

# UNAVOIDABLE PARALLEL MINORS OF REGULAR MATROIDS

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*To the memory of Tom Brylawski, who contributed so much to matroid theory.*

ABSTRACT. We prove that, for each positive integer  $k$ , every sufficiently large 3-connected regular matroid has a parallel minor isomorphic to  $M^*(K_{3,k})$ ,  $M(W_k)$ ,  $M(K_k)$ , the cycle matroid of the graph obtained from  $K_{2,k}$  by adding paths through the vertices of each vertex class, or the cycle matroid of the graph obtained from  $K_{3,k}$  by adding a complete graph on the vertex class with three vertices.

## 1. INTRODUCTION

For 3-connected graphs, the collections of unavoidable parallel and unavoidable series minors were determined by Chun, Ding, Oporowski, and Vertigan [3] and by Oporowski, Oxley, and Thomas [8]. In this paper, we combine these results with Seymour's decomposition theorem for regular matroids [12] to determine the collection of unavoidable parallel minors for the class of 3-connected regular matroids. In particular, we prove that the last collection is precisely the union of the collections of unavoidable parallel minors for the classes of 3-connected graphic and 3-connected cographic matroids. The collections of unavoidable minors for binary 3-connected matroids and for all 3-connected matroids were determined in [6, 7]. From the first of these, one can determine the collection of unavoidable minors for regular 3-connected matroids, although this result had been obtained earlier by Ding and Oporowski [5]. We would like to extend our main theorem to find the unavoidable parallel minors for the class of binary 3-connected matroids, but this will require some new ideas.

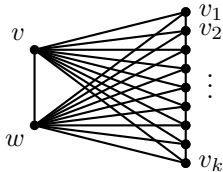
Our terminology for matroids and graphs generally follows [9] and [4]. If  $M$  and  $N$  are both matroids or are both graphs,  $N$  is a *parallel minor* of  $M$  if  $N$  can be obtained from  $M$  by a sequence of moves each consisting of contracting an element (in the graph case, an edge) or deleting an element that is in a 2-element circuit. When  $M$  and  $N$  are both matroids,  $N$  is a *series minor* of  $M$  if  $N^*$  is a parallel minor of  $M^*$ . If  $G$  and  $H$  are graphs and  $H$  is a parallel minor of  $G$ , then  $M(H)$  is a parallel minor of  $M(G)$ . Conversely, when  $G$  and  $H$  are loopless 3-connected graphs, if  $M(H)$  is a parallel minor of  $M(G)$ , then  $H$  is a parallel minor of  $G$ .

Let  $M$  be a matroid with ground set  $E$  and rank function  $r$ . The simplification of  $M$  will be denoted by  $\text{si}(M)$ . The *connectivity function*  $\lambda_M$  of  $M$  is defined for all subsets  $X$  of  $E$  by  $\lambda_M(X) = r(X) + r(E - X) - r(M)$ . Equivalently,  $\lambda_M(X) = r(X) + r^*(X) - |X|$ . Thus  $\lambda_M(X) = \lambda_{M^*}(X)$ . For a positive integer  $m$ , when  $\lambda_M(X) < m$ , a partition  $(X, Y)$  of  $E$  is an  *$m$ -separation* if  $\min\{|X|, |Y|\} \geq m$  and is a *vertical  $m$ -separation* if  $\min\{r(X), r(Y)\} \geq m$ . A matroid is  *$n$ -connected*

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FIGURE 1. A double fan graph  $DF_k$ .

if, for all  $m < n$ , it has no  $m$ -separations [13]. A 3-connected matroid is *internally 4-connected* if it has no 3-separation  $(X, Y)$  with  $\min\{|X|, |Y|\} \geq 4$ . A matroid  $M$  is *vertically 3-connected* if it is loopless and has no vertical 1-separations and no vertical 2-separations. Note that this adds the requirement that  $M$  be loopless to the usual definition of vertical 3-connectedness. Thus  $M$  is vertically 3-connected if and only if  $\text{si}(M)$  is 3-connected and  $M$  is loopless.

In the following theorem, the main result of the paper,  $\mathcal{W}_k$  denotes the  $k$ -spoked wheel,  $K'_{i,j}$  is the bipartite graph  $K_{i,j}$  together with a complete graph on the vertex class of  $i$  vertices, and  $DF_k$  is a double fan, as shown in Figure 1.

**Theorem 1.1.** *There is a function  $f_{1.1}$  such that, for each integer  $k$  exceeding three, every 3-connected regular matroid with at least  $f_{1.1}(k)$  elements has a parallel minor isomorphic to  $M(K'_{3,k})$ ,  $M^*(K_{3,k})$ ,  $M(\mathcal{W}_k)$ ,  $M(DF_k)$ , or  $M(K_k)$ .*

By using duality, we immediately obtain the set of unavoidable series minors of 3-connected regular matroids. We denote the dual of the double fan  $DF_k$  by  $V_k$ .

**Corollary 1.2.** *There is a function  $f_{1.2}$  such that, for each integer  $k$  exceeding three, every 3-connected regular matroid with at least  $f_{1.2}(k)$  elements has a series minor isomorphic to  $M^*(K'_{3,k})$ ,  $M(K_{3,k})$ ,  $M(\mathcal{W}_k)$ ,  $M(V_k)$ , or  $M^*(K_k)$ .*

From either of the last two results, we can deduce the following result of Ding and Oporowski [5] which shows that the collection of unavoidable minors of 3-connected regular matroids is the union of the collections of unavoidable minors for the classes of 3-connected graphic and 3-connected cographic matroids.

**Corollary 1.3.** *There is a function  $f_{1.3}$  such that, for each integer  $k$  exceeding three, every 3-connected regular matroid with at least  $f_{1.3}(k)$  elements has a minor isomorphic to  $M(K_{3,k})$ ,  $M^*(K_{3,k})$ , or  $M(\mathcal{W}_k)$ .*

By a result of Seymour, stated below as Theorem 2.1, an internally 4-connected regular matroid with at least eleven elements is graphic or cographic. This means that the sets of unavoidable parallel minors and unavoidable series minors of internally 4-connected regular matroids can be immediately determined by combining results in [3] and [8] that determine the sets of unavoidable parallel minors and unavoidable series minors, respectively, of internally 4-connected graphs.

The proof of Theorem 1.1 contains numerous technicalities but the basic method is standard. By Seymour's decomposition theorem, a large 3-connected regular matroid can be decomposed in a tree-like fashion into pieces each of which is graphic or cographic. If any of these pieces is large enough, then we can apply the known results on unavoidable parallel minors in 3-connected graphic matroids and in 3-connected cographic matroids. Thus we may assume that all the pieces are small, so the tree is large and therefore contains a long path or a vertex of high degree. In both of these cases, we can find a parallel minor of the desired type.

## 2. PRELIMINARIES

In this section, we introduce some more terminology and prove some lemmas that will be used in the proof of the main theorem, which appears in the next section. Much of what we do here is concerned with finding a tree-like decomposition of a regular matroid. Of particular importance is the operation of generalized parallel connection of matroids, which was introduced and examined in detail by Tom Brylawski [2]. We shall only use one special case of this operation.

For binary matroids  $M_1$  and  $M_2$  with ground sets  $E_1$  and  $E_2$  such that  $E_1 \cap E_2 = \Delta$  and  $M_1|_{\Delta}$  and  $M_2|_{\Delta}$  are triangles, the *generalized parallel connection* of  $M_1$  and  $M_2$  with respect to  $\Delta$ , written  $P_{\Delta}(M_1, M_2)$ , is the matroid with ground set  $E_1 \cup E_2$  in which  $F$  is a flat if and only if  $F \cap E_i$  is a flat of  $M_i$  for each  $i$ . Then  $P_{\Delta}(M_2, M_1) = P_{\Delta}(M_1, M_2)$ . Moreover, one can show that if  $\text{cl}$ ,  $\text{cl}_1$ , and  $\text{cl}_2$  are the closure operators of  $P_{\Delta}(M_1, M_2)$ ,  $M_1$ , and  $M_2$ , then, for every subset  $X$  of  $E_1 \cup E_2$ ,

$$\text{cl}(X) = \text{cl}_1([X \cup \text{cl}_2(X \cap E_2)] \cap E_1) \cup \text{cl}_2([X \cup \text{cl}_1(X \cap E_1)] \cap E_2). \quad (1)$$

This correction to [9, Exercise 12.4.5] appears in the errata to that book available at the second author's website and in the second edition of the book [10].

When  $M_1$  and  $M_2$  both have at least seven elements and  $\Delta$  does not contain a cocircuit of  $M_1$  or  $M_2$ , Seymour [12] defined the *3-sum*,  $M_1 \oplus_{\Delta} M_2$ , of  $M_1$  and  $M_2$  to be the matroid  $P_{\Delta}(M_1, M_2) \setminus \Delta$ . In much of what we do, it will be convenient to work with generalized parallel connections rather than 3-sums because of the additional constraints that must be satisfied in order for the latter to be defined. The generalized parallel connection across a triangle of two graphic matroids is easily seen to be graphic. Hence so is their 3-sum. Note, however, that the 3-sum of two cographic matroids need not be cographic. For example, the non-cographic matroid  $R_{12}$  can be written as a 3-sum of  $M(K_5 \setminus e)$  and  $M^*(K_{3,3})$  (see, for example, [9, Exercise 1(ii), p. 440]). When  $G_1$  and  $G_2$  are graphs and both have  $\Delta$  as a vertex bond,  $P_{\Delta}(M^*(G_1), M^*(G_2))$  and  $P_{\Delta}(M^*(G_1), M^*(G_2)) \setminus \Delta$  are easily shown to be cographic. Hence so is  $M^*(G_1) \oplus_{\Delta} M^*(G_2)$  when it is defined.

The next theorem was proved by Seymour [12]. The matroid  $R_{10}$  is the 10-element matroid that can be represented over  $GF(2)$  by the matrix whose columns consist of all 5-tuples with exactly three ones (see [9, p. 518]).

**Theorem 2.1.** *Let  $M$  be a 3-connected regular matroid. Then at least one of the following holds:*

- (i)  $M$  is graphic;
- (ii)  $M$  is cographic;
- (iii)  $M \cong R_{10}$ ; or
- (iv) there are regular matroids  $M_1$  and  $M_2$  such that  $E(M_1) \cap E(M_2) = \Delta$ , where  $\Delta$  is a triangle of both  $M_1$  and  $M_2$ , and  $M = M_1 \oplus_{\Delta} M_2$ ; and, for each  $i$  in  $\{1, 2\}$ ,
  - (a)  $M_i$  is 2-connected and, for every 2-separation  $(X, Y)$  of it, either  $X$  or  $Y$  has exactly two elements and meets  $\Delta$ , so  $\text{si}(M_i)$  is 3-connected;
  - (b)  $M_i$  is isomorphic to a minor of  $M$ ; and
  - (c)  $|E(M_i) - \text{cl}_{M_i}(\Delta)| \geq 6$  and  $|E(\text{si}(M_i))| \geq 9$ .

The proof of our main result will require us to carefully consider both the matroids that are built up by a sequence of generalized parallel connections across disjoint triangles, and the matroids we get by deleting all of these triangles. We

now formally describe these constructions. Let  $M_1$  and  $M_2$  be binary matroids with  $E(M_1) \cap E(M_2) = \Delta_2$ , where  $\Delta_2$  is a triangle of both  $M_1$  and  $M_2$ . Let  $P(M_1, M_2)$  and  $(M_1, \Delta_2, M_2)$  be  $P_{\Delta_2}(M_1, M_2)$  and  $P_{\Delta_2}(M_1, M_2) \setminus \Delta_2$ . Now assume, for some  $n \geq 3$ , that  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$  and  $P(M_1, M_2, \dots, M_{n-1})$  have been defined, that

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}) = P(M_1, M_2, \dots, M_{n-1}) \setminus (\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_{n-1}),$$

and that the flats of  $P(M_1, M_2, \dots, M_{n-1})$  are those subsets  $F$  of its ground set such that  $F \cap E(M_i)$  is a flat of  $M_i$  for all  $i$  in  $\{1, 2, \dots, n-1\}$ . Let  $M_n$  be a binary matroid whose ground set meets that of  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$  in a set  $\Delta_n$  that is a triangle of both  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$  and  $M_n$ . Define

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n) = P_{\Delta_n}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}), M_n) \setminus \Delta_n$$

and  $P(M_1, M_2, \dots, M_n) = P_{\Delta_n}(P(M_1, M_2, \dots, M_{n-1}), M_n)$ . Then one easily checks that  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n) = P(M_1, M_2, \dots, M_n) \setminus (\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_n)$  and that the flats of  $P(M_1, M_2, \dots, M_n)$  are those subsets  $F$  of its ground set such that  $F \cap E(M_i)$  is a flat of  $M_i$  for all  $i$  in  $\{1, 2, \dots, n\}$ . It will be convenient to abbreviate  $P(M_1, M_2, \dots, M_n)$  as  $M_{[n]}^P$ . Observe that the construction guarantees that  $\Delta_2, \Delta_3, \dots, \Delta_n$  are disjoint.

**Lemma 2.2.** *If  $\text{si}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n))$  is 3-connected, then  $\text{si}(M_i)$  is 3-connected for all  $i$ .*

*Proof.* By definition,  $\text{si}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n))$  is

$$\text{si}(P_{\Delta_n}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}), M_n) \setminus \Delta_n).$$

Assume that  $\text{si}(P_{\Delta_2}(M_1, M_2) \setminus \Delta_2)$  is 3-connected. If we can show that both  $\text{si}(M_1)$  and  $\text{si}(M_2)$  are 3-connected, then the result will follow by induction. For some  $k$  in  $\{1, 2\}$ , suppose that  $(X, Y)$  is a vertical  $k$ -separation of  $M_1$ . Without loss of generality, we may assume that  $|X \cap \Delta_2| \geq 2$ . Then

$$r(X \cup \Delta_2) + r(Y - \Delta_2) - r(M_1) \leq r(X) + r(Y) - r(M_1) \leq k - 1.$$

Now, by [9, Lemma 8.2.10],

$$\begin{aligned} & r((X \cup E(M_2)) - \Delta_2) + r(Y - \Delta_2) - r(P_{\Delta_2}(M_1, M_2) \setminus \Delta_2) \\ & \leq r(X \cup E(M_2) \cup \Delta_2) + r(Y - \Delta_2) - r(P_{\Delta_2}(M_1, M_2)) \\ & \leq [r(X \cup \Delta_2) + r(M_2) - r(\Delta_2)] + r(Y - \Delta_2) - [r(M_1) + r(M_2) - r(\Delta_2)] \\ & = r(X \cup \Delta_2) + r(Y - \Delta_2) - r(M_1) \leq k - 1. \end{aligned}$$

Thus  $P_{\Delta_2}(M_1, M_2) \setminus \Delta_2$  has a vertical  $k$ -separation; a contradiction. Therefore  $M_1$  is vertically 3-connected and, by symmetry, so is  $M_2$ .  $\square$

The next lemma will be helpful in the proof of Lemma 2.4, where we use Seymour's theorem to obtain a sequential decomposition of a regular matroid.

**Lemma 2.3.** *Let  $M_1$  and  $M_2$  be binary matroids whose ground sets meet in a set  $\Delta_2$  that is a triangle of both matroids. If  $\Delta_3$  is a triangle of  $P_{\Delta_2}(M_1, M_2) \setminus \Delta_2$ , then, for some  $\{i, j\} = \{1, 2\}$ , either*

- (i)  $\Delta_3 \subseteq E(M_i)$ ; or

- (ii)  $|\Delta_3 \cap E(M_i)| = 2$  and  $|\Delta_3 \cap E(M_j)| = 1$ , and the element  $c$  of  $\Delta_3 \cap E(M_j)$  is parallel to some element  $g$  of  $M_i$ . Moreover, if  $M'_j$  and  $M'_i$  are obtained by deleting  $c$  from  $M_j$ , and adding  $c$  in parallel to  $g$  in  $M_i$ , then  $P_{\Delta_2}(M'_1, M'_2) = P_{\Delta_2}(M_1, M_2)$ , while  $\text{si}(M'_1) = \text{si}(M_1)$  and  $\text{si}(M'_2) = \text{si}(M_2)$ .

*Proof.* Let  $E_1 = E(M_1)$  and  $E_2 = E(M_2)$ . We may assume that  $|\Delta_3 \cap E_1| = 2$  and  $|\Delta_3 \cap E_2| = 1$ . Then, in  $P_{\Delta_2}(M_1, M_2)$ , the intersection of  $\text{cl}(E_1)$  and  $\text{cl}(E_2)$  is  $\text{cl}(\Delta_2)$ . Thus the element  $c$  of  $\Delta_3 \cap E(M_2)$  is parallel to some element of  $\text{cl}(\Delta_2)$ , and the lemma follows.  $\square$

**Lemma 2.4.** *Let  $M$  be a vertically 3-connected regular matroid such that  $\text{si}(M)$  has at least six elements and is not isomorphic to  $R_{10}$ . Then either  $M$  is graphic or cographic, or, for some  $n \geq 2$ , there is a sequence  $M_1, M_2, \dots, M_n$  of graphic and cographic matroids such that  $M = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  where, for all  $i$  with  $2 \leq i \leq n$ , the triangle  $\Delta_i \subseteq E(M_j)$  for some  $j < i$ , and all of  $\text{si}(M_1), \text{si}(M_2), \dots, \text{si}(M_n)$  are 3-connected having at least nine elements.*

*Proof.* We shall assume that  $M$  is simple since it suffices to prove the lemma in that case. We proceed by induction on  $|E(M)|$ . Since  $M$  is regular, if  $|E(M)| \leq 9$ , then either  $M$  is graphic, or  $M$  is isomorphic to  $M^*(K_{3,3})$  and so is cographic. In both cases, the lemma holds. Now suppose that the lemma holds for matroids with fewer than  $k$  elements and let  $|E(M)| = k \geq 10$ .

Assume that  $M$  is neither graphic nor cographic. Then, by Theorem 2.1,  $M$  is the 3-sum of some matroids  $N_1$  and  $N_2$ , where both  $\text{si}(N_1)$  and  $\text{si}(N_2)$  are 3-connected having at least nine elements. Choose such a 3-sum decomposition in which  $|E(N_2)|$  is minimized. Let  $\Delta$  be the common triangle of  $N_1$  and  $N_2$ . We may assume that  $\Delta \subseteq E(\text{si}(N_i))$  for each  $i$ .

Since  $N_2$  has a triangle, it is not isomorphic to  $R_{10}$ . Suppose  $\text{si}(N_2)$  is not graphic or cographic. Then, by Theorem 2.1,  $N_2$  is the 3-sum of matroids  $N'_2$  and  $N''_2$  across a common triangle  $\Delta'$  where each of  $\text{si}(N'_2)$  and  $\text{si}(N''_2)$  is 3-connected and contains at least nine elements. As  $\Delta$  is a triangle of  $P_{\Delta'}(N'_2, N''_2) \setminus \Delta'$ , Lemma 2.3 implies that, without altering  $\text{si}(N'_2)$  or  $\text{si}(N''_2)$ , we can assume that  $\Delta \subseteq E(N'_2)$ . Then, by comparing flats, we can show that  $P_{\Delta}(N_1, P_{\Delta'}(N'_2, N''_2)) = P_{\Delta'}(P_{\Delta}(N_1, N'_2), N''_2)$ , so  $M = (N_1 \oplus_{\Delta} N_2) \oplus_{\Delta'} N''_2$ . By Lemma 2.2,  $\text{si}(N_1 \oplus_{\Delta} N_2)$  is 3-connected; a contradiction, since  $|E(N_2)|$  was chosen to be minimal.

We may now assume that  $\text{si}(N_2)$  is graphic or cographic. Hence so is  $N_2$ . By the inductive hypothesis,  $N_1 = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  and the desired conditions hold. Now  $\Delta$  is a triangle of  $N_1$ . Pick the smallest integer  $k$  such that  $\Delta \subseteq E((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_k, M_k))$ . Then  $\Delta$  meets  $E(M_k)$ .

Suppose that  $|\Delta \cap E(M_k)| \geq 2$ . Then, by moving at most one element of  $\Delta$  from being parallel to an element of  $\Delta_k$  in  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{k-1}, M_{k-1})$  to being parallel to that element of  $\Delta_k$  in  $M_k$ , we ensure that  $\Delta \subseteq E(M_k)$ , as desired.

It remains to consider when  $\Delta \cap E(M_k)$  contains a single element, say  $c$ . Then, by Lemma 2.3 again, we move  $c$  from being parallel to an element of  $\Delta_k$  in  $M_k$  to being parallel with that element in  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{k-1}, M_{k-1})$ . We now have  $\Delta \subseteq E((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{k-1}, M_{k-1}))$  and we can repeat the above process until we eventually obtain  $\Delta \subseteq E(M_i)$  for some  $i$ . Thus the lemma holds.  $\square$

Let  $M$  be a vertically 3-connected regular matroid having at least six elements. If  $M = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  for some  $n \geq 2$ , we call  $(M_1, \Delta_2, M_2, \Delta_3, \dots,$

$\Delta_n, M_n$ ) a *good decomposition* of  $M$  if, for all  $i$  with  $2 \leq i \leq n$ , the triangle  $\Delta_i \subseteq E(M_j)$  for some  $j < i$ . Also, we view  $(M)$  as a good decomposition of  $M$ .

Two disjoint triangles  $X_1$  and  $X_2$  in a binary matroid are *parallel* if  $r(X_1 \cup X_2) = 2$ . Recall that a regular matroid  $M$  is vertically 3-connected if  $\text{si}(M)$  is 3-connected and  $M$  is loopless. For a good decomposition  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  of a vertically 3-connected regular matroid, define the *associated tree*  $T$  to have vertex set  $\{M_1, M_2, \dots, M_n\}$  and edge set  $\{\Delta_2, \Delta_3, \dots, \Delta_n\}$  where  $\Delta_i$  joins  $M_i$  to the vertex  $M_j$  with  $j < i$  such that  $\Delta_i \subseteq E(M_j)$ . We shall sometimes write  $M_T$  for  $M$ . Note that this labelling means that, for every path  $M_{i_1} M_{i_2} \dots M_{i_k}$  in  $T$ , there is a  $j$  in  $\{1, 2, \dots, k\}$  such that  $i_1 > i_2 > \dots > i_j$  and  $i_j < i_{j+1} < \dots < i_k$ . The reader may find some features of the tree disconcerting. For example, the matroids labelling two non-adjacent vertices may contain triangles that are parallel in  $M_{[n]}^P$ . In spite of this apparent shortcoming, this tree will be adequate for our needs.

**Lemma 2.5.** *Let  $M$  be a vertically 3-connected regular matroid for which  $|E(\text{si}(M))| \geq 9$  and  $\text{si}(M) \not\cong R_{10}$ . Let  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  be a good decomposition of  $M$  and  $M_i M_j$  be an edge of the associated tree with  $j < i$ . Then*

$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_j, (M_j, \Delta_i, M_i), \Delta_{j+1}, \dots, M_{i-1}, \Delta_{i+1}, M_{i+1}, \dots, \Delta_n, M_n)$   
*is a good decomposition of  $M$ . Moreover,  $\text{si}((M_j, \Delta_i, M_i))$  is 3-connected.*

*Proof.* We shall show first that

$$\begin{aligned} (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_j, (M_j, \Delta_i, M_i), \Delta_{j+1}, \dots, \Delta_{i-1}, M_{i-1}) \\ = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_i, M_i). \end{aligned} \quad (2)$$

Now  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_i, M_i)$  is obtained from  $P(M_1, M_2, \dots, M_i)$  by deleting  $\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_i$ . Moreover,  $P(M_1, M_2, \dots, M_i)$  has, as its flats, those sets  $F$  such that  $F \cap E(M_s)$  is a flat of  $M_s$  for all  $s$  with  $1 \leq s \leq i$ . The matroid on the left-hand side of (2) is obtained from  $P(M_1, M_2, \dots, M_{j-1}, P_{\Delta_i}(M_j, M_i) \setminus \Delta_i, M_{j+1}, \dots, M_{i-1})$  by deleting  $\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_{i-1}$ . Thus it is obtained from  $P(M_1, M_2, \dots, M_{j-1}, P_{\Delta_i}(M_j, M_i), M_{j+1}, \dots, M_{i-1})$  by deleting  $\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_i$ . The flats of the last parallel connection coincide with the flats of  $P(M_1, M_2, \dots, M_i)$ . Hence (2) holds. It follows that  $M$  has the decomposition specified in the lemma, and one easily checks that this decomposition is good. Finally,  $\text{si}((M_j, \Delta_i, M_i))$  is 3-connected by Lemma 2.2.  $\square$

We shall repeatedly use the following routine consequence of the last lemma.

**Corollary 2.6.** *Let  $T$  be a tree associated with a vertically 3-connected matroid  $M$ . Delete an edge  $M_a M_b$  of  $T$  and let  $T_a$  be the component of the resulting forest that contains  $M_a$ . A new tree associated with  $M$  can be obtained from  $T$  by contracting the edges of  $T_a$ , one by one, each time labelling the composite vertex that results from contracting the edge  $\Delta$  joining  $M_i$  and  $M_j$  by  $(M_j, \Delta, M_i)$ .*

When we have a good decomposition of a regular matroid  $M$ , the next two lemmas will be useful in obtaining good decompositions of certain minors of  $M$ .

**Lemma 2.7.** *Let  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  be a good decomposition of a regular matroid  $M$ . For  $e$  in  $E(M_i) - (\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_n)$ , if  $e \in \text{cl}_{M_{[n]}^P}(\Delta_j)$  for some  $j$ , then  $e \in \text{cl}_{M_i}(\Delta_k)$  for some  $k$  in  $\{2, 3, \dots, n\}$  where  $\Delta_k \subseteq E(M_i)$ .*

*Proof.* Choose  $j$  to be the smallest integer  $t$  for which  $e \in \text{cl}_{M_{[n]}^P}(\Delta_t)$ . If  $\Delta_j \subseteq E(M_i)$ , then the result holds with  $j = k$ . Thus we may assume that  $\Delta_j \not\subseteq E(M_i)$  so  $\Delta_j \cap E(M_i) = \emptyset$  and  $j \neq i$ . Now  $e$  is parallel in  $M_{[n]}^P$  to some element of  $\Delta_j$ .

Assume  $j < i$ . Then  $e \in \text{cl}_{M_{[i]}^P}(\Delta_j)$  so, in  $M_{[i]}^P$ , the element  $e$  is in the intersection of  $\text{cl}(E(M_i))$  and  $\text{cl}(E(P(M_1, M_2, \dots, M_{i-1})))$ . Hence  $e \in \text{cl}_{M_{[i]}^P}(\Delta_i)$ . Thus  $e \in \text{cl}_{M_i}(\Delta_i)$  and the result holds with  $k = i$ .

We may now assume that  $j > i$  so  $j \geq 2$ . We know that  $\Delta_j \subseteq E(M_j)$  and  $\Delta_j \subseteq E(M_s)$  for some  $s < j$ . If  $s < i$ , then, it follows, as above, that  $e \in \text{cl}_{M_i}(\Delta_i)$ . Hence we may assume that  $s > i$ . Then  $e \in \text{cl}_{M_{[s]}^P}(\Delta_j)$  so  $e \in \text{cl}_{M_{[s]}^P}(\Delta_s)$  and hence  $e \in \text{cl}_{M_{[n]}^P}(\Delta_s)$ . But  $s < j$ ; a contradiction.  $\square$

**Lemma 2.8.** *Let  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  be a good decomposition of a regular matroid  $M$ . For  $e$  in  $E(M_i) - (\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_n)$ , if  $e \in \text{cl}_{M_{[n]}^P}(E(M_j))$  for some  $j \neq i$ , then  $e \in \text{cl}_{M_i}(\Delta_k)$  for some  $k$  in  $\{2, 3, \dots, n\}$  where  $\Delta_k \subseteq E(M_i)$ .*

*Proof.* First we show the following.

**2.8.1.** *The lemma holds if  $e \in \text{cl}_{M_{[q+1]}^P}(E(M_j)) - \text{cl}_{M_{[q]}^P}(E(M_j))$  for some  $q$  with  $j \leq q < n$ .*

By definition,  $M_{[q+1]}^P = P_{\Delta_{q+1}}(M_{[q]}^P, M_{q+1})$ . Suppose  $E(M_j) \cap E(M_{q+1}) \neq \emptyset$ . Then the construction of  $M$  means that  $E(M_j) \cap E(M_{q+1}) = \Delta_{q+1}$ . Thus, by (1),  $\text{cl}_{M_{[q+1]}^P}(E(M_j)) = \text{cl}_{M_{[q]}^P}(E(M_j)) \cup \text{cl}_{M_{q+1}}(\Delta_{q+1})$ , so  $e \in \text{cl}_{M_{q+1}}(\Delta_{q+1})$ . Hence  $e \in \text{cl}_{M_{[q+1]}^P}(\Delta_{q+1})$ , so  $e \in \text{cl}_{M_{[n]}^P}(\Delta_{q+1})$  and the lemma follows by Lemma 2.7. Hence 2.8.1 holds.

Now assume that  $j > i$ . If  $e \notin \text{cl}_{M_{[j]}^P}(E(M_j))$ , then, since  $e \in \text{cl}_{M_{[n]}^P}(E(M_j))$ , the lemma follows by 2.8.1. Hence we may assume that  $e \in \text{cl}_{M_{[j]}^P}(E(M_j))$ . Then  $e \in E(M_i) \cap \text{cl}_{M_{[j]}^P}(E(M_j))$ . Hence  $e \in \text{cl}_{M_{[j]}^P}(\Delta_j)$ , so  $e \in \text{cl}_{M_{[n]}^P}(\Delta_j)$  and again the lemma follows by Lemma 2.7.

Finally, assume that  $j < i$ . By 2.8.1, we may assume that  $e \in \text{cl}_{M_{[i]}^P}(E(M_j))$ . But  $e \in E(M_i)$ , so  $e \in \text{cl}_{M_{[i]}^P}(E(M_j)) \cap \text{cl}_{M_{[i]}^P}(E(M_i)) \subseteq \text{cl}_{M_{[i]}^P}(\Delta_i)$ . Thus  $e \in \text{cl}_{M_{[n]}^P}(\Delta_i)$  and the lemma follows by Lemma 2.7.  $\square$

**Corollary 2.9.** *Let  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  be a good decomposition of a regular matroid  $M$ . For some  $i$  in  $\{1, 2, \dots, n\}$ , let  $N_i$  be a minor of  $M_i$  such that if  $\Delta_j \subseteq E(M_i)$  for some  $j$  in  $\{2, 3, \dots, n\}$ , then  $\Delta_j$  is a triangle of  $N_i$ . Then*

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, M_{i-1}, \Delta_i, N_i, \Delta_{i+1}, M_{i+1}, \dots, \Delta_n, M_n)$$

*is a good decomposition of a minor of  $M$ .*

*Proof.* It suffices to prove this when  $N_i$  is  $M_i \setminus e$  or  $M_i/e$  for some element  $e$ . In this case, the result follows without difficulty using the last lemma and properties of the generalized parallel connection [2] summarized in [9, Proposition 12.4.16].  $\square$

Let  $A$  and  $B$  be parallel triangles in a loopless binary matroid  $N$ . Then  $N|(A \cup B)$  is a *double triangle*. We call  $N$  a *multi- $K_4$  with respect to  $A$  and  $B$*  if  $\text{si}(N) = M(K_4)$ ; and we call  $N$  a *multi-triangle with respect to  $A$  and  $B$*  if  $r(N) = 2$  and  $N$  contains at least one element not in  $A \cup B$ .

The following result is an immediate consequence of the Scum Theorem.

**Lemma 2.10.** *If a binary matroid  $M$  has as a minor a multi-triangle or a multi- $K_4$  with respect to two parallel triangles  $A$  and  $B$ , then  $E(M)$  has a subset  $Y$  such that  $M/Y$  is, respectively, a multi-triangle or a multi- $K_4$  with respect to  $A$  and  $B$ .*

The next lemma [10] was proved by Jim Geelen and is useful for finding a double triangle as a parallel minor of a 3-connected graphic or cographic matroid. If  $X$  and  $Y$  are disjoint subsets of the ground set of a matroid  $M$ , we define  $\kappa_M(X, Y)$  to be  $\min\{\lambda_M(Z) : X \subseteq Z \subseteq E(M) - Y\}$ .

**Lemma 2.11.** *Let  $C$  and  $X$  be disjoint sets in a matroid  $M$  such that  $C$  is a circuit and  $\kappa_M(C, X) = 2$ . Then there are elements  $a, b$ , and  $c$  of  $C$  and a minor  $N$  of  $M$  that has  $\{a, b, c\}$  as a circuit and  $X \cup \{a, b, c\}$  as its ground set such that  $\kappa_N(\{a, b, c\}, X) = 2$ .*

**Lemma 2.12.** *Let  $M$  be a vertically 3-connected regular matroid for which  $|E(\text{si}(M))| \geq 9$  and  $\text{si}(M) \not\cong R_{10}$ . Let  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$  be a good decomposition of  $M$  such that each  $\text{si}(M_i)$  has at least nine elements. Let  $T$  be the tree associated with this decomposition. Let  $T'$  be a connected subgraph of  $T$ . Then  $T'$  is a tree associated with the matroid  $M'$  that labels the one vertex that results after all the edges of  $T'$  are contracted. Moreover,  $\text{si}(M')$  is a 3-connected matroid that is isomorphic to a parallel minor of  $M$ .*

*Proof.* It suffices to prove the lemma in the case that  $T' = T - M_i$  for some vertex  $M_i$  of degree one. Let  $M_j$  be the neighbor of  $M_i$  in  $T$  and let  $\Delta_k$  be the triangle common to  $M_i$  and  $M_j$ . By Corollary 2.6,  $M = P_{\Delta_k}(M_i, M_j) \setminus \Delta_k$  where  $M'_j$  labels the vertex other than  $M_i$  in the graph that is obtained by contracting every edge of  $T$  other than  $M_i M_j$ . By Lemma 2.2,  $\text{si}(M'_j)$  is 3-connected. We may assume that the only 2-circuits of  $M_i$  meet  $\text{cl}_{M_i}(\Delta_k)$ .

Because the vertex  $M_i$  has degree one in  $T$ , in  $M_{[n]}^P$ , the intersection of the closures of  $E(M_i)$  and  $E(M_1) \cup \dots \cup E(M_{i-1}) \cup E(M_{i+1}) \cup \dots \cup E(M_n)$  is the closure of  $\Delta_k$ . Let  $Y_i = E(M_i) - \text{cl}_{M_i}(\Delta_k)$ . Then  $|Y_i| \geq 6$  so, as  $M_i$  is regular and cosimple,  $r^*(Y_i) \geq 3$ . Now  $2 = \lambda_{M_i}(\Delta_k) = \lambda_{M_i}(Y_i) = r(Y_i) + r^*(Y_i) - |Y_i|$ . Thus  $r(Y_i) < |Y_i|$  so  $Y_i$  contains a circuit  $C$ . By Lemma 2.11, there are elements  $a, b$ , and  $c$  of  $C$  and a minor  $N_i$  of  $M_i$  that has  $\{a, b, c\}$  as a circuit and  $\Delta_k \cup \{a, b, c\}$  as its ground set such that  $\kappa_{N_i}(\{a, b, c\}, \Delta_k) = 2$ . Thus  $2 = \lambda_{N_i}(\{a, b, c\}) = r(\{a, b, c\}) + r(\Delta_k) - r(N_i) \leq r(\Delta_k) \leq 2$ , so equality holds throughout and  $r(\Delta_k) = r(N_i) = 2$ . Therefore  $N_i$  is a double triangle that is a minor of  $M_i$ . Hence, by the Scum Theorem, since  $M_i$  is binary,  $N_i$  is a parallel minor of  $M_i$ . Then  $(N_i, \Delta_k, M'_j)$  is isomorphic to  $M'_j$  and the latter is a parallel minor of  $M$ . The lemma now follows using Corollary 2.9.  $\square$

The next lemma is from an unpublished paper [5] of Ding and Oporowski. The proof is given here for completeness.

**Lemma 2.13.** *Let  $G$  be a 3-connected simple graph containing distinct 3-element bonds  $S_1$  and  $S_2$ . Then one of the following occurs.*

- (i)  $S_1$  and  $S_2$  are both vertex bonds.
- (ii)  $G$  has a subgraph  $H$  that is a subdivision of  $K_4$  such that  $H$  has a degree-three vertex  $v$  so that  $S_1 \cup S_2$  is contained in the union of the minimal paths in  $H$  from  $v$  to the other degree-three vertices of  $H$ .

*Proof.* Let  $S_1 = \{e_1, f_1, g_1\}$  and  $S_2 = \{e_2, f_2, g_2\}$ . Either  $S_1 \cap S_2 = \emptyset$  or  $|S_1 \cap S_2| = 1$ . In each case, since  $G$  is 3-connected,  $S_2 - S_1$  is a bond of  $G \setminus S_1$ , and  $S_1 - S_2$  is



a bond of  $G \setminus S_2$ . Let  $A$  be the component of  $G \setminus S_1$  that does not contain  $S_2 - S_1$ , and let  $C$  be the component of  $G \setminus S_2$  that does not contain  $S_1 - S_2$ . Then  $A$  and  $C$  are vertex disjoint.

Suppose  $A$  contains no cycles. Then  $A$  is a tree and, since  $G$  is 3-connected, all the leaves of  $A$  must meet edges of  $S_1$ . Assume that  $A$  contains an edge. Then  $A$  has at least two vertices of degree one, so  $G$  has a vertex of degree at most two; a contradiction. Hence  $A$  contains no edges, and  $S_1$  is a vertex bond. Likewise, if  $C$  contains no cycles, then  $S_2$  is a vertex bond.

We may now assume that  $A$  or  $C$ , say  $A$ , contains a cycle  $D$ , otherwise (i) holds. Take a vertex  $v$  in  $V(C)$ . By Menger's Theorem,  $G$  contains three paths from  $v$  to  $V(D)$  that have no internal vertices in  $V(D)$  and that are disjoint except that all contain  $v$ . Each such path contains exactly one edge of  $S_1$  and exactly one edge of  $S_2$ . The union of these three paths with  $D$  is a subdivision of  $K_4$  satisfying (ii).  $\square$

### 3. THE PROOF OF THE MAIN THEOREM

The following theorem is well-known (see, for example, [4]).

**Theorem 3.1.** *There is an integer-valued function  $f_{3.1}$  such that, for each positive integer  $d$ , every tree with at least  $f_{3.1}(d)$  vertices has an induced subgraph isomorphic to  $K_{1,d}$  or a path with  $d$  vertices.*

The next two theorems [3, 8] will be crucial in the proof of Theorem 1.1.

**Theorem 3.2.** *There is an integer-valued function  $f_{3.2}$  such that, for each integer  $k$  exceeding three, every 3-connected graph with at least  $f_{3.2}(k)$  vertices has a parallel minor isomorphic to  $K'_{3,k}$ ,  $\mathcal{W}_k$ ,  $DF_k$ , or  $K_k$ .*

**Theorem 3.3.** *There is an integer-valued function  $f_{3.3}$  such that, for each integer  $k$  exceeding two, every 3-connected graph with at least  $f_{3.3}(k)$  vertices has a subgraph that is isomorphic to a subdivision of  $V_k$ ,  $\mathcal{W}_k$ , or  $K_{3,k}$ .*

We will also use the following result of Oxley [11].

**Lemma 3.4.** *Let  $N$  be a 3-connected binary matroid having rank and corank at least three and suppose  $\{x, y, z\} \subseteq E(N)$ . Then  $N$  has a minor isomorphic to  $M(K_4)$  whose ground set contains  $\{x, y, z\}$ .*

The proof of our main result will occupy the rest of the paper.

*Proof of Theorem 1.1.* Let  $k$  be an integer exceeding three. Let  $f_{3.2}$  and  $f_{3.3}$  be the functions described in Theorems 3.2 and 3.3, respectively. Let  $s = f_{3.2}(k) + f_{3.3}(k) + 11$ . Let  $m = \lceil (k+2)\frac{1}{3}f_{3.3}(k) \rceil + 2$  and  $l = \max\{\binom{s}{3}(k+2), 2(2m+1)\}$ . Let  $t = (s-1)f_{3.1}(l)$ . Set  $f_{1.1}(k) = t$ . Let  $M$  be a 3-connected regular matroid with at least  $t$  elements. Then  $t \geq 11$ .

By Lemma 2.4,  $M$  has a good decomposition into matroids each of which is graphic or cographic and has a 3-connected simplification with at least nine elements. By Lemma 2.5, we retain a good decomposition satisfying these additional conditions if we contract, one by one, the edges between vertices labelling graphic matroids. Let the resulting good decomposition be  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ , and let  $T$  be the tree associated with this decomposition.

By Lemma 2.2, for each  $i$ , the matroid  $\text{si}(M_i)$  is 3-connected. Suppose that some such  $\text{si}(M_i)$  has at least  $s$  elements. By Lemma 2.12,  $\text{si}(M_i)$  is isomorphic to

a parallel minor  $N$  of  $M$ . If  $N$  is graphic, then, by Theorem 3.2,  $M$  has a parallel minor isomorphic to  $M(K'_{3,k})$ ,  $M(\mathcal{W}_k)$ ,  $M(DF_k)$ , or  $M(K_k)$ , and the theorem holds. If, instead,  $N$  is cographic, then, by Theorem 3.3,  $N^*$  has a series minor isomorphic to  $M(K_{3,k})$ ,  $M(V_k)$ , or  $M(\mathcal{W}_k)$ . Thus  $N$ , and hence  $M$ , has a parallel minor isomorphic to  $M^*(K_{3,k})$ ,  $M(DF_k)$ , or  $M(\mathcal{W}_k)$ , and again the theorem holds.

We may now assume that no vertex of  $T$  labels a matroid whose simplification has at least  $s$  elements. As  $|E(M)| \leq \sum_{i=1}^n |E(\text{si}(M_i))|$ , we have  $n > \frac{t}{s-1} = f_{3.1}(l)$ .

Suppose next that  $T$  contains a vertex  $M_i$  of degree at least  $l$ . We will show that  $M$  has a parallel minor isomorphic to  $M(K'_{3,k})$ . Since  $\text{si}(M_i)$  has fewer than  $s$  elements,  $\text{si}(M_i)$  has fewer than  $\binom{s}{3}$  triangles. As  $M_i$  has degree at least  $l$ , for some triangle  $S$  in  $\text{si}(M_i)$ , at least  $l/\binom{s}{3}$  of the matroids labelling vertices adjacent with the vertex  $M_i$  have a triangle whose union with  $S$  has rank 2 in  $M_{[n]}^P$ . We may assume that  $\text{si}(M_i)$  is labelled so that  $S = \Delta_h$  for some  $h$ . Clearly  $j > i$  for all but at most one neighbor  $M_j$  of  $M_i$  in  $T$ ; and  $\Delta_h$  is contained in the ground set of a unique neighbor of  $M_i$  in  $T$ . By definition,  $l/\binom{s}{3} \geq k+2$ . Take a subgraph  $T'$  of  $T$  induced by  $M_i$  and  $k$  of its higher-indexed neighbors,  $M_{i_1}, M_{i_2}, \dots, M_{i_k}$ , that contain triangles that are parallel to and so disjoint from  $\Delta_h$ . By Lemma 2.12, the simplification of the matroid  $M'$  associated with  $T'$  is isomorphic to a parallel minor  $Q$  of  $M$ . We relabel  $M_i, M_{i_j}, \Delta_{i_j}$ , and  $\Delta_h$  as  $M_0, M_j, \Delta_j$ , and  $\Delta_0$ . Then  $V(T') = \{M_0, M_1, \dots, M_k\}$  and  $M_0$  has  $\Delta_0, \Delta_1, \dots, \Delta_k$  as parallel triangles.

By Lemma 3.4, for all  $j$  in  $\{1, 2, \dots, k\}$ , the matroid  $M_j$  has an  $M(K_4)$ -minor  $M'_j$  having  $\Delta_j$  as a triangle. Because  $M_j$  has no Fano-minor, by the Scum Theorem,  $M'_j$  is a parallel minor of  $M_j$ . Take two distinct elements  $d_1$  and  $d_2$  in  $\Delta_0$  and extend  $\{d_1, d_2\}$  to a basis  $B$  of  $M_0$ . Let  $M'_0 = M_0/(\text{cl}_{M_0}(B - \{d_1, d_2\}))$ . Then  $\Delta_0 \subseteq E(M'_0)$ . Therefore, if  $i \geq 1$ , for every parallel deletion that is done in  $M_i$  to produce  $M'_i$ , there is a corresponding parallel deletion in  $Q$ . It follows by Corollary 2.9 that  $(M'_0, \Delta_1, M'_1, \Delta_2, \dots, \Delta_k, M'_k)$  is a parallel minor  $N$  of  $Q$ . Moreover,  $\text{si}(N)$  can be obtained by identifying a triangle in each of  $k$  matroids isomorphic to  $M(K_4)$ , so  $\text{si}(N) \cong M(K'_{3,k})$ . Hence  $M$  has a parallel minor isomorphic to  $M(K'_{3,k})$ .

We may now suppose that every vertex of  $T$  has degree at most  $l-1$ . By Theorem 3.1,  $T$  contains a path  $M_{i_1}M_{i_2}\dots M_{i_l}$  with  $l$  vertices. By construction, there is some index  $j$  such that  $i_1 > i_2 > \dots > i_j$  and  $i_j < i_{j+1} < \dots < i_l$ . Now  $\frac{l}{2} \geq 2m+1$ , so  $T$  contains a path  $T'$  with at least  $2m+1$  vertices such that the indices on the vertices are increasing. As no two adjacent vertices of this path label graphic matroids, by removing vertices from the ends of the path, we can get a path  $T'$  with  $2m$  vertices such that the first vertex of  $T'$  labels a non-graphic matroid. We relabel the vertices of  $T'$  so that  $T' = M_1M_2\dots M_{2m}$  and relabel each edge  $M_iM_{i+1}$  as  $\Delta_{i+1}$ . Let  $M' = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$  and  $\bar{M} = \text{si}(M')$ . By Lemma 2.12,  $\bar{M}$  is 3-connected and is isomorphic to a parallel minor of  $M$ . We can modify the decomposition we have for  $M'$  to obtain a good decomposition for  $\bar{M}$  by *deleting superfluous parallel elements*. Specifically, we replace each  $M_i$  by its restriction to the set  $(E(\bar{M}) \cap E(M_i)) \cup (\Delta_i \cup \Delta_{i+1})$ . Note that  $\Delta_1$  and  $\Delta_{2m+1}$  do not exist so we take these sets to be empty. This process gives us a good decomposition of  $\bar{M}$  for which we shall retain the labelling  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$ .

Next we prove two lemmas to deal with this kind of situation. Let  $N$  be a 3-connected regular matroid having  $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$  as a good decomposition such that the associated tree is a path  $N_1N_2\dots N_d$ ; each  $\text{si}(N_i)$  has at least nine elements and is graphic or cographic, with no two consecutive matroids

being graphic; and  $N_1$  is not graphic. We call such a good decomposition a *fine decomposition* of  $N$ . Note that, in a fine decomposition, every non-trivial parallel class of each  $N_i$  meets  $\Delta_i$  or  $\Delta_{i+1}$ . When  $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$  is a fine decomposition of  $N$ , if  $1 < i < d$ , we denote  $(N_1, \Delta_2, N_2, \dots, \Delta_{i-1}, N_{i-1})$  and  $(N_{i+1}, \Delta_{i+2}, N_{i+2}, \dots, \Delta_d, N_d)$  by  $\hat{N}_{i-1}$  and  $\check{N}_{i+1}$ . As a graph, the *triangular prism* consists of the vertices and edges of the eponymous polyhedron. This graph is the planar dual of the graph  $K_5 \setminus e$ .

**Lemma 3.5.** *Let  $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$  be a fine decomposition of a 3-connected regular matroid. For all  $i$  with  $1 < i < d$ , one of the following occurs:*

- (i)  $N_i$  is graphic and  $E(N_i)$  has a subset  $Y_i$  such that  $N_i/Y_i$  is a multi-triangle with respect to  $\Delta_i$  and  $\Delta_{i+1}$ ;
- (ii)  $N_i$  is the cycle matroid of a triangular prism, and  $N_{i-1}$  and  $\hat{N}_{i-1}$  have no triads meeting  $\Delta_i$ , while  $N_{i+1}$  and  $\check{N}_{i+1}$  have no triads meeting  $\Delta_{i+1}$ ;
- (iii)  $N_i$  is not graphic and  $N_i = M^*(G_i)$  for some graph  $G_i$  where  $\Delta_i$  and  $\Delta_{i+1}$  are vertex bonds of  $G_i$ ; or
- (iv)  $N_i$  is cographic but not graphic and  $E(N_i)$  has a subset  $Y_i$  such that  $N_i/Y_i$  is a multi- $K_4$  with respect to  $\Delta_i$  and  $\Delta_{i+1}$ .

*Proof.* If  $\Delta_i$  and  $\Delta_{i+1}$  are parallel in  $N_i$ , then Lemma 3.4 implies that  $E(N_i)$  has a subset  $Y_i$  such that  $N_i/Y_i$  is a multi- $K_4$  with respect to  $\Delta_i$  and  $\Delta_{i+1}$ . Thus (i) or (iv) holds depending on whether  $N_i$  is graphic or not. We may now assume that  $\Delta_i$  and  $\Delta_{i+1}$  are not parallel in  $N_i$ .

Suppose that  $N_i$  is graphic and let  $G_i$  be the 3-connected graph such that  $M(G_i) = N_i$ . By Menger's Theorem,  $G_i$  has three vertex-disjoint paths,  $P_1$ ,  $P_2$ , and  $P_3$ , from  $V(\Delta_i)$  to  $V(\Delta_{i+1})$ .

We assume first that  $G_i \setminus (E(\Delta_i) \cup E(\Delta_{i+1}))$  has a component  $C$  that contains at least two of the chosen paths. Then  $G_i \setminus (E(\Delta_i) \cup E(\Delta_{i+1}))$  contains a path  $R$  with ends in two different chosen paths and no other vertices in any chosen path. Evidently,  $G_i$  has a multi-triangle as a minor whose restriction to each of  $E(\Delta_i)$  and  $E(\Delta_{i+1})$  is a triangle. By Lemma 2.10,  $E(N_i)$  contains a set  $Y_i$  such that  $N_i/Y_i$  is a multi-triangle with respect to  $\Delta_i$  and  $\Delta_{i+1}$ , and (i) holds.

We may now assume that  $G_i \setminus (E(\Delta_i) \cup E(\Delta_{i+1}))$  has three disjoint components each containing one chosen path. Since  $G_i$  is 3-connected, no  $P_i$  has an internal vertex since its ends do not form a vertex cut. Thus  $V(G_i) = V(P_1) \cup V(P_2) \cup V(P_3)$ . If  $G_i$  has a non-trivial parallel class, then this class meets  $\Delta_i$  or  $\Delta_{i+1}$ , and (i) holds with  $Y_i = P_1 \cup P_2 \cup P_3$ . Thus we may assume that  $G_i$  is simple. Then  $|E(G_i)| = |E(\text{si}(N_i))| \geq 9$ , and it follows that  $G_i$  is a triangular prism.

Let  $\{x_1, x_2, x_3\} = E(N_i) - (\Delta_i \cup \Delta_{i+1})$ . By Lemma 2.5,  $N_{i-1} \oplus_{\Delta_i} N_i$  and  $\hat{N}_{i-1} \oplus_{\Delta_i} N_i$  have no series pairs. Since every pair of elements in a triangle of  $N_i$  is contained in a triad, it follows that  $N_{i-1}$  and  $\hat{N}_{i-1}$  have no triads meeting  $\Delta_i$ . Similarly,  $N_{i+1}$  and  $\check{N}_{i+1}$  have no triads meeting  $\Delta_{i+1}$ , and (ii) holds.

We may now assume that  $N_i$  is not graphic. Then  $N_i$  is cographic and so too is  $\text{si}(N_i)$ . Hence  $\text{si}(N_i) = M^*(H_i)$  for some 3-connected simple graph  $H_i$ . Now  $\Delta_i$  and  $\Delta_{i+1}$  are not parallel in  $N_i$ . Thus  $r(\Delta_i \cup \Delta_{i+1})$  is 3 or 4. Hence we can choose  $H_i$  so that either both  $\Delta_i$  and  $\Delta_{i+1}$  label bonds of it, or so that  $\Delta_i$  and  $(\Delta_{i+1} - e_{i+1}) \cup e_i$  label bonds of it where  $\{e_i, e_{i+1}\}$  is a circuit of  $N_i$  with each  $e_j$  in  $\Delta_j$ . Consider the bonds  $\Delta_i$  and  $\Delta'_{i+1}$  of  $H_i$  where  $\Delta'_{i+1}$  is  $\Delta_{i+1}$  or  $(\Delta_{i+1} - e_{i+1}) \cup e_i$ . Suppose first that both  $\Delta_i$  and  $\Delta'_{i+1}$  are vertex bonds. Then, by replacing edges of  $H_i$  by

paths if necessary, we can get a graph  $G_i$  such that  $N_i = M^*(G_i)$  and  $\Delta_i$  and  $\Delta_{i+1}$  are both vertex bonds of  $G_i$ . Thus (iii) holds.

It remains to consider when  $\Delta_i$  or  $\Delta'_{i+1}$  is not a vertex bond of  $H_i$ . By Lemma 2.13,  $H_i$  has a subgraph  $J$  that is a subdivision of  $K_4$  such that  $J$  has a degree-three vertex  $v$  so that  $\Delta_i \cup \Delta'_{i+1}$  is contained in the union of the minimal paths in  $J$  from  $v$  to the other degree-three vertices of  $J$ . If  $\Delta'_{i+1} \neq \Delta_{i+1}$ , form  $J'$  from  $J$  by replacing  $e_i$  by a 2-edge path  $\{e_i, e_{i+1}\}$ ; otherwise let  $J'$  be  $J$ . Then  $M^*(J')$  is a minor of  $N_i$ . By Lemma 2.10,  $E(N_i)$  has a subset  $Y_i$  such that  $N_i/Y_i$  is a multi- $K_4$  with respect to  $\Delta_i$  and  $\Delta_{i+1}$ , and (iv) holds.  $\square$

We will say that  $N_i$  is *type (i)* if it meets the conditions of (i) in the preceding lemma. Likewise, we will say that  $N_i$  is *type (ii)*, *type (iii)*, or *type (iv)* if it meets the conditions of (ii), (iii), or (iv), respectively. The goal of the next lemma is to eliminate the graphic matroids in a fine decomposition. The strategy of the proof is as follows. Suppose that  $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$  is a fine decomposition of a 3-connected regular matroid and that  $N_i$  is graphic for some  $i$  other than 1 or  $d$ . By the preceding lemma,  $N_i$  is type (i) or type (ii). In the latter case, it is straightforward to eliminate  $N_i$  by replacing it by a double-triangle. But if  $N_i$  is type (i), then replacing  $N_i$  with the multi-triangle  $N_i/Y_i$  may create a series pair in the underlying matroid. In particular, this will occur if every pair of elements in  $\Delta_i$  is in a triad in both  $N_{i-1}$  and  $N_{i+1}$ , and  $N_i/Y_i$  contains exactly seven elements. When such a series pair arises, we will need to contract an element, say  $a$ , from this pair to preserve the vertical 3-connectivity of the matroid we are working with.

**Lemma 3.6.** *Let  $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$  be a fine decomposition of a 3-connected regular matroid  $N$ . For some  $i$  with  $2 \leq i \leq d-1$ , suppose  $N_1, N_2, \dots, N_{i-1}$  are not graphic. When  $N_i$  is type (i), let  $N'_i$  be a contraction of  $N_i$  that is a multi-triangle with respect to  $\Delta_i$  and  $\Delta_{i+1}$ . When  $N_i$  is type (ii), let  $N'_i$  be the double triangle obtained by contracting each element not in a triangle of  $N_i$ . Then either  $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_i, N'_i, \Delta_{i+1}, \dots, \Delta_d, N_d)$  is vertically 3-connected, or there is an element  $a$  of  $E(N_j) - (\text{cl}_{N_j}(\Delta_j) \cup \text{cl}_{N_j}(\Delta_{j+1}))$  for some  $j \leq i-1$  such that*

$$(N_1, \Delta_2, N_2, \dots, \Delta_j, N_j/a, \Delta_{j+1}, \dots, N_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \dots, \Delta_d, N_d)$$

*is vertically 3-connected, and  $N_j/a$  is not graphic.*

*Proof.* By Lemma 2.12, both  $\hat{N}_{i-1}$  and  $\tilde{N}_{i+1}$  are vertically 3-connected. We show first that:

**3.6.1.** *Either  $(\hat{N}_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \tilde{N}_{i+1})$  is vertically 3-connected, or that there is an element  $a$  of  $E(\hat{N}_{i-1}) - \Delta_i$  such that  $(\hat{N}_{i-1}/a, \Delta_i, N'_i, \Delta_{i+1}, \tilde{N}_{i+1})$  is vertically 3-connected.*

Now  $N'_i$  is either a double triangle with ground set  $\Delta_i \cup \Delta_{i+1}$ , or it is obtained from this matroid by adding some elements in parallel with elements of  $\Delta_{i+1}$ . In both cases, we let  $\hat{N}'_{i-1} = \hat{N}_{i-1} \oplus_{\Delta_i} N'_i$ . Then  $\hat{N}'_{i-1}$  may be obtained from  $\hat{N}_{i-1}$  by relabelling the elements of  $\Delta_i$  by the appropriate elements in  $\Delta_{i+1}$  and, when  $N'_i$  is type (i), adding some non-empty set of elements in parallel with those of  $\Delta_{i+1}$ . Let  $\bar{N}$  be the matroid  $P_{\Delta_{i+1}}(\hat{N}'_{i-1}, \tilde{N}_{i+1})$ . Then every non-trivial parallel class of  $\bar{N}$  meets  $\Delta_{i+1}$ . Let  $\Delta_{i+1} = \{x, y, z\}$ . We shall distinguish the following two cases:

- (a) no element of  $\Delta_{i+1}$  is in a non-trivial parallel class of  $\bar{N}$ ; and
- (b) some element, say  $z$ , of  $\Delta_{i+1}$  is in a non-trivial parallel class of  $\bar{N}$ .

Observe that if  $N_i$  is type (i), then (b) holds.

Assume first that (a) holds. Then  $N_i$  is type (ii), so  $\hat{N}'_{i-1}$  has no triad meeting  $\Delta_{i+1}$  because  $\hat{N}_{i-1}$  has no triad meeting  $\Delta_i$ . Moreover,  $\bar{N}$  is simple and, since it is the generalized parallel connection across a triangle of two 3-connected matroids, it too is 3-connected. Let  $C^*$  be a cocircuit of  $\bar{N}$  meeting  $\Delta_{i+1}$ . Then  $|C^* \cap \Delta_{i+1}| = 2$ . Furthermore, as  $C^* \cap E(\hat{N}'_{i-1})$  and  $C^* \cap E(\check{N}_{i+1})$  contain cocircuits of  $\hat{N}'_{i-1}$  and  $\check{N}_{i+1}$ , it follows that both  $|C^* \cap E(\hat{N}'_{i-1})|$  and  $|C^* \cap E(\check{N}_{i+1})|$  exceed 3, so  $|C^*| \geq 6$ . Thus, if  $Z \subseteq \Delta_{i+1}$ , then  $\bar{N} \setminus Z$  has no 2-cocircuits. Hence  $\bar{N} \setminus x$  is cosimple. Since  $\bar{N} \setminus x$  has a non-minimal 2-separation, it follows, by a well-known result of Bixby [1] (see also [9, Proposition 8.4.6]), that  $\bar{N} \setminus x$  is 3-connected. Similarly,  $\bar{N} \setminus x/y$  and  $\bar{N} \setminus x, y/z$  have non-minimal 2-separations, so  $\bar{N} \setminus x, y$  is 3-connected and then so is  $\bar{N} \setminus x, y, z$ . Hence, in case (a),  $(\hat{N}_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$  is vertically 3-connected.

Now assume that (b) holds. Then  $\bar{N}$  has  $\{e, z\}$  as a 2-circuit for some element  $e$ , so  $\text{si}(\bar{N} \setminus z)$  is 3-connected. We shall show next that  $\text{si}(\bar{N} \setminus z, y)$  is 3-connected. Suppose not. Then  $y$  is not in a 2-circuit of  $\bar{N}$ . Clearly  $\text{si}(\bar{N} \setminus z)/y$  has a non-minimal 2-separation. Thus, by Bixby's Lemma,  $\text{co}(\text{si}(\bar{N} \setminus z) \setminus y)$  is 3-connected, that is,  $\text{co}(\text{si}(\bar{N} \setminus z, y))$  is 3-connected. As  $\text{si}(\bar{N} \setminus z, y)$  is not 3-connected,  $\text{si}(\bar{N} \setminus z) \setminus y$  has a 2-cocircuit. Thus  $\text{si}(\bar{N} \setminus z)$  has a triad  $C^*$  containing  $y$ . As each of  $\text{si}(\hat{N}'_{i-1})$  and  $\text{si}(\check{N}_{i+1})$  is a restriction of  $\text{si}(\bar{N} \setminus z)$ , and either  $C^* \cap E(\text{si}(\hat{N}'_{i-1}))$  or  $C^* \cap E(\text{si}(\check{N}_{i+1}))$  has exactly two elements, we deduce that  $\text{si}(\hat{N}'_{i-1})$  or  $\text{si}(\check{N}_{i+1})$  has a cocircuit with at most two elements; a contradiction. Thus  $\text{si}(\bar{N} \setminus z, y)$  is indeed 3-connected.

Now  $\text{si}(\bar{N} \setminus z, y)/x$  has a non-minimal 2-separation. Thus, by Bixby's Lemma again,  $\text{co}(\text{si}(\bar{N} \setminus z, y) \setminus x)$  is 3-connected. As  $\text{si}(\bar{N} \setminus z, y, x) \cong \text{si}(P(\hat{N}'_{i-1}, \check{N}_{i+1}) \setminus \Delta_{i+1})$ , we assume that  $\text{si}(\bar{N} \setminus z, y, x)$  is not 3-connected, otherwise the lemma holds. Then

**3.6.2.**  $\bar{N}$  has no 2-circuit containing  $x$  or  $y$ .

As  $\text{si}(\bar{N} \setminus z, y)$  is 3-connected,  $\bar{N}$  has no 2-circuit containing  $x$ . By symmetry,  $\bar{N}$  has no 2-circuit containing  $y$ .

Now  $\text{si}(\bar{N} \setminus z, y)$  must have a triad containing  $x$ . Assume that  $\{a, b, x\}$  and  $\{c, d, x\}$  are such triads. Then their symmetric difference is a disjoint union of cocircuits of  $\text{si}(\bar{N} \setminus z, y)$ . Thus  $\{a, b\} \cap \{c, d\} = \emptyset$ . Now  $\text{si}(\bar{N} \setminus z) \setminus y$  is 3-connected. Therefore  $\{a, b, x, y\}$  and  $\{c, d, x, y\}$  contain cocircuits of  $\text{si}(\bar{N} \setminus z)$  containing  $\{a, b, x\}$  and  $\{c, d, x\}$ . By considering the intersections of these cocircuits with  $E(\text{si}(\hat{N}'_{i-1}))$  and  $E(\text{si}(\check{N}_{i+1}))$ , we see that each such cocircuit has four elements. Moreover, we may assume that the first contains  $\{a, c\}$  and the second contains  $\{b, d\}$ . Thus  $\{a, x, y\}$  and  $\{c, x, y\}$  are cocircuits of  $\text{si}(\hat{N}'_{i-1})$ . Hence  $\text{si}(\hat{N}'_{i-1})$  has a cocircuit contained in  $\{a, c\}$ ; a contradiction. We deduce that  $\text{si}(\bar{N} \setminus z, y)$  has exactly one triad, say  $\{a, b, x\}$ , containing  $x$ . Moreover, we may assume that  $\{a, x, y\}$  and  $\{b, x, y\}$  are triads of  $\text{si}(\hat{N}'_{i-1})$  and  $\text{si}(\check{N}_{i+1})$ , respectively.

**3.6.3.**  $\hat{N}'_{i-1}$  has no 2-circuit containing  $a$ .

If  $a$  is in a 2-circuit of  $\hat{N}'_{i-1}$ , then, by 3.6.2,  $a$  is parallel to  $z$ . Thus  $\{a, x, y\}$  is both a triangle and a triad of  $\text{si}(\hat{N}'_{i-1})$ ; a contradiction.

By 3.6.2 and 3.6.3,  $\{a, x, y\}$  is a triad of  $\hat{N}'_{i-1}$ . Since  $\{a, b\}$  is the only 2-cocircuit of  $\text{si}(\hat{N}'_{i-1} \oplus_{\Delta_{i+1}} \check{N}_{i+1})$ , the matroid  $\text{si}(\hat{N}'_{i-1} \oplus_{\Delta_{i+1}} \check{N}_{i+1})/a$  is 3-connected, so  $\text{si}((\hat{N}'_{i-1}/a) \oplus_{\Delta_{i+1}} \check{N}_{i+1})$  is 3-connected. This completes the proof of 3.6.1.

Observe that the construction of  $\hat{N}'_{i-1}$  means that we can label the triangle  $\Delta_i$  of  $N_{i-1}$  by  $\{x_i, y_i, z_i\}$  where  $\{x, x_i\}$ ,  $\{y, y_i\}$ , and  $\{z, z_i\}$  are circuits of  $N'_i$ . Clearly  $\hat{N}_{i-1}$  can be obtained from  $\hat{N}'_{i-1}$  by first relabelling the elements  $x, y$ , and  $z$  of the latter as  $x_i, y_i$ , and  $z_i$  and then deleting some elements that are parallel to  $x_i, y_i$ , or  $z_i$ . By 3.6.2 and 3.6.3, none of  $a, x$ , or  $y$  is in a 2-circuit of  $\hat{N}'_{i-1}$ . Hence none of  $a, x_i$ , or  $y_i$  is in a 2-circuit of  $\hat{N}_{i-1}$ . Moreover, as  $\{a, x, y\}$  is a triad of  $\hat{N}'_{i-1}$ , and  $\text{si}(\hat{N}_{i-1})$  is 3-connected,  $\{a, x_i, y_i\}$  is a triad of  $\hat{N}_{i-1}$ .

For all  $p$  with  $2 \leq p \leq i-1$ , let  $\Delta_p = \{x_p, y_p, z_p\}$ . Now  $\hat{N}_{i-1} = P_{\Delta_{i-1}}(\hat{N}_{i-2}, N_{i-1}) \setminus \Delta_{i-1}$ . Since  $\{a, x_i, y_i\}$  is a triad of  $\hat{N}_{i-1}$ , either  $\{a, x_i, y_i\}$  is a triad of  $N_{i-1}$ ; or  $\{a, x_i, y_i\} \cup Z$  is a cocircuit of  $P_{\Delta_{i-1}}(\hat{N}_{i-2}, N_{i-1})$  for some 2-element subset  $Z$  of  $\Delta_{i-1}$ . In the latter case, we may assume that  $Z = \{x_{i-1}, y_{i-1}\}$ . Then  $\{a, x_{i-1}, y_{i-1}\}$  contains, and so is, a cocircuit of  $\hat{N}_{i-2}$ . By repeating this argument, we deduce that, for some  $j$  with  $1 \leq j \leq i-1$ , after possibly relabelling the elements of  $\Delta_{j+1}$ , we have  $\{a, x_{j+1}, y_{j+1}\}$  as a triad of  $N_j$ .

Next we shall show that  $a$  is not in the closure of  $\Delta_j$  or  $\Delta_{j+1}$  in  $N_j$ . Note that, when  $j = 1$ , the set  $\Delta_j$  is empty. We have  $\{a, x_{j+1}, y_{j+1}\}$  as a triad of  $N_j$ . If  $N_j$  has a circuit containing  $a$  and contained in  $a \cup \Delta_j$ , then we contradict orthogonality. If  $N_j$  has a circuit containing  $a$  and contained in  $a \cup \Delta_{j+1}$ , then  $a$  is parallel to some element of  $\Delta_{j+1}$ . Thus  $\text{si}(N_j)$  has a 2-cocircuit, a contradiction since  $\text{si}(N_j)$  is 3-connected having at least nine elements.

We now show that  $N_j/a$  is not graphic. Assume it is and let  $G$  be a graph such that  $M(G) = N_j^*$ . Since  $\{a, x_{j+1}, y_{j+1}\}$  is a triad of  $N_j$ , it is a triangle of  $G$ . As  $\{x_{j+1}, y_{j+1}, z_{j+1}\}$  is a triad of  $G$ , the vertex  $v$  common to  $x_{j+1}$  and  $y_{j+1}$  has degree 3. Since  $N_j$  is not graphic,  $G$  has a minor isomorphic to  $K_5$  or  $K_{3,3}$ . Assume first that  $G$  has a  $K_{3,3}$ -minor. Since  $K_{3,3}$  is cubic,  $G$  contains a subgraph  $H$  that is a subdivision of  $K_{3,3}$ . As  $M^*(G \setminus a)$  is graphic,  $G \setminus a$  has no  $K_{3,3}$ -minor. Thus  $a$  is in  $H$ . Since  $H$  has no triangles, at most one of  $x_{j+1}$  and  $y_{j+1}$  is in  $H$ . Either  $v$  has degree two in  $H$ , or  $v$  is not in  $V(H)$ . In each case, by interchanging  $x_{j+1}$  and  $y_{j+1}$  if necessary, we get that  $G/x_{j+1}$  has a  $K_{3,3}$ -minor. But  $\{a, y_{j+1}\}$  is a cycle of  $G/x_{j+1}$ , so  $G/x_{j+1} \setminus a$  has a  $K_{3,3}$ -minor. Hence so does  $G \setminus a$ ; a contradiction.

We may now assume that  $G$  has a  $K_5$ -minor. Then  $G$  has five disjoint connected subgraphs  $G_1, G_2, G_3, G_4$ , and  $G_5$  that together contain all of the vertices in  $G$  and such that  $G$  has at least one edge between every pair of these subgraphs. Suppose first that  $a$  is in  $G_1$ . Then two of the three neighbors of  $v$  are in  $G_1$ , and we may assume that  $v$  is in  $G_1$ . Hence  $x_{j+1}$  and  $y_{j+1}$  are in  $G_1$ . Then  $G_1 \setminus a$  is connected, since  $\{a, x_{j+1}, y_{j+1}\}$  is a triangle, and  $G \setminus a$  contains a minor isomorphic to  $K_5$ ; a contradiction. Finally, assume that  $a$  is a  $G_1$ - $G_2$ -edge. If  $x_{j+1}$  or  $y_{j+1}$  is a  $G_1$ - $G_2$ -edge, then  $G \setminus a$  has a minor isomorphic to  $K_5$ . In the exceptional case, without loss of generality, we may assume that  $x_{j+1}$  is a  $G_2$ - $G_3$ -edge and  $y_{j+1}$  is a  $G_3$ - $G_1$ -edge. Then  $v$  is in  $G_3$ . Since  $v$  has degree three in  $G$ , it has degree one in the graph  $G_3$ . Hence  $G_3 - v$  is a connected graph and, for each  $i$  in  $\{4, 5\}$ , there is an edge of  $G$  with one end in  $G_3 - v$  and the other in  $G_i$ . We contract the subgraphs  $G_1, G_2, G_3 - v, G_4$ , and  $G_5$  to vertices  $v_1, v_2, v_3, v_4$ , and  $v_5$ , respectively, and delete the edge  $a$ . The resulting 6-vertex graph has  $K_{3,3}$  as a subgraph, where the vertex classes are  $\{v_1, v_2, v_3\}$  and  $\{v_4, v_5\}$ . Thus  $G \setminus a$  has a  $K_{3,3}$ -minor; a contradiction. We conclude that  $N_j/a$  is not graphic and the lemma is proved.  $\square$

Now returning to the proof of the main theorem, recall that, immediately before Lemma 3.5, we showed that we could obtain a fine decomposition  $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$  of a 3-connected matroid  $\bar{M}$  that is isomorphic to a parallel minor of  $M$ . Each  $M_i$  with  $1 < i < 2m$  satisfies one of (i)-(iv) of Lemma 3.5.

Suppose that some matroid in the set  $\{M_1, M_2, \dots, M_{2m-1}\}$  is graphic. In that case, let  $M_i$  be the lowest-indexed such matroid. Then  $i > 1$ , so  $M_i$  labels a type (i) or type (ii) matroid. By Lemma 3.6, we may contract elements from  $M_i$  to obtain a matroid  $M'_i$  that is a double triangle or a multi-triangle containing  $\Delta_i$  and  $\Delta_{i+1}$ , and we may contract at most one element of some  $M_j$  with  $j \leq i - 1$  to obtain a non-graphic matroid  $M''_j$  such that

$$(M_1, \Delta_2, M_2, \dots, \Delta_j, M''_j, \Delta_{j+1}, \dots, M_{i-1}, \Delta_i, M'_i, \Delta_{i+1}, \dots, \Delta_{2m}, M_{2m}) \quad (3)$$

is vertically 3-connected. Now let  $M''_{i-1}$  be  $M''_j$  when  $j = i - 1$  and let  $M''_{i-1} = M_{i-1}$  when  $j < i - 1$ . Then  $M''_{i-1}$  is cographic but not graphic. Hence  $(M''_{i-1}, \Delta_i, M'_i)$  is also cographic but not graphic. Thus, in (3), when we remove  $\Delta_i$  and  $M'_i$ , and replace  $M''_{i-1}$  by  $(M''_{i-1}, \Delta_i, M'_i)$ , we get a good decomposition of a vertically 3-connected matroid whose simplification is a parallel minor  $\bar{M}'$  of  $M$ . We can convert this good decomposition into a fine decomposition for  $\bar{M}'$  by deleting superfluous parallel elements. This means that we can repeat the above process. Thus, from our original fine decomposition, we eliminate graphic matroids one by one, beginning with the lowest-indexed such matroid. After each such move, we recover a fine decomposition of a 3-connected parallel minor of  $M$ . Since no two consecutive matroids in  $M_1, M_2, \dots, M_{2m}$  are graphic and  $M_1$  is non-graphic, we eventually obtain a fine decomposition for which the corresponding path has at least  $m + 1$  vertices, where each vertex except possibly the last labels a cographic matroid that is not graphic. If this path ends in a graphic matroid, that matroid has been unaltered in the above process and so its simplification has at least nine elements. Hence we can apply Lemma 2.12 and remove at least one vertex from the end of this path to obtain a path  $Q$  with  $m$  vertices each of which is labelled by a cographic matroid that is not graphic. Again by deleting superfluous parallel elements, we may assume that  $M_Q$ , which is a parallel minor of  $M$ , is simple. Relabel  $Q$  as  $N_1 N_2 \dots N_m$ . By Lemma 3.5, each  $N_i$  with  $1 < i < m$  is type (iii) or type (iv).

Recall that  $m = \lceil (k + 2) \frac{1}{3} f_{3.3}(k) \rceil + 2$ . Suppose that the interior vertices of  $Q$  contain a subpath  $Q'$  of at least  $\lfloor \frac{1}{3} f_{3.3}(k) \rfloor$  vertices each of which is labelled by a type (iii) matroid. Then it is not difficult to check that the associated matroid  $M_{Q'}$  is cographic. Because each  $\text{si}(N_i)$  has at least nine elements,  $\text{si}(M_{Q'})$  has at least  $f_{3.3}(k)$  elements and, by Lemma 2.12,  $M_{Q'}$  is vertically 3-connected. Since  $DF_k$  is the dual of  $V_k$ , we deduce by Theorem 3.3, that  $M$  has a parallel minor isomorphic to  $M(DF_k)$ ,  $M(\mathcal{W}_k)$ , or  $M^*(K_{3,k})$ . Hence, in this case, Theorem 1.1 holds.

We may now assume that every interior subpath of  $Q$  with at least  $\frac{1}{3} f_{3.3}(k)$  vertices contains a vertex labelled by a type (iv) matroid. Thus  $Q$  has at least  $\lfloor (m - 2) / (\frac{1}{3} f_{3.3}(k)) \rfloor$  vertices that are labelled by type (iv) matroids, so  $Q$  has at least  $k + 2$  such vertices.

We now modify each  $N_i$  with  $1 < i < m$  to produce  $N'_i$  as follows. If  $N_i$  is type (iv), we let  $N'_i = N_i / Y_i$ , where  $N_i / Y_i$  is a multi- $K_4$  with respect to  $\Delta_i$  and  $\Delta_{i+1}$ . Now suppose  $N_i$  is type (iii). Then  $N_i = M^*(G_i)$  for some graph  $G_i$  that has  $\Delta_i$  and  $\Delta_{i+1}$  as vertex bonds. By Menger's Theorem,  $G_i$  has a subgraph  $H_i$  that is a subdivision of  $K_{2,3}$  where  $\Delta_i$  and  $\Delta_{i+1}$  are vertex bonds of  $H_i$ . Thus  $N_i$  has, as a

minor, a double triangle with ground set  $\Delta_i \cup \Delta_{i+1}$ . Hence, by the Scum Theorem, for some subset  $Y_i$  of  $E(N_i)$ , the matroid  $N_i/Y_i$  is either this double triangle or a multi-triangle with respect to  $\Delta_i$  and  $\Delta_{i+1}$ . In this case, we let  $N'_i = N_i/Y_i$ .

Let  $R = N'_2 N'_3 \dots N'_{m-1}$ . Using Corollary 2.9 and Lemma 2.12, we can show that  $\text{si}(M_R)$  is a parallel minor of  $\text{si}(M_Q)$ . Furthermore,  $M_R$  may be obtained by identifying at least  $k + 2$  copies of  $M(K_4)$  across a triangle and either deleting elements from the common triangle or adding elements parallel with the elements in the common triangle. Evidently  $M_R$ , and hence  $M$ , has a parallel minor isomorphic to  $M(K'_{3,k})$ , and this completes the proof of the theorem.  $\square$

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