 1-blocks over $GF(5)$. The first is the simplification of $\overline{T}_{M(K_4)}(M(K_6))$. We note that both the first and second have characteristic polynomial equal to $(\lambda-1)(\lambda-4)(\lambda-5)$, while the third has characteristic polynomial $(\lambda-1)(\lambda-3)(\lambda-5)$. Each of these three matroids is the simplification of some quotient of $M(K_6)$.

In connection with his constructions, Whittle raised several questions. Call a tangential k-block over $GF(q)$ *normal* if it is the simplification of a quotient of $M(K_{q^k+1})$. Not every tangential k-block is normal; for example, $M^*(P_{10})$ is a non-normal 2-block over $GF(2)$.

6.4.31. Problem

(i) Are there any non-normal supersolvable tangential k-blocks?
Less strongly:
(ii) Are there any non-normal tangential k-blocks with modular hyperplanes?
Conversely:
(iii) Does every normal tangential k-block have a modular hyperplane?
More strongly:
(iv) Is every normal tangential k-block supersolvable?

Another interesting unsolved problem raised by Whittle (1985) is the following.

6.4.32. Problem. Do all tangential k-blocks contain a spanning bond?

Given that tangential blocks are much more abundant than was once thought, the approach to classifying such objects has somewhat changed. Whittle (1987) showed that one group of well-behaved tangential blocks is those with modular hyperplanes. He also showed (Whittle, 1989a, b) that if $|A| \geqslant 2$, then $Q'_r(A)$ has no modular hyperplanes unless $r = 3$ and $|A| = 2$; that $M^*(P_{10})$ has no modular hyperplanes; and that a q-lift of a matroid M has no modular hyperplanes if and only if M has no modular hyperplanes. On combining these observations with Propositions 6.4.15 and 6.4.20, we deduce that there are tangential k-blocks over $GF(q)$ with no modular hyperplanes for all prime powers q and all positive integers k except when k is 1 and q is 2 or 3. Interestingly, it is precisely in the exceptional cases just noted that the problem of finding all tangential k-blocks over $GF(q)$ has been solved. Indeed, Whittle (1989b) asserts that the existence of tangential blocks without modular hyperplanes lies at the heart of the problem of determining all tangential blocks.

6.4.C. Bounding the Critical Exponent for Classes of Matroids

In this section we survey a number of results and conjectures concerned with determining an upper bound on the critical exponent of a matroid M when M is in some class of matroids characterized by excluded minors. Most of this work has appeared since 1980 and is related to the following conjecture of Brylawski (1975c).

6.4.33. Conjecture. *If M is a loopless matroid coordinatizable over $GF(q)$ and M has no minor isomorphic to $M(K_4)$, then $c(M; q) \leqslant 2$.*

Restated in terms of k-blocks this conjecture asserts that every tangential 2-block over $GF(q)$ has a minor isomorphic to $M(K_4)$.

The following very general extension of this conjecture was proposed by Whittle (1985).

6.4.34. Conjecture. *Every tangential k-block over $GF(q)$ has a minor isomorphic to $M(K_{k+2})$.*

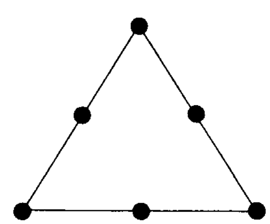
This conjecture is easily seen to be true for $k = 1$. In support of the general case of the conjecture, Whittle (1987) has proved the following result.

6.4.35. Theorem. *A tangential k-block over $GF(q)$ that has a modular hyperplane has $M(K_{k+2})$ as a submatroid.*

Return now to Brylawski's conjecture. It is certainly true for $q = 2$. To see this, recall from the proof of Proposition 6.4.20 that a loopless matroid with no minor isomorphic to $U_{2,4}$ or $M(K_4)$ is isomorphic to $M(\Gamma)$ for some series–parallel network Γ. By Proposition 6.4.21, Γ is 3-colorable. Hence Γ is certainly 4-colorable, so $p(M(\Gamma); 4) > 0$ and hence $c(M(\Gamma); 2) \leqslant 2$.

For larger values of q, the conjecture is much more difficult. The following resolution of the conjecture for the case $q = 3$ and partial result for the case $q = 4$ were obtained by Oxley (1987b) as consequences of non-trivial structure theorems for the classes of matroids involved. The matroid $\mathcal{W}^3$ is the rank 3 whirl; an affine representation for it is shown in Figure 6.9.

Figure 6.9.

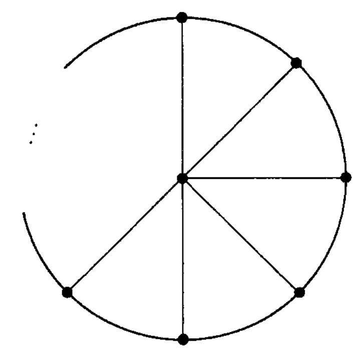


6.4.36. Proposition. *Let M be a loopless ternary matroid having no minor isomorphic to $M(K_4)$. Then $c(M; 3) \leqslant 2$.*

6.4.37. Proposition. *Let M be a loopless matroid that is coordinatizable over $GF(4)$. Suppose that M has no minor isomorphic to $M(K_4)$ or $\mathcal{W}^3$. Then $c(M; 4) \leqslant 2$.*

The graph K_4 is isomorphic to the 3-spoked wheel graph $\mathcal{W}_3$ where $\mathcal{W}_r$ is shown in Figure 6.10. The polygon matroids of the wheel graphs are of fundamental structural importance in the class of matroids (see, for example, Tutte, 1966b), and the following conjecture of Oxley is an alternative strengthening of Brylawski's conjecture in the case $q = 2$.

Figure 6.10.



6.4.38. Conjecture. *Let M be a loopless binary matroid having no minor isomorphic to $M(\mathcal{W}_r)$. Then $c(M; 2) \leqslant r - 1$.*

The truth of Conjecture 6.4.34 would imply the truth of this conjecture. However, both conjectures seem very difficult. Conjecture 6.4.38 holds when $r = 3$ since it is equivalent to the case $q = 2$ of Conjecture 6.4.33, and we showed above that the latter is true. Moreover, Oxley proved Conjecture 6.4.38 in general when $r = 4$ (Oxley, 1987a) and for regular matroids when $r = 5$ (Oxley, 1989a). In the latter special case, the stronger bound $c(M; 2) \leqslant r - 2$ holds. Again both these results were derived from results on the structure of the relevant classes of matroids.

Prior to Whittle's advancing Conjecture 6.4.34, Walton & Welsh (1980) had proposed the following:

6.4.39. Conjecture. *If M is a loopless binary matroid having no minor isomorphic to $M(K_5)$, then $c(M; 2) \leqslant 3$.*

Welsh (1979) also offered the weaker conjecture that there is a fixed positive integer k so that, for all loopless binary matroids M having no minor isomorphic to $M(K_5)$, $c(M; 2) \leqslant k$. The corresponding weakenings of each of Conjectures 6.4.34 and 6.4.38 are both open and would certainly be sensible starting points for the conjectures themselves. We shall describe next some results of Kung that resolve Welsh's conjecture as well as the weakened form of Conjecture 6.4.33 that seeks only a fixed bound on $c(M; q)$ for all loopless matroids M that are coordinatizable over $GF(q)$ and have no minor isomorphic to $M(K_4)$.

Let $\mathscr{G}$ be a class of geometries that is closed under deletion. Kung defines its *size function*, $h(\mathscr{G}, r)$, to be the function with domain $D = \{r \in \mathbb{N}: \mathscr{G}$ contains a rank r geometry$\}$ for which $h(\mathscr{G}, r) = \max\{|E(M)|: M \in \mathscr{G}, r(M) = r\}$.

The *growth rate*, $g(\mathscr{G}, r)$, of $\mathscr{G}$ is defined, for all positive integers r in D, by

$$g(\mathscr{G}, r) = h(\mathscr{G}, r) - h(\mathscr{G}, r-1).$$

The *maximum growth rate* $g(\mathscr{G})$ of $\mathscr{G}$ is $\max\{g(\mathscr{G}, r): r \in D - \{0\}\}$, provided this maximum exists, and is infinite otherwise.

The following conjecture is due to Kung (1986a). A class of geometries is *minor-closed* if every geometry that is a minor of a member of the class is also in the class.

6.4.40. Conjecture. *Let $\mathscr{G}$ be a minor-closed class of geometries coordinatizable over $GF(q)$. Then the maximum growth rate of $\mathscr{G}$ is finite if and only if $\max\{c(M; q): M \in \mathscr{G}\}$ is finite.*

In one direction this conjecture is proved by the following result (Kung, 1986a). The other direction remains open.

6.4.41. Proposition. *Let $\mathscr{G}$ be a class of geometries coordinatizable over $GF(q)$ and closed under deletion. Suppose that the size function of $\mathscr{G}$ satisfies $h(\mathscr{G}, r) \leqslant cr$ for some integer c. Then $c(M; q) \leqslant c$ for all M in $\mathscr{G}$.*

Proof. Let $M(E)$ be a member of $\mathscr{G}$. As $\mathscr{G}$ is closed under deletion, for all subsets E' of E, $|E'| \leqslant cr(E')$. Thus, by Edmonds' covering theorem (1965b), E can be partitioned into c independent sets. Since independent sets are affine, it follows by (6.50) that $c(M; q) \leqslant c$. □

Kung (1986a, 1987) has proved a number of results on growth rates of various classes of geometries and from these has deduced, using the last result, bounds on the critical exponents of members of these classes. The next two propositions are examples of such results. The first proves the conjecture of Welsh stated after Conjecture 6.4.39. In each, M is a loopless binary matroid.

6.4.42. Proposition. *If M has no minor isomorphic to $M(K_5)$, then $c(M; 2) \leqslant 8$.*

6.4.43. Proposition. *If M has no minor isomorphic to $M(K_{3,3})$, then $c(M; 2) \leqslant 10$.*

The proofs of these results are long and involve operations on the bond graphs of binary geometries. For a binary geometry M having a hyperplane H, the *bond graph* $\Gamma(H, M)$ is the labelled graph defined as follows. The vertex set of $\Gamma(H, M)$ is the set of points of M not in H. Two vertices a and b are joined by an edge ab if there is a third point c on the line of M spanned by a and b. As $\{a, b, c\}$ is a circuit of M having an odd number of elements and M is binary, $\{a, b, c\}$ cannot be contained in the bond E–H of M. Thus $c \in H$. The edge ab of $\Gamma(H, M)$ is labelled by c.

An approach to Conjecture 6.4.39 that provides an alternative to that offered by Proposition 6.4.42 is to try to retain the original bound, but to prove the result for a subclass of the original class. The next two results are of this form. The first is due to Walton & Welsh (1980), the second to Kung (1986a). As above, M is a loopless binary matroid.

6.4.44. Proposition. *Suppose that M has no minor isomorphic to $M(K_5)$ or F_7^*. Then $c(M; 2) \leqslant 3$.*

For an outline of the proof of this proposition, see Exercise 6.55.

6.4.45. Proposition. *Suppose that M has no minor isomorphic to $M(K_5)$ or F_7. Then $c(M; 2) \leqslant 3$.*

The following result of Kung (1988) employs a modification of the technique used to prove Propositions 6.4.42, 6.4.43, and 6.4.45 to obtain a partial result toward Conjecture 6.4.33.

6.4.46. Proposition. *Let M be a loopless matroid coordinatizable over $GF(q)$ and having no $M(K_4)$-minor. Then $c(M; q) \leqslant 6q^3$.*

We conclude this section with a solution to the critical problem for the class of transversal matroids. Brylawski (1975c) proved that, for a loopless principal transversal matroid M that is coordinatizable over $GF(q)$, $c(M; q) \leqslant 2$. He also proposed the following extension of that result. Recall that a gammoid is a minor of a transversal matroid.

6.4.47. Conjecture. *Let M be a loopless gammoid coordinatizable over $GF(q)$. Then $c(M; q) \leqslant 2$.*

As $M(K_4)$ is not a gammoid, this conjecture is weaker than Conjecture 6.4.33. Moreover, since, the latter holds for $q=2$ and $q=3$, so does the former. The following result of Whittle (1984) verifies Brylawski's conjecture for transversal matroids.

6.4.48. Proposition. *Let M be a loopless transversal matroid coordinatizable over $GF(q)$. Then $c(M; q) \leqslant 2$.*

To prove this proposition we shall need to recall some basic facts about transversal matroids. Let $M(E)$ be such a matroid and $(A_1, A_2, \ldots, A_n)$ be a presentation of M, that is, the independent sets of M are the partial transversals of this family of sets. A *cyclic flat* of M is a flat that is a union of circuits. From Corollary 5.1.3 of White (1987), we deduce that if F is a proper cyclic flat of M, then

$$F = \cap(E - A_i), \tag{6.57}$$

where the intersection is taken over all i for which $F \cap A_i \neq \varnothing$.

The proof of Proposition 6.4.48 will use the next result of Bondy & Welsh (1972) and a lemma.

6.4.49. Proposition. *If $M(E)$ is a rank r transversal matroid, then M has a presentation $(A_1, A_2, \ldots, A_r)$ such that each A_i is a bond of M.*

6.4.50. Lemma. *Let $M(E)$ be a simple matroid coordinatizable over $GF(q)$. If E' is a subset of E that intersects all cyclic flats of M, then $c(M(E); q) \leqslant c(M(E'); q) + 1$.*

Proof. We identify $M(E)$ with a submatroid of $PG(r-1, q)$ to which it is isomorphic. Let $c(M(E'); q) = k$. Then, by Corollary 6.4.3, there are hyperplanes, $H_1, H_2, \ldots, H_k$, of $PG(r-1, \hat{q})$ *such that* $\left(\bigcap_{i=1}^{k} H_i\right) \cap E' = \varnothing$. If $\left(\bigcap_{i=1}^{k} H_i\right) \cap E$ contains a circuit C, then, since $\left(\bigcap_{i=1}^{k} H_i\right) \cap E$ is a flat of M, it contains $\mathrm{cl}_M(C)$, the closure in M of C. This set contains a point of E', hence so must $\left(\bigcap_{i=1}^{k} H_i\right) \cap E'$; a contradiction. We conclude that $\left(\bigcap_{i=1}^{k} H_i\right) \cap E$ is independent in M. Therefore the restriction of M to this set is affine, that is, $c(M(E-E'); q) = 1$. But, by Proposition 6.4.4, $c(M; q) \leqslant c(M(E'); q) + c(M(E-E'); q)$. Hence, $c(M; q) \leqslant c(M(E'); q) + 1$, as required. □

Proof of Proposition 6.4.48. We may assume that M is simple and that $r(M) = r$. Thus we can identify $M(E)$ with a submatroid of $PG(r-1, q)$ to which it is isomorphic. By Proposition 6.4.49, M has a presentation $(A_1, A_2, \ldots, A_r)$ such that each A_i is a bond of M. Let $I = \{1, 2, \ldots, r\}$ and, for all i in I, let H_i be the hyperplane of $PG(r-1, q)$ that is spanned by $E - A_i$.

We suppose first that $\bigcap_{i \in I} H_i = \varnothing$. For all j in I, consider $\bigcap_{i \in I - j} H_i$. As $\bigcap_{i \in I} H_i = \varnothing$ and H_j is a modular flat of $PG(r-1, q)$, $\bigcap_{i \in I-j} H_i$ has rank one and so contains a single point, say x_j. It follows, without difficulty, that $X = \{x_1, x_2, \ldots, x_r\}$ is independent in $PG(r-1, q)$. Hence X is a basis of $PG(r-1, q)$. Let N be the submatroid of $PG(r-1, q)$ on $E \cup X$. We shall show next that X meets every cyclic flat of N. Assume that F is such a flat and that $F \cap X = \varnothing$. Then $r(F) \leqslant r-1$, so F is a proper cyclic flat of M. Hence by (6.57), for some subset J of I,

$$F = \bigcap_{j \in J} (E - A_j) = E \cap \left(\bigcap_{j \in J} H_j\right).$$

But $\mathrm{cl}_N(F) = (E \cup X) \cap \mathrm{cl}_M(F) = (E \cup X) \cap \left(E \cap \left(\bigcap_{j \in J} H_j\right)\right) = (E \cup X) \cap \left(\bigcap_{j \in J} H_j\right)$.
Thus $\mathrm{cl}_N(F)$ contains $\{x_i\colon i \notin J\}$, that is, $\mathrm{cl}_N(F) \cap X \neq \varnothing$. Since $\mathrm{cl}_N(F) = F$, this is a contradiction. We conclude that X does indeed meet all cyclic flats of N. Thus, by Lemma 6.4.50, $c(N; q) \leqslant c(N(X); q) + 1$. But X is independent, so $N(X)$ is affine. Hence $c(N; q) \leqslant 2$, and so, by Proposition 6.4.4, $c(M; q) \leqslant 2$. This completes the proof in the case that $\bigcap_{i \in I} H_i = \varnothing$.

Now suppose that $\bigcap_{i \in I} H_i \neq \varnothing$ and choose x from $\bigcap_{i \in I} H_i$. This time we let N be the submatroid of $PG(r-1, q)$ on $E \cup x$. A similar argument to the above again shows that $c(N; q) \leqslant 2$ and hence that $c(M; q) \leqslant 2$. □

6.5. Linear Codes

In this section, we consider the work of Greene, Dowling, Jaeger, and Rosenstiehl & Read that applies Tutte–Grothendieck techniques to various problems related to linear codes.

To begin, recall that an $[n, r]$ *linear code* C over $GF(q)$ is an r-dimensional subspace of the n-dimensional vector space $V(n, q)$ over $GF(q)$. We call r and n, respectively, the *dimension* and *length* of C. If U is an $r \times n$ matrix over $GF(q)$, the rows of which form a basis for C, then U is called a *generator matrix* for C. It is straightforward to check that, for such a matrix U, the matroid on the columns of U depends only on C and not on U. We denote this matroid by $M(C)$. The *dual code* C^* of C is defined by $C^* = \{\mathbf{v} \in V(n, q)\colon \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \text{ in } C\}$. Evidently C^* is an $[n, n-r]$ linear code. Moreover, it follows by Proposition 5.4.1 of White (1986) that $M(C^*)$ is isomorphic to $M^*(C)$, the dual of $M(C)$.

The members of a linear code C are called *codewords*. If $\mathbf{v}$ is the codeword $(v_1, v_2, \ldots, v_n)$, its *weight* $w(\mathbf{v})$ is the cardinality of its support, that is, $w(\mathbf{v}) = |\{v_i: v_i \neq 0\}|$. Dimension and length are two of the three fundamental parameters associated with a linear code C. The third of these parameters is the *distance* d of C. This is defined to be $\min\{w(\mathbf{v}): \mathbf{v} \in C - \mathbf{0}\}$. Evidently d is the size of a smallest bond in $M(C)$, or equivalently, the size of a smallest circuit in $M^*(C)$. It is therefore easily determined from $t(M(C))$ using the formula

$$d = n - r + 1 - \max\{j: b_{ij} > 0, \text{ for some } i > 0\}. \qquad (6.58)$$

As a much deeper application of T–G techniques, we next present Greene's (1976) result that the distribution of codeweights in a linear code is a generalized T–G invariant. For a linear code C, the *codeweight polynomial* $A(C; q, z)$ of C is defined by

$$A(C; q, z) = \sum_{\mathbf{v} \in C} z^{w(\mathbf{v})}.$$

Thus, if a_i is the number of codewords $\mathbf{v}$ in C having weight i, then

$$A(C; q, z) = \sum_{i=0}^{n} a_i z^i.$$

The proofs below of Propositions 6.5.1 and 6.5.4 will use the following notation. If W is a subspace of $V(n, q)$, then W_0 will denote the subspace consisting of those vectors in W whose first entry is zero; $\hat{W}$ will denote the vector space obtained from W by removing the first entry of every vector. By convention, $\hat{W}_0 = \hat{W}$ where $W = W_0$.

6.5.1. Proposition.

$$A(C; q, z) = (1-z)^r z^{n-r} t\left(M(C); \frac{1+(q-1)z}{1-z}, \frac{1}{z}\right).$$

Proof. Let $f(M(C)) = A(C; q, z)$. We shall show that f is well defined and that it is a generalized T–G invariant for which $\sigma = z$ and $\tau = 1 - z$. We begin by noting that f is well defined if C has length 1 for, in that case,

$$f(M(C)) = \begin{cases} 1, & \text{if } M(C) \text{ is a loop,} \\ 1 + (q-1)z, & \text{if } M(C) \text{ is an isthmus.} \end{cases}$$

Assume that f is well defined if C has length less than m, and suppose that C has length m where $m \geqslant 2$.

Let U be a generator matrix for C and suppose that the element e of $M(C)$ is neither a loop nor an isthmus. Without loss of generality, we may assume that e corresponds to the first column of U. Let U' be the matrix obtained by row reducing U so that the first entry in the first column is 1 and all other entries in this column are 0. Clearly U' is also a generator matrix for C. From considering the matrix U', we easily deduce that

$$M(\hat{C}_0) = M(C)/e \quad \text{and} \quad M(\hat{C}) = M(C) - e. \qquad (6.59)$$

Now consider the map $g: C - C_0 \to \hat{C} - \hat{C}_0$ that removes the first entry of each codeword, that is, $g((v_1, \mathbf{v}')) = \mathbf{v}'$. We show next that

$$g \text{ is a bijection.} \qquad (6.60)$$

First observe that if $(v_1, \mathbf{v}')$ and $(u_1, \mathbf{v}')$ are both in C for some distinct v_1 and u_1, then $(v_1 - u_1, \mathbf{0}) \in C$. But this implies the contradiction that e is an isthmus of $M(C)$. Hence the image of g is indeed $\hat{C} - \hat{C}_0$ and (6.60) holds. By definition, $A(C) = \sum_{\mathbf{v} \in C} z^{w(\mathbf{v})}$. Thus

$$A(C) = \sum_{(v_1, \mathbf{v}') \in C_0} z^{w((v_1, \mathbf{v}'))} + \sum_{(v_1, \mathbf{v}') \in C - C_0} z^{w((v_1, \mathbf{v}'))} = \sum_{\mathbf{v}' \in \hat{C}_0} z^{w(\mathbf{v}')} + z \sum_{(v_1, \mathbf{v}') \in C - C_0} z^{w(\mathbf{v}')}.$$

Therefore, by (6.60),

$$A(C) = \sum_{\mathbf{v}' \in \hat{C}_0} z^{w(\mathbf{v}')} + z \sum_{\mathbf{v}' \in \hat{C} - \hat{C}_0} z^{w(\mathbf{v}')} = z \sum_{\mathbf{v}' \in \hat{C}} z^{w(\mathbf{v}')} + (1-z) \sum_{\mathbf{v}' \in \hat{C}_0} z^{w(\mathbf{v}')}$$
$$= zA(\hat{C}) + (1-z)A(\hat{C}_0).$$

Hence, by (6.59) and the induction assumption, if e is neither a loop nor an isthmus of $M(C)$, then

$$A(C) = zf(M(C) - e) + (1-z)f(M(C)/e). \qquad (6.61)$$

Now suppose that e is a loop of $M(C)$. Then $A(C) = A(\hat{C}_0)$, so, by the induction assumption,

$$A(C) = f(L)f(M(C)/e). \qquad (6.62)$$

Finally, if e is an isthmus of $M(C)$, then C is the direct sum of $\hat{C}$ with a one-dimensional space. Hence $A(C) = (1 + (q-1)z)A(\hat{C})$ so, by the induction assumption,

$$A(C) = f(I)f(M(C) - e). \qquad (6.63)$$

On combining (6.61)–(6.63), we conclude by induction that f is well defined. Moreover, the same equations imply that f is a generalized T–G invariant. The proposition now follows easily by Corollary 6.2.6. □

We now sketch an alternative derivation of Proposition 6.5.1 that mimics the treatment of two-variable coloring in section 6.3.F. As there, T denotes the $(n+1) \times (n+1)$ matrix with $T(i, j) = \binom{j}{i}$ where i and j both range over the set $\{0, 1, 2, \ldots, n\}$; and M_{KC} is the $(n+1) \times (r+1)$ matrix with rows and columns indexed by $\{0, 1, 2, \ldots, n\}$ and $\{0, 1, 2, \ldots, r\}$, respectively, such that

$M_{KC}(i, j)$ is the number of i-element subsets of $M(C)$ having corank j, that is, rank $r-j$. Now, recall that $A(C) = \sum_{i=0}^{r} a_i z^i$. Then, as in (6.43), we have the matrix equation

$$M_{KC} \cdot \mathbf{q} = T \cdot \mathbf{c}_q \tag{6.64}$$

where $\mathbf{q} = (1, q, q^2, \ldots, q^r)^t$ and $\mathbf{c}_q = (a_n, a_{n-1}, \ldots, a_0)^t$. Both sides of (6.64) are column vectors with rows indexed by i where $0 \leqslant i \leqslant n$. To verify (6.64), we observe that, for fixed i, the corresponding entries in these column vectors count the number of pairs $(P, \mathbf{w})$ where P is an i-element subset of the set of columns of the generator matrix U of C and $\mathbf{w}$ is a codeword that has entry 0 in every column corresponding to P. The left-hand side of (6.64) chooses P first according to its corank, while the right-hand side chooses $\mathbf{w}$ first according to its number of zero entries.

Proposition 6.5.1 can be derived from (6.64) by arguing as in section 6.3.F. We leave the details of this to the reader as an exercise. Again, as for two-variable coloring, we can invert Proposition 6.5.12 to obtain

$$t(M(C); x, y) = \frac{y^n}{(y-1)^r} A\left(C; (x-1)(y-1), \frac{1}{y}\right). \tag{6.65}$$

Hence we can recover the Tutte polynomial of a vector matroid M if we know its associated linear code over sufficiently many finite fields. For example, if M is coordinatizable over $GF(q)$, we could use the fields $GF(q^i)$ for $1 \leqslant i \leqslant r$.

The next result is the celebrated MacWilliams duality formula for linear codes (MacWilliams, 1963; Greene, 1976). It can be proved analogously to the proof of Proposition 6.3.27. We leave the details to the reader.

6.5.2. Proposition.

$$A(C^*; q, z) = \frac{(1+(q-1)z)^n}{q^r} A\left(C; q, \frac{1-z}{1+(q-1)z}\right).$$

Apart from Greene's work just described, the other pioneering work in the matroid invariant theory of linear codes was by Dowling (1971). He showed that a fundamental problem of coding theory, that of finding the maximum possible dimension r for a linear code over $GF(q)$ having length n and distance at least d, is a special case of the critical problem for matroids.

The precise statement of Dowling's result will require another definition. The *punctured Hamming ball*, $H_q(n, d-1)$, consists of all non-zero vectors of $V(n, q)$ having fewer than d non-zero coordinates. Evidently C is a maximum-dimension length n linear code having distance at least d if and only if C is a maximum-dimension subspace of $V(n, q)$ containing no member of $H_q(n, d-1)$. Thus the problem of maximizing the dimension of a code of distance at least d is equivalent to the problem of finding the critical exponent of the vector matroid on $H_q(n, d-1)$. In fact:

6.5.3. Proposition. (*Dowling, 1971*) *If r is the maximum dimension of a linear code over $GF(q)$ having length n and distance at least d, and c is the critical exponent of the vector matroid on $H_q(n, d-1)$, then $r = n - c$.*

Proof. This follows directly from the definition of the critical exponent. □

Let $G_q(n, d-1)$ be the simplification of the vector matroid on $H_q(n, d-1)$. Then $G_q(n, 2)$ is $Q_n(GF(q)^*)$, the rank n Dowling geometry based on the multiplicative group of $GF(q)$ (see section 6.4.B). When $q = 2$, $G_q(n, 2)$ is isomorphic to the polygon matroid of K_{n+1}. In general, the calculation of the characteristic polynomial of $G_q(n, m)$ is difficult except in the cases where m is 1, 2, $n-1$, or n, when the geometry is supersolvable (see Exercise 6.78).

Next we turn our attention to an interesting T–G invariant for *binary codes*, that is, linear codes over $GF(2)$. This invariant was discovered for graphs by Rosenstiehl & Read (1978), and Jaeger (1989b) noted that their result could be extended to binary matroids. The dimension of a vector space V will be denoted by dim V.

6.5.4. Proposition. *Let C be a binary code of length n. Then*

(i) $$t(M(C); -1, -1) = (-1)^n 2^{\dim(C \cap C^*)}.$$

Hence

(ii) $$|t(M(C); -1, -1)| = |C \cap C^*|.$$

Proof. The proof in Rosenstiehl & Read (1978) was in graph-theoretic terms. We generalize these ideas to binary vector spaces. Let $h(M(C)) = (-1)^{n(C)} 2^{\dim(C \cap C^*)}$ where $n(C)$ is the length of C. We shall show that h is a well defined T–G invariant. First note that h is well defined if $n(C) = 1$ for, in that case, $C \cap C^* = \{\mathbf{0}\}$ and so $h(M(C)) = -1$. Assume that h is well defined if $n(C) < m$ and let $n(C) = m \geqslant 2$.

Let $B = B(C) = C \cap C^*$ and suppose that the element e of $M(C)$ is neither a loop nor an isthmus. We may assume that e corresponds to the first column of a generator matrix U for C. If $\mathbf{x} \in \hat{C}$, then either $(1, \mathbf{x})$ or $(0, \mathbf{x})$ is in C, but not both, otherwise $(1, \mathbf{0}) \in C$, and e is an isthmus of $M(C)$.

Now observe that

$$\textit{either } B = B_0, \textit{ or, for some vector } \mathbf{x} \textit{ having first entry } 1, B = B_0 + \langle \mathbf{x} \rangle. \tag{6.66}$$

In view of (6.59), we shall write $C - e$ for $\hat{C}$, and C/e for $\hat{C}_0$. One easily checks that $(C-e)^*$ *is* C^*/e, that is,

$$(C-e)^* = (\widehat{C^*})_0.$$

Thus if $\mathbf{x} \in B(C-e) = (C-e) \cap (C-e)^*$, then $(0, \mathbf{x}) \in C^*$, while either $(0, \mathbf{x}) \in C$ or $(1, \mathbf{x}) \in C$, but not both. Dually, if $\mathbf{y} \in B(C/e)$, then $(0, \mathbf{y}) \in C$, while either $(0, \mathbf{y}) \in C^*$ or $(1, \mathbf{y}) \in C^*$, but not both. The proof of the next result is straightforward and is left as an exercise. □

6.5.5. Lemma. *Either*

(i) *for some* $\mathbf{x}$ *in* $B(C-e)$, $(1, \mathbf{x}) \in C$ *and so* $B(C-e) = \hat{B}_0 + \langle \mathbf{x} \rangle$;

or

(ii) *for all* $\mathbf{x}$ *in* $B(C-e)$, $(0, \mathbf{x}) \in C$ *and* $B(C-e) = \hat{B}_0$.

Now suppose that $\dim B = k$. We shall show that one of the following three possibilities must occur.

$$\dim B(C-e) = k-1 = \dim B(C/e). \tag{6.67}$$

$$\dim B(C-e) = k+1, \ \dim B(C/e) = k. \tag{6.68}$$

$$\dim B(C-e) = k, \ \dim B(C/e) = k+1. \tag{6.68*}$$

First we note that each statement in the following is equivalent to its successor.

(1) $B = B_0$.
(2) $(1, \mathbf{0}) \in B^*$.
(3) $(1, \mathbf{0}) \in (C \cap C^*)^* = C + C^*$.
(4) For some $\mathbf{z}$,
  (a) $(1, \mathbf{z}) \in C$ and $(0, \mathbf{z}) \in C^*$; or
  (b) $(1, \mathbf{z}) \in C^*$ and $(0, \mathbf{z}) \in C$.

Suppose that $B \neq B_0$. Then (i) cannot occur otherwise $(1, \mathbf{x}) \in C$ and $(0, \mathbf{x}) \in C^*$ so $(1, \mathbf{0}) \in C + C^*$; a contradiction. Thus $B(C-e) = \hat{B}_0$ and, dually, $B(C/e) = \hat{B}_0$. Hence $\dim B(C-e) = \dim \hat{B}_0 = \dim B(C/e)$. But, by (6.66), as $B \neq B_0$, $\dim B = \dim B_0 + 1 = \dim \hat{B}_0 + 1$. Thus if $B \neq B_0$, then (6.67) occurs.

We may now assume that $B = B_0$. Then, from above, (4a) or (4b) occurs. In the former case, for some $\mathbf{z}_1$, $(1, \mathbf{z}_1) \in C$ and $(0, \mathbf{z}_1) \in C^*$, so $\mathbf{z}_1 \in B(C-e)$. In the latter case, for some $\mathbf{z}_2$, $(0, \mathbf{z}_2) \in C$ and $(1, \mathbf{z}_2) \in C^*$ so $\mathbf{z}_2 \in B(C/e)$. Moreover, exactly one of (4a) and (4b) occurs; otherwise, for some $\mathbf{z}_1$ and $\mathbf{z}_2$, both $(1, \mathbf{z}_1)$ and $(0, \mathbf{z}_2)$ are in C and $(0, \mathbf{z}_1)$ and $(1, \mathbf{z}_2)$ are in C^*. Hence $(1, \mathbf{z}_1) \cdot (1, \mathbf{z}_2) = 0$ and $(0, \mathbf{z}_1) \cdot (0, \mathbf{z}_2) = 0$, so $1 = 0$; a contradiction.

Suppose that (4a) occurs. Then, as $\mathbf{z}_1 \in B(C-e)$, we have, by (i), that $\dim B(C-e) = \dim \hat{B}_0 + 1 = \dim B_0 + 1 = \dim B + 1$. Moreover, as (4b) does not occur, the dual of Lemma 6.5.5 implies that $B(C/e) = \hat{B}_0$, so $\dim B(C/e) = \dim \hat{B}_0 = \dim B_0 = \dim B$. Hence, if (4a) occurs, then so does (6.68) and, by duality, if (4b) occurs, so does (6.68*). We conclude that one of (6.67), (6.68), and (6.68*) must occur. It is routine to check that, in each case,

$$(-1)^n 2^{\dim B} = (-1)^{n(C-e)} 2^{\dim B(C-e)} + (-1)^{n(C/e)} 2^{\dim B(C/e)}.$$

Hence, by (6.59) and the induction assumption, if e is neither a loop nor an isthmus of $M(C)$, then

$$(-1)^{n(C)} 2^{\dim B} = h(M(C)-e) + h(M(C)/e).$$

It is also easy to check that

$$(-1)^{n(C)} 2^{\dim B} = \begin{cases} h(I) h(M(C)-e) & \text{if } e \text{ is an isthmus,} \\ h(L) h(M(C)-e) & \text{if } e \text{ is a loop.} \end{cases}$$

From the last two equations, we deduce by induction that h is well defined and, moreover, that h is a T–G invariant. As $h(I) = h(L) = -1$, it follows by Theorem 6.2.2 that $h(M(C)) = t(M(C); -1, -1)$. □

6.5.6. Corollary. *Let* C *be a binary code. Then* $C \cap C^*$ *is trivial if and only if* $M(C)$ *has an odd number of bases.*

Proof. For an arbitrary matroid M, if both $t(M; 1, 1)$ and $t(M; -1, -1)$ are evaluated modulo 2, we obtain the same result. But, by 6.2.11(i), $t(M; 1, 1)$ is the number of bases of M. Using these observations, the result follows easily from Proposition 6.5.4. □

On putting $(q, z) = (4, -1)$ in Proposition 6.5.1 and using Proposition 6.5.4, we get that, for a binary code C,

$$2^{k+r} = |c_e - c_o| \tag{6.69}$$

where $k = \dim(C \cap C^*)$ and c_e and c_o are the number of even- and odd-weight codewords, respectively, in the linear code over $GF(4)$ that is generated by C.

A consequence of (6.69) is that $C \subseteq C^*$ if and only if, in the code over $GF(4)$ that is generated by C, all codewords have even weight. We leave the verification of this to the reader noting that one can derive this directly from the fact that $C \subseteq C^*$ if and only if, in a binary generator matrix U for C, any two rows are orthogonal over $GF(2)$.

In contrast to the binary case, if C is a linear code over $GF(q)$ for $q \geqslant 4$ then $\dim(C \cap C^*)$ is not, in general, a matroid invariant. For example, consider the following representation of the 4-point line over $GF(13)$ where $a \notin \{0, 1\}$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & a \end{bmatrix}.$$

We leave the reader to check that

$$\dim(C \cap C^*) = \begin{cases} 1 & \text{if } a \in \{6, 8\}, \\ 0 & \text{otherwise.} \end{cases}$$

When $q = 3$, Jaeger (1989c) showed that $(\sqrt{3})^{\dim(C \cap C^*)}$ is the modulus of the complex number $t(M(C); j, j^2)$ where $j = e^{2\pi i/3}$.

6.6. Other Tutte Invariants

The bibliography at the end of this chapter indicates how widespread the use of T–G techniques has been in combinatorics. We have tried to provide some indication of this in this chapter. However, the large number of diverse applications, together with our desire to avoid providing a superficial skim through the theory, has meant that some important topics have been omitted. To compensate partially for this we shall, in this short section, briefly survey some of the other applications. Still more applications are touched on in the exercises and we hope that the extensive bibliography will provide the reader with adequate opportunities for further reading on matroid invariant theory. One particularly exciting recent development has seen the application of T–G techniques in knot theory where a family of new polynomial invariants has been discovered. We shall not discuss this rapidly growing area here, but instead, refer the interested reader to Kauffman's survey (1988) of the area and to the following papers in the bibliography: Jaeger (1988c), Jaeger, Vertigan & Welsh (1990), Jones (1985), Kauffman (1987, 1988), Lickorish (1988), Lipson (1986), Thistlethwaite (1987, 1988a, b), Traldi (1989), and Vertigan (1990).

6.6.A. Basis Activities

We consider here what is essentially the most basic group invariant. It was introduced for graphs by Tutte in his founding work (1954) and was later extended to matroids by Crapo (1969). For a matroid M, recall that $t(M; 1, 1)$ enumerates the bases of M. Thus, it is clear that we may partition $\mathscr{B}(M)$, the set of bases of M, into blocks $\mathscr{B}_{ij}$ where $|\mathscr{B}_{ij}| = b_{ij}$, the coefficient of $x^i y^j$ in $t(M; x, y)$. We now describe one way to obtain such a partition.

First we linearly order the ground set E of M by relabeling the elements of E as $1, 2, \ldots, n$. For a basis B of M, the *internal activity*, $\imath(B)$, of B is equal to the number of elements e of B for which e is the least element in the unique bond contained in $(E - B) \cup e$. Similarly, the *external activity*, $\varepsilon(B)$, of B is the number of elements e of $E - B$ for which e is the least element in the unique circuit contained in $B \cup e$. On letting $\mathscr{B}_{ij}$ be the number of bases of M of internal activity i and external activity j, we obtain the desired partition of $\mathscr{B}(M)$, that is,

$$b_{ij} = |\{B \in \mathscr{B}(M) \colon \imath(B) = i,\ \varepsilon(B) = j\}|. \tag{6.70}$$

To verify this, one shows that if $f(M) = |\mathscr{B}_{ij}|$, then provided the greatest element n is neither a loop nor an isthmus,

$$f(M) = f(M - n) + f(M/n). \tag{6.71}$$

One striking consequence of (6.70) is the fact that $|\{B \in \mathscr{B}(M) \colon \imath(B) = i, \varepsilon(B) = j\}|$ does not depend upon the particular ordering chosen for E.

Whitney (1932) introduced another concept that has been fruitfully developed in this context. A *broken circuit* is an independent set that is obtained from a circuit by deleting its least element. Evidently a basis has external activity equal to zero if and only if it contains no broken circuits. Now define $p^+(M; \lambda)$ by

$$p^+(M; \lambda) = t(M; \lambda + 1, 0). \tag{6.72}$$

By (6.20), if w_i is the coefficient of λ^{r-i} in the characteristic polynomial $p(M; \lambda)$ of M, then

$$p^+(M; \lambda) = \sum_{i=0}^{r} |w_i| \lambda^{r-i}.$$

The numbers w_i, $0 \leqslant i \leqslant r$, are called the *Whitney numbers of the first kind*. A detailed discussion of their properties can be found in Aigner (1987). It is not difficult to show, by verifying (6.71) for $f(M) = |w_i|$, that

$$|w_i| = |\{I \subseteq E \colon |I| = i \text{ and } I \text{ contains no broken circuits}\}|. \tag{6.73}$$

Hence, by (6.72),

$$t(M; 2, 0) \text{ is } \sum_{i=0}^{r} |w_i|, \text{ the number of subsets of } E \text{ that contain no broken circuits.} \tag{6.74}$$

A bijective proof of the relationship between (6.70) and (6.73) is given in Brylawski (1977c), and other properties of the associated invariants may be found in Beissinger (1982), Björner (1980, 1982), Brylawski (1977b, 1982), Brylawski & Oxley (1980, 1981), Wilf (1976), and Zaslavsky (1983). Among these are toplogical properties of the broken-circuit complex, the simplicial complex whose simplices are the subsets that contain no broken circuits. Berman (1977) gives an activity-theoretic interpretation of b_{ij} for acyclic orientations. A generalization of his result to a three-variable polynomial has been given by Las Vergnas (1978; 1984).

6.6.B. Hyperplane Arrangements

A finite set of hyperplanes in Euclidean d-space $\mathbb{E}^d$ is called an *arrangement of hyperplanes*. Such an arrangement decomposes $\mathbb{E}^d$, and various counting problems associated with this decomposition have been extensively studied. See Brylawski (1976, 1985), Cordovil (1980, 1982, 1985), Cordovil & Silva (1985, 1987), Cordovil, Las Vergnas & Mandel (1982), Greene (1977), Greene & Zaslavsky (1983), Las Vergnas (1977), Winder (1966), Zaslavsky (1975a, 1976, 1977, 1979, 1981a, b, 1983), Buck (1943), Orlik (1989), Schläfi (1950), and Stanley (1980). Moreover, Zaslavsky (1975b) has given a comprehensive treatment of such problems. In this section we briefly survey some of his results which can be derived using T–G techniques.

Let $\{H_1, H_2, \ldots, H_n\}$ be a set of hyperplanes in $\mathbb{E}^d$ and consider the set of intersections $\left\{\bigcap_{i\in I} H_i\colon I \subseteq \{1, 2, \ldots, n\}\right\}$. This set is partially ordered by reverse inclusion. In general, this poset P need not be a geometric lattice, although it will be if no intersection of hyperplanes is parallel to another hyperplane, or, more precisely, if the following condition holds (Exercise 6.87).

Whenever $J \subseteq \{1, 2, \ldots, n\}$ and $\bigcap_{i\in J} H_i$ contains a line, $\bigcap_{i\in J} H_i$ meets H_j for all j in $\{1, 2, \ldots, n\} - J$. (6.75)

In the results that follow we assume that P is a geometric lattice and we let M denote the simple matroid for which P is the lattice of flats. For the generalizations of these results to arbitrary arrangements of hyperplanes in projective as well as Euclidean space, we refer the reader to Zaslavsky's paper (1975b).

When the hyperplanes $H_1, H_2, \ldots, H_n$ are removed from $\mathbb{E}^d$, the remainder of the space falls into components, each a d-dimensional open polyhedron. We call these polyhedra *regions* of the arrangement. Such regions may be bounded or unbounded. An arrangement of hyperplanes is called *central* if the hyperplanes have non-empty common intersection.

6.6.1. Proposition. *The total number of regions of a non-central arrangement is* $t(M; 2, 0) - t(M; 1, 0) = t(M; 2, 0) - |\mu(M)|$.

6.6.2. Proposition. *The number of bounded regions of a non-central arrangement is* $t(M; 1, 0) = |\mu(M)|$.

For central arrangements, the situation is a little different. In particular, (6.75) always holds for such an arrangement so the poset of hyperplane intersections will certainly be a geometric lattice in this case. It is not difficult to show that a central arrangement has no bounded regions. On the other hand:

6.6.3. Proposition. *The number of (unbounded) regions in a central arrangement is* $t(M; 2, 0)$.

Given a central arrangement $\{H_1, H_2, \ldots, H_n\}$ having associated matroid M, suppose we perturb one of the hyperplanes, say H_i, by translation from its initial position. Let H_i' be the perturbation of H_i.

6.6.4. Proposition. *The numbers of bounded regions of the arrangements* $\{H_1, H_2, \ldots, H_n\} \cup \{H_i'\}$ *and* $(\{H_1, H_2, \ldots, H_n\} \cup \{H_i'\}) - \{H_i\}$ *are the same and are equal to* $\beta(M)$.

As a further development of these ideas, Greene & Zaslavsky (1983) have given an interpretation in terms of arrangements for the coefficients of the characteristic polynomial. Moreover, many of the above results can be generalized to oriented matroids (see, for example, Cordovil, Las Vergnas & Mandel, 1982; Las Vergnas, 1975a; 1984). Finally, we note that Cordovil (1980) showed that a conjecture of Grünbaum on the minimum number of regions of a pseudoline arrangement in the real projective plane can be deduced from certain general inequalities for the Whitney numbers.

6.6.C. Separation of Points by Hyperplanes

We now consider various results that are obtained by projectively dualizing the results of the previous subsection. When this is done, the image of a hyperplane is a point and the image of a region is a topologically connected family of hyperplanes. Suppose E is a finite subset of $\mathbb{E}^d$ and M is the affine matroid induced on E. We consider those subsets E' of E that can be separated from their complements $E - E'$ by some hyperplane of $\mathbb{E}^d$. The number of such *hyperplane-separable subsets* is a T–G invariant. In fact:

6.6.5. Proposition. *The number of hyperplane-separable subsets of E equals* $t(M; 2, 0)$.

To verify the fundamental T–G recursion in this case, one considers an extreme point e of E. Then every separation of a subset E' of $E - e$ gives a separation of either E' or $E' \cup e$. Both E' and $E' \cup e$ are hyperplane-separable if and only if E' can be separated from $E - (E' \cup e)$ by a hyperplane through e. But separations of the latter type are in one-to-one correspondence with separations in M/e, where here one projects from e onto a hyperplane H which is in general position further from e than any point of $E - e$.

Similar arguments to the above can be used to establish the following results.

6.6.6. Proposition. *Let C be the convex hull of a basis B of M and suppose that $C \cap E = B$. Then the number of subsets of E that can be separated from their complements by a hyperplane intersecting C is given by* $t(M; 2, 0) - 2t(M; 1, 0) = t(M; 2, 0) - 2|\mu(M)|$.

6.6.7. Proposition. *Let ε be sufficiently small and B_ε be an epsilon ball in $\mathbb{E}^d$ centered at an element e of M that is neither a loop nor an isthmus. Then the* *number of subsets of E that can be separated by a hyperplane that passes through B_ε is given by* $t(M; 2, 0) - 2b_{10} = t(M; 2, 0) - 2\beta(M)$.

To conclude this section, we note that, apart from their close links with hyperplane dissections, hyperplane separations are also closely related to acyclic orientations of graphs. Details of this relationship, including combinatorial correspondences between the objects involved, can be found in Brylawski (1985), Greene (1977), and Greene & Zaslavsky (1983).

6.6.D. Intersection Theory

We saw in sections 6.3.F and 6.5 that the coboundary polynomial $\bar{\chi}(M; \lambda, v)$ enumerates generalized colorings as well as codeweights. We now put these two facts into a general framework: the combinatorial structure of the way an embedded matroid intersects the flats of its ambient geometry. More details of this *intersection theory* can be found in Brylawski (1979b, 1981b).

A matroid $M(E)$ is said to be *embedded into a geometry* $G(T)$ if there is a mapping $f: E \to T \cup \hat{0}$ such that $r_M(E') = r_G(f(E'))$ for all subsets E' of E. Equivalently, the simplification of M is a subgeometry of G. The element $\hat{0}$ here serves merely as the image of any loops in M.

A rank r geometry G is called *upper combinatorially uniform* if it has the same number $W_2(i, j)$ of flats of corank j in every upper interval of rank i. The numbers $W_2(i, j)$ are called the (doubly indexed) *Whitney numbers* of G *of the second kind*. The equation $W_2(r, r-k) = W_k$ relates these numbers to the (singly indexed) Whitney numbers W_k of the second kind discussed by Aigner (1987). The reader should note that these doubly indexed Whitney numbers of the second kind differ from their namesakes W_{ij}, which were studied by Greene & Zaslavsky (1983). Examples of upper combinatorially uniform geometries include finite affine and projective geometries, where the Whitney numbers are Gaussian coefficients; Boolean algebras, where the Whitney numbers are binomial coefficients; polygon matroids of complete graphs, where the Whitney numbers are Stirling numbers of the second kind; and perfect matroid designs, that is, matroids in which flats of the same rank have equal cardinalities. In the last case, a formula for $W_2(i, j)$ in terms of the sizes of the flats can be found in Brini (1980), Brylawski (1979b, 1982), and Young, Murty & Edmonds (1970).

The *intersection matrix* $I_G(M)$ of the embedding of $M(E)$ into the upper combinatorially uniform geometry G is defined by $I_G(M; i, j) = |\{F: F$ is a corank j flat of G with $|f^{-1}(F)| = i\}|$, that is, $I_G(M; i, j)$ counts the flats of G of corank j that contain i points of E. The *intersection polynomial* $i_G(M; u, v)$ is then given by

$$i_G(M; u, v) = \sum_i \sum_j I_G(M; i, j) u^i v^j.$$

The principal idea of intersection theory is that the numbers $I_G(M; i, j)$ do not depend upon the embedding but only on the Whitney numbers of G and the Tutte polynomial of M. Indeed, if W_2 denotes the matrix of Whitney numbers $W_2(i, j)$ of G, then we have the matrix equation

$$T^{-1} \cdot M_{KC} \cdot W_2 = I_G(M) \tag{6.76}$$

where $T^{-1}(i, j) = (-1)^{i+j}\binom{j}{i}$ for all i, j in $\{0, 1, \ldots, |E|\}$ and M_{KC} is the cardinality-corank matrix of M (see section 6.5). After seeing similar arguments in sections 6.3.F and 6.5, the reader will not be surprised that (6.76) is derived by counting in two different ways: it is not difficult to check that

$$(T \cdot I_G)(i, j) = (M_{KC} \cdot W_2)(i, j) = N(i, j) \tag{6.77}$$

where $N(i, j) = |\{(F, E'): F$ is a corank j flat of G, $E' \subseteq E$, $|E'| = i$, and $E' \subseteq f^{-1}(F)\}|$. Hence $N(i, j)$ is the number of ordered pairs consisting of a flat of corank j and a subset of size i embedded in the flat.

The intersection polynomial $i_G(M; u, v)$ is not a T–G invariant. However, it does satisfy the same recursion as the coboundary polynomial $\bar{\chi}$:

$$i_G(M) = i_G(M - e) + (u - 1) i_{\overline{G/f(e)}}(M/e) \tag{6.78}$$

where e is neither a loop nor an isthmus of M, and M/e is embedded into $\overline{G/f(e)}$, the simplification of $G/f(e)$. The formula (6.78) is derived from the following recursion:

$$I_G(M; i, j) = I_G(M - e; i, j) + I_{\overline{G/f(e)}}(M/e; i-1, j) - I_{\overline{G/f(e)}}(M/e; i, j). \tag{6.79}$$

To verify (6.79), observe that $I_G(M - e; i, j)$ counts the i-point flats of M that do not contain e, together with the $(i+1)$-point flats of M containing e. But there are precisely $I_{\overline{G/f(e)}}(M/e; i, j)$ flats of the latter type. Since $I_{\overline{G/f(e)}}(M/e; i-1, j)$ counts those i-point flats that do contain e, (6.79) follows.

On combining (6.76) and the matrix-theoretic proof of 6.3.26, we get the polynomial equation

$$i_G(M; u, v) = (u - 1)^r t\left(M; \frac{u + \lambda - 1}{u - 1}, u\right)\bigg|_{\lambda^i \mapsto \sum_j W_2(i, j) v^j}. \tag{6.80}$$

Here we use the same evaluation of the Tutte polynomial as in 6.3.26 and then replace λ^i in the resulting polynomial for $\bar{\chi}(M)$ by $\sum_j W_2(i, j) v^j$. To justify this, compare (6.43) and (6.77).

For any upper combinatorially uniform geometry, G, every rank k upper interval has the same characteristic polynomial. Denoting this polynomial by $p_k(G; \lambda)$, we see that $p_r(G; \lambda) = p(G; \lambda)$. Now let $W_1(i, j)$ denote the coefficient of λ^j in $p_i(G; \lambda)$, where both i and j are chosen from the set $\{0, 1, 2, \ldots, r\}$. We call the numbers $W_1(i, j)$ the (doubly indexed) *Whitney numbers of the first kind*. The equation $W_1(r, r-k) = w_k$ relates these numbers to the (singly indexed) Whitney numbers w_k of the first kind which are discussed in section 6.6.A and in more detail by Aigner (1987). As with the Whitney numbers of the second kind, the numbers $W_1(i, j)$ differ from the numbers $w_{i,j}$, which Greene & Zaslavsky (1983) call 'doubly indexed Whitney numbers of the first kind'.

One can show that W_1 and W_2 are inverse matrices (Exercise 6.92b). When G is the polygon matroid of a complete graph, this inverse relationship is precisely the relationship between the Stirling numbers of the first and second kinds. When G is a Boolean algebra B, the inverse relationship is that exploited earlier between the binomial coefficient matrix T and the signed binomial coefficient matrix T^{-1}. Indeed, one easily checks that $p_i(B; \lambda) = (\lambda - 1)^i$, so that, in this case, $W_1(i, j) = (-1)^{i+j}\binom{j}{i}$, that is, $W_1(i, j) = T^{-1}(i, j)$. Dowling (1973b) was the first to prove the formula for evaluating $p_i(G; \lambda)$ from an inverse matrix. This formula also appears in a slightly more general form in Brylawski (1979b).

Using the fact that W_1 is the inverse of W_2, we get immediately from (6.76) that

$$M_{KC} = T \cdot I_G(M) \cdot W_1. \tag{6.81}$$

Moreover, by inverting (6.80) and using (6.45), it can be shown that

$$\bar{\chi}(M; u, \lambda) = i_G(M; u, v)|_{v^i \mapsto p_i(M; \lambda)}. \tag{6.82}$$

The above theory can be used in a straightforward manner to compute, for example $i_{G'}(M)$ from $i_G(M)$ where G' is another upper combinatorially uniform geometry in which M is embedded (Brylawski, 1979b). Hence, the intersection numbers for an embedding of an n-vertex graph Γ into K_n yield those for a linear representation of M_Γ. Further, $i_G(M)$ can be computed from $i_G(M^*)$, this result being the intersection analog of the MacWilliams duality formula. Also, if G is a perfect matroid design and G' is a subgeometry of G, then one can compute $t(G')$ from $t(G - G')$. To see this, we note that if a_j is the size of each corank j flat of G, then

$$I_G(G'; i, j) = I_G(G - G'; a_j - i, j).$$

Some examples of the above calculations are given in the exercises.

Among the other applications of intersection theory are various reconstruction results for the Tutte polynomial. In particular, Proposition 5.1 of Brylawski (1982) enables one to reconstruct $t(M)$ from the Tutte polynomials of its single-element contractions. With $S_{KC}(M; x, y)$ equal to the cardinality-corank polynomial of $M(E)$, we have the straightforward formula

$$S_{KC}(M; x, y) = \int \left(\sum_{e \in E} S_{KC}(M/e; x, y) \right) dx + y^{r(M)} \tag{6.83}$$

where $\int dx$ is the formal integral operator: $\int x^k y^m dx = \dfrac{x^{k+1}y^m}{k+1}$.

Brylawski (1981c) reconstructed $t(M)$ from the multiset of isomorphism classes of hyperplanes of M; in the graphic case (1981b), he reconstructed $t(M_\Gamma)$ from the multiset of isomorphism classes of single-vertex deletions of Γ. To make the latter calculation and to find the intersection numbers for graph complements, a finer graph invariant called the *polychromate* was introduced. For a graph Γ having m vertices and n edges, this is defined by

$$\hat{\chi}(\Gamma; y, z_1, z_2, \ldots, z_m) = \sum_i y^i \sum_\pi M(\Gamma, i, \pi)\mathbf{z}^\pi. \tag{6.84}$$

The second sum here is over all integer partitions π of m, while if $\pi = 1^{a_1}2^{a_2} \ldots m^{a_m}$, then $\mathbf{z}^\pi = z_1^{a_1}z_2^{a_2} \ldots z_m^{a_m}$. Further, $M(\Gamma, i, \pi)$ denotes the number of vertex partitions of type π in which there are exactly i edges of Γ joining two vertices in the same class.

It is an easy matter to reconstruct the polychromate $\hat{\chi}$ from vertex or edge deletions, as well as to determine its behavior under the addition of isolated vertices or the taking of complements. Moreover,

$$i_{K_m}(\Gamma; u, v) = \hat{\chi}(\Gamma; u, v, v, \ldots, v). \tag{6.85}$$

One can use these ideas to construct non-isomorphic graphic matroids of arbitrarily high connectivity having the same polychromate, and hence the same Tutte polynomial (see Exercise 6.19).

As a final application of intersection theory, we sketch suggestively similar statements and proofs of two extremal theorems, one due to Bose & Burton (1966), the other to Turán (see, for example, Erdös, 1967). Another proof of the former is outlined in Exercise 6.65. In both these theorems, the ambient geometry is a supersolvable upper combinatorially uniform geometry G. Each theorem has two parts: the first part asserts that a smallest subset of G that meets every modular flat of rank $c + 1$ has the same size as a smallest flat of corank c; the second part asserts that every such smallest subset of G is a smallest flat of corank c. Our proof will be only of the second part of these theorems, the characterization of the extremal subsets. Since any flat of corank c meets all modular flats of rank $c + 1$, to prove this second part it suffices to show that every extremal subset is a flat.

In the Bose–Burton theorem, the ambient geometry is $PG(d, q)$ and the assertion of the second part of the theorem is that if a subset A of the points of this projective space has the same size as a subspace F of rank $c + 1$ and A meets every subspace of corank c, then $A \cong F \cong PG(c, q)$.

In Turán's theorem, the ambient geometry is $M(K_n)$. To state the theorem we shall need some more notation. For natural numbers p and n, the *Turán graph* $T_p(n)$ is the unique n-vertex p-partite graph for which every vertex class has either $\left\lfloor \frac{n}{p} \right\rfloor$ or $\left\lceil \frac{n}{p} \right\rceil$ members. It is not difficult to check that $T_p(n)$ has the largest number of edges among all complete p-partite graphs on n vertices (Exercise 6.91). Hence a smallest corank c flat F in $M(K_n)$ contains $\binom{n}{2} - |E(T_{c+1}(n))|$ edges. The second part of Turán's theorem asserts that if a subset A of the edges of K_n has this minimum number $|F|$ of edges and A meets every $(c+2)$-clique, then A is isomorphic to the complement in K_n of $T_{c+1}(n)$.

Both of these theorems have equivalent formulations in terms of a subset of the ambient geometry that is of maximum size with respect to the property of not containing any modular flat of corank $c+1$. Indeed, Turán's theorem is probably more commonly stated in this way.

For fixed c, the common proof of the second part of these two theorems is by induction on $r(G)-(c+1)$, each result being trivial when this quantity is 0. Let F be as in the theorems and X be a subset of G of size $|G-F|$ which, like $G-F$, does not contain any rank $(c+1)$ modular flats. Then all the single-element contractions X/e of X are essentially extremal. It then follows by induction that X/e is isomorphic to $(G-F)/e$. Thus we may employ (6.83) to show that X and $G-F$ have the same Tutte polynomial and the same intersection matrix. The only flat of G of corank c that avoids $G-F$ is F itself, that is, $I_G(G-F;\, 0, c)=1$. Hence $I_G(X;\, 0, c)$ is also 1 so X must avoid, and therefore be complementary to, a flat of the same corank and size as F.

We shall illustrate this idea further in the graphical case by looking in more detail at what happens when $c=1$. We leave the task of completing the details in the projective case as an exercise. Evidently $|E(T_2(n))| = \left\lfloor \frac{n^2}{4} \right\rfloor$. Now take a subset X of $E(K_n)$ that has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges and no triangles. We shall also let X denote the subgraph of K_n induced by this set of edges. Let e be an edge of X with endpoints u and v. Then

$$E(X/e) = \left\lfloor \frac{n^2}{4} \right\rfloor - 1 = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 2,$$

and none of the edges of X/e is multiple. If $\bar{v}$ is the vertex of X/e formed by identifying u and v, then

$$|E(X/e - \bar{v})| = |E(X - \{u, v\})| = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 2 - \deg \bar{v}.$$

But $X - \{u, v\}$ contains no triangles and therefore has at most $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ edges. Hence $\deg \bar{v} = n-2$ and $|E(X/e - \bar{v})| = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor$. Thus, by the induction assumption, $X/e - \bar{v}$ is isomorphic to $T_2(n-2)$ and $\bar{v}$ is adjacent to every vertex of this graph. Thus the isomorphism type and intersection matrix of X/e are determined for all e in X. Hence X itself is determined and is isomorphic to $T_2(n)$.

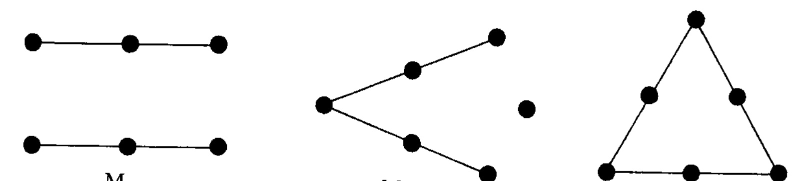
Exercises

Those with one asterisk are more difficult; those with two asterisks are unsolved.

Section 6.2

6.1. (a) Determine the Tutte polynomials of the three matroids M_1, M_2, and M_3 for which affine representations are shown below.

Figure 6.11.



M_1 M_2 M_3

(b) (Brylawski, 1972b) Show that M_1 and M_2 are the unique smallest pair of non-isomorphic matroids with the same Tutte polynomial.

* (c) (Brylawski, 1972b) Determine when two rank 3 geometries have the same Tutte polynomial.

6.2. Define a function t^* on the class $\mathscr{M}$ of all matroids by $t^*(M) = t(M^*;\, x, y)$. Show that t^* is a T–G invariant and use this to give an alternative proof of the fact that $t(M^*;\, x, y) = t(M;\, y, x)$.

6.3. (a) Show that, for $m \geq 2$,

$$t(U_{1,m};\, x, y) = x + y + y^2 + \ldots + y^{m-1}.$$

(b) Determine $t(U_{r,m};\, x, y)$.

6.4. If a_{ij} is the coefficient of $x^i y^j$ in $S(M;\, x, y)$, prove that it is a group invariant and determine its value in terms of the coefficients of $t(M;\, x, y)$.

6.5. Let F be a field and σ, τ, γ, and δ be non-zero elements of F. Suppose that f is a function from $\mathscr{M}$ into $F[x, y]$ having the following properties.

(1) $f(I;\, x, y) = x$ and $f(L;\, x, y) = y$.

If e is an element of the matroid M, then

(2) $f(M;\, x, y) = \sigma f(M-e;\, x, y) + \tau f(M/e;\, x, y)$ if e is neither an isthmus nor a loop,

(3) $f(M;\, x, y) = \gamma f(M(e);\, e, y) f(M-e;\, x, y)$ if e is an isthmus, and

(4) $f(M;\, x, y) = \delta f(M(e);\, x, y) f(M-e;\, x, y)$ if e is a loop.

(a) Find an example to show that, in order for f to be well defined, we must have $\gamma = \delta$.

(b) Show that if $\gamma = \delta$, then

$$f(M;\,x,\,y)=\gamma^{-1}\sigma^{|E|-r(E)}\tau^{r(E)}t\left(M;\,\frac{x\gamma}{\tau},\,\frac{y\gamma}{\sigma}\right).$$

(c) If $\gamma=\delta$, show that

(5) $f(M_1\oplus M_2;\,x,\,y)=\gamma f(M_1;\,x,\,y)f(M_2;\,x,\,y)$

and that (b) still holds if this condition replaces (3) and (4).

6.6. Use Theorem 6.2.2 to prove 6.2.11(ii).

6.7. Prove Proposition 6.2.20.

6.8. For each of the following determine whether there is a matroid having the specified polynomial as its Tutte polynomial. If there is such a matroid, determine whether or not it is unique.

(a) $x^2+x+3y+y^2$.

(b) $x^3+2x^2+x+x^2y+3xy+xy^2+y+2y^2+y^3$.

(c) $x^4+4x^3+7x^2+3x^2y+7xy+6xy^2+3xy^3+xy^4$.

(d) $x^3+3x^2+2xy+y^2$.

(e) $x^3+3x^2y+2x^2y^2+x^2y^3+3xy^2+4xy^3+3xy^4+xy^5+y^3+2y^4+2y^5+y^6$.

6.9. Let $S(M;\,x,\,y)=\sum_i\sum_j a_{ij}x^iy^j$ where $S(M;\,x,\,y)$ is the rank generating polynomial of $M(E)$.

(a) Show that

$$\max\{i\colon a_{ij}>0 \text{ for some } j\}=r(M),$$
$$\max\{j\colon a_{ij}>0 \text{ for some } i\}=n(M)$$

and $0\leqslant a_{ij}\leqslant\binom{m}{\lfloor m/2\rfloor}$ for all i and j where $m=|E|$.

(b) Use (a) to deduce that, corresponding to matroids on m elements there are at most $2^{(m+1)^3/4}$ distinct rank generating polynomials.

(c) If $g(m)$ denotes the number of non-isomorphic geometries on a set of cardinality m, then Knuth (1974) has shown that

$$g(M)\geqslant\frac{1}{m!}2^{\left\lceil\binom{m}{\lfloor m/2\rfloor}/2m\right\rceil}.$$

Use this result with (b) to deduce that, for any number N, there are at least N non-isomorphic geometries having the same Tutte polynomial.

6.10. Let H be a circuit and a hyperplane in the matroid $M(E)$. Let $\mathscr{B}'$ consist of the set of bases of M together with the set H.

(a) Show that $\mathscr{B}'$ is the set of bases of a matroid M' on E.

(b) Use the rank generating polynomial to show that $t(M';\,x,\,y)=t(M;\,x,\,y)-xy+x+y$.

6.11. Show that

$$t(M(K_4);\,x,\,y)=x^3+3x^2+2x+4xy+2y+3y^2+y^3,$$
$$t(M(K_5);\,x,\,y)=x^4+6x^3+11x^2+6x+10x^2y+20xy+15xy^2+5xy^3+6y+15y^2+15y^3+10y^4+4y^5+y^6,$$

and

$$t(M(K_{3,3});\,x,\,y)=x^5+4x^4+10x^3+11x^2+5x+9x^2y+15xy+6xy^2+5y+9y^2+5y^3+y^4.$$

6.12. (Brylawski, 1982) Let $\mathscr{G}$ be the set of isomorphism classes of geometries and R be a commutative ring. If M is a matroid, $\bar{M}$ denotes its simplification. A *geometric T–G invariant* is a function f from $\mathscr{G}$ into R such that if e is an element of a geometry G, then

(1) $f(G)=f(G(e))f(G-e)$ if e is an isthmus, and

(2) $f(G)=f(G-e)+f(\overline{G/e})$ otherwise.

(a) Give a non-trivial example of a geometric T–G invariant.

(b) Show that if G is a geometry, then $f(G)=t(G;\,f(I),\,0)$.

6.13. For a matroid M having Tutte polynomial $t(M;\,x,\,y)=\sum_{i=0}^{r(M)}\sum_{j=0}^{n(M)}b_{ij}x^iy^j$, show that $\sum_{j=0}^{n(M)}b_{r(M)-1,j}=n(M)$ and $\sum_{i=0}^{r(M)}b_{i,n(M)-1}=r(M)$.

6.14. (Oxley, 1983a) Let f be a generalized T–G invariant and suppose that $f(I)=x$ and $f(L)=\sigma+\tau$ (see 6.2.6(ii)).

(a) Prove that if H is a hyperplane of $M(E)$ and $E-H=\{x_1,\,x_2,\,\ldots,\,x_k\}$, then

$$f(M)=\sigma^{k-1}(x+(k-1)\tau)f(M(H))+\tau^2\sum_{j=2}^{k}\sum_{i=1}^{j-1}\sigma^{j-2}f((M-x_1,\,x_2,\,\ldots,\,x_{i-1},\,x_{i+1},\,\ldots,\,x_{j-1})/x_i,\,x_j).$$

(b) State the corollary of this result obtained by taking $f(M)=p(M;\,\lambda)$.

(c) Interpret (a) when x, σ, and τ are all equal to one.

(d) Use (b) to obtain an identity relating $\beta(M)$ and $\beta(M(H))$.

6.15. A matroid M_1 is a *series–parallel extension* of a matroid M_2 if M_1 can be obtained from M_2 by repeated application of the operations of series and parallel extension.

(a) Show that if M_1 is a series–parallel extension of the loopless matroid M_2, then $\beta(M_1)=\beta(M_2)$.

(b) (Crapo, 1967) If $M(\mathscr{W}_r)$ is the polygon matroid of the r-spoked wheel and $\mathscr{W}^r$ is the rank r whirl, show that $\beta(M(\mathscr{W}_r))=r-1$ and $\beta(\mathscr{W}^r)=r$.

A matroid $M(E)$ is *3-connected* if $M(E)$ is connected and there is no partition $\{X,\,Y\}$ of S such that $|X|,\,|Y|\geqslant 2$ and $r(X)+r(Y)-r(M)=1$.

(c) (Seymour, 1980) Prove that a connected matroid M is not 3-connected if and only if M is the 2-sum of two matroids on at least three elements.

(d) Use Exercise 6.1 to show that 3-connectedness is not a Tutte invariant.

(e) (Oxley, 1982a) Let M be a matroid with $\beta(M)=k>1$. Prove that either

(1) M is a series–parallel extension of a 3-connected matroid N such that $\beta(N)=k$, or

(2) M is the 2-sum of two matroids each having $\beta<k$.

* (f) (Oxley, 1982a) Let N be a minor of the matroid M and suppose that $\beta(N)=\beta(M)>0$. Prove that if N is 3-connected, then M is a series–parallel extension of N.

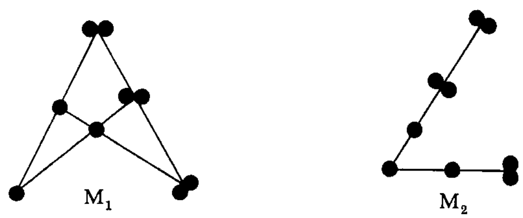
(g) Prove that $\beta(M)=2$ if and only if M is a series–parallel extension of the 4-point line, $U_{2,4}$, or $M(K_4)$.

(h) Prove that $\beta(M)=3$ if and only if M is a series–parallel extension of $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, $M(\mathscr{W}_4)$, or $\mathscr{W}^3$.

* (i) Determine all matroids M for which $\beta(M)=4$.

6.16. Find two geometries G and G' with the same Tutte polynomial where G is isomorphic to its dual but G' is not. (Hint: Form two distinct matroids from $AG(3, 2)$ each retaining seven circuit-hyperplanes, and use Exercise 6.10.)

Figure 6.12.



6.17. (a) Show that for M_1 and M_2, shown in Figure 6.12, $t(M_1^*)=t(M_2^*)$ but that M_1^* is Hamiltonian, that is, has a spanning circuit, whereas M_2^* is not.

(b) Show that the size of a smallest circuit in a matroid is a Tutte invariant.

** (c) Is the size of a largest circuit in a graph a Tutte invariant? (Recently Schwärzler, 1991, has answerede this question in the negative.)

6.18. (Brylawski, 1975a) Prove that if $p(M;\lambda)=\sum_{k=0}^{r} c_k\lambda^k$ and $T(M)$ is the truncation of M, then

$$p(T(M);\lambda)=\sum_{k=1}^{r} c_k\lambda^{k-1}+c_0.$$

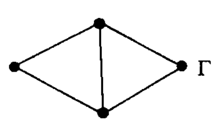
Section 6.3.A

6.19. (a) Find two simple graphs Γ and Δ such that $\chi_\Gamma(M)=\chi_\Delta(M)$, but M_Γ and M_Δ have different Tutte polynomials.

* (b) (Tutte, 1974) Now find two simple graphs Γ and Δ such that $t(M_\Gamma; x, y)=t(M_\Delta; x, y)$ but M_Γ and M_Δ are not isomorphic.

* (c) (Brylawski, 1981b) Show that Γ and Δ can be chosen in (b) to have arbitrarily high connectivity.

Figure 6.13.



6.20. For $\lambda=1, 2, 3, 4$, and 5, find the number of proper λ-colorings of the graph Γ shown in Figure 6.13, and use these numbers to calculate $\chi_\Gamma(\lambda)$ for arbitrary λ.

6.21. If $\mathscr{W}_n$ is the n-spoked wheel, show that its chromatic polynomial is $\lambda[(\lambda-2)^n+(-1)^n(\lambda-2)]$.

6.22. (a) Let K_n-e be the graph obtained from K_n by deleting an edge. Show that its chromatic polynomial is $\lambda(\lambda-1)(\lambda-2)\ \dots\ (\lambda-n+3)(\lambda-n+2)^2$.

(b) If e and f are adjacent edges of K_n, find the chromatic polynomial of K_n-e-f.

(c) Show that the chromatic polynomials of $K_{3,3}$ and $K_{3,4}$ are

$$\lambda(\lambda-1)(\lambda^4-8\lambda^3+28\lambda^2-47\lambda+31)$$

and

$$\lambda(\lambda-1)(\lambda^5-11\lambda^4+55\lambda^3-147\lambda^2+204\lambda-115),$$

respectively.

Section 6.3.B

6.23. If Γ is a graph, find the degree of its flow polynomial.

6.24. Show that the flow polynomials of $K_{3,3}$ and K_5 are

$$(\lambda-1)(\lambda-2)(\lambda^2-6\lambda+10)$$

and

$$(\lambda-1)(\lambda^2-4\lambda+5)(\lambda^3-5\lambda^2+11\lambda-9),$$

respectively.

6.25. If Γ is a connected planar graph and Γ^* is a geometric dual of Γ, give a one-to-λ correspondence between the set of nowhere-zero λ-flows of Γ and the set of proper λ-colorings of Γ^*.

6.26. Let Γ be a cubic graph. Show that the number of proper edge 3-colorings of Γ equals $(-1)^{r(M_\Gamma^*)}t(M_\Gamma; 0, -3)$. (Hint: Relate the coloring to a nowhere-zero flow over an appropriate group.)

6.27. (Negami, 1987) The two 3-variable polynomials $f(\Gamma; t, x, y)$ and $f^*(\Gamma; t, x, y)$ are recursively defined for graphs as follows:

(1) $f(\bar{K}_n)=t^n$ for all $n\geqslant 1$ where $\bar{K}_n$ is the complement of K_n;

(2) $f(\Gamma)=yf(\Gamma-e)+xf(\Gamma/e)$ for all e in $E(\Gamma)$;

and

(3) $f^*(\bar{K}_n)=t^n$ for all $n\geqslant 1$;

(4) $f^*(\Gamma)=xf^*(\Gamma-e)+yf^*(\Gamma/e)$ for all edges e of Γ that are not loops or isthmuses;

(5) $f^*(\Gamma)=(x+ty)f^*(\Gamma-e)$ if e is a loop;

(6) $f^*(\Gamma)=(x+y)f^*(\Gamma/e)$ if e is an isthmus.

(a) (Negami, 1987) Prove that if Γ has $k(\Gamma)$ components, then

$$f(\Gamma; (x-1)(y-1), 1, y-1)=(y-1)^{|V(\Gamma)|}(x-1)^{k(\Gamma)}t(M_\Gamma; x, y).$$

(b) (Oxley, 1989a) Prove that

$$f(\Gamma; t, x, y)=\left(\frac{ty}{x}\right)^{k(\Gamma)}\left(\frac{x}{y}\right)^{|V(\Gamma)|}y^{|E(\Gamma)|}t\left(M_\Gamma; 1+\frac{ty}{x}, 1+\frac{x}{y}\right)$$

and

$$f^*(\Gamma; t, x, y) = \left(\frac{tx}{y}\right)^{k(\Gamma)}\left(\frac{y}{x}\right)^{|V(\Gamma)|} x^{|E(\Gamma)|} t\left(M_\Gamma; 1+\frac{x}{y}, 1+\frac{ty}{x}\right).$$

(c) Prove that

$$f^*(\Gamma; t, x, y) = t^{k(\Gamma)-|V(\Gamma)|} f(\Gamma; t, ty, x).$$

(d) Show that

$$f(\Gamma; \lambda, -1, 1) = \chi_\Gamma(\lambda)$$

and

$$\lambda^{-|V(\Gamma)|} f(\Gamma; \lambda, \lambda, -1) = \chi_\Gamma^*(\lambda).$$

(e) Let Γ be a plane graph and Γ^* be a geometric dual of Γ. Show that

$$f^*(\Gamma) = f(\Gamma^*).$$

Section 6.3.C

6.28. Prove that if a graph Γ has a nowhere-zero n-flow, then Γ has a nowhere-zero $(n+1)$-flow.

6.29. (Jaeger, 1976b) Let the complement of a spanning tree in a connected graph Γ be a *cotree*.

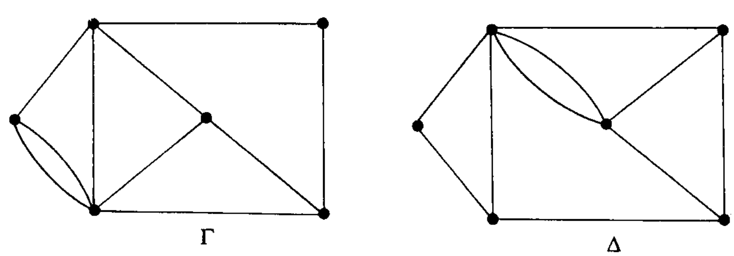
(a) Prove that if $E(\Gamma)$ is the union of m cotrees, then Γ has a nowhere-zero $\mathbb{Z}_2^m$-flow.

(b) Use Edmonds' covering theorem for matroids (Edmonds, 1965b) to prove that every 3-edge connected graph is the union of three cotrees.

(c) Use (a) and (b) (and not Theorem 6.3.10) to prove that every bridgeless graph has a nowhere-zero 8-flow.

(d) Deduce that every bridgeless graph can be covered by three Eulerian subgraphs, where the latter is a subgraph whose edge set can be partitioned into circuits.

Figure 6.14.



(e) Use the same technique that was used to prove (c) to show that every 4-edge connected graph has a nowhere-zero 4-flow.

6.30. (a) (Tutte, 1974) Consider the two graphs shown in Figure 6.14. Show that the Tutte polynomials of their polygon matroids are equal but that these polygon matroids are non-isomorphic.

(b) Show that any orientation of Γ has 48 nowhere-zero $\mathbb{Z}$-flows taking values in $[-2, 2]$, but any orientation of Δ has 52 such flows.

6.31. Prove Lemma 6.3.12.

6.32. Prove that S-closure is a closure operator.

6.33. (Walton, 1981) Let the graph Γ be formed from a set of 2-connected planar graphs by taking 3-sums. Prove that Γ has a nowhere-zero 4-flow.

Section 6.3.D

6.34. (Greene & Zaslavsky, 1983) A *totally cyclic orientation* of a graph Γ is an orientation θ of Γ such that every edge of Γ_θ is in some directed cycle. If Γ is planar and Γ^* is a geometric dual of Γ, find a combinatorial correspondence between the set of acyclic orientations of Γ and the set of totally cyclic orientations of Γ^*.

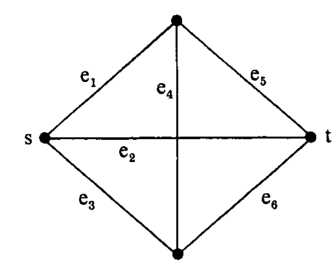
6.35. (Greene & Zaslavsky, 1983) If uv is an edge of a graph Γ and $N(\Gamma)$ denotes the number of acyclic orientations of Γ having u as the unique source and v as the unique sink, prove that

$$N(\Gamma) = \beta(M_\Gamma).$$

Section 6.3.E

6.36. Let Γ be the graph shown in Figure 6.15.
If edge e_i has retention probability p_i, and $p_1 = \frac{3}{4}$, $p_2 = \frac{5}{16}$, $p_3 = \frac{1}{4}$, $p_4 = \frac{3}{8}$, $p_5 = \frac{11}{16}$, and $p_6 = \frac{15}{16}$, construct a graph $\tilde{\Gamma}$ from Γ so that every edge of $\tilde{\Gamma}$ has retention probability $\frac{1}{2}$ and the probability that there is an (s, t)-path in Γ is equal to the probability that there is an (s, t)-path in $\tilde{\Gamma}$.

Figure 6.15.



6.37. Supply the argument that proves 6.3.21(ii).

*6.38. If $\Pr(\mathscr{D}(M)) = \sum_{i=1}^{|M|} b_i p^i$, determine the coefficients b_i.

6.39. In $M = M_d(E \dot\cup d)$, suppose the retention probability p_i of every point e_i of E is equal to the finite binary decimal p. Derive the formula for $\Pr(\mathscr{D}(M))$ from the corresponding formula when $p = \frac{1}{2}$. By taking limits give another proof of 6.3.21(i).

6.40. (a) Complete the proof of Lemma 6.3.24.

(b) Let $M_{(k)}$ be the matroid that is obtained from M by replacing every non-isthmus of M by k elements in series and replacing every isthmus by k isthmuses. Find $t(M_{(k)}; x, y)$ in terms of $t(M; x, y)$.

* (c) (Brylawski, 1982, Proposition 4.10) Let M_d be a pointed matroid in which d is neither a loop nor an isthmus. Let M be a matroid on the set $\{p_1, p_2, \ldots, p_k\}$. The *tensor product* $M \otimes M_d$ is the matroid N_k formed from M as follows: let $N_0 = M$ and, for $i = 1, 2, \ldots, k$, let N_i be the 2-sum of the basepointed matroids (N_{i-1}, p_i) and (M_d, d), this 2-sum being $(N_{i-1}/p_i) \oplus (M_d/d)$ if p_i is a loop, and $(N_{i-1}\backslash p_i) \oplus (M_d \backslash d)$ if p_i is an isthmus.

If $t_P(M_d) = x'f(x, y) + y'g(x, y)$, show that

$$t(M \otimes M_d; x, y) = (f(x, y))^{|M| - r(M)}(g(x, y))^{r(M)} t\left(M; \frac{(x-1)f + g}{g}, \frac{f + (y-1)g}{f}\right).$$

(d) Deduce Lemma 6.3.24 and part (b) from part (c).

Section 6.3.F

6.41. Complete the details of each of the proofs of Proposition 6.3.26.

6.42. Verify that $T^{-1}(i, j) = (-1)^{i+j}\binom{j}{i}$ where T^{-1} is the inverse of the matrix T for which $T(i, j) = \binom{j}{i}$ for all i, j in $\{0, 1, 2, \ldots, n\}$.

6.43. Argue directly from Theorem 6.2.2 to show that

$$t(M; x, y) = (y-1)^{-r}\bar{\chi}(M; (x-1)(y-1), y).$$

6.44. Show that if M has rank r,
(a) $\bar{\chi}(M; 0, \lambda) = p_M(\lambda)$;
(b) $\bar{\chi}(M; \lambda, v) = S_{KC}(M; \lambda - 1, v)$; and
(c) $S_{KC}(M; u, v) = u^r S\left(M; \frac{v}{u}, u\right)$.

Section 6.3.G

6.45. Use deletion–contraction arguments to prove 6.3.28 and 6.3.29.

6.46. (a) Find the medial graphs of each of K_2, K_3, and a planar embedding of K_4.
(b) Show that the medial graph of an n-cycle C_n is $C_n^{(2)}$, the graph obtained from C_n by doubling every edge.
(c) Find the medial graph of $C_n^{(2)}$.
(d) If Δ is the graph of the octahedron, find a graph Γ such that $\Gamma_m = \Delta$.

6.47. Prove that if Γ is a plane graph and Γ^* is its geometric dual, then $\Gamma_m = (\Gamma^*)_m$.

6.48. Show how deletion and contraction of an edge e in a plane graph Γ correspond to decomposing Γ_m into two smaller 4-regular graphs.

Section 6.4.A

6.49. Prove Corollary 6.4.11.

6.50. Let $M(E)$ be a rank r binary matroid.
(a) Prove that the following statements are equivalent.
(1) M is affine.
(2) If ϕ is a coordinatization of M in $V(r, 2)$, then there is a linear functional f on $V(r, 2)$ that distinguishes $\phi(E)$.
(3) All circuits of M have even cardinality.
(4) All hyperplane complements of M^* have even cardinality.
(5) There is a partition of E into bonds of M.
(b) Assume that $M(E) \cong M_\Gamma$ for a graph Γ. Prove that (1)–(5) are equivalent to (6) Γ is 2-colorable.
(c) Assume that $M(E) \cong M_\Gamma^*$ for a connected graph Γ. Prove that (1)–(5) are equivalent to (7) Γ is Eulerian.

*6.51. Prove Proposition 6.4.12.

6.52. Show that if $c(M; q) = k$, but $c(M - e; q) < k$ for all e, then $c(M/e; q) < k$ for all e.

6.53. For a loopless matroid M, there are several ways one may attempt to define the chromatic number. Two possibilities are

$$\chi(M) = \min\{j \in \mathbb{Z}^+ : p(M; j) > 0\}$$

and

$$\pi(M) = \min\{j \in \mathbb{Z}^+ : p(M; j + k) > 0 \text{ for } k = 0, 1, 2, \ldots\}.$$

(a) Evidently $\chi(M) \leqslant \pi(M)$. Give examples to show that $\pi(M) - \chi(M)$ can become arbitrarily large.
(b) Show that if $T \subseteq E(M)$, then $\chi(M(T))$ can exceed $\chi(M)$ and $\pi(M(T))$ can exceed $\pi(M)$.
(c) Let M be a geometry and $\mathscr{C}^*(M)$ be its set of bonds. Prove that

$$\pi(M) \leqslant 1 + \max_{C^* \in \mathscr{C}^*(M)} |C^*|.$$

(d) (Heron, 1972b) Prove that, for a matroid $M(E)$, $\pi(M) \leqslant |E| - r + 2$.
(e) (Lindström, 1978) Prove that if M is a regular matroid, then $\chi(M) = \pi(M)$.
*(f) (Walton, 1981) Extend (e) to show that if M is binary and has no minor isomorphic to F_7, then $\chi(M) = \pi(M)$.
(g) (Oxley, 1978a) Prove that for a regular matroid M having $\mathscr{R}(M)$ as its set of simple submatroids,

$$\pi(M) \leqslant 1 + \max_{N \in \mathscr{R}(M)} \left(\min_{C^* \in \mathscr{C}^*(N)} |C^*| \right).$$

(h) Deduce from (g) the following result of Lindström (1978). If a loopless regular matroid can be covered by bonds of size less than n, then $\pi(M) \leqslant n$.
*(i) (Oxley, 1978a) Use (g) to prove that if M is a connected regular geometry, then

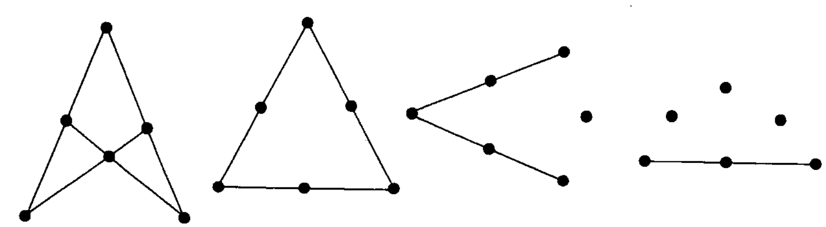
$$\pi(M) \leqslant \max_{C^* \in \mathscr{C}^*(M)} |C^*|$$

unless M is an odd circuit or an isthmus.

*6.54. (Walton, 1981) With $\pi(M)$ as defined in the previous exercise, prove that if M is loopless and representable over $GF(q)$, then, provided M has no minor isomorphic to any of the four matroids shown in Figure 6.16,

$$\pi(M) \leqslant q + 1.$$

Figure 6.16.



6.55.* (a) (Brylawski, 1975a) Let M be the generalized parallel connection of the matroids M_1 and M_2 across the modular flat X. Prove that

$$p(M; \lambda) = p(M_1; \lambda)p(M_2; \lambda)[p(M_1(X); \lambda)]^{-1}.$$

(b) (Walton & Welsh, 1980) Let M be the 2-sum of matroids M_1 and M_2 with basepoint z. Show that

$$p(M; \lambda) = (\lambda - 1)^{-1} p(M_1; \lambda) p(M_2; \lambda) + p(M_1/z; \lambda) p(M_2/z; \lambda).$$

(c) (Walton & Welsh, 1980) Let M_1 and M_2 be binary matroids and $X = \{a, b, c\}$ be a 3-circuit of M_1 and M_2. Let M be the generalized parallel connection of M_1 and M_2 across X, and N be the 3-*sum* of M_1 and M_2, that is, $N = M - X$. Show that

$$p(N; \lambda) = \frac{p(M_1; \lambda) p(M_2; \lambda)}{(\lambda - 1)(\lambda - 2)} + p((M - \{a, b\})/c; \lambda) + p((M - a)/b; \lambda) + p(M/a; \lambda).$$

Let $\mathscr{F} = EX(M_1, M_2, \ldots, M_n)$ be the class of matroids having no minor isomorphic to any one of $M_1, M_2, \ldots, M_n$. A *splitter* N for $\mathscr{F}$ is a 3-connected member of $\mathscr{F}$ such that if $M \in \mathscr{F}$ and M has a minor isomorphic to N, then $M \cong N$.

(d) Let $\mathscr{F}$ be a class of matroids closed under isomorphism and the taking of minors. Prove that if N is a splitter for $\mathscr{F}$, then N^* is a splitter for $\mathscr{F}^* = \{M : M^* \in \mathscr{F}\}$.

(e) (Walton & Welsh, 1982) Let $\mathscr{A}$ be a class of matroids and $\pi(\mathscr{A}) = \max\{\pi(M)\colon M$ is a loopless member of $\mathscr{A}\}$. Prove that if $\pi(EX(N, M_1, M_2, \ldots, M_n)) = k$ and N is a splitter for $EX(M_1, M_2, \ldots, M_n)$, then $\pi(EX(M_1, M_2, \ldots, M_n)) \leqslant \max\{k, \pi(N)\}$.

(f) Show that Seymour's 6-flow theorem (6.3.10) is equivalent to the assertion that, for a loopless member M of $EX(F_7, F_7^*, M(K_5), M(K_{3,3}), \pi(M) \leqslant 6$.

*(g) (Seymour, 1980) Prove that F_7 is a splitter for $EX(U_{2,4}, F_7^*)$, that $M(K_5)$ is a splitter for $EX(U_{2,4}, F_7, F_7^*, M(K_{3,3}))$, and that R_{10} is a splitter for $EX(U_{2,4}, F_7^*)$.

*(h) (Walton & Welsh, 1980) Prove the following:

$$\pi(EX(U_{2,4}, F_7, M(K_5))) \leqslant 6;$$
$$\pi(EX(U_{2,4}, F_7^*, M(K_5))) \leqslant 6;$$
$$\pi(EX(U_{2,4}, F_7, M(K_{3,3}))) \leqslant 6;$$
$$\pi(EX(U_{2,4}, F_7^*, M(K_{3,3}))) \leqslant 6.$$

(i) Show that the Four Color theorem is equivalent to the assertion that

$$\pi(EX(U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M(K_5), M(K_{3,3}))) = 4.$$

(j) Use (i) and Wagner's theorem (1964) that the case $k = 4$ of Hadwiger's conjecture is equivalent to the Four Color theorem to prove that

$$\pi(EX(U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M(K_5))) = 4.$$

(k) Prove the following:

$$\pi(EX(U_{2,4}, F_7, M^*(K_{3,3}), M(K_5))) = 4;$$
$$\pi(EX(U_{2,4}, F_7^*, M^*(K_{3,3}), M(K_5))) = 5;$$
$$\pi(EX(U_{2,4}, F_7, M^*(K_{3,3}), M(K_{3,3}))) = 5;$$
$$\pi(EX(U_{2,4}, F_7^*, M^*(K_{3,3}), M(K_{3,3}))) = 5.$$

(l) Deduce from (k) that a loopless graph having no subgraph contractible to $K_{3,3}$ is 5-colorable.

(m) Deduce from (k) that a graph without isthmuses having no subgraph contractible to $K_{3,3}$ has a nowhere-zero 4-flow.

(n) Use (m) to show that a cubic graph without isthmuses having no subgraph contractible to $K_{3,3}$ has edge-chromatic number 3.

*6.56. (Walton, 1981) Use Seymour's decomposition result for regular matroids to prove that if $M \in EX(U_{2,4}, F_7)$, then $p(M; \lambda) \geqslant 0$ for all λ in $\mathbb{Z}^+$.

6.57. (a) Show that $p(PG(n, q); \lambda) = \sum_{i=0}^{n+1} (\lambda - q^i)$.

(b) Show that if $M = AG(n, q)$, then

$$p(M; \lambda) = (\lambda - 1)[\lambda^n - (q^n - 1)\lambda^{n-1} + (q^n - 1)(q^{n-1} - 1)\lambda^{n-2} + \ldots + (-1)^k (q^n - 1)(q^{n-1} - 1) \ldots (q^{n-k+1} - 1)\lambda^{n-k} + \ldots + (-1)^n (q^n - 1)(q^{n-1} - 1) \ldots (q - 1)].$$

6.58. Find three non-isomorphic matroids each having characteristic polynomial equal to $(\lambda - 1)(\lambda - 3)^2$.

Section 6.4.B

6.59. (a) Find all minimal 1-blocks over $GF(2)$.

** (b) Find all minimal 1-blocks over $GF(3)$.

6.60. Prove that if M and N are minimal k-blocks over $GF(q)$, then so is their series connection.

6.61. Prove Proposition 6.4.14.

6.62. Check that the matroid N_4 in Example 6.4.22 is isomorphic to $AG(3,2)$.

6.63. (a) (Oxley, 1980) Let M be coordinatizable over $GF(q)$ and C^* be a bond of M such that $c(M; q) - 1 = c(M - C^*; q) = k$. Prove that $|C^*| \geqslant q^k$.

(b) Show that if M is a tangential k-block and C^* is a bond of M having exactly q^k elements, then $E - C^*$ is a modular hyperplane of M.

6.64. (a) (Walton, 1981) Prove that a tangential k-block over $GF(q)$ is 3-connected.

(b) Give an example of a 3-connected minimal t-block that is not a tangential t-block.

(c) (Walton, 1981) Prove that a tangential k-block over $GF(2)$ is not a 3-sum.

6.65. (Mullin & Stanton, 1979) A $(q, m. k)$-matroid is a submatroid of $PG(k-1, q)$ having rank k and critical exponent greater than m. A minimal (q, m, k)-matroid is a (q, m, k)-matroid for which no submatroid is also a (q, m, k)-matroid. Let $\eta(q, m, k)$ denote the least number of elements in a (q, m, k)-matroid.

* (a) (Oxley, 1979a) Prove that $\eta(q, m, k) = \dfrac{q^{m+1} - 1}{q - 1} + k - m - 1$.

(b) (Brylawski, 1975c) Show that if $r, q > 2$ and $U_{r,n}$ is coordinatizable over $GF(q)$, then $c(U_{r,n}; q) = 1$.

* (c) Prove that a (q, m, k)-matroid having $\eta(q, m, k)$ elements is isomorphic to $PG(m, q) \oplus U_{k-m-1,k-m-1}$ for $q = 2$ and $m \geqslant 2$, and for $q > 2$ and $m \geqslant 1$.

(d) Deduce from (c) the following result of Bose & Burton (1966). Let M be a loopless matroid coordinatizable over $GF(q)$ and suppose M has critical exponent greater than m. Then M has at least $\frac{q^{m+1}-1}{q-1}$ elements. Moreover, if M has exactly $\frac{q^{m+1}-1}{q-1}$ elements, then $M \cong PG(m, q)$.

6.66. Use an argument involving coloring and the critical exponent to prove that $AG(3, 2)$ has no minor isomorphic to $M(K_4)$.

6.67. (Whittle, 1985) Recall that if M and N are matroids having a common ground set E, then N is a *quotient* of M if every flat of N is also a flat of M. Let M be a tangential k-block over $GF(q)$ and N be a loopless quotient of M that is coordinatizable over $GF(q)$. Let $r(M) - r(N) = m$. Assume that N has a proper non-empty flat F such that
(1) F is a modular flat of M;
(2) $r_M(F) - r_N(F) = m$; and
(3) for every proper flat F' of $N(F)$, $p(N(F)/F'; q^k) > 0$.
* (a) Prove that the simplification of N is a tangential k-block over $GF(q)$.
(b) Deduce Theorem 6.4.27 from (a).
(c) Deduce Theorem 6.4.29 from (a).

6.68. Suppose that the matroid M is represented over $GF(q)$ by a matrix A. We call M' a *$GF(q)$-vector quotient* of M (see, for example, White, 1986, 7.4.8) if M' can be obtained from M by adjoining a linearly independent set of columns to A and then contracting those elements of the resulting matroid that correspond to the newly adjoined columns.
(a) Show that if M' is a $GF(q)$-vector quotient of M, then $c(M'; q) \geqslant c(M; q)$.
* (b) (Jaeger, 1981) Prove the following analog of Hajós' theorem (1961) characterizing all graphs of chromatic number at least k. If M is coordinatizable over $GF(q)$, then $c(M; q) \geqslant k$ if and only if M has as a restriction a matroid that can be constructed from copies of $PG(k-1, q)$ by a sequence of series connections and $GF(q)$-vector quotients.

Section 6.4.C

6.69. Both K_4 and K_5 have two different types of hyperplanes. Find the four bond graphs that arise from K_4 and K_5.

6.70. Add the argument omitted in the last paragraph of the proof of Proposition 6.4.48.

6.71. Give an example of a transversal matroid coordinatizable over $GF(q)$ having critical exponent 2.

6.72.**(a) (Jaeger, 1982) Let I_1 and I_2 be independent sets in $PG(r-1, q)$ and M be the submatroid on $I_1 \cup I_2$. For which values of q is $c(M; q) = 1$?
**(b) If $I_1, I_2, \ldots, I_k$ are independent sets in $PG(r-1, q)$, when is $c(I_1 \cup I_2 \cup \ldots \cup I_k; q) \leqslant k$? When is k best-possible here?

Section 6.5

6.73. (a) Show that $M(C)$ depends only on C and not on the generator matrix U for C.
(b) Show that $M(C^*) \cong M^*(C)$.

6.74. Verify (6.58).

6.75. Complete the derivation of Proposition 6.5.1 from (6.64).

6.76. Prove Proposition 6.5.2.

6.77. Let C be a binary code. Use Corollary 6.5.6 to verify that, in the code over $GF(4)$ that is generated by C, all codewords have even weight.

6.78. (a) Calculate $p_M(\lambda)$ when M is $G_q(n, 2)$ and when M is $G_q(n, n-1)$.
(b) Calculate $p_M(\lambda)$ when M is $G_2(5, 3)$.
(c) Can the critical exponent of $G_2(n+1, 2k+1)$ be computed directly from that of $G_2(n, 2k)$? (They are the same.)

6.79. Calculate the codeweight polynomial $A(C)$ for
(a) the projective code C_p where $M(C_p) = PG(r-1, q)$;
(b) the dual C_p^* of C_p (This is the *Hamming code*.);
(c) the optimal code C where $M(C) = U_{r,n}$.

6.80. (a) Prove that if M is coordinatizable over $GF(q)$, then, provided q is sufficiently large, $C(M)$ has codewords of every possible weight.
(b) Prove that, for $q = 2$, $A(C(M))$ always has a_n or a_{n-1} equal to zero unless M has an isthmus.

6.81. With $A(C; q, z) = \sum_{i=0}^{n} a_i z^i$, define $T_j = \frac{\sum_i a_{n-i}\binom{i}{j}}{q^{r-j}}$. Use (6.64) to prove (a)–(c).
(a) T_j is always an integer.
(b) $T_0 = 1$.
(c) T_1 determines the number of loops in $M(C)$.
If C^* has distance d, calculate T_i for $i \leqslant d$.

6.82. (Asano, Nishizeki, Saito & Oxley, 1984) Let U be an $r \times n$ matrix over $GF(q)$. The *chain-group N generated* by U is the linear code C generated by U, a *chain* being a codeword in C. The support $\sigma(f)$ of a chain f is the support of the corresponding codeword. Let $M = M(C)$ and E denote the set of elements of M.
(a) Prove that $c(M; q) \leqslant k$ if and only if N contains k chains $f_1, f_2, \ldots, f_k$ such that $E = \sum_{i=1}^{k} \sigma(f_i)$.
Suppose that $S \subseteq T \subseteq E(M)$. Prove that
(b) $c(M(S); q) \leqslant k$ if and only if N contains k chains $f_1, f_2, \ldots, f_k$ such that $S \subseteq \bigcup_{i=1}^{k} \sigma(f_i)$;
(c) $c(M/(E-S); q) \leqslant k$ if and only if N contains k chains $f_1, f_2, \ldots, f_k$ such that $S = \bigcup_{i=1}^{k} \sigma(f_i)$;
(d) $c(M(T)/(T-S); q) \leqslant k$ if and only if $N(T)$, the set of restrictions to T of chains in N, contains k chains $f_1', f_2', \ldots, f_k'$ such that $S = \bigcup_{i=1}^{k} \sigma(f_i')$.
(e) Use (b)–(d) to give another proof of Proposition 6.4.5.

6.83. (Jaeger, 1989a) Let C be a binary code and E be the ground set of $M(C)$. Let U^* be a generator matrix for C^*. If $e \in E$, then to *double e in series*, one replaces the column of U^* corresponding to e by two copies of this column. If $F \subseteq E$, let $U^*:F$ be the matrix obtained from U^* by doubling every element of F in series. Let $C^*:F$ be the code generated by $U^*:F$ and let $C:F$ be $(C^*:F)^*$.

(a) Show that, for e in E, $M(C:\{e\})$ is obtained from $M(C)$ by adding an element in series with e unless e is an isthmus of $M(C)$; in the exceptional case, $M(C:\{e\})$ is obtained from $M(C)$ by adjoining another isthmus.

Let N be the number of ordered pairs (X_1, X_2) of subsets of E for which $X_1 \cup X_2 = E$ and each X_i is a disjoint union of circuits of $M(C)$.

(b) Show that N equals the number of ordered pairs of codewords of C^*, the union of whose supports is E.

* (c) Prove that

$$N = (-\tfrac{1}{2})^r \sum_{F \subseteq E} (-1)^{|F|}(-2)^{\dim B(C:F)}$$

where $r = r(M(C))$.

* (d) Prove that

$$N = (-1)^{r(M^*(C))} t(M(C); 0, -3).$$

(e) Use (c), (d), and Proposition 6.5.4 to prove that, for a binary matroid M having ground set E and rank r,

$$t(M; 0, -3) = (\tfrac{1}{2})^r \sum_{F \subseteq E} t(M:F; -1, -1).$$

Here $M:F$ is obtained from M by adjoining an element in series to each non-isthmus element of F, and adjoining an isthmus to M for each isthmus of M in F.

(f) Let M be a rank r matroid on E and α, β, and γ be numbers with $1+\gamma \neq 0$ and $1+\gamma(\alpha+1) \neq 0$. Using induction, prove that

$$t\left(M; \frac{\alpha+\gamma\alpha^2}{1+\gamma}, \frac{\beta+\gamma(\alpha+\beta)}{1+\gamma(\alpha+1)}\right) = \left(\frac{1}{1+\gamma}\right)^r \left(\frac{1}{1+\gamma(\alpha+1)}\right)^{|E|-r} \sum_{F \subseteq E} \gamma^{|F|} t(M:F; \alpha, \beta).$$

(g) If C is a binary code, show that

$$|F| + \dim(B(C:F)) \geqslant \dim B(C).$$

(h) Use (f), (g) and Proposition 6.5.4 to prove that, if M is a binary matroid, then $t(M; 3, 3) = kt(M; -1, -1)$ for some odd integer k.

(i) Show that the Four Color theorem is equivalent to the statement that if M is the polygon matroid of a planar loopless graph, then $t(M; -3, 0)$ is non-zero with the sign of $(-1)^{r(M)}$.

(j) Use (f) and (i) to prove that if M is the polygon matroid of a planar loopless graph, then , for all γ in $(-1, 0]$, $t\left(M; \frac{-3+9\gamma}{1+\gamma}, \frac{3\gamma}{2\gamma-1}\right)$ is non-zero with the sign of $(-1)^{r(M)}$.

Section 6.6.A

6.84. Prove that if $f(M) = |\mathscr{B}_{ij}(M)|$ and the greatest element n of M is neither a loop nor an isthmus, then $f(M) = f(M-n) + f(M/n)$.

6.85. Prove 6.73.

6.86. (Brylawski, 1977c) Let M be a matroid of rank r on $\{1, 2, \ldots, n\}$. Show that the number of ways to color the elements of M with $\{1', 2', \ldots, n'\}$ so that no broken circuit is colored entirely with $1'$ is equal to $\lambda^{n-r}p^+(M; \lambda)$.

Section 6.6.B

6.87. Show that if (6.75) holds, then the poset of hyperplane intersections is a geometric lattice.

6.88. (Buck, 1943) Suppose that the hyperplanes $H_1, H_2, \ldots, H_n$ are in general position in $\mathbb{E}^d$. Show that

(a) the number of regions of this arrangement is $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{d}$;

(b) the number of bounded regions of this arrangement is $\binom{n-1}{d}$.

6.89. Let Γ be a loopless graph having vertex set $\{1, 2, \ldots, d\}$. For each edge $e = ij$ of Γ, let H_e be the hyperplane $\{(x_1, x_2, \ldots, x_n): x_i = x_j\}$ of $\mathbb{E}^d$. Show that the following hold.

(a) For the arrangement $\{H_e: e \in E(\Gamma)\}$, the poset of hyperplane intersections is isomorphic to the lattice of flats of M_Γ.

(b) There is a bijection between the set of acyclic orientations of Γ and the regions of the arrangement $\{H_e: e \in E(\Gamma)\}$ determined as follows: the region corresponding to the acyclic orientation α of Γ is $\{(x_1, x_2, \ldots, x_d): x_i < x_j$ if α directs the edge ij of Γ from i to $j\}$.

Section 6.6.C

6.90. Let E be a set of n points in general position in $\mathbb{E}^d$. Show that the number of hyperplane-separable subsets of E is $\left[2\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{d}\right]$. (Zaslavsky, 1975b, p. 72, discusses the history of this result.)

Section 6.6.D

6.91. Show that the Turán graph $T_p(n)$ has the largest number of edges among all complete p-partite graphs on n vertices.

6.92. (a) For G equal to $PG(r-1, q)$ and $AG(r-1, q)$, find the matrices W_1 and W_2 (see Exercise 6.57).

* (b) (Brylawski, 1979b) Prove that, for any upper combinatorially uniform geometry G, the matrices W_1 and W_2 are inverses of each other.

6.93. (Brylawski, 1979b) Show that if M_{KC} is the cardinality-corank matrix of the matroid M, then the corresponding matrix M^*_{KC} for M^* satisfies

$$M^*_{KC}(i, j) = M_{KC}(n-i, i+j+r-n)$$

for all i in $\{0, 1, \ldots, n\}$ and all j in $\{0, 1, \ldots, n-r\}$.

6.94. (Brylawski, 1981b) Let Γ and Δ be the graphs shown in Figure 6.14.

(a) Show that, for each of the matroids M_Γ and M_Δ, the matrix M_{KC} is the following.

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 44 & 1 & 0 \\ 0 & 0 & 108 & 12 & 0 & 0 \\ 0 & 151 & 58 & 1 & 0 & 0 \\ 98 & 142 & 12 & 0 & 0 & 0 \\ 151 & 58 & 1 & 0 & 0 & 0 \\ 108 & 12 & 0 & 0 & 0 & 0 \\ 44 & 1 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(b) Show that for $M(K_6)$ the matrix W_2 is the following.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

(c) Find W_1 for $M(K_6)$.

(d) Use (6.76) to show that, for M in $\{M_\Gamma, M_\Delta\}$, $I_{M(K_6)}(M)$ is the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 8 & 6 & 1 \\ 0 & 0 & 10 & 24 & 8 & 0 \\ 0 & 2 & 28 & 24 & 1 & 0 \\ 0 & 4 & 24 & 8 & 0 & 0 \\ 0 & 7 & 19 & 1 & 0 & 0 \\ 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6.95. (Brylawski, 1979b) Let G and G' be upper combinatorially uniform geometries into which the matroid M is embedded. Show that

$$I_{G'}(M) = I_G(M) \cdot W_1 \cdot W_2'$$

where W_1 is the matrix of doubly indexed Whitney numbers of the first kind of G, and W_2' is the matrix of doubly indexed Whitney numbers of the second kind of G'.

6.96. (Brylawski, 1979b) Let M_1 be the matroid that is represented over any field by the following matrix:

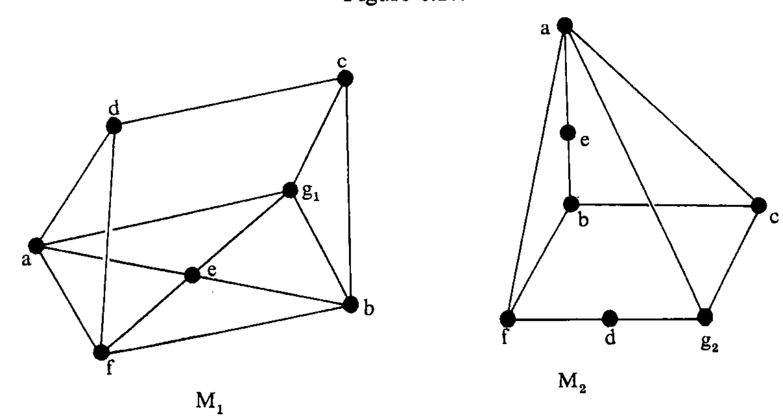
$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g_1 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Let M_2 be the matroid that is represented over any field except F_2 by the following matrix, where $\alpha \neq 0, 1$.

$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g_2 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & \alpha \end{array}\right] \end{array}$$

(a) Show that M_1 and M_2 have the affine embeddings shown in Figure 6.17.

Figure 6.17.



(b) Show that, for both M_1 and M_2, the matrix M_{KC} is the following.

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 21 & 0 & 0 \\ 0 & 33 & 2 & 0 & 0 \\ 24 & 11 & 0 & 0 & 0 \\ 20 & 1 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(c) Show that $t(M_1; x, y) = t(M_2; x, y)$.

(d) Show that if $M \in \{M_1, M_2\}$, then $I_{PG(3,3)}(M)$ is the following matrix.

$$\begin{bmatrix} 0 & 2 & 58 & 33 & 1 \\ 0 & 7 & 55 & 7 & 0 \\ 0 & 17 & 15 & 0 & 0 \\ 0 & 7 & 2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

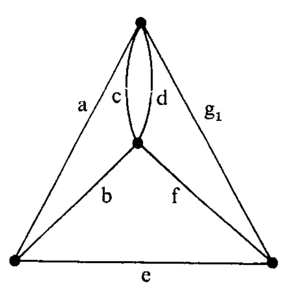
(e) Show by using (6.76) and also by using the result of Exercise 6.95 that $I_{PG(3,2)}(M_1)$ is the following matrix.

$$\begin{bmatrix} 0 & 0 & 5 & 8 & 1 \\ 0 & 1 & 13 & 7 & 0 \\ 0 & 2 & 15 & 0 & 0 \\ 0 & 5 & 2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(f) Why is the matrix $I_{PG(3,2)}(M_2)$ undefined?

(g) Show that M_1^* is isomorphic to the polygon matroid of the graph shown in Figure 6.18.

Figure 6.18.



(h) Use Exercise 6.93 to find M_{KC} for M_1^* and then (6.76) to show that $I_{M(K_4)}(M_1^*)$ is the following matrix. Check your calculations by finding $I_{M(K_4)}(M_1^*)$ directly from the definition.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(i) Find M_{KC} for M_1^c, where M_1^c is the matroid $PG(3, 2) - X$ where X is the set of columns of the matrix representing M_1.

6.97. (Brylawski, 1981b) Show that the graphs Γ and Δ in Figure 6.14, which have the same Tutte polynomial, have different polychromates.

References

Aigner, M. (1987). Whitney numbers, in N. White (ed.). *Combinatorial Geometries*, Encyclopedia of Mathematics and Its Applications 29, pp. 139–60. Cambridge University Press.

Appel, K. & Haken, W. (1976). Every planar map is four-colorable, *Bull. Amer. Math. Soc.* **82**, 711–12.

Arrowsmith, D. K. & Jaeger, F. (1982). On the enumeration of chains in regular chain-groups, *J. Comb. Theory Ser. B* **32**, 75–89.

Asano, T., Nishizeki, T., Saito, N. & Oxley, J. (1984). A note on the critical problem for matroids, *Europ. J. Comb.* **5**, 93–7.

Baclawski, K. (1975). Whitney numbers of geometric lattices, *Adv. Math.* **16**, 125–38.

Baclawski, K. (1979). The Möbius algebra as a Grothendieck ring, *J. Algebra* **57**, 167–79.

Bari, R. A. & Hall, D. W. (1977). Chromatic polynomials and Whitney's broken circuits, *J. Graph Theory* **1**, 269–75.

Barlotti, A. (1965). Some topics in finite geometrical structures, Institute of Statistics Mimeograph Series 439, Department of Statistics, University of North Carolina, Chapel Hill, North Carolina.

Barlotti, A. (1966). Bounds for *k*-caps in *PG*(*r*, *q*) useful in the theory of error-correcting codes, Institute of Statistics Mimeograph Series 484.2, Department of Statistics, University of North Carolina, Chapel Hill, North Carolina.

Barlotti, A. (1980). Results and problems in Galois geometry, in J. Srivastava (ed.), *Combinatorial Mathematics, Optimal Designs and their Applications*, Ann. Discrete Math. 6, pp. 1–5. North-Holland, Amsterdam.

Beissinger, J. S. (1982). On external activity and inversion in trees, *J. Comb. Theory Ser. B* **33**, 87–92.

Bender, E., Viennot, G. & Williamson, S. G. (1984). Global analysis of the delete–contract recursion for graphs and matroids, *Linear and Multilinear Algebra* **15**, 133–60.

Berman, G. (1977). The dichromate and orientations of a graph, *Can. J. Math.* **29**, 947–56.

Berman, G. (1978). Decomposition of graph functions, *J. Comb. Theory Ser. B* **25**, 151–65.

Biggs, N. (1974). *Algebraic Graph Theory*, Cambridge University Press.

Biggs, N. (1979). Resonance and reconstruction, in B. Bollobás (ed.), *Surveys in Combinatorics, Proceedings of the Seventh British Combinatorial Conference*, pp. 1–21. Cambridge University Press.

Biggs, N. L., Lloyd, E. K. & Wilson, R. J. (1976). *Graph Theory: 1736–1936*, Oxford University Press.

Birkhoff, G. D. (1912–13). A determinant formula for the number of ways of coloring a map. *Ann. Math.* (2) **14**, 42–46.

Birkhoff, G. D. (1913). The reducibility of maps, *Amer. J. Math.* **35**, 115–28.

Birkhoff, G. D. (1930). On the number of ways of coloring a map, *Proc. Edinburgh Math. Soc.* (2) **2**, 83–91.

Birkhoff, G. D. & Lewis, D. C. (1946). Chromatic polynomials, *Trans. Amer. Math. Soc.* **60**, 355–451.
Bixby, R. E. (1975). A composition for matroids, *J. Comb. Theory Ser. B* **18**, 59–73.
Bixby, R. E. (1977). Kuratowski's and Wagner's theorems for matroids, *J. Comb. Theory Ser. B* **22**, 31–53.
Björner, A. (1980). Some matroid inequalities, *Discrete Math.* **31**, 101–3.
Björner, A. (1982). On the homology of geometric lattices, *Algebra Universalis* **14**, 107–28.
Blake, I. F. & Mullin, R. C. (1976). *An Introduction to Algebraic and Combinatorial Coding Theory*, Academic Press, New York.
Bland, R. G. & Las Vergnas, M. (1978). Orientability of matroids, *J. Comb. Theory Ser. B* **24**, 94–123.
Bollobás, B. (1976). *Extremal Graph Theory*, Academic Press, London.
Bondy, J. A. & Hemminger, R. L. (1977). Graph reconstruction – a survey, *J. Graph Theory* **1**, 227–68.
Bondy, J. A. & Murty, U. S. R. (1976). *Graph Theory with Applications*, Macmillan, London; American Elsevier, New York.
Bondy, J. A. & Welsh, D. J. A. (1972). Some results on transversal matroids and constructions for identically self-dual matroids, *Quart. J. Math. Oxford* (2) **23**, 435–51.
Bose, R. C. & Burton, R. B. (1966). A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and MacDonald codes, *J. Comb. Theory* **1**, 96–104.
Bosman, O. (1982). Nowhere-zero flows in graphs, Honours Thesis, Australian National University.
Brini, A. (1980). A class of rank-invariants for perfect matroid designs, *Europ. J. Comb.* **1**, 33–8.
Broadbent, S. R. & Hammersley, J. M. (1957). Percolation processes I. Crystals and mazes, *Proc. Camb. Phil. Soc.* **53**, 629–41.
Brooks, R. L. (1941). On colouring the nodes of a network, *Proc. Camb. Phil. Soc.* **37**, 194–7.
Brouwer, A. E. & Schrijver, A. (1978). The blocking number of an affine space, *J. Comb. Theory Ser. A* **24**, 251–3.
Bruen, A. A. & de Resmini, M. (1983). Blocking sets in affine planes, *Combinatorics '81* (Rome, 1981), North-Holland Mathematics Studies 78, pp. 169–75. North-Holland, Amsterdam.
Bruen, A. A. & Thas, J. A. (1977). Blocking sets, *Geom. Dedicata* **6**, 193–203.
Brylawski, T. (1971). A combinatorial model for series–parallel networks, *Trans. Amer. Math. Soc.* **154**, 1–22.
Brylawski, T. (1972a). The Tutte–Grothendieck ring, *Algebra Universalis* **2**, 375–88.
Brylawski, T. (1972b). A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171**, 235–82.
Brylawski, T. (1974). Reconstructing combinatorial geometries, in R. A. Bari & F. Harary (eds), *Graphs and Combinatorics*, Lecture Notes in Mathematics 406, pp. 226–35. Springer-Verlag, Berlin.
Brylawski, T. (1975a). Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203**, 1–44.
Brylawski, T. (1975b). On the nonreconstructibility of combinatorial geometries, *J. Comb. Theory Ser. B* **19**, 72–6.
Brylawski, T. (1975c). An affine representation for transversal geometries, *Stud. Appl. Math.* **54**, 143–60.
Brylawski, T. (1976). A combinatorial perspective on the Radon convexity theorem, *Geom. Dedicata* **54**, 459–66.
Brylawski, T. (1977a). A determinantal identity for resistive networks, *SIAM J. Appl. Math.* **32**, 443–9.
Brylawski, T. (1977b). Connected matroids with the smallest Whitney numbers, *Discrete Math.* **18**, 243–52.
Brylawski, T. (1977c). The broken-circuit complex, *Trans. Amer. Math. Soc.* **234**, 417–33.
Brylawski, T. (1977d). Geometrie combinatorie e loro applicazioni, University of Rome lecture series (unpublished).
Brylawski, T. (1977e). Funzioni di Möbius, University of Rome lecture series (unpublished).
Brylawski, T. (1979a). Teoria dei codici e matroidi, University of Rome lecture series (unpublished).
Brylawski, T. (1979b). Intersection theory for embeddings of matroids into uniform geometries, *Stud. Appl. Math.* **61**, 211–44.

Brylawski, T. (1980). The affine dimension of the space of intersection matrices, *Rend. Mat.* (6) **13**, 59–68.
Brylawski, T. (1981a). Matroidi coordinabili, University of Rome lecture series (unpublished).
Brylawski, T. (1981b). Intersection theory for graphs, *J. Comb. Theory Ser. B* **30**, 233–46.
Brylawski, T. (1981c). Hyperplane reconstruction of the Tutte polynomial of a geometric lattice, *Discrete Math.* **35**, 25–38.
Brylawski, T. (1982). The Tutte polynomial. Part 1: General theory, in A. Barlotti (ed.), *Matroid Theory and Its Applications, Proceedings of the Third Mathematics Summer Center* (C.I.M.E., 1980), pp. 125–275. Liguori, Naples.
Brylawski, T. (1985). Coordinatizing the Dilworth truncation, in L. Lovász & A. Recski (eds), *Matroid Theory*, Colloq. Math. Soc. János Bolyai 40, pp. 61–95. North-Holland, New York.
Brylawski, T. (1986). Blocking sets and the Möbius function, in *Combinatorica*, Symposia Mathematica 28, pp. 231–49. Academic Press, New York.
Brylawski, T. & Kelly, D. G. (1978). Matroids and combinatorial geometries, in G.-C. Rota (ed.), *Studies in Combinatorics*, pp. 179–217. Mathematical Association of America, Washington, D.C.
Brylawski, T. & Kelly, D. G. (1980). *Matroids and Combinatorial Geometries*, Carolina Lecture Series 8, Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina.
Brylawski, T. & Lucas, T. D. (1976). Uniquely representable combinatorial geometries, in *Colloquio Internazionale Sulle Teorie Combinatorie* (Roma, 1973), Atti dei Convegni Lincei 17, Tomo I, pp. 83–104. Accad. Naz. Lincei, Rome.
Brylawski, T. & Oxley, J. G. (1980). Several identities for the characteristic polynomial of a combinatorial geometry, *Discrete Math.* **31**, 161–70.
Brylawski, T. & Oxley, J. G. (1981). The broken-circuit complex: its structure and factorizations, *Europ. J. Comb.* **2**, 107–21.
Brylawski, T., Lo Re, P. M., Mazzocca, F. & Olanda, D. (1980). Alcune applicazioni della teoria dell'intersezione alle geometrie di Galois, *Richerche Mat.* **29**, 65–84.
Buck, R. C. (1943). Partition of space, *Amer. Math. Monthly* **50**, 541–4.
Cardy, S. (1973). The proof of and generalisations to a conjecture by Baker and Essam, *Discrete Math.* **4**, 101–22.
Cartier, P. (1981). Les arrangements d'hyperplans: Un chapitre de géométrie combinatoire, in *Séminaire Bourbaki, Vol. 1980/81 Exposés 561–578*, Lecture Notes in Mathematics, 901, pp. 1–22. Springer-Verlag, Berlin.
Chaiken, S. (1989). The Tutte polynomial of a ported matroid, *J. Comb. Theory Ser. B* **46**, 96–117.
Cordovil, R. (1979). Contributions à la théorie des géométries combinatories, Thesis, l'Université Pierre et Marie Curie, Paris.
Cordovil, R. (1980). Sur l'évaluation $t(M; 2, 0)$ du polynôme de Tutte d'un matroïde et une conjecture de B. Grünbaum relative aux arrangements de droites du plan. *Europ. J. Comb.* **1**, 317–22.
Cordovil, R. (1982). Sur les matroïdes orientés de rang 3 et les arrangements du pseudodroites dans le plan projectif réel, *Europ. J. Comb.* **3**, 307–18.
Cordovil, R. (1985). A combinatorial perspective on the non-Radon partitions, *J. Comb. Theory Ser. A* **38**, 38–47; erratum **40**, 194.
Cordovil, R. & Silva, I. P. (1985). A problem of McMullen on the projective equivalences of polytopes, *Europ. J. Comb.* **6**, 157–61.
Cordovil, R. & Silva, I. P. (1987). Determining a matroid polytope by non-Radon partitions, *Linear Algebra Appl.* **94**, 55–60.
Cordovil, R., Las Vergnas, M. & Mandel, A. (1982). Euler's relation, Möbius functions, and matroid identities, *Geom. Dedicata* **12**, 147–62.
Cossu, A. (1961). Su alcune proprietà dei (k, n)-archi di un piano proiettivo sopra un corpo finito, *Rend. Mat.* (5) **20**, 271–77.
Crapo, H. H. (1966). The Möbius function of a lattice, *J. Comb. Theory* **1**, 126–31.
Crapo, H. H. (1967). A higher invariant for matroids, *J. Comb. Theory* **2**, 406–17.
Crapo, H. H. (1968a). Möbius inversions in lattices, *Arch. Math. (Basel)* **19**, 595–607.
Crapo, H. H. (1968b). The joining of exchange geometries, *J. Math. Mech.* **17**, 837–52.
Crapo, H. H. (1969). The Tutte polynomial, *Aequationes Math.* **3**, 211–29.
Crapo, H. H. (1970). Chromatic polynomials for a join of graphs, in P. Erdös, A. Rényi &

V. Sós (eds), *Combinatorial Theory and its Applications*, Colloq. Math. Soc. János Bolyai 4, pp. 239–45. North-Holland, Amsterdam.
Crapo, H. H. (1971). Constructions in combinatorial geometries, Notes, National Science Foundation Advanced Science Seminar in Combinatorial Theory, Bowdoin College, Maine (unpublished).
Crapo, H. H. & Rota, G.-C. (1970). *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, preliminary edition, M.I.T. Press, Cambridge, Mass.
Cunningham, W. H. & Edmonds, J. (1980). A combinatorial decomposition theory, *Can. J. Math.* **32**, 734–65.
d'Antona, O. & Kung, J. P. S. (1980). Coherent orientations and series–parallel networks, *Discrete Math.* **32**, 95–8.
Datta, B. T. (1976a). On tangential 2-blocks, *Discrete Math.* **21**, 1–22.
Datta, B. T. (1976b). Nonexistence of six-dimensional tangential 2-blocks, *J. Comb. Theory Ser. B* **21**, 171–93.
Datta, B. T. (1979). On Tutte's conjecture for tangential 2-blocks, in J. A. Bondy & U. S. R. Murty (eds), *Graph Theory and Related Topics*, pp. 121–32. Academic Press, New York.
Datta, B. T. (1981). Nonexistence of seven-dimensional tangential 2-blocks, *Discrete Math.* **36**, 1–32.
Dawson, J. E. (1984). A collection of sets related to the Tutte polynomial of a matroid, in K. M. Koh & H. P. Yap (eds), *Graph Theory*, Lecture Notes in Mathematics 1073, pp. 193–204. Springer-Verlag, Berlin.
Deza, M. (1977). On perfect matroid designs, in *Construction and Analysis of Designs* (Japanese), Proceedings of a Symposium, pp. 98–108. Research Institute for Mathematical Sciences, Kyoto University, Kyoto.
Deza, M. (1984). *Perfect matroid designs*, Reports of the Department of Mathematics of the University of Stockholm, Sweden, No. 8.
Deza, M. & Singhi, N. M. (1980). Some properties of perfect matroid designs, in J. Srivastava (ed.), *Combinatorial Mathematics, Optimal Designs and their Applications*, Ann. Discrete Math. 6, pp. 57–76. North-Holland, Amsterdam.
Dirac, G. A. (1952). A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27**, 85–92.
Dirac, G. A. (1957). A theorem of R. L. Brooks and a conjecture of H. Hadwiger, *Proc. London Math. Soc.* (3) **7**, 161–95.
Dirac, G. A. (1964). Generalisations of the five colour theorem, *Theory of Graphs and Its Applications* (Smolenice, 1963), pp. 21–7. Academic Press, New York.
Dowling, T. A. (1971). Codes, packing, and the critical problem, *Atti del Convegno di Geometria Combinatoria e sue Applicazioni*, pp. 209–24. Institute of Mathematics, University of Perugia, Perugia.
Dowling, T. A. (1973a). A class of geometric lattices based on finite groups, *J. Comb. Theory* **13**, 61–87.
Dowling, T. A. (1973b). A q-analog of the partition lattice, in J. N. Srivastava *et al.* (eds), *A Survey of Combinatorial Theory*, pp. 101–15. North-Holland, Amsterdam.
Dowling, T. A. & Wilson, R. M. (1974). The slimmest geometric lattices, *Trans. Amer. Math. Soc.* **196**, 203–15.
Edelman, P. (1984a). A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes, *Trans. Amer. Math. Soc.* **283**, 617–32.
Edelman, P. H. (1984b). The acyclic sets of an oriented matroid, *J. Comb. Theory Ser. B* **36**, 26–31.
Edmonds, J. (1965a). Lehman's switching game and a theorem of Tutte and Nash-Williams, *J. Res. Nat. Bur. Stand. Section B* **69**, 73–7.
Edmonds, J. (1965b). Minimum partition of a matroid into independent sets, *J. Res. Nat. Bur. Stand. Section B* **69**, 67–72.
Erdös, P. (1967). Extremal problems in graph theory, in F. Harary (ed.), *A Seminar on Graph Theory*, pp. 54–9. Holt, Rinehart & Winston, New York.
Essam, J. W. (1971). Graph theory and statistical physics, *Discrete Math.* **1**, 83–112.
Farrell, E. J. (1979). On a general class of graph polynomials, *J. Comb. Theory Ser. B* **26**, 111–22.
Goldman, J. & Rota, G.-C. (1969). The number of subspaces of a vector space, in W. T. Tutte (ed.), *Recent Progress in Combinatorics*, pp. 75–83. Academic Press, New York.
Good, I. J. & Tideman, T. N. (1977). Stirling numbers and a geometric structure from voting

theory, *J. Comb. Theory Ser. A* **23**, 34–45.
Greene, C. (1973). On the Möbius algebra of a partially ordered set, *Adv. Math.* **10**, 177–87.
Greene, C. (1975). An inequality for the Möbius function of a geometric lattice, *Stud. Appl. Math.* **54**, 71–4.
Greene, C. (1976). Weight enumeration and the geometry of linear codes, *Stud. Appl. Math.* **55**, 119–28.
Greene, C. (1977). Acyclic orientations (notes from the talk), in M. Aigner (ed.), *Higher Combinatorics*, pp. 65–8. Reidel, Dordrecht.
Greene, C. & Zaslavsky, T. (1983). On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions and orientations of graphs, *Trans. Amer. Math. Soc.* **280**, 97–126.
Greenwell, D. L. & Hemminger, R. L. (1969). Reconstructing graphs, in G. Chartrand & S. F. Kapoor (eds), *The Many Facets of Graph Theory*, Lecture Notes in Mathematics 110, pp. 91–114. Springer-Verlag, Berlin.
Hadwiger, H. (1943). Über eine Klassification der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich* **88**, 133–42.
Hajós, G. (1961). Über eine Konstruktion nicht färbbärer Graphen, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg A* **10**, 116–17.
Hall, D. W., Siry, J. W. & Vanderslice, B. R. (1959). The chromatic polynomial of the truncated icosahedron, *Proc. Amer. Math. Soc.* **66**, 405–7.
Heron, A. P. (1972a). Matroid polynomials, in D. J. A. Welsh & D. R. Woodall (eds), *Combinatorics*, pp. 164–203. Institute of Mathematics and Its Applications, Southend-on-Sea, U.K.
Heron, A. P. (1972b). Some topics in matroid theory, D. Phil. Thesis, Oxford.
Huseby, A. B. (1989). Domination theory and the Crapo β-invariant, *Networks* **19**, 135–49.
Jaeger, F. (1976a). Balanced valuations and flows in multigraphs, *Proc. Amer. Math. Soc.* **55**, 237–42.
Jaeger, F. (1976b). On nowhere-zero flows in multigraphs, in C. Nash-Williams & J. Sheehan (eds), *Proceedings of the Fifth British Combinatorial Conference*, Congressus Numerantium 15, pp. 373–9. Utilitas Mathematica, Winnipeg.
Jaeger, F. (1979). Flows and generalized coloring theorems in graphs, *J. Comb. Theory Ser. B* **26**, 205–16.
Jaeger, F. (1981). A constructive approach to the critical problem for matroids, *Europ. J. Comb.* **2**, 137–44.
Jaeger, F. (1982). Problem, Matroid Theory (Szeged, 1982), Colloq. Math. Soc. János Bolyai 40 (unpublished).
Jaeger, F. (1983). Geometrical aspects of Tutte's 5-flow conjecture, in *Graphs and Other Combinatorial Topics*, pp. 124–33. Teubner, Leipzig.
Jaeger, F. (1985). On five-edge-colorings of cubic graphs and nowhere-zero flow problems, *Ars Comb.* **20**, 229–44.
Jaeger, F. (1988a). On Tutte polynomials and cycles of plane graphs, *J. Comb. Theory Ser. B* **44**, 127–46.
Jaeger, F. (1988b). Nowhere-zero flow problems, in L. W. Beineke & R. J. Wilson (eds), *Selected Topics in Graph Theory. 3*, pp. 71–95. Academic Press, London.
Jaeger, F. (1988c). On Tutte polynomials and link polynomials, *Proc. Amer. Math. Soc.* **103**, 647–54.
Jaeger, F. (1989a). On edge-colorings of cubic graphs and a formula of Roger Penrose, in L. D. Andersen (ed.), *Graph Theory in Memory of G. A. Dirac*, Ann. Discrete Math. 41, pp. 267–80. North-Holland, Amsterdam.
Jaeger, F. (1989b). On Tutte polynomials of matroids representable over $GF(q)$, *Europ. J. Comb.* **10**, 247–55.
Jaeger, F. (1989c). Tutte polynomials and bicycle dimension of ternary matroids, *Proc. Amer. Math. Soc.* **107**, 17–25.
Jaeger, F., Vertigan, D. L. & Welsh, D. J. A. (1990). On the computational complexity of the Jones and Tutte polynomials, *Math. Proc. Camb. Phil. Soc.* **108**, 35–53.
Jambu, M. & Terao, H. (1984). Free arrangements of hyperplanes and supersolvable lattices, *Adv. Math.* **52**, 248–58.
Joyce, D. (1984). Generalized chromatic polynomials, *Discrete Math.* **50**, 51–62.
Jones, V. F. R. (1985). A polynomial invariant for knots via von Neumann algebras,

Bull. Amer. Math. Soc. **12**, 103–11.
Kahn, J. & Kung, J. P. S. (1980). Varieties and universal models in the theory of combinatorial geometries, *Bull. Amer. Math. Soc.* **3**, 857–8.
Kászonyi, L. (1978a). An example for geometries having property $K(4)$ and being not four-colourable, in A. Hajnal & V. Sós (eds), *Combinatorics*, Colloq. Math. Soc. János Bolyai 18, pp. 635–8. North-Holland, New York.
Kászonyi, L. (1978b). On half-planar geometries, in A. Hajnal & V. Sós (eds), *Combinatorics*, Colloq. Math. Soc. János Bolyai 18, pp. 639–51. North-Holland, New York.
Kauffman, L. H. (1987). *On Knots*, Annals of Mathematical Studies 115, Princeton University Press.
Kauffman, L. H. (1988). New invariants in the theory of knots, *Amer. Math. Monthly* **95**, 195–242.
Kelly, D. G. & Rota, G.-C. (1973). Some problems in combinatorial geometry, in J. N. Srivastava *et al.* (eds), *A Survey of Combinatorial Theory*, pp. 309–13. North-Holland, Amsterdam.
Knuth, D. E. (1974). The asymptotic number of geometries, *J. Comb. Theory Ser. A* **17**, 398–401.
Kung, J. P. S. (1980). The Rédei function of a relation, *J. Comb. Theory Ser. A* **29**, 287–96.
Kung, J. P. S. (1986a). Growth rates and critical exponents of classes of binary combinatorial geometries, *Trans. Amer. Math. Soc.* **293**, 837–59.
Kung, J. P. S. (1986b). *A Source Book in Matroid Theory*, Birkhäuser, Boston.
Kung, J. P. S. (1987). Excluding the cycle geometries of the Kuratowski graphs from binary geometries, *Proc. London Math. Soc.* (3) **55**, 209–42.
Kung, J. P. S. (1988). The long-line graph of a combinatorial geometry: 1. Excluding $M(K_4)$ and the $(q+2)$-point line as minors, *Quart. J. Math. Oxford* (2) **39**, 223–34.
Kung, J. P. S., Murty, M. R. & Rota, G.-C. (1980). On the Rédei zeta function, *J. Number Theory* **12**, 421–36.
Las Vergnas, M. (1975a). Matroïdes orientables, *C.R. Acad. Sci. Paris Sér. A* **280**, 61–4.
Las Vergnas, M. (1975b). Extensions normales d'une matroïde, polynôme de Tutte d'un morphisme, *C.R. Acad. Sci. Paris Sér. A* **280**, 1479–82.
Las Vergnas, M. (1977). Acyclic and totally cyclic orientations of combinatorial geometries, *Discrete Math.* **20**, 51–61.
Las Vergnas, M. (1978). Sur les activités des orientations d'une géométrie combinatoire, *Colloque Mathématiques Discrètes: Codes et Hypergraphs* (Brussels 1978), Cahiers Centre Études Recherche Opérations 20, pp. 293–300. Brussels.
Las Vergnas, M. (1979). On Eulerian partitions of graphs, in R. J. Wilson (ed.), *Graph Theory and Combinatorics*, Research Notes in Mathematics 34, pp. 62–75. Pitman, San Francisco.
Las Vergnas, M. (1980). On the Tutte polynomial of a morphism of matroids, in M. Deza & I. G. Rosenberg (eds), *Combinatorics 79*, Part 1, Ann. Discrete Math. 8, pp. 7–20. North-Holland, Amsterdam.
Las Vergnas, M. (1981). Eulerian circuits of 4-valent graphs imbedded in surfaces, in L. Lovász & V. Sós (eds), *Algebraic Methods in Graph Theory*, Colloq. Math. Soc. János Bolyai 25, pp. 451–78. North-Holland, New York.
Las Vergnas, M. (1983). Le polynôme de Martin d'un graphe Eulérien, in C. Berge *et al.* (eds), *Combinatorial Mathematics*, Ann. Discrete Math. 17, pp. 397–411. North-Holland, Amsterdam.
Las Vergnas, M. (1984). The Tutte polynomial of a morphism of matroids, II: Activities of orientations, in J. A. Bondy & U. S. R. Murty (eds), *Progress in Graph Theory*, pp. 367–80. Academic Press, New York.
Las Vergnas, M. (1988). On the evaluation at (3, 3) of the Tutte polynomial of a graph, *J. Comb. Theory Ser. B* **44**, 367–72.
Lee, L. A. (1975). On chromatically equivalent graphs, Thesis, George Washington University.
Lickorish, W. B. R. (1988). Polynomials for links, *Bull. London Math. Soc.* **20**, 558–88.
Lindner, C. C. & Rosa, A. (1978). Steiner quadruple systems – a survey, *Discrete Math.* **22**, 147–81.
Lindström, B. (1978). On the chromatic number of regular matroids, *J. Comb. Theory Ser. B* **24**, 367–9.
Lipson, A. S. (1986). An evaluation of a link polynomial, *Math. Proc. Camb. Phil. Soc.* **100**, 361–4.
Lucas, T. D. (1974). Properties of rank-preserving weak maps, *Bull. Amer. Math. Soc.* **80**, 127–31.
Lucas, T. D. (1975). Weak maps of combinatorial geometries, *Trans. Amer. Math. Soc.* **206**, 247–79.
MacWilliams, F. J. (1963). A theorem on the distribution of weights in a systematic code, *Bell. Sys. Tech. J.* **42**, 79–94.
Martin, P. (1977). Enumérations eulériennes dans les multigraphes et invariants de Tutte–Grothendieck, Thesis, Grenoble.
Martin, P. (1978). Remarkable valuation of the dichromatic polynomial of planar multigraphs. *J. Comb. Theory Ser. B* **24**, 318–24.
Mason, J. (1972). Matroids: unimodal conjectures and Motzkin's theorem, in D. J. A. Welsh & D. R. Woodall (eds), *Combinatorics*, pp. 207–21. Institute of Mathematics and Its Applications, Southend-on-Sea, U.K.
Mason, J. (1977). Matroids as the study of geometrical configurations, in M. Aigner (ed.), *Higher Combinatorics*, pp. 133–76. Reidel, Dordrecht.
Matthews, K. R. (1977). An example from power residues of the critical problem of Crapo and Rota, *J. Number Theory* **9**, 203–8.
Minty, G. J. (1966). On the axiomatic foundations of the theories of directed linear graphs, electrical networks, and network programming, *J. Math. Mech.* **15**, 485–520.
Minty, G. J. (1967). A theorem on three-coloring the edges of a trivalent graph, *J. Comb. Theory* **2**, 164–7.
Mullin, R. C. & Stanton, R. G. (1979). A covering problem in binary spaces of finite dimension, in J. A. Bondy & U. S. R. Murty (eds), *Graph Theory and Related Topics*, pp. 315–27. Academic Press, New York.
Murty, U. S. R. (1971). Equicardinal matroids, *J. Comb. Theory* **11**, 120–6.
Nash-Williams, C. St. J. A. (1966). An application of matroids to graph theory, in *Theory of Graphs*, International Symposium (Rome), pp. 263–5. Dunod, Paris.
Negami, S. (1987). Polynomial invariants of graphs, *Trans. Amer. Math. Soc.* **299**, 601–22.
Ore, O. (1967). *The Four-Color Problem*, Academic Press, New York.
Orlik, P. (1989). *Introduction to Arrangements*, C.B.M.S. Regional Conference Series 72, American Mathematical Society, Providence, Rhode Island.
Orlik, P. & Solomon, L. (1980). Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56**, 167–89.
Oxley, J. G. (1978a). Colouring, packing, and the critical problem, *Quart. J. Math. Oxford* (2) **29**, 11–22.
Oxley, J. G. (1978b). Cocircuit coverings and packings for binary matroids, *Math. Proc. Camb. Phil. Soc.* **83**, 347–51.
Oxley, J. G. (1979a). A generalization of a problem of Mullin and Stanton for matroids, in A. F. Horadam & W. D. Wallis (eds), *Combinatorial Mathematics VI*, Lecture Notes in Mathematics 748, pp. 92–7. Springer-Verlag, Berlin.
Oxley, J. G. (1979b). On cographic regular matroids, *Discrete Math.* **25**, 89–90.
Oxley, J. G. (1980). On a covering problem of Mullin and Stanton for binary matroids, *Aequationes Math.* **20**, 104–12.
Oxley, J. G. (1982a). On Crapo's beta invariant for matroids, *Stud. Appl. Math.* **66**, 267–77.
Oxley, J. G. (1982b). A note on half-planar geometries, *Period. Math. Hungar.* **13**, 137–9.
Oxley, J. G. (1983a). On a matroid identity, *Discrete Math.* **44**, 55–60.
Oxley, J. G. (1983b). On the numbers of bases and circuits in simple binary matroids, *Europ. J. Comb.* **4**, 169–78.
Oxley, J. G. (1987a). The binary matroids with no 4-wheel minor, *Trans. Amer. Math. Soc.* **301**,

63–75.
Oxley, J. G. (1987b). A characterization of the ternary matroids with no $M(K_4)$-minor, *J. Comb. Theory Ser. B* **42**, 212–49.
Oxley, J. G. (1989a). The regular matroids with no 5-wheel minor, *J. Comb. Theory Ser. B* **46**, 292–305.
Oxley, J. G. (1989b). A characterization of certain excluded-minor classes of matroids, *Europ. J. Comb.* **10**, 275–9.
Oxley, J. G. (1989c). A note on Negami's polynomial invariants for graphs, *Discrete Math.* **76**, 279–81.
Oxley, J. G. (1990). On an excluded-minor class of matroids, *Discrete Math.* **82**, 35–52.
Oxley, J. G. & Welsh, D. J. A. (1979a). On some percolation results of J. M. Hammersley, *J. Appl. Prob.* **16**, 526–40.
Oxley, J. G. & Welsh, D. J. A. (1979b). The Tutte polynomial and percolation, in J. A. Bondy & U. S. R. Murty (eds), *Graph Theory and Related Topics*, pp. 329–39. Academic Press, New York.
Oxley, J. G., Prendergast, K. & Row, D. H. (1982). Matroids whose ground sets are domains of functions, *J. Austral. Math. Soc. Ser. A* **32**, 380–7.
Penrose, R. (1971). Applications of negative dimensional tensors, in D. Welsh (ed.), *Combinatorial Mathematics and Its Applications*, pp. 221–44. Academic Press, London.
Pezzoli, L. (1984). On *D*-complementation, *Adv. Math.* **51**, 226–39.
Provan, J. S. & Ball, M. O. (1984). Computing network reliability in time polynomial in the number of cuts, *Oper. Res.* **32**, 516–26.
Purdy, G. B. (1979). Triangles in arrangements of lines, *Discrete Math.* **25**, 157–63.
Purdy, G. B. (1980). On the number of regions determined by *n* lines in the projective plane, *Geom. Dedicata* **9**, 107–9.
Rado, R. (1978). Monochromatic paths in graphs, in B. Bollobás (ed.), *Advances in Graph Theory*, Ann. Discrete Math. 3, pp. 191–4. North-Holland, Amsterdam.
Read, R. C. (1968). An introduction to chromatic polynomials, *J. Comb. Theory* **4**, 52–71.
Rosenstiehl, P. (1975). Bicycles et diagonales des graphes planaires, in *Colloques sur la Théorie des Graphes* (Paris, 1974), Cahiers Centre Études Recherche Opérations 17, pp. 365–83. Brussels.
Rosenstiehl, P. & Read, R. C. (1978). On the principal edge tripartition of a graph, in B. Bollobás (ed.), *Advances in Graph Theory*, Ann. Discrete Math. 3, pp. 195–226. North-Holland, Amsterdam.
Rota, G.-C. (1964). On the foundations of combinatorial theory I: Theory of Möbius functions, *Z. Wahrsch. verw. Gebiete* **2**, 340–68.
Rota, G.-C. (1967). Combinatorial analysis as a theory, Hedrick Lectures, Mathematical Association of America summer meeting (Toronto, 1967) (unpublished).
Rota, G.-C. (1971). Combinatorial theory, old and new, in *Proceedings of the International Mathematics Congress* (Nice, 1970), 3, pp. 229–33. Gauthier-Villars, Paris.
Satyanarayana, A. & Tindell, R. (1987). Chromatic polynomials and network reliability, *Discrete Math.* **67**, 57–79.
Scafati Tallini, M. (1966). (*k*, *n*)-archi di un piano grafico finito, con particolare riguardo a quelli condue caratteri, Nota I, II, *Rend. Acc. Naz. Lincei* (8) **40**, 812–18, 1020–5.
Scafati Tallini, M. (1973). Calotte di tipo (*m*, *n*) in uno spazia di Galois *S*(*r*, *q*), *Rend Acc. Naz. Lincei* (8) **53**, 71–81.
Schläfi, L. (1950). *Gesammelte mathematische Abhandlungen*, Band I, Birkhäuser, Basel.
Schwärzler, W. (1991). Being Hamiltonian is not a Tutte invariant, *Discrete Math.* (to appear).
Segre, B. (1961). *Lectures on Modern Geometry*, Edizioni Cremonese, Rome.
Seymour, P. D. (1979). On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. London Math. Soc.* (3) **38**, 423–60.
Seymour, P. D. (1980). Decomposition of regular matroids, *J. Comb. Theory Ser. B* **28**, 305–59.
Seymour, P. D. (1981a). Nowhere-zero 6-flows, *J. Comb. Theory Ser. B* **30**, 130–5.
Seymour, P. D. (1981b). On Tutte's extension of the four-colour problem, *J. Comb. Theory Ser. B* **31**, 82–94.
Seymour, P. D. (1981c). Some applications of matroid decomposition, in L. Lovász & V. Sós (eds), *Algebraic Methods in Graph Theory*, Collōq. Math. Soc. János Bolyai 25, pp. 713–26. North-Holland, New York.
Seymour, P. D. (1991). Matroid structure, in *Handbook of Combinatorics*, R. Graham, M. Grötschel & L. Lovász (eds) (to appear).
Seymour, P. D. & Welsh, D. J. A. (1975). Combinatorial applications of an inequality from statistical mechanics, *Math. Proc. Camb. Phil. Soc.* **77**, 485–97.
Shank, H. (1975). The theory of left–right paths, in A. P. Street & W. D. Wallis (eds), *Combinatorial Mathematics III*, Lecture Notes in Mathematics 452, pp. 42–54. Springer-Verlag, Berlin.
Shepherd, G. C. (1974). Combinatorial properties of associated zonotopes, *Can. J. Math.* **26**, 302–21.
Smith, C. A. B. (1969). Map colourings and linear mappings, in D. J. A. Welsh (ed.), *Combinatorial Mathematics and Its Applications*, pp. 259–83. Academic Press, London.
Smith, C. A. B. (1972). Electric currents in regular matroids, in D. J. A. Welsh & D. R. Woodall (eds), *Combinatorics*, pp. 262–85. Institute of Mathematics and Its Applications, Southend-on-Sea, U.K.
Smith, C. A. B. (1974). Patroids, *J. Comb. Theory* **16**, 64–76.
Smith, C. A. B. (1978). On Tutte's dichromatic polynomial, in B. Bollobás (ed.), *Advances in Graph Theory*, Ann. Discrete Math. 3, pp. 247–57. North-Holland, Amsterdam.
Stanley, R. P. (1971a). Modular elements of geometric lattices, *Algebra Universalis* **1**, 214–17.
Stanley, R. P. (1971b). Supersolvable semimodular lattices, in *Proceedings of the Conference on Möbius Algebras*, pp. 80–142. University of Waterloo, Ontario.
Stanley, R. P. (1972). Supersolvable lattices, *Algebra Universalis* **2**, 197–217.
Stanley, R. P. (1973a). A Brylawski decomposition for finite ordered sets, *Discrete Math.* **4**, 77–82.
Stanley, R. P. (1973b). Acyclic orientations of graphs, *Discrete Math.* **5**, 171–8.
Stanley, R. P. (1980). Decompositions of rational convex polytopes, in J. Srivastava (ed.), *Combinatorial Mathematics, Optimal Designs and Their Applications*, Ann. Discrete Math. 6, pp. 333–42. North-Holland, Amsterdam.
Stanley, R. P. (1986). *Enumerative Combinatorics, Vol. I*, Wadsworth & Brooks/Cole, Monterey, California.
Szekeres, G. & Wilf, H. (1968). An inequality for the chromatic number of a graph, *J. Comb. Theory* **4**, 1–3.
Terao, H. (1980). Arrangement of hyperplanes and their freeness, I and II, *J. Faculty of Sci., Univ. Tokyo, Sci. IA*, **27**, 293–312, 313–20.
Terao, H. (1981). Generalized exponents of a free arrangement of hyperplanes and Shepard–Todd–Brieskorn formula, *Invent. Math.* **63**, 159–79.
Thistlethwaite, M. B. (1987). A spanning tree expansion of the Jones polynomial, *Topology* **26**, 297–309.
Thistlethwaite, M. B. (1988a). Kauffman's polynomial and altering links, *Topology* **27**, 311–18.
Thistlethwaite, M. B. (1988b). On the Kauffman polynomial of an adequate link, *Invent. Math.* **93**, 285–96.
Traldi, L. (1989). A dichromatic polynomial for weighted graphs and link polynomials, *Proc. Amer. Math. Soc.* **106**, 279–86.
Tutte, W. T. (1947). A ring in graph theory, *Proc. Camb. Phil. Soc.* **43**, 26–40.
Tutte, W. T. (1954). A contribution to the theory of chromatic polynomials, *Can. J. Math.* **6**, 80–91.
Tutte, W. T. (1956). A class of abelian groups, *Can. J. Math.* **8**, 13–28.
Tutte, W. T. (1958). A homotopy theorem for matroids I, II, *Trans. Amer. Math. Soc.* **88**, 144–60, 161–74.
Tutte, W. T. (1959). Matroids and graphs, *Trans. Amer. Math. Soc.* **90**, 527–52.
Tutte, W. T. (1965). Lectures on matroids, *J. Res. Nat. Bur. Stand. Section B* **69**, 1–47.
Tutte, W. T. (1966a). On the algebraic theory of graph colorings, *J. Comb. Theory* **1**, 15–50.

Tutte, W. T. (1966b). Connectivity in matroids, *Can. J. Math.* **18**, 1301–24.
Tutte, W. T. (1967). On dichromatic polynomials, *J. Comb. Theory* **2**, 301–20.
Tutte, W. T. (1969a). Projective geometry and the 4-color problem, in W. Tutte (ed.), *Recent Progress in Combinatorics*, pp. 199–207. Academic Press, New York.
Tutte, W. T. (1969b). A geometrical version of the four color problem, in R. C. Bose & T. A. Dowling (eds), *Combinatorial Mathematics and its Applications*, pp. 553–61. University of North Carolina Press, Chapel Hill, North Carolina.
Tutte, W. T. (1974). Codichromatic graphs, *J. Comb. Theory* **16**, 168–75.
Tutte, W. T. (1976). The dichromatic polynomial, in C. Nash-Williams & J. Sheehan (eds), *Proceedings of the Fifth British Combinatorial Conference*, Congressus Numerantium 15, pp. 605–35. Utilitas Mathematics, Winnipeg.
Tutte, W. T. (1979). All the king's men (a guide to reconstruction), in J. A. Bondy & U. S. R. Murty (eds), *Graph Theory and Related Topics*, pp. 15–33. Academic Press, New York.
Tutte, W. T. (1980a). 1-factors and polynomials, *Europ. J. Comb.* **1**, 77–87.
Tutte, W. T. (1980b). Rotors in graph theory, in J. Srivastava (ed.), *Combinatorial Mathematics, Optimal Designs and Their Applications*, Ann. Discrete Math. 6, pp. 343–7. North-Holland, Amsterdam.
Tutte, W. T. (1984). *Graph Theory*, Encyclopedia of Mathematics and Its Applications 21, Cambridge University Press.
Van Lint, J. H. (1971). *Coding Theory*, Lecture Notes in Mathematics 201, Springer-Verlag, Berlin.
Veblen, O. (1912). An application of modular equations in Analysis Situs, *Ann. Math.* (2) **14**, 86–94.
Vertigan, D. L. (1991). The computational complexity of Tutte invariants for planar graphs (to appear).
Wagner, K. (1964). Beweis einer Abschwächung der Hadwiger-Vermutung, *Math. Ann.* **153**, 139–41.
Walton, P. N. (1981). Some topics in combinatorial theory, D. Phil. Thesis, Oxford.
Walton, P. N. & Welsh, D. J. A. (1980). On the chromatic number of binary matroids, *Mathematika* **27**, 1–9.
Walton, P. N. & Welsh, D. J. A. (1982). Tangential 1-blocks over *GF*(3), *Discrete Math.* **40**, 319–20.
Welsh, D. J. A. (1969). Euler and bipartite matroids, *J. Comb. Theory* **6**, 375–77.
Welsh, D. J. A. (1971). Combinatorial problems in matroid theory, in D. J. A. Welsh (ed.), *Combinatorial Mathematics and Its Applications*, pp. 291–307. Academic Press, London.
Welsh, D. J. A. (1976). *Matroid Theory*, Academic Press, London.
Welsh, D. J. A. (1977). Percolation and related topics, *Science Prog.* **64**, 65–83.
Welsh, D. J. A. (1979). Colouring problems and matroids, in B. Bollobás (ed.), *Surveys in Combinatorics, Proceedings of the Seventh British Combinatorial Conference*, pp. 229–57. Cambridge University Press.
Welsh, D. J. A. (1980). Colourings, flows, and projective geometry, *Nieuw Archief Wiskunde* (3) **28**, 159–76.
Welsh, D. J. A. (1982). Matroids and combinatorial optimisation, in A. Barlotti (ed.), *Matroid Theory and Its Applications, Proceedings of the Third Mathematics Summer Center* (C.I.M.E., 1980), pp. 323–416. Liguori, Naples.
Welsh, D. J. A. (1988). Matroids and their applications, in L. W. Beineke & R. J. Wilson (eds), *Selected Topics in Graph Theory*, **3**, pp. 43–70. Academic Press, London.
White, N. (1972). The critical problem and coding theory, research paper, SPS-66 Vol. III, section 331, Jet Propulsion Lab., Pasadena, California.
White, N. (ed.) (1986). *Theory of Matroids*, Encyclopedia of Mathematics and Its Applications 26, Cambridge University Press.
White, N. (ed.) (1987). *Combinatorial Geometries*, Encyclopedia of Mathematics and Its Applications 29, Cambridge University Press.
Whitney, H. (1932). A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38**, 572–9.

Whitney, H. (1933a). The coloring of graphs. *Ann. Math.* (2) **33**, 688–718.
Whitney, H. (1933b). 2-isomorphic graphs, *Amer. J. Math.* **55**, 245–54.
Whitney, H. (1933c). A set of topological invariants for graphs, *Amer. J. Math.* **55**, 231–5.
Whitney, H. (1935). On the abstract properties of linear dependence, *Amer. J. Math.* **57**, 509–33.
Whittle, G. P. (1984). On the critical exponent of transversal matroids, *J. Comb. Theory Ser. B* **37**, 94–5.
Whittle, G. P. (1985). Some aspects of the critical problem for matroids, Ph.D. Thesis, University of Tasmania.
Whittle, G. P. (1987). Modularity in tangential k-blocks, *J. Comb. Theory Ser. B* **42**, 24–35.
Whittle, G. P. (1988). Quotients of tangential k-blocks, *Proc. Amer. Math. Soc.* **102**, 1088–98.
Whittle, G. P. (1989a). Dowling group geometries and the critical problem, *J. Comb. Theory Ser. B* **47**, 80–92.
Whittle, G. P. (1989b). q-lifts of tangential k-blocks, *J. London Math. Soc.* (2) **39**, 9–15.
Wilf, H. S. (1976). Which polynomials are chromatic? in *Colloquio Internazionale sulle Teorie Combinatorie* (Roma, 1973), Atti dei Convegni Lincei 17, Tomo I, pp. 247–56. Accad. Naz. Lincei, Rome.
Winder, R. O. (1966). Partitions of N-space by hyperplanes, *SIAM J. Appl. Math.* **14**, 811–18.
Woodall, D. R. & Wilson, R. J. (1978). The Appel–Haken proof of the four-color theorem, in L. W. Beineke & R. J. Wilson (eds), *Selected Topics in Graph Theory*, pp. 83–101. Academic Press, London.
Young, P. O. & Edmonds, J. (1972). Matroid designs, *J. Res. Nat. Bur. Stand. Section B* **72**, 15–44.
Young, P., Murty, U. S. R. & Edmonds, J. (1970). Equicardinal matroids and matroid designs, in *Combinatorial Mathematics and Its Applications*, pp. 498–542. University of North Carolina, Chapel Hill, North Carolina.
Zaslavsky, T. (1975a). Counting the faces of cut-up spaces, *Bull. Amer. Math. Soc.* **81**, 916–18.
Zaslavsky, T. (1975b). Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.*, No. 154.
Zaslavsky, T. (1976). Maximal dissections of a simplex, *J. Comb. Theory Ser. A* **20**, 244–57.
Zaslavsky, T. (1977). A combinatorial analysis of topological dissections, *Adv. Math.* **25**, 267–85.
Zaslavsky, T. (1979). Arrangements of hyperplanes; matroids and graphs, in *Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing*, Congressus Numerantium 24, pp. 895–911. Utilitas Mathematica, Winnipeg.
Zaslavsky, T. (1981a). The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88**, 88–105.
Zaslavsky, T. (1981b). The slimmest arrangements of hyperplanes: II. Basepointed geometric lattices and Euclidean arrangements, *Mathematika* **28**, 169–90.
Zaslavsky, T. (1982a). Signed graphs, *Discrete Appl. Math.* **4**, 47–74.
Zaslavsky, T. (1982b). Signed graph coloring, *Discrete Math.* **39**, 215–28.
Zaslavsky, T. (1982c). Chromatic invariants of signed graphs, *Discrete Math.* **42**, 287–312.
Zaslavsky, T. (1982d). Bicircular geometry and the lattice of forests of a graph, *Quart. J. Math. Oxford* (2) **33**, 493–511.
Zaslavsky, T. (1983). The slimmest arrangements of hyperplanes: I. Geometric lattices and projective arrangements, *Geom. Dedicata* **14**, 243–59.
Zaslavsky, T. (1985a). Geometric lattices of structured partitions I: Gain-graphic matroids and group-valued partitions, manuscript.
Zaslavsky, T. (1985b). Geometric lattices of structured partitions II: Lattices of group-valued partitions based on graphs and sets, manuscript.
Zaslavsky, T. (1987). The Möbius function and the characteristic polynomial, in N. White (ed.), *Combinatorial Geometries*, Encyclopedia of Mathematics and Its Applications 29, pp. 114–38. Cambridge University Press.
Zaslavsky, T. (1991a). Biased graphs IV: Geometrical realizations, *J. Comb. Theory Ser. B*, (to appear).
Zaslavsky, T. (1991b). Biased graphs III: Chromatic and dichromatic invariants, *J. Comb. Theory Ser. B* (to appear).