Unavoidable cycle-contraction minors of large 2-connected graphs

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Abstract

It is well known that every sufficiently large connected graph has, as an induced subgraph, K_n, $K_{1,n}$, or an n-vertex path. A 2023 paper of Allred, Ding, and Oporowski identified a set of unavoidable induced subgraphs of sufficiently large 2-connected graphs. In this paper, we establish a dual version of this theorem by focusing on the minors obtained by contracting cycles.

1 Introduction

For graph and matroid terminology not explicitly defined here, we follow Bondy and Murty [2] and Oxley [6]. In particular, we allow graphs to have loops and parallel edges; a graph is *simple* if it has neither. A *bond* in a graph is a minimal edge cut.

For an integer k exceeding one, a graph G is *k-connected* if $|V(G)| \geq k$ and, whenever u and v are distinct vertices of G, there are at least k pairwise internally disjoint uv-paths. In particular, for $n \geq 2$, the *bond graph* B_n that consists of two vertices joined by n parallel edges is 2-connected. Observe that our definition of a k-connected graph is broader than that of many authors who require that $|V(G)| \geq k+1$ for such a graph G. A graph H with at least two vertices is *k-edge-connected* if $H \setminus Z$ is connected for all subsets Z of $E(H)$ with $|Z| < k$. We note that a single-vertex graph is neither 2-connected nor 2-edge-connected.

1.1 Unavoidable families

The following theorem of Ramsey [8] has led to the development of a widely studied area of combinatorics.

Theorem 1.1. *Let r be a positive integer. There is an integer $R(r)$ such that every graph on at least $R(r)$ vertices has, as an induced subgraph, either K_r or its complement $\overline{K_r}$.*

Ramsey's Theorem identifies a family of graphs, namely, $\{K_r, \overline{K_r}\}$, such that every sufficiently large graph must have a member of this family as an induced

subgraph. There are many other results of this type. For example, a well-known theorem (see, for instance, [4, Proposition 9.4.1]) states that every sufficiently large connected graph has, as an induced subgraph, K_n, $K_{1,n}$, or an n-vertex path. Allred, Ding, and Oporowski [1] determined a family of 2-connected graphs such that every sufficiently large 2-connected graph has a member of this family as an induced subgraph. The aim of this paper is to establish a dual of Allred, Ding, and Oporowski's result.

To distinguish our setting from other problems in Ramsey theory, we refer to such results as a search for *unavoidable families*. Let $\mathcal{F}$ be a family of graphs. An *unavoidable-families characterization* consists of a subfamily $\mathcal{U}$ of $\mathcal{F}$ and a relation $\leq$ such that, for every sufficiently large graph G in $\mathcal{F}$, there is a large graph H in $\mathcal{U}$ such that $H \leq G$. To describe what "sufficiently large" means, one typically specifies the existence of a function such as $R(r)$ in Theorem 1.1. While much work has gone into improving the upper and lower bounds for such functions, this paper, like many others in the area, concerns itself only with the existence of such functions.

While the three theorems mentioned above characterize unavoidable induced subgraphs for different connectivity classes, such characterizations for topological minors have been given for 3- and internally-4-connected graphs [5]. For parallel minors, such characterizations are given for 1-, 2-, 3-, and internally-4-connected graphs [3]. In the next section, we introduce cycle-contraction minors of graphs and note how such minors are related through duality to induced subgraphs. The main result of this paper is an unavoidable-families characterization for cycle-contraction minors of 2-connected graphs.

1.2 cc-minors

A graph K is a *cycle-contraction minor* or *cc-minor* of a graph J if there is a sequence $J_0, J_1, \ldots, J_n$ of graphs such that $(J_0, J_n) = (J, K)$ and, for each $i \in \{1, 2, \ldots, n\}$, there is a cycle C_{i-1} of J_{i-1} such that $J_i = J_{i-1}/C_{i-1}$. In this paper, we determine a list of loopless 2-connected graphs such that every sufficiently large 2-connected graph has a member of the list as a cc-minor. Unless otherwise stated, every cycle contraction we perform is accompanied by the contraction of all of the loops it creates, where the contraction of a loop has the same effect as the deletion of the corresponding edge. Because each loop is itself a cycle, each contraction of a loop is an example of the contraction of a cycle.

Let G be a graph and let H be an induced subgraph of G. Clearly, H can be obtained from G by a sequence of operations, each consisting of either deleting a bond from the current graph or deleting an isolated vertex. When G is a plane graph with planar dual G^*, the planar dual H^* of H is obtained from G^* by a sequence of operations, each consisting of contracting a cycle in the current graph. Although, in the planar setting, the dual operation of deleting a bond is contracting a cycle, we need more care in specifying the dual of the operation of taking an induced subgraph. This is because a graph obtained from G by repeatedly deleting bonds and isolated vertices need not be an induced subgraph of G. For example,

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in C_4, a matching consisting of two edges e and f forms a bond, yet $C_4 \setminus \{e, f\}$ is not an induced subgraph of C_4. In Section 2, we shall prove the following result, which links induced subgraphs and cc-minors via duality.

Lemma 1.2. *Let G be a loopless 2-connected plane graph. A graph H is a 2-connected induced subgraph of G if and only if H^* is a 2-connected cc-minor of G^*.*

The statements of both our main result and of the theorem of Allred, Ding, and Oporowski [1] will rely on Tutte's tree decomposition result for 2-connected graphs, which we shall introduce next.

Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $E(G_1) \cap E(G_2) = \{e\}$ where e is neither a loop nor a cut edge of G_1 or G_2. The graph $G_1 \cup G_2$ is the *parallel connection* of G_1 and G_2 with *basepoint* e. The graph obtained from $G_1 \cup G_2$ by deleting e is the *2-sum*, $G_1 \bigoplus_2 G_2$, of G_1 and G_2 with *basepoint* e.

A *graph-labeled tree* is a tree T with vertex set $\{G_1, G_2, \ldots, G_k\}$ for some positive integer k, satisfying the following conditions for all distinct $i, j \in \{1, 2, \ldots, k\}$:

(i) G_i is a graph;

(ii) if G_i and G_j are joined by an edge e of T, then $E(G_i) \cap E(G_j) = \{e\}$ and e is neither a loop nor a cut edge of G_i or G_j; and

(iii) if G_i and G_j are non-adjacent, then $E(G_i) \cap E(G_j)$ is empty.

We call $G_1, G_2, \ldots, G_k$ the *vertex labels* of T; for each i in $\{1, 2, \ldots, k\}$, the edges in $E(G_i) \cap E(T)$ are called the *basepoints* of G_i.

Let e be an edge of a graph-labeled tree T and suppose e joins the vertices H_1 and H_2. If we contract e from T and relabel by $H_1 \bigoplus_2 H_2$ the vertex that results by identifying the endpoints of e leaving all other edge and vertex labels unchanged, then we get a new graph-labeled tree, T/e.

A *tree decomposition* of a loopless 2-connected graph G is a graph-labeled tree T such that if $V(T) = \{G_1, G_2, \ldots, G_k\}$ and $E(T) = \{e_1, e_2, \ldots, e_{k-1}\}$, then

(i) $E(G) = (E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)) - \{e_1, e_2, \ldots, e_{k-1}\}$;

(ii) $|E(G_i)| \geq 3$ for all i unless $|E(G)| < 3$, in which case, $k = 1$ and $G_1 = G$; and

(iii) G is the graph that labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$.

Tutte [10] proved that every 2-connected graph has a tree decomposition in which the vertex labels are restricted. However, such a tree decomposition need not be unique (see, for example, [6, p.308]).

Theorem 1.3. *Let G be a loopless 2-connected graph. Then G has a tree decomposition in which every vertex label is a simple 3-connected graph, a copy of K_3, or a copy of B_3. Moreover, each vertex label is isomorphic to a minor of G.*

Next we use tree decompositions to state Allred, Ding, and Oporowski's [1] identification of a set of unavoidable 2-connected induced subgraphs of large simple 2-connected graphs.

Theorem 1.4. *Let k be an integer exceeding two. Then there is an integer $f(k)$ such that every simple 2-connected graph with at least $f(k)$ vertices has, as an induced subgraph, one of*

(i) K_k*;*

(ii) *a subdivision of $K_{2,k}$;*

(iii) *a graph that is obtained from a subdivision of $K_{2,k}$ by adding an edge joining the two degree-k vertices; or*

(iv) *a k-vertex graph having a tree decomposition whose underlying tree is a path P, such that each vertex of P is labeled by*

 (a) *a copy of K_4 in which the basepoints form a matching, or*

 (b) *a copy of K_3 or B_3,*

 where neither end of P is labeled by B_3, and no two consecutive vertices of P are labeled by B_3.

The following theorem is the main result of this paper. A graph H is a *parallel extension* of a graph G if H can be obtained from G by, for each non-loop edge e in G, deleting e and joining the ends of e by a non-empty set of parallel edges. Similarly, a graph J is a *parallel-path extension* of a graph K if J can be obtained from K by, for each non-loop edge e in K, deleting e and adding a non-empty set of internally disjoint paths, each of which contains at least one edge and connects the ends of e.

Theorem 1.5. *Let r be a positive integer. There is an integer $g(r)$ such that every loopless 2-connected graph G with $|E(G)| \geq g(r)$ has, as a cc-minor, a parallel-path extension of a graph having a tree decomposition whose underlying tree is a path on at least r vertices, where each vertex is labeled either by a copy of K_3 or B_3, or by a copy of K_4 in which the basepoints form a matching.*

We note that, since cycle contraction does not preserve simplicity, we do not require a cc-minor to be simple. Consequently, for certain large graphs, such as large cliques, the unavoidable cc-minor in our list is a bond graph consisting of many parallel edges. As will be explained in Section 2, a bond graph admits a tree decomposition whose underlying tree is a path, with each vertex labeled by B_3.

1.3 A matroid perspective

While the current paper focuses mainly on cc-minors of graphs, this notion can be naturally generalized to matroids. A reader who is only interested in graphs

may ignore this subsection. A reader familiar with matroids will find that this subsection provides additional motivation for the material in this paper. While a comprehensive treatment of matroids may be found in [6], we refer to [7] for a concise overview of the definitions needed to understand the following section.

A matroid N is a *circuit-contraction minor* or *cc-minor* of a matroid M if there is a sequence $M_0, M_1, \ldots, M_n$ of matroids such that $(M_0, M_n) = (M, N)$ and, for each $i \in \{1, 2, \ldots, n\}$, there is a circuit C_{i-1} of M_{i-1} with $M_i = M_{i-1}/C_{i-1}$.

On the other hand, an *induced restriction* of a matroid M is a matroid N obtained from M by restricting to a flat. Not only do induced restrictions and circuit-contraction minors extend the notions of induced subgraphs and cycle-contraction minors to matroids, but the graph duality described in Lemma 1.2 is, in fact, a special case of the following.

Proposition 1.6. *A matroid N is an induced restriction of a matroid M if and only if N^* is a cc-minor of M^*.*

Proof. Suppose N is an induced restriction of M. Then there is a sequence $M_0, M_1, \ldots, M_n$ of matroids such that $(M_0, M_n) = (M, N)$ and, for each $i \in \{1, 2, \ldots, n\}$, the matroid M_i is obtained from M_{i-1} by restricting to a hyperplane H_{i-1} of M_{i-1}. Taking duals, we obtain a sequence $M_0^*, M_1^*, \ldots, M_n^*$ of matroids such that $(M_0^*, M_n^*) = (M^*, N^*)$ and, for each $i \in \{1, 2, \ldots, n\}$, the matroid M_i^* is obtained from M_{i-1}^* by contracting the circuit $E(M_{i-1}) - H_{i-1}$. Thus, N^* is a cc-minor of M^*.

The converse follows by reversing the argument just given. $\qquad\square$

One of our motivations is to extend Theorem 1.4 to regular matroids by addressing the following.

Problem 1. *Find a set of unavoidable induced restrictions for sufficiently large 2-connected simple regular matroids.*

Seymour [9] showed that every regular matroid can be constructed from graphic matroids, cographic matroids, and copies of the 10-element matroid R_{10}. The allowable operations in this construction are direct sums, 2-sums, and 3-sums. While Theorem 1.4 addresses the case in which a regular matroid has a large graphic matroid as an induced restriction, Theorem 1.5 addresses the case in which a large cographic matroid appears. Thus, Theorem 1.5 is not only the dual of Theorem 1.4, but also a crucial step toward solving Problem 1.

Since Theorem 1.5 answers Problem 1 for cographic matroids by solving the dual problem for graphic matroids, we observe that the requirement that $M^*(G)$ be simple is equivalent to the requirement that the matroid $M^*(G)$ has no circuit of size less than three. Equivalently, the graph G has no bond of size less than three, that is, G is 3-edge-connected. It is this equivalence that explains why Theorem 1.5 does not confine attention to simple graphs.

2 Preliminaries

This section presents some definitions along with a proof of Lemma 1.2. Specifically, we define a class of graphs called r-templates, that have tree decompositions exhibiting a particular structure. Additionally, we motivate the study of cc-minors, showing how such minors relate to induced subgraphs and minors.

2.1 r-templates

Figure 1 provides an example of a 2-connected graph G with a tree decomposition, as described in Theorem 1.3. Although, for an arbitrary 2-connected graph G, such a graph-labeled tree may exhibit an arbitrary structure, we introduce a class of 2-connected graphs that have tree decompositions with specific, notable features.
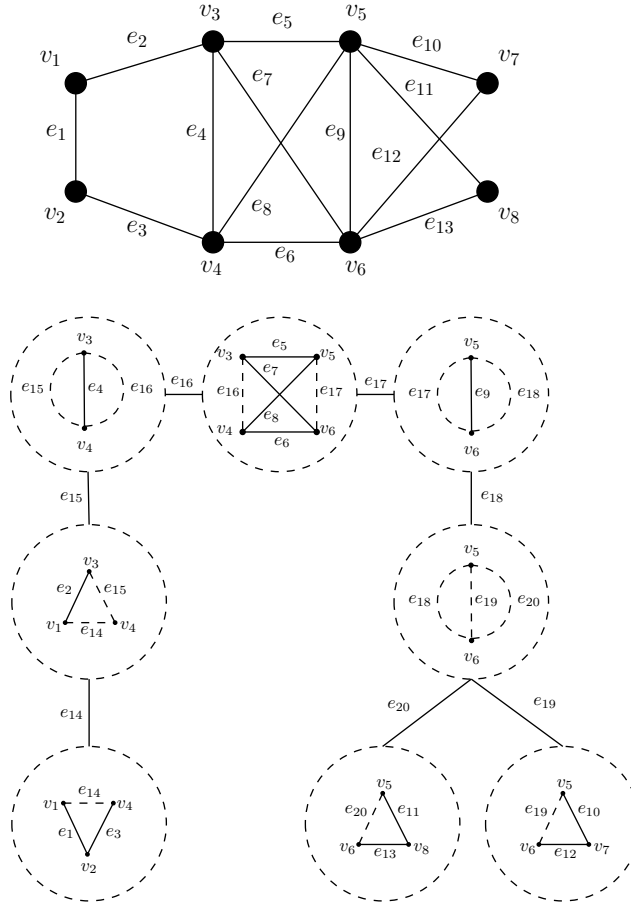
Figure 1: A 2-connected graph G and its tree decomposition

A *fan graph* F_n is a simple graph that is obtained from an n-vertex path $v_1 v_2 \ldots v_n$ by joining each v_i to a new vertex u. We call each uv_i edge a *spoke*. In particular, each edge joining u with v_1 or v_n is an *outer spoke*. For positive integers $t_1, t_2, \ldots, t_n$, we obtain a *fan-type graph* $F_{t_1,t_2,\ldots,t_n}$ by replacing each spoke uv_i of F_n by t_i parallel edges. In Figure 2, we show three different examples of fan-type graphs. Note that bond graphs are considered to be fan-type graphs, since they arise from single-vertex paths.
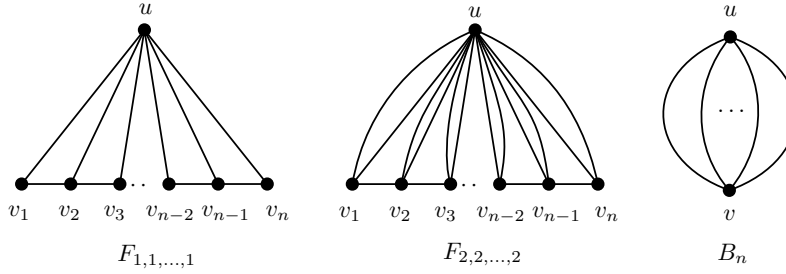
Figure 2: Three examples of fan-type graphs

An *r-template* is a 2-connected graph G that can be obtained from an r-vertex path P_r by using the following operations.

(i) Label each vertex of P_r by K_4, K_3, or B_3. Such vertex labels are called *parts*.

(ii) For each part, pick distinct basepoints, one for each of its adjacent parts. Moreover, if a part that is labeled by K_4 is adjacent to two other parts, then we always pick two non-adjacent edges as basepoints in that K_4.

(iii) Apply 2-sums across the specified basepoints.

Figure 3 shows the possible parts for templates, and Figure 4 shows the construction process of a sample 6-template. The following are some special examples of templates.
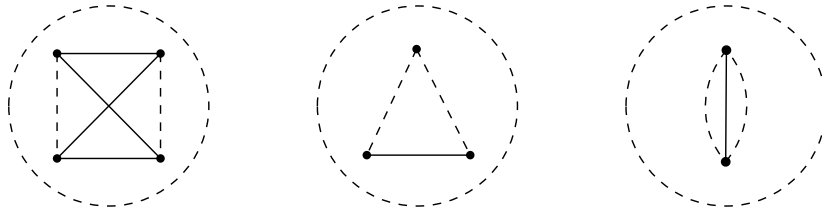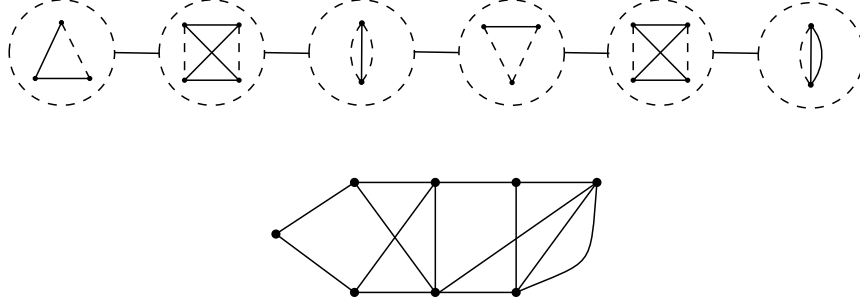

Figure 3: Possible parts of templates

Figure 4: A sample 6-template

(i) A fan graph F_n is a $(2n-3)$-template in which the parts alternate between K_3 and B_3, beginning and ending with K_3.

(ii) A fan-type graph $F_{t_1, t_2, \ldots, t_n}$ other than B_2 is a $((\sum_{i=1}^{n} t_i) + n - 3)$-template.

(iii) For $n \geq 3$, a bond graph B_n is an $(n-2)$-template for which every part is a B_3.

(iv) For $n \geq 3$, a cycle C_n is an $(n-2)$-template for which every part is a K_3.

The main result of this paper asserts that a set of unavoidable cc-minors of a sufficiently large 2-connected, loopless graph consists of parallel-path extensions of large templates. Before presenting the proofs, we provide motivation for studying cc-minors in the context of graph theory.

2.2   cc-minors of graphs

Graph relations, including induced subgraphs and minors, have garnered significant attention in various research areas. However, the study of cc-minors in graphs remains relatively unexplored. Although this relation is novel in certain respects, it has a strong connection to other graph relations. Here, we demonstrate how cc-minors relate to induced subgraphs through duality.

Proof of Lemma 1.2. First observe that, the graph H is 2-connected if and only if its dual H^* is 2-connected.

Now assume that H is a 2-connected induced subgraph of G. Then H can be obtained from G by consecutively deleting some vertices $v_1, v_2, \ldots, v_m$. Let $G_0 = G$ and, for each $i \in \{1, 2, \ldots, m\}$, let $G_i = G_{i-1} - v_i$. Since H is a connected subgraph of G, we may choose the deletion order so that, for each $i \in \{1, 2, \ldots, m\}$, the vertex v_i is not a cut vertex of G_{i-1} and therefore the set of edges of G_{i-1} meeting v_i is a bond B_i. As B_i is a cycle in the dual of G_{i-1}, we see that $H^* = G_m^* = G^*/B_1/B_2/\ldots/B_m$ is a cc-minor of G^*.

Conversely, assume that H^* is a 2-connected cc-minor of G^*. Let $Z = E(G^*) - E(H^*)$. Because H^* is obtained from G^* by repeatedly contracting cycles, we have

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$H^* = G^*/Z$. By the construction of H^*, every edge of Z is in a cycle of G^* that is contained in Z. Since H^* is 2-connected, for each cycle C in the plane graph G^* such that $C \subseteq Z$, either all of the edges in the interior of C are in Z or all of the edges in the exterior of C are in Z, but not both; otherwise H^* would have a cut vertex. In the first case, we color the faces in the interior of C red. In the second case, we color the faces in the exterior of C red. In particular, if $F \subseteq Z$ and F is a cycle bounding a face of G^*, then that face is colored red. Now, in G, consider the set R of vertices that correspond to the red faces of G^*. Because every edge in Z is contained in a red face, contracting Z in G^* has the same effect as contracting all red faces. Moreover, the dual of contracting a face is deleting the corresponding vertex in the dual. Therefore, deleting the vertices of R from G gives the dual of the graph H^*. $\square$

Rather than defining a graph minor through the local operations of deletion and contraction, it can also be characterized by its global structure.

Proposition 2.1. *A graph G has a graph H as a minor if and only if G has a set $\{G_v : v \in V(H)\}$ of disjoint connected subgraphs and a set $\{f_e : e \in E(H)\}$ of distinct edges that is disjoint from $\cup_{v \in V(H)} E(G_v)$ such that, for every edge $e \in E(H)$ having ends u and v, the ends of f_e are contained in G_u and G_v, respectively.*

We can characterize a cc-minor in a similar way. We note that, for the next proposition only, when a cycle contraction generates a loop, we do not require the resulting loop to be contracted.

Proposition 2.2. *A loopless graph G contains a graph H as a cc-minor if and only if G has*

(i) *a collection $\{G_v : v \in V(H)\}$ of disjoint subgraphs such that*

(a) *each G_v is either a 2-edge-connected subgraph of G or a single-vertex subgraph of G, and*

(b) *$\bigcup_{v \in V(H)} V(G_v) = V(G)$;*

and

(ii) *a set $\{f_e : e \in E(H)\}$ of distinct edges in G that is the complement of $\bigcup_{v \in V(H)} E(G_v)$ such that, for every edge $e \in E(H)$ with endpoints u and v, the endpoints of f_e lie in G_u and G_v, respectively.*

The proof of Proposition 2.2 is deferred to Section 4.

3 Internally disjoint XY-paths

Let G be a graph and X, Y be two disjoint set of vertices of G. An XY-*path* P is an xy-path such that there are vertices x and y for which $V(P) \cap X = \{x\}$ and

$V(P) \cap Y = \{y\}$. When H and K are disjoint subgraphs of G, a $V(H)V(K)$-path will sometimes be called an HK-path. Two XY-paths P_1 and P_2 are *internally disjoint* if $(V(P_1) \cap V(P_2)) - X - Y = \emptyset$.

Lemma 3.1. *Let G be a graph, and let X and Y be two disjoint sets of vertices in G. For a cycle C in G such that $V(C) \cap X \neq \emptyset$ and $V(C) \cap Y = \emptyset$, define $X' = V(G[X \cup V(C)]/C)$. If there are k internally disjoint XY-paths in G, then there are at least k internally disjoint $X'Y$-paths in G/C.*

Proof. Let $P_1, P_2, \ldots, P_k$ be k internally disjoint XY-paths in G. Since contracting C can be achieved by repeatedly contracting single edges of C, it suffices to establish the following assertion.

3.1.1. *If $G' = G/e$ where e joins u and v, with $\{u,v\} \cap X \neq \emptyset$ and $\{u,v\} \cap Y = \emptyset$, then G' contains k internally disjoint $X'Y$-paths, where $X' = V(G[X \cup \{u,v\}]/e)$.*

If $\{u,v\} \subseteq X$ or if $u \in X$ and $v \notin V(P_i)$ for all $i \in \{1,2,\ldots,k\}$, then $P_1, P_2, \ldots, P_k$ are internally disjoint $X'Y$-paths in G'. Now, assume $u \in X$ and $v \in V\left(\bigcup_{i \in \{1,2,\ldots,k\}} P_i\right) - X$. Since $P_1, P_2, \ldots, P_k$ are internally disjoint and $v \notin Y$, there is exactly one path, say P_1, containing v. Let y denote the unique vertex in $Y \cap V(P_1)$, and let P_1' be the yv-subpath of P_1. Then $P_1', P_2, \ldots, P_k$ form k internally disjoint $X'Y$-paths in G'. This completes the proof of 3.1.1, and the lemma follows. □

Lemma 3.2. *For a positive integer k, let G be a k-edge-connected graph, and let H be a cc-minor of G with at least two vertices. Then H is k-edge-connected.*

Proof. It suffices to show that if e is an edge of a graph G with at least three vertices, then G/e is k-edge-connected whenever G is k-edge-connected. This is an immediate consequence of the fact that every bond of G/e is also a bond of G. □

4 cc-minors of 2-edge-connected graphs

In this section, we determine a set of unavoidable cc-minors in 2-edge-connected graphs.

Lemma 4.1. *Let G be a graph. Then the following hold.*

  (i) *If G is 2-edge-connected and C is a cycle of G, then G/C is either a single-vertex graph or is 2-edge-connected.*

  (ii) *If G is not 2-edge-connected and C is a cycle of G, then G/C is not 2-edge-connected.*

  (iii) *If G is 2-edge-connected, then $G/E(G)$, which is isomorphic to K_1, is a cc-minor of G.*

  (iv) *If $|V(G)| \geq 2$ and K_1 is a cc-minor of G, then G is 2-edge-connected.*

Proof. If G is 2-edge-connected, then it has no cut edge. By Lemma 3.2, each cc-minor of G also has no cut edge, thus proving (i).

If G is not 2-edge-connected, then either G is a single-vertex graph and all the cycles of G are loops, or G has a cut edge e. In either case, if C is a cycle of G, then G/C is not 2-edge-connected, which proves (ii).

Suppose that G is 2-edge-connected and define $G_0 = G$. If G_i contains an edge e, then G_i has a cycle C_i that contains e. Let $G_{i+1} = G_i/C_i$. This process generates a sequence $G_0, G_1, \ldots, G_q$ such that G_q is connected and $E(G_q) = \emptyset$. Hence $G_q \cong K_1$, proving (iii).

If G has a cut edge, then, by Lemma 3.2, every cc-minor of G will also have a cut edge, and thus it can never be isomorphic to K_1. Thus (iv) holds. □

Using Lemma 4.1, we define the operation of *contracting a 2-edge-connected subgraph F of G* as performing a sequence of cycle contractions equivalent to contracting all of the edges in F.

Corollary 4.2. *If F is a 2-edge-connected subgraph of G, then $G/E(F)$ is a cc-minor of G.*

Next we prove a characterization of cc-minors.

Proof of Proposition 2.2. If (i) and (ii) hold, then, by Lemma 4.1(iii), G has H as a cc-minor. Note that $H = G/\big(\cup_{v \in V(H)} E(G_v)\big)$. To prove the converse, suppose that $H = G/\{e_1, e_2, \ldots, e_k\}$. Let J be the subgraph of G induced by the set $\{e_1, e_2, \ldots, e_k\}$ of edges. It suffices to show that each component of J is 2-edge-connected. Since none of $e_1, e_2, \ldots, e_k$ is a loop, each component of J has at least two vertices. Moreover, each component has K_1 as a cc-minor and hence, by Lemma 4.1(iv), is 2-edge-connected. □

The next lemma identifies a set of unavoidable cc-minors in 2-connected graphs when preserving a specified edge. Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $E(G_1) \cap E(G_2) = \{e\}$, where e is neither a loop nor a cut edge in G_1 or G_2. Recall that the graph $G_1 \cup G_2$ is the parallel connection of G_1 and G_2 with basepoint e. More generally, let $G_1, G_2, \ldots, G_n$ be a collection of graphs such that, for all distinct i and j in $\{1, 2, \ldots, n\}$, we have $V(G_i) \cap V(G_j) = \{u, v\}$ and $E(G_i) \cap E(G_j) = \{e\}$, where e is neither a loop nor a cut edge in any G_k. The union $G_1 \cup G_2 \cup \cdots \cup G_n$ is called the *parallel connection* of $G_1, G_2, \ldots, G_n$ with basepoint e. If v is a vertex of degree two in a graph G and v does not meet a loop of G, then, by *suppressing v*, we mean deleting v and adding an edge between its two neighbors.

Lemma 4.3. *Let G be a 2-edge-connected graph, and let e be a non-loop edge of G. Then G has a cc-minor H that is the parallel connection, with basepoint e, of a collection of cycles containing e.*

Proof. Let $G_0 = G$ and let $e = uv$. If G_i has a cycle C_i that does not contain both u and v, define $G_{i+1} = G_i/C_i$. Repeating this process generates a sequence

$G_0, G_1, \ldots, G_k$ of graphs such that u and v remain distinct vertices in G_k, and every cycle in G_k contains both u and v. We will show that G_k is a parallel connection, with basepoint e, of a collection of cycles containing e.

Let w be a vertex of G_k that is not in $\{u, v\}$. We first prove that $d(w) = 2$. By Lemma 3.2, G_k is 2-edge-connected, so $d(w) \geq 2$. Now, w does not meet a loop of G_k. Let g be an edge meeting w. Since g is not a cut edge of G_k, there is a cycle C in G_k containing g, and C must contain both u and v. Suppose $d(w) \geq 3$. Then there is an edge f that meets w but is not in C. Let x be the other endpoint of f. By the choice of G_k, we see that $x \notin V(C)$. As G_k is 2-edge-connected, $G_k - f$ is connected. Choose P as a shortest path in $G_k - f$ from x to a vertex in C. Let P^+ be the path in G_k that consists of P and the edge f. Note that $E(P^+) \cap E(C) = \emptyset$ and $V(P^+) \cap V(C) = \{w, y\}$ for some $y \in V(C)$. Since P^+ cannot contain both u and v, there is a cycle in G_k that does not contain both u and v, a contradiction. Hence $d(w) = 2$.

Let A be the set of degree-2 vertices in G_k. If we suppress $A - \{u, v\}$, then for some $n \geq 2$, the resulting graph will be a bond graph B_n with vertex set $\{u, v\}$. Hence, G_k is the parallel connection, with basepoint e, of a collection of $n-1$ cycles containing e. $\qquad\square$

Combining Lemma 3.2 and Lemma 4.3, we obtain the following result.

Corollary 4.4. *Let G be a 3-edge-connected graph, and let e be a non-loop edge of G. Then G has a cc-minor H that is a bond graph B_n containing e, for some $n \geq 3$.*

5 Classes closed under cc-minors

Let $\mathcal{F}_1$ be the class of loopless connected graphs. For each positive integer $k > 1$, let $\mathcal{F}_k$ be the class consisting of all loopless k-edge-connected graphs along with the single-vertex graph K_1. The next proposition follows immediately from Lemma 3.2 and Lemma 4.1(i).

Proposition 5.1. *For every positive integer k, the class $\mathcal{F}_k$ is closed under cycle contraction.*

Clearly, $\mathcal{F}_{k+1} \subseteq \mathcal{F}_k$ for each positive integer k. Now we provide a forbidden cc-minor characterization of $\mathcal{F}_k$ for each k in $\{1, 2, 3\}$.

Theorem 5.2. *The following statements hold for a loopless graph G.*

(i) *G is in $\mathcal{F}_1$ if and only if G does not have a forest with at least two components as a cc-minor.*

(ii) *G is in $\mathcal{F}_2$ if and only if G is in $\mathcal{F}_1$ and G does not have a tree with at least one edge as a cc-minor.*

(iii) *G is in $\mathcal{F}_3$ if and only if G is in $\mathcal{F}_2$ and G does not have a cycle of size at least two as a cc-minor.*

Proof. First we observe the following.

(a) A forest with at least two components is not in $\mathcal{F}_1$.

(b) A tree with at least one edge is not in $\mathcal{F}_2$.

(c) A cycle of size at least two is not in $\mathcal{F}_3$.

Since each $\mathcal{F}_k$ is closed under cycle contractions, to prove the theorem, it remains to show that the graphs in (a)-(c) are the only obstructions to membership of $\mathcal{F}_k$ for k in $\{1, 2, 3\}$.

First, suppose G is not connected. Let F be a cc-minor of G obtained by repeatedly contracting cycles until no cycles remain. Evidently, F is a forest that has the same number of components as G. Thus, F is a forest with at least two components, which confirms (i).

Now, suppose $G \in \mathcal{F}_1 - \mathcal{F}_2$. Clearly, G has a cut edge e. Let H be a cc-minor of G obtained by repeatedly contracting cycles until no cycles remain. Evidently, H is a tree that contains e, which confirms (ii).

Finally, suppose that $G \in \mathcal{F}_2 - \mathcal{F}_3$. Since G is not 3-edge-connected, G has a bond $\{e, f\}$. Let J be a cc-minor of G obtained by repeatedly contracting cycles that contain neither e nor f until no such cycles remain. By Lemma 3.2, J is 2-edge-connected. Since $\{e, f\}$ is a bond of J, every cycle of J contains either none or all of $\{e, f\}$. Therefore, by the construction of J, every cycle in J contains both e and f. Suppose C_1 and C_2 are two distinct cycles of J. Then $C_1 \Delta C_2$, which equals $(C_1 \cup C_2) - (C_1 \cap C_2)$, is a non-empty disjoint union of cycles of J. However, $\{e, f\} \nsubseteq C_1 \Delta C_2$, a contradiction. Therefore, J is a cycle, which confirms (iii). $\square$

The next theorem characterizes $\mathcal{F}_k$ for all $k > 3$.

Theorem 5.3. *Let k be an integer exceeding three. A loopless graph G is in $\mathcal{F}_k$ if and only if G is in $\mathcal{F}_{k-1}$ and G does not have a cc-minor isomorphic to B_{k-1}.*

Proof. Clearly $\mathcal{F}_{k-1} \subseteq \mathcal{F}_k$ and $B_{k-1} \notin \mathcal{F}_k$. To prove the converse, suppose that G belongs to $\mathcal{F}_{k-1} - \mathcal{F}_k$. Let $\{x_1, x_2, \ldots, x_{k-1}\}$ be a bond of G, and let H be a cc-minor of G obtained by repeatedly contracting cycles that do not contain any edge in $\{x_1, x_2, \ldots, x_{k-1}\}$ until no such cycles remain. Evidently, $H \setminus \{x_1, x_2, \ldots, x_{k-1}\}$ consists of two components, T_1 and T_2, each of which is a tree. Since H is $(k-1)$-edge-connected, we have $d_H(v) \geq k - 1$ for all $v \in V(H)$. Let l be a leaf of T_i for some $i \in \{1, 2\}$. In H, the leaf l is incident with at least $k - 2$ edges from $\{x_1, x_2, \ldots, x_{k-1}\}$. However, for $k > 3$, we have $2(k-2) > k-1$. Therefore, T_i has at most one leaf for each $i \in \{1, 2\}$. Thus, both T_1 and T_2 must be single-vertex graphs, and we conclude that $H \cong B_{k-1}$, which confirms the theorem. $\square$

6   cc-minors of 3-connected graphs

In this section, we determine a set of unavoidable cc-minors of 3-connected graphs. By a *simplification* of a graph G, we mean a simple graph that is obtained from G by deleting all the loops and deleting all but one edge from each maximal set of parallel edges.

Theorem 6.1. *Let G be a simple 3-connected graph, and let e and f be two distinct edges in G. Then G has a cc-minor H containing e and f such that one of the following holds.*

(i) *For some $n \geq 3$, the graph H is isomorphic to a bond graph B_n containing e and f; or*

(ii) *H is isomorphic to a fan-type graph of which e and f are distinct outer spokes that are not parallel; or*

(iii) *a simplification of H has e and f as non-adjacent edges and is isomorphic to K_4. Moreover, if an edge g of H is not parallel to e or f, then g is not parallel to any edge in $E(H)$.*
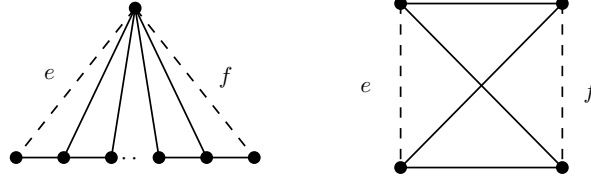

Figure 5: Possible simplifications of H that fall under cases (ii) and (iii).

Proof. Let e have ends x_1 and x_2, and let f have ends y_1 and y_2. First, we prove the following.

6.1.1. *If e and f are adjacent, then G has a cc-minor H such that H is isomorphic to B_n, for some $n \geq 3$, and H has e and f as distinct edges.*

We may assume that $x_1 = y_1$ and $x_2 \neq y_2$. Since G is 3-connected, $G - x_1$ is 2-connected and hence 2-edge-connected. By Lemma 4.1 (iii), contracting $E(G - x_1)$ forms a cc-minor that is isomorphic to a bond graph that contains e and f. Moreover, by Lemma 3.2, H has at least three edges. Hence, 6.1.1 holds.

Now we assume e and f are not adjacent. For two distinct vertices u and v, a Θ-*graph on* (u, v) consists of three internally disjoint uv-paths. For each $(x, y) \in \{x_1, x_2\} \times \{y_1, y_2\}$, a Θ-graph on (x, y) that contains both e and f is classified as *type-A* if one of its xy-paths contains both e and f. It is *type-B* if e and f belong to two different xy-paths of the Θ-graph. Examples of these types
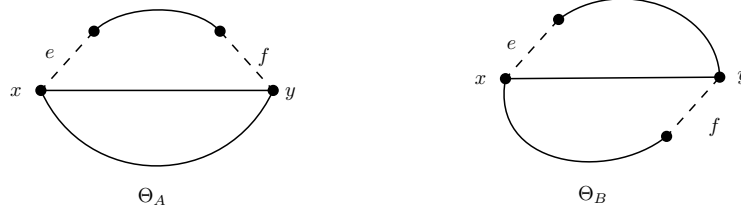
Figure 6: A type-A Θ-graph Θ_A and a type-B Θ-graph Θ_B

of graphs are shown in Figure 6, where e and f represent single edges, while the other lines in the diagram correspond to paths.

Next we prove the following.

6.1.2. *For each pair $(x, y) \in \{x_1, x_2\} \times \{y_1, y_2\}$, there is a Θ-graph on (x, y) that contains e and f.*

Without loss of generality, assume $x = x_1$ and $y = y_1$. By Menger's Theorem, there are three internally disjoint $x_1 y_1$-paths that form a Θ-graph on (x, y). Choose Θ_0 to be a Θ-graph on (x, y) that contains the maximal number of members of $\{e, f\}$. We may assume that $e \notin E(\Theta_0)$. Since $G - x_1$ is connected, there is an $x_2 y_1$-path in $G - x_1$ whose vertices and edges, in order, are $v_1 e_1 v_2 \ldots e_{k-1} v_k$, where $v_1 = x_2$ and $v_k = y_1$. In G, we adjoin the edge e to the beginning of this path to form a path P. Let i be the smallest index such that $v_i \in V(\Theta_0)$.

Suppose $v_i \neq y_1$. Then v_i lies on an $x_1 y_1$-path Q in Θ_0. Note that the $x_1 v_i$-subpath Q' of Q does not use the edge f. Replacing Q' by the $x_1 v_i$-subpath of P, we obtain a Θ-graph that violates the choice of Θ_0. Thus $v_i = y_1$. Choose an $x_1 y_1$-path R in Θ_0 such that $f \notin E(R)$, and let Θ_1 be the Θ-graph obtained by replacing R with P. Then Θ_1 violates the choice of Θ_0. The contradiction completes the proof of 6.1.2.

Next we show that if G contains a type-A Θ-graph, then the theorem holds.

6.1.3. *If G has a cc-minor G' that contains a type-A Θ-graph Θ_A as a subgraph, then G has a cc-minor H containing e and f such that H is either isomorphic to a fan-type graph, with e and f as distinct non-parallel outer spokes, or H is isomorphic to B_n for some $n \geq 3$.*

By Lemma 3.2, every cc-minor of G is 3-edge-connected. Thus, we may assume that Θ_A is 3-edge-connected. Without loss of generality, we may also assume that Θ_A is a Θ-graph on (x_1, y_1). Observe that Θ_A has a cycle C that does not contain e or f. Contracting C in G' has the effect of identifying all the vertices in $V(C)$ as a single vertex c and then deleting all the edges in $E(C)$. Since e and f are contained in a cycle of G/C, there is a maximal 2-connected subgraph L of G'/C containing both e and f.

Since G'/C has no cut edges, every edge of G'/C is in a 2-connected subgraph of G'/C. Now, by Lemma 4.1 (iii), we may contract the edges of all of the maximal 2-connected subgraphs of G'/C except L. The resulting graph is isomorphic to L, so we continue referring to it as L. Let $L_0 = L$. For $i \geq 0$, if L_i^- contains a cycle C_i, let $L_{i+1} = L_i/C_i$. This process produces a sequence $L_0, L_1, \ldots, L_s$ of graphs such that L_s^- is a tree T. Let P be the $x_2 y_2$-path in T. Note that T may be a single vertex if x_2 and y_2 have been identified.

For $j \geq s$, if $L_j^- \neq P$, then there is a leaf $l_j \in V(L_j^-) - V(P)$. By Lemma 3.2, $d(l_j) \geq 3$, so there is a cycle O_j consisting of two edges with ends c and l_j. Let $L_{j+1} = L_j/O_j$. This results in a sequence $L_s, L_{s+1}, \ldots, L_{s+t}$ such that $L_{s+t} - c = P$. In L_{s+t}, since each vertex of P has degree at least three, there is at least one cp-edge for each $p \in V(P)$. Thus, L_{s+t} is a fan-type graph with e and f as two outer spokes. Moreover, e and f are not parallel unless L_{s+t} is a bond graph with at least three edges. Hence 6.1.3 holds.

In view of 6.1.3, we may now assume that

6.1.4. *G has no cc-minor that contains a type-A Θ-graph.*

By 6.1.2, G has a type-B Θ-graph Θ_B as a subgraph. Without loss of generality, suppose Θ_B is on (x_1, y_1). Throughout the following argument, for any type-B Θ-graph on (x_1, y_1), we denote the $x_2 y_1$-path as P_1, the $x_1 y_1$-path as P_2, and the $x_1 y_2$-path as P_3, as shown in Figure 7.

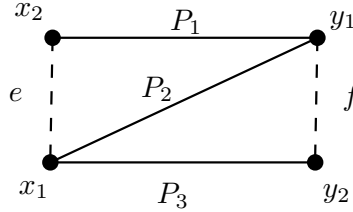
Figure 7: Θ_B

A graph is a *four-path connector* if it consists of the edges e and f, along with four internally disjoint paths, P_1, P_2, P_3, and P_4, that connect the vertex pairs $\{x_2, y_1\}$, $\{x_1, y_1\}$, $\{x_1, y_2\}$, and $\{x_2, y_2\}$, respectively. Moreover, each path P_i contains at least one edge for every $i \in \{1, 2, 3, 4\}$. Throughout the remainder of this proof, in any four-path connector, the labels of these four paths will be consistent with Figure 8. Next, we show the following.

6.1.5. *The graph G has a cc-minor G_s that contains a four-path connector K as a subgraph.*

Let $(G_0, \Theta_0) = (G, \Theta_B)$. For $i \geq 0$, define Q_i to be a path in G_i such that

(i) Q_i connects x_2 to a vertex v_i on P_1 such that all vertices of Q_i, except for its two ends, are not in Θ_i, or
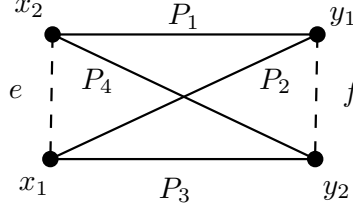
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Figure 8: A four-path connector

(ii) Q_i connects y_2 to a vertex v_i on P_3 such that all vertices of Q_i, except for its two ends, are not in Θ_i.

If some Q_i satisfying (i) exists, then let C_i be the cycle formed by Q_i and the x_2v_i-subpath of P_1. Similarly, if some Q_i satisfying (ii) exists, then let C_i be the cycle formed by Q_i and the y_2v_i-subpath of P_3. Define $(G_{i+1}, \Theta_{i+1}) = (G_i/C_i, \Theta_i/(E(\Theta_i) \cap E(C_i)))$. This process produces a sequence $(G_0, \Theta_0), (G_1, \Theta_1), \ldots, (G_s, \Theta_s)$ such that, in G_s, no path satisfies the condition that defines Q_i. Note that, in this process, Q_i is never an x_2y_1-path; otherwise, we would obtain a type-A Θ-graph on (x_2, y_1) having as its paths, Q_i, P_1, and the path with edge set $\{e, f\} \cup E(P_3)$, which contradicts 6.1.4. Thus, in G_s, the path P_1 retains at least one edge. Similarly, in G_s, the path P_3 also retains at least one edge.
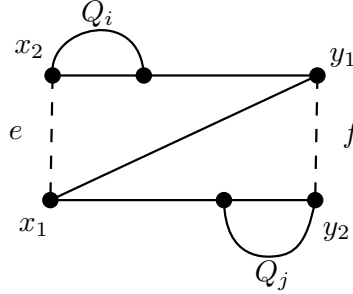

Figure 9: Paths similar to Q_i or Q_j will not appear in G_s.

Let $\mathcal{P}_x$ be the collection of paths in G_s that start at x_2, end at a vertex in $V(\Theta_s)$, and are *internally disjoint from* Θ_s, meaning they are vertex-disjoint from Θ_s except at their endpoints. Similarly, let $\mathcal{P}_y$ be the collection of paths that start at y_2, end at a vertex in $V(\Theta_s)$, and are internally disjoint from Θ_s.

If there is a path $P \in \mathcal{P}_x$ that ends at a vertex, say w, in $V(P_2) - \{x_1\}$, then there is a type-A Θ-graph on (x_2, y_1) (see Figure 10), which contradicts 6.1.4. Similarly, we may assume that $\mathcal{P}_y$ does not contain any path that ends at a vertex in $V(P_2) - \{y_1\}$.

Note that G_s is obtained from G by repeatedly contracting cycles containing exactly one of $\{x_2, y_2\}$. Because G is 3-connected, by Lemma 3.1, there are at
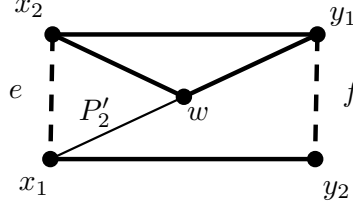
Figure 10: Deleting the edges of P_2' results in a type-A Θ-graph.

least three internally disjoint x_2y_2-paths in G_s. However, if all paths in $\mathcal{P}_x$ end at x_1, then every x_2y_2-path in G_s contains either x_1 or the neighbor of x_2 on P_1, contradicting the existence of three internally disjoint x_2y_2-paths. Therefore, by the choice of G_s, we may assume $\mathcal{P}_x$ contains a path P_x that ends at a vertex in $V(P_3) - \{x_1\}$. By symmetry, we may also assume $\mathcal{P}_y$ contains a path P_y that ends at a vertex in $V(P_1) - \{y_1\}$. If P_x and P_y are internally disjoint, then there is a type-A Θ-graph on (x_2, y_2) (see Figure 11), which contradicts 6.1.4.

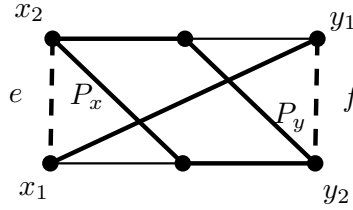

Figure 11: The thickened paths form a type-A Θ-graph.

We may now assume that P_x and P_y are not internally disjoint. Therefore, there is an x_2y_2-path P_4 in G_s that is internally disjoint from Θ_s. Let K be the subgraph of G_s consisting of Θ_s and P_4. Hence 6.1.5 holds.

A four-path connector F in a graph J is *spanning* if $V(F) = V(J)$.

6.1.6. *The graph G_s has a cc-minor G_s' that contains a spanning four-path connector K'.*

It is easy to check that, in the process of obtaining G_s from G, every cycle that has been contracted intersects K. In order to prove 6.1.6, it suffices to prove the following.

6.1.7. *Let D be a cc-minor of G that contains a non-spanning four-path connector F. Moreover, if $D = G/C_1/C_2/\ldots/C_n$ where C_i is a cycle of $G/C_1/C_2/\ldots/C_{i-1}$ for all $i \in \{1, 2, \ldots, n\}$, assume that $V(C_i) \cap V(F) \neq \emptyset$ for all such i. Then D has a cycle C_{n+1} such that if $D' = D/C_{n+1}$, then D' has a four-path connector F' for which $|V(D) - V(F)| > |V(D') - V(F')|$. In particular, $V(C_{n+1}) \cap V(F) \neq \emptyset$.*

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Assume that this fails. By Menger's Theorem and Lemma 3.1, we know that, for each $v \in V(D) - V(F)$, there are at least three internally disjoint vF-paths. If two such paths have the same end in $V(F)$, then they form a cycle C such that $V(C)$ meets $V(F)$. It is straightforward to see that 6.1.7 holds when $D' = D/C$. Thus, we may now assume that there are three internally disjoint vF-paths with distinct endpoints a, b, and c in $V(F)$. Next, we show the following.

6.1.8. *None of the paths P_1, P_2, P_3, P_4 contains more than one of a, b, and c.*

Suppose that $\{a, b\} \subseteq V(P_i)$ for some $i \in \{1, 2, 3, 4\}$. Without loss of generality, suppose that $\{a, b\} \subseteq V(P_1)$. Observe that if $\{a, b\} = \{x_2, y_1\}$, then D contains a type-A Θ-graph, which contradicts 6.1.4. Hence, we may assume that at least one member of $\{a, b\}$ is an internal vertex of P_1. Let C_v be the cycle formed by the va-path, the ab-subpath of P_1, and the bv-path (see Figure 12 (a)). Then the graphs $D' = D/C_v$ and $F' = F/(E(F) \cap E(C_v))$ satisfy 6.1.7, a contradiction. Thus, 6.1.8 holds.

Next we show that at least two of a, b, and c are not in $\{x_1, x_2, y_1, y_2\}$. Suppose, without loss of generality, that $\{a, b\} \subseteq \{x_1, x_2, y_1, y_2\}$. By 6.1.8, we know $\{a, b\} \in \{\{x_1, x_2\}, \{y_1, y_2\}\}$. Suppose that $\{a, b\} = \{x_1, x_2\}$. Then there is a path in $\{P_1, P_2, P_3, P_4\}$ that contains c and one of a and b, contradicting 6.1.8. By symmetry, $\{a, b\} \neq \{y_1, y_2\}$ and hence at least two of a, b, and c are not in $\{x_1, x_2, y_1, y_2\}$.

Without loss of generality, we assume that $a \in V(P_1) - \{x_2, y_1\}$. Moreover, by 6.1.8, we know that at least one of b and c belongs to $V(F) - (V(P_1) \cup V(P_3))$. By symmetry, we assume $b \in V(P_2) - \{x_1, y_1\}$. Let C_v be the cycle formed by the va-path, the ay_1-subpath of P_1, the y_1b-subpath of P_2, and the bv-path (see Figure 12 (b)). Then the graphs $D' = D/C_v$ and $F' = F/(E(F) \cap E(C_v))$ satisfy 6.1.7, a contradiction. Thus 6.1.7 holds, and 6.1.6 follows immediately.
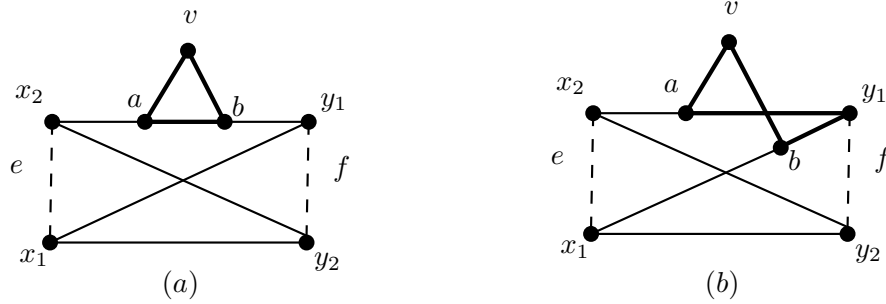

Figure 12: Contracting the thickened cycles absorbs v into F

Next, we prove the following.

6.1.9. *Suppose W is a cc-minor of G that contains a spanning four-path connector R and there is a vertex $u \in V(W) - \{x_1, x_2, y_1, y_2\}$. Then W has a cc-minor W'*

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containing a spanning four-path connector R' such that $|V(W) - \{x_1, x_2, y_1, y_2\}| >$
$|V(W') - \{x_1, x_2, y_1, y_2\}|$.

Without loss of generality, suppose $u \in V(P_1) - \{x_2, y_1\}$. By Lemma 3.2, the graph W is 3-edge-connected, and $d_W(u) \geq 3$. Therefore, u is incident to an edge $uw \notin E(R)$. Suppose u and w lie on two non-adjacent paths of R (say, P_1 and P_3 under our assumption). Figure 13 shows that when $w \notin \{x_1, y_2\}$, when $w = x_1$, and when $w = y_2$, the graph W contains a type-A Θ-graph, a contradiction to 6.1.4.

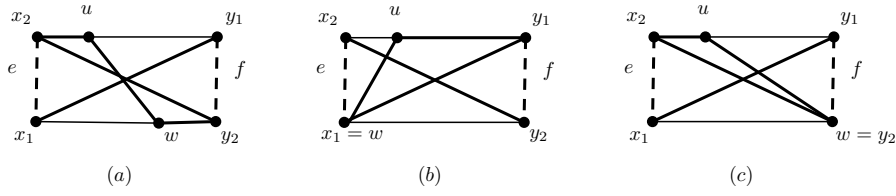

Figure 13: The thickened paths, together with the edges e and f, form type-A Θ-graphs in each case.

We may now assume that $w \in V(P_1 \cup P_2 \cup P_4) - \{x_1, y_2\}$.

(a) Suppose that $w \in V(P_1)$, as shown in Figure 14 (a). Let C_u be the cycle formed by the edge uw and the wu-subpath of P_1.

(b) Suppose that $w \in V(P_2) - \{x_1, y_1\}$ (or symmetrically $w \in V(P_4) - \{x_2, y_2\}$), as shown in Figure 14 (b). Let C_u be the cycle formed by the edge uw, the wz-subpath of P_2, and the zu-subpath of P_1.

In each case, let $W' = W/C_u$ and $R' = R/(E(C_u) \cap E(R))$. Then R' is a spanning four-path connector of W', so 6.1.9 holds.
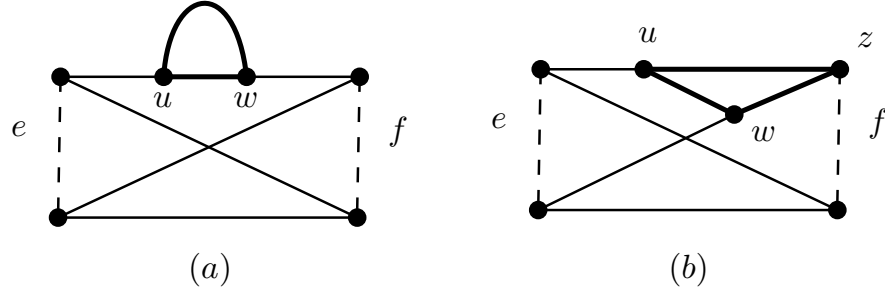

Figure 14: Two cases of a uw-edge

Applying 6.1.9 inductively on G'_s and K', we conclude that G'_s has a cc-minor H containing a four-path connector. Moreover, $V(H) = \{x_1, x_2, y_1, y_2\}$. It is not difficult to see that the edges e and f are non-adjacent in H and that the simplification of H is isomorphic to K_4. Suppose that there is a pair of parallel edges g and h in $E(H)$ that are not parallel to e or f. Because the simplification of

H is isomorphic to K_4, there is a 4-cycle U that contains e, f and g. However, the graph with edge set $U \cup h$ is a type-A Θ-graph, a contradiction to 6.1.4. Hence, Theorem 6.1 holds. $\qquad\square$

7 cc-minors of large 3-connected graphs

In this section, we determine a set of unavoidable cc-minors of sufficiently large 3-connected graphs.

Theorem 7.1. *For every integer $t \geq 3$, there is a function $f_{7.1}(t)$ such that if a simple 3-connected graph G has more than $f_{7.1}(t)$ edges, then G has a fan-type graph $F_{t_1,t_2,\ldots,t_n}$ as a cc-minor such that $\sum_{i=1}^{n} t_i \geq t$.*

Before beginning the proof of Theorem 7.1, we present three lemmas. The first is a Ramsey-type result for 3-connected graphs; the second is a result for weighted trees, where we use the latter as auxiliary graphs in our analysis.

Let k be an integer exceeding two. Figure 15 shows three families of graphs that we now describe. The *k-rung ladder* L_k has vertices $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_k$ form paths in the listed order, and v_i is adjacent to u_i for each $i \in \{1, 2, \ldots, k\}$. The graph V_k is obtained from L_k by adding an edge between v_1 and v_k and contracting the edges joining u_1 to v_1 and u_k to v_k. The *k-spoke wheel* is denoted by W_k. Oporowski, Oxley, and Thomas [5] characterized a set of unavoidable structures of large 3-connected graphs as follows.
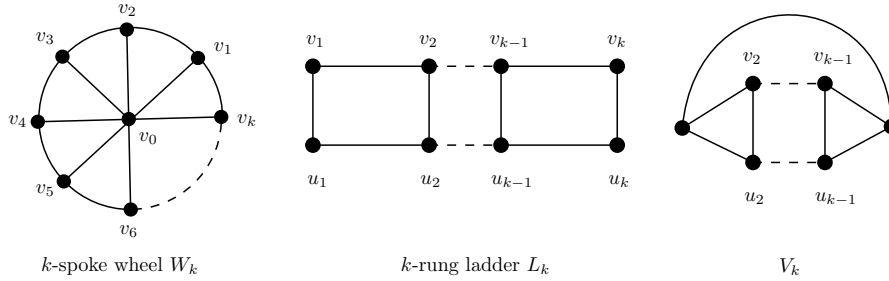

Figure 15: Some important graphs for Lemma 7.2

Lemma 7.2. *For every integer $k \geq 3$, there is a function $f_{7.2}(k)$ such that every 3-connected graph with at least $f_{7.2}(k)$ vertices contains a subgraph isomorphic to a subdivision of one of W_k, V_k, and $K_{3,k}$.*

A *weighted tree* is a tree T together with a weight function w such that each vertex v is assigned a non-negative integer-valued weight $w(v)$, and $w(A) = \sum_{v \in A} w(v)$ for each $A \subseteq V(T)$.

In the next lemma, we use the notion of the *center* of a graph. This is the set of vertices with the smallest maximum distance to other vertices. It is well known

that, in a tree, the center consists of either a single vertex or two adjacent vertices of the tree.

Lemma 7.3. *For every integer $t > 1$, there is a function $f_{7.3}(t)$ such that every weighted tree T with $w(V(T)) > f_{7.3}(t)$ contains one of the following:*

(i) *a vertex v such that $d(v) > t$;*

(ii) *a path P such that $|V(P)| > t$; or*

(iii) *a path P such that $w(V(P)) > t$.*

Proof. We prove that $f_{7.3}(t) = \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor + 1} t^i$ satisfies the condition. Let T be a tree. We may assume that both the maximum degree and the number of vertices in a longest path do not exceed t otherwise (i) or (ii) holds. By grouping the vertices of T based on their distance from a center vertex of T, we see that $|V(T)| \leq \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} t^j$. However, since $w(T) \geq 1 + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor + 1} t^i$, there must be a vertex v such that $w(v) > t$. Therefore, each path P containing v satisfies $w(V(P)) > t$, so (iii) holds. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 7.1. Let H be a subgraph of G. An edge e of $E(G) - E(H)$ is an H-*bridge* if e is incident with at least one vertex of H. We call each connected component of $G - V(H)$ an H-*island*. Note that an H-bridge is either

(i) an edge having both vertices in $V(H)$, or

(ii) an edge having one vertex in $V(H)$ and one vertex in an H-island.

We first prove the following.

7.3.1. *If G has a cycle C and a C-island I such that there are at least $f_{7.3}(t)$ C-bridges between C and I, then G has a fan-type graph $F_{t_1,t_2,\ldots,t_n}$ as a cc-minor where $\sum_{i=1}^{n} t_i \geq t$.*

First, observe that $G - V(I)$ does not have a cut edge. Thus we can contract all of the edges of $G - V(I)$ by successively contracting a sequence of cycles. The resulting graph G' is obtained from $G[V(C) \cup V(I)]$ by contracting C. We denote by c the vertex that results by identifying all of the vertices of C. Note that $d_{G'}(c) \geq f_{7.3}(t)$ and the neighbors of c in G' are contained in $V(I)$. Let $G_0 = G'$. If $G_i - c$ has a cycle C_i, define $G_{i+1} = G_i / C_i$. This process results in a sequence $G_0, G_1, \ldots, G_s$ such that $G_s - c$ is a tree T.

Now, define a weight function w on $V(T)$ by, for each vertex v of T, letting $w(v)$ be the number of edges joining c and v. Clearly, $w(V(T)) = d_{G'}(c) \geq f_{7.3}(t)$. By Lemma 7.3, T has a subgraph T' that is one of the following:

(i) a vertex v such that $d(v) > t$;

(ii) a path P such that $|V(P)| > t$; or

(iii) a path P such that $w(V(P)) > t$.

Let $T_0 = T$. Assume that we have defined a sequence $(G_s, T_0), (G_{s+1}, T_1), \ldots,$ (G_{s+i}, T_i) where each G_{s+j} is 3-edge-connected having the tree T_j as a subgraph. If $T_i \neq T'$, then T_i has a leaf $l \notin V(T')$. Since G_{s+i} is 3-edge-connected, there are two edges joining c and l that form a cycle O_i in G_{s+i}. Define $G_{s+i+1} = G_{s+i}/O_i$ and $T_{i+1} = T - l$. Repeating this process, we eventually obtain a pair (G_{s+h}, T_h) with $T_h = T'$. By the choice of T', we see that if T' is a vertex of degree more than t in G_{s+h}, then G_{s+h} is a bond graph with more than t edges; if T' is a path on more than t vertices, then G_{s+h} is a fan-type graph with more than t sets of parallel spokes; and if T' is a path P such that $w(V(P)) > t$, then G_{s+h} is a fan-type graph with more than t spokes. Therefore, 7.3.1 holds.

To complete the proof of the theorem, we shall show that the required result holds for the function $f_{7.1}(t) = \binom{f_{7.2} \circ f_{7.3}(t)}{2}$. Since G is simple and has more than $\binom{f_{7.2} \circ f_{7.3}(t)}{2}$ edges, G has more than $f_{7.2} \circ f_{7.3}(t)$ vertices. By Lemma 7.2, G has a subgraph isomorphic to a subdivision of one of W_k, V_k, or $K_{3,k}$ where $k = f_{7.3}(t)$. In each of these three cases, let C be the bold cycle and I be the C-island containing the white vertices, as shown in Figure 16. It is straightforward to verify that the choices of C and I satisfy the conditions in 7.3.1. Hence, by 7.3.1, G has a fan-type graph $F_{t_1, t_2, \ldots, t_n}$ as a cc-minor such that $\sum_{i=1}^{n} t_i \geq t$. $\qquad\square$
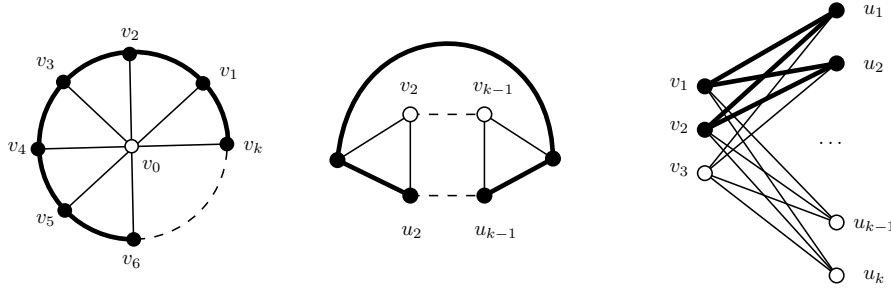

Figure 16: C and a C-island I in each of W_k, V_k, and $K_{3,k}$

8   Proof of Theorem 1.5

Before we present the proof of Theorem 1.5, we prove two lemmas.

Lemma 8.1. *If H is a cc-minor of a loopless graph G and G' is a parallel-path extension of G, then G' has a cc-minor H' that is a parallel-path extension of H.*

Proof. By Proposition 2.2, there is a collection $\{G_1, G_2, \ldots, G_k\}$ of disjoint 2-edge-connected subgraphs of G such that $H = G/(\cup_{i=1}^{k} E(G_k))$. Now G' is obtained from G by replacing some edges with internally disjoint paths joining their ends. For each $i \in \{1, 2, \ldots, k\}$, let G_i' be the graph that is obtained from G_i by replacing all such

edges in G_i with the same set of internally disjoint paths joining their ends as in G'. Clearly, G_i' is 2-edge-connected for all $i \in \{1, 2, \ldots, k\}$. Let $H' = G'/(\cup_{i=1}^{k} E(G_i'))$. It is straightforward to check that H' is a parallel-path extension of H. $\qquad \square$

Lemma 8.2. *Let G be the 2-sum of two loopless graphs G_1 and G_2 on the basepoint b. Suppose that, for each $i \in \{1, 2\}$, H_i is a cc-minor of G_i that has b as a non-loop edge. If H is the 2-sum of H_1 and H_2 on the basepoint b, then H is a cc-minor of G.*

Proof. By Proposition 2.2, for each i in $\{1, 2\}$, there is a collection $\mathcal{J}_i$ of disjoint 2-edge-connected subgraphs of G_i such that $H_i = G_i/(\cup_{J \in \mathcal{J}_i} E(J))$. Because $E(H_1) \cap E(H_2) = \{b\}$ and $b \notin E(J)$ for each J in $\mathcal{J}_1 \cup \mathcal{J}_2$, we know $\mathcal{J}_1 \cup \mathcal{J}_2$ is a collection of edge-disjoint 2-edge-connected subgraphs of G. By Corollary 4.2, we deduce that H, which equals $G/(\cup_{J \in \mathcal{J}_1 \cup \mathcal{J}_2} E(J))$, is a cc-minor of G. $\qquad \square$

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1.5. Note that proving Theorem 1.5 is equivalent to proving the following.

8.3.1. *Let r be a positive integer. There is an integer $g(r)$ such that every loopless 2-connected graph G with $|E(G)| \geq g(r)$ has a cc-minor H that is a parallel-path extension of a template with at least r parts.*

We shall show that G has a parallel-path extension of a template with at least r parts as a cc-minor when $g(r) = \sum_{i=1}^{r} \left(f_{7.1}(r+2) \right)^i$ where $f_{7.1}$ is the function whose existence was established in Theorem 7.1. Let T be a tree decomposition of G such that each vertex of T is either a simple 3-connected graph, or K_3, or B_3, as described in Theorem 1.3. First, we show the following.

8.3.2. *If there is a vertex $G_v \in V(T)$ such that $|E(G_v)| > f_{7.1}(r+2)$, then 8.3.1 holds.*

Note that $|f_{7.1}(t)| \geq t$ for all $t \geq 3$, so, for any positive integer r, we have $|E(G_v)| > f_{7.1}(r+2) \geq 3$. Thus, G_v cannot be isomorphic to K_3 or B_3. Therefore, G_v must be a simple 3-connected graph. Note that each component of $T - G_v$ is a tree decomposition for a 2-connected graph. Let $\{J_1, J_2, \ldots, J_n\}$ be the collection of such graphs. Then G can be obtained by repeatedly gluing each J_i to G_v via a 2-sum on the basepoint b_i where $\{b_i\} = E(J_i) \cap E(G_v)$ for all $i \in \{1, 2, \ldots, n\}$. By Lemma 4.3, for each $i \in \{1, 2, \ldots, n\}$, the graph J_i has a cc-minor J_i' that is a parallel connection, with basepoint b_i, of a collection of cycles containing b_i. Thus, G has a cc-minor G_v' that is obtained by, for each i in $\{1, 2 \ldots, n\}$, gluing J_i' to G_v via a 2-sum on the basepoint b_i. It is straightforward to verify that G_v' is a parallel-path extension of G_v.

By Lemma 8.1, it suffices to show that G_v has a cc-minor that is a template with at least r parts. By Theorem 7.1, G_v has a cc-minor that is a fan-type graph with at least $r + 2$ spokes, which constitutes a template with at least r parts.

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Hence 8.3.2 holds.

We may now assume that $|E(G_v)| \leq f_{7.1}(r+2)$ for every vertex G_v in $V(T)$. First we show the following.

8.3.3. *T has a path with at least r vertices.*

Since the basepoints are deleted after 2-sums, for each $G_v \in V(T)$, we have $d_T(G_v) \leq |E(G_v)|$. Therefore, we conclude that $d_T(G_v) \leq f_{7.1}(r+2)$ for every $G_v \in V(T)$. For any two vertices G_u and G_v, the *distance $d(G_u, G_v)$* between them is the number of edges of the shortest $G_u G_v$-path in T. For an arbitrary vertex G_w in $V(T)$, we have the following.

8.3.4. *For each non-negative integer h,*

$$|\{G_v \in V(T) : d(G_w, G_v) = h\}| \leq \left(f_{7.1}(r+2)\right)^h.$$

Since $|E(G)| \geq g(r)$ and $|E(G_v)| \leq f_{7.1}(r+2)$ for each vertex G_v in $V(T)$, the tree T has at least $\frac{g(r)}{f_{7.1}(r+2)}$ vertices. As $g(r) = \sum_{i=1}^{r} \left(f_{7.1}(r+2)\right)^i$, we deduce that T has at least $\sum_{j=0}^{r-1} \left(f_{7.1}(r+2)\right)^j$ vertices. Therefore, by 8.3.4, there is a vertex $G_q \in V(T)$ such that $d(G_w, G_q) \geq r-1$. Hence 8.3.3 holds.

Let P be a path $G_1 G_2 \ldots G_r$ in T. Note that P is a graph-labeled tree representing a graph G_P obtained from the graphs $G_1, G_2, \ldots, G_r$ by applying a sequence of 2-sums. Note that each component of $T - V(P)$ is a tree decomposition for a 2-connected graph. Let $\{F_1, F_2, \ldots, F_m\}$ be the collection of these graphs. Then G can be obtained by, for each i in $\{1, 2, \ldots, m\}$, gluing F_i to G_P via a 2-sum on the basepoint p_i where $\{p_i\} = E(F_i) \cap E(G_P)$. By Lemma 4.3, for each $i \in \{1, 2, \ldots, m\}$, the graph F_i has a cc-minor F_i' that is a parallel connection, with basepoint p_i, of a collection of cycles containing p_i. Thus, G has a cc-minor G_P' that is obtained by, for each i in $\{1, 2, \ldots, m\}$, gluing F_i' to G_P via a 2-sum on the basepoint p_i. It is straightforward to verify that G_P' is a parallel-path extension of G_P. By Lemma 8.1, it remains only to show that G_P has a cc-minor that is a template with at least r parts.

For each $i \in \{1, 2, \ldots, r-1\}$, let e_i be the unique edge in $E(G_i) \cap E(G_{i+1})$ that is used as the basepoint between G_i and G_{i+1} in the construction of G_P. For each $i \in \{2, 3, \ldots, r-1\}$, by Theorem 6.1, G_i has a cc-minor G_i' that is

(i) a bond graph containing e_{i-1} and e_{i+1}; or

(ii) a fan-type graph containing e_{i-1} and e_{i+1} as distinct outer spokes that are not parallel; or

(iii) a parallel extension of K_4 that has e_{i-1} and e_{i+1} as non-adjacent edges.

For $i \in \{1, r\}$, if G_i is isomorphic to K_3 or B_3, let $G_i' = G_i$. Now suppose that G_1 is not isomorphic to K_3 or B_3. Then, by Corollary 4.4, G_1 has a cc-minor G_1' that

contains e_1 and is isomorphic to a bond graph with at least three edges. Define G'_r symmetrically when G_r is not isomorphic to K_3 or B_3. Let G'_P be the graph that is obtained by applying a sequence of 2-sums to the graphs $G'_1, G'_2, \ldots, G'_r$, using the edges $e_1, e_2, \ldots, e_{r-1}$ as basepoints. Applying Lemma 8.2 inductively, we see that G'_P is a cc-minor of G_P. It is straightforward to verify that G'_P is a template on at least r parts. $\qquad\square$

Acknowledgments

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