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Cocircuit coverings and packings for binary matroids

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If M is an arbitrary loopless matroid with ground set $E(M)$ and rank function ρ, then let $\alpha(M)$ be the minimum size of a set of cocircuits of M, whose union is $E(M)$, and let $\beta(M)$ be the maximum size of a set of pairwise disjoint cocircuits of M. The following conjecture is based on Gallai's theorem that the vertex-stability and vertex-covering numbers of a graph G sum to the number of vertices of G (1).

CONJECTURE 1. (Welsh (see (4)).) *If M has $k(M)$ components, then*

$$\alpha(M)+\beta(M) \leqslant \rho(M)+k(M).$$

In this paper we prove this conjecture when M is binary.

The matroid terminology used here will in general follow Welsh (5). However, if $T = \{x_1, x_2, \ldots, x_m\}$ we shall denote the restriction of M to $S\backslash T$ by $M\backslash T$ or

$$M\backslash x_1, x_2, \ldots, x_m$$

and the contraction of M to $S\backslash T$ by M/T or $M/x_1, x_2, \ldots, x_m$. The symmetric difference of sets A and B will be denoted by $A \triangle B$, and $\mathscr{C}(M)$ and $\mathscr{C}^*(M)$ will denote respectively the set of circuits of M and the set of cocircuits of M. A flat of M of rank one will be called a *point* of M.

The next three results, which are well known (see, for example, ((5), theorems 5.1.1, 2.1.6 and 10.1.3)), will be used frequently in the proof of the main theorem.

LEMMA 1. *Let C^* be a cocircuit of a matroid M and let x and y be distinct elements of C^*. Then there exists a circuit C of M such that $C \cap C^* = \{x, y\}$.*

LEMMA 2. *For a matroid M, if C is a circuit and C^* is a cocircuit, then*

(*a*) $|C \cap C^*| \neq 1$.

Moreover, if M is binary, then

(*b*) $|C \cap C^*|$ *is even.*

LEMMA 3. *If C_1 and C_2 are circuits of a binary matroid M, then $C_1 \triangle C_2$ is a disjoint union of circuits.*

If M is a matroid having no coloops, then let $\alpha^*(M) = \alpha(M^*)$ and $\beta^*(M) = \beta(M^*)$. We now prove that the dual of Conjecture 1, and hence Conjecture 1, holds for binary matroids.

THEOREM 1. *Let M be a binary matroid having no coloops. Then*

$$\alpha^*(M)+\beta^*(M) \leqslant |E(M)|-\rho(M)+k(M). \tag{1}$$

Proof. We argue by induction on $|E(M)|$. Clearly (1) holds for $|E(M)| = 1$. Assume it holds for $|E(M)| < n$ and suppose that $|E(M)| = n$. We may also assume that M is connected, for otherwise the result follows easily by the induction assumption.

Assume that M has a set of p pairwise parallel elements ($p \geqslant 2$). If $p = n$, then $M \cong U_{1,n}$ and (1) is easily verified.

LEMMA 4. *If $\{x_1, x_2, \ldots, x_p\}$ is a set of pairwise parallel elements of M and $3 \leqslant p < n$, then* (1) *holds by the induction assumption.*

Proof. Clearly $k(M\backslash x_1, x_2) = k(M)$,

$$\rho(M\backslash x_1, x_2) = \rho(M) \quad \text{and} \quad |E(M\backslash x_1, x_2)| = |E(M)|-2.$$

Moreover, since M is connected, $\alpha^*(M\backslash x_1, x_2)+1 \geqslant \alpha^*(M)$. The lemma will be proved if we can show that

$$\beta^*(M\backslash x_1, x_2) \geqslant \beta^*(M)-1. \tag{2}$$

To verify this, suppose that $\{C_1, C_2, \ldots, C_{\beta^*}\}$ is a maximal set of pairwise disjoint circuits of M. If either $\left|\left(\bigcup_{i=1}^{\beta^*} C_i\right) \cap \{x_1, x_2\}\right| \leqslant 1$ or $\{x_1, x_2\} = C_i$ for some i, then (2) is immediate. The only other possibility is that $x_1 \in C_i$ and $x_2 \in C_j$ for i and j distinct members of $\{1, 2, \ldots, \beta^*\}$. But $\{x_1, x_2\} \in \mathscr{C}(M)$ and hence $(C_j\backslash x_2) \cup x_1 \in \mathscr{C}(M)$. It follows that there is a circuit of M, and hence of $M\backslash x_1, x_2$, contained in $(C_i \cup ((C_j\backslash x_2) \cup x_1))\backslash x_1$. Therefore (2) holds and so Lemma 4 is proved. ∎

By the above we may assume that

(3) No point of M contains more than two elements of $E(M)$.

We now distinguish three cases.

(I) M has a cocircuit of size two.

(II) Every cocircuit of M contains at least three elements but M has a cocircuit containing only two points.

(III) Every cocircuit of M contains at least three points.

Case I. If $\{c, d\}$ is a cocircuit of M, then $k(M/c) = k(M)$ and $\rho^*(M/c) = \rho^*(M)$. Moreover, by Lemma 2(*a*), $\alpha^*(M/c) = \alpha^*(M)$ and $\beta^*(M/c) = \beta^*(M)$. The result follows by applying the induction assumption to M/c.

Case II. Let C_1^* be a cocircuit of M containing at least three elements but only two points. Then by (3), either

(i) $C_1^* = \{u, v, w\}$, where $\{u, v\}$ and $\{w\}$ are points of M; or

(ii) $C_1^* = \{u, v, w, x\}$, where $\{u, v\}$ and $\{w, x\}$ are points of M.

Case II (i). By Lemma 2(*a*), since M is connected, M/w is connected. Moreover, $\rho^*(M/w) = \rho^*(M)$ and $\beta^*(M/w) \geqslant \beta^*(M)$. We shall show that $\alpha^*(M/w) \geqslant \alpha^*(M)$ from which (1) will follow by induction. Suppose that $\{C_1, C_2, \ldots, C_t\}$ is a minimal collection of circuits of M/w whose union is $E(M/w)$. If for some i in $\{1, 2, \ldots, t\}$, $C_i \cup w$ is a circuit of M, then (1) holds. Therefore suppose that for all $1 \leqslant i \leqslant t$, C_i is a circuit of M. The next lemma completes the proof for case II (i).

LEMMA 5. *Let $\{c, d, e\}$ be a cocircuit of a binary connected matroid N, where $\{d, e\}$ and $\{c\}$ are points of N. If $\{C_1, C_2, \ldots, C_t\}$ is a minimal collection of circuits of N/c covering $E(N/c)$ and $\{C_1, C_2, \ldots, C_t\} \subseteq \mathscr{C}(N)$, then there is a set of t circuits of N covering $E(N)$ such that c is in exactly two of these circuits.*

Proof. It is straightforward to show that N has a cocircuit D^* such that D^* contains d but not c. Then by Lemma 2(*a*), $D^* \cap \{c, d, e\} = \{d, e\}$. Hence $D^* \in \mathscr{C}^*(N/c)$.

Since $\{C_1, C_2, \ldots, C_t\} \subseteq \mathscr{C}(N)$, we have by Lemma 2(*a*) that $\{d, e\} \in \{C_1, C_2, \ldots, C_t\}$, say $\{d, e\} = C_1$. Thus $C_i \cap \{d, e\} = \varnothing$ for $2 \leqslant i \leqslant t$. Since D^* properly contains $\{d, e\}$, for some j in $\{2, 3, \ldots, t\}$, say $j = 2$, $C_j \cap D^* \neq \varnothing$. Now, since N is connected, there is a circuit of N containing c and intersecting C_2. Among such circuits choose one, say F_1, such that $|F_1 \backslash C_2|$ is minimal. Since $F_1 \cap \{c, d, e\} \neq \varnothing$, we have by Lemma 2(*a*) that F_1 contains exactly one of d and e, say d. Consider $F_1 \triangle C_2$. This contains a circuit F_2 containing d and c. By the choice of F_1 it follows that $F_2 \backslash C_2 = F_1 \backslash C_2$. But $F_2 \backslash C_2 = F_2 \cap F_1$. Therefore by Lemma 3, $C_2 \backslash F_1 = F_2 \backslash F_1$ and so $F_2 = F_1 \triangle C_2$. But now

$$\{(F_1 \backslash d) \cup e, F_2, C_3, \ldots, C_t\}$$

is a set of circuits of N covering $E(N)$ and c is in both $(F_1 \backslash d) \cup e$ and F_2. ∎

Case II (ii). M/w has two components: a loop, $\{x\}$, and $M/w \backslash x$. Thus

$$k(M) = k(M/w) - 1.$$

Moreover, $\beta^*(M/w) \geqslant \beta^*(M)$. We show that

$$\alpha^*(M) \leqslant \alpha^*(M/w) - 1, \tag{4}$$

from which (1) follows using the induction assumption.

Let $\{C_1, C_2, \ldots, C_t\}$ be a minimal collection of circuits of M/w covering $E(M/w)$. Then $\{x\} \in \{C_1, C_2, \ldots, C_t\}$, say $\{x\} = C_t$. If for distinct elements i and j of $\{1, 2, \ldots, t-1\}$, $C_i \cup w$ and $C_j \cup w$ are circuits of M, then $C_i \cup w$ and $C_j \cup x$ are circuits of M and (4) holds. Next suppose that there is exactly one element i of $\{1, 2, \ldots, t-1\}$ such that $C_i \cup w$ is a circuit of M. Then, as

$$x \notin \bigcup_{i=1}^{t-1} C_i,$$

Lemma 2(*a*) implies that $\{u, v\} \in \{C_1, C_2, \ldots, C_{t-1}\}$, say $\{u, v\} = C_1$. Now $|(C_i \cup w) \cap C_1^*|$ is non-zero and hence exceeds one, and $x \notin (C_i \cup w) \cap C_1^*$. Therefore C_i contains one of u and v, say u. But then $(C_i \backslash u) \cup v$ is a circuit of M/w and so, by Lemma 2(*a*),

$$(C_i \backslash u) \cup v \cup w$$

is a circuit of M. Thus $(C_i\backslash u) \cup v \cup x$ is a circuit of M and

$$\{(C_i\backslash u) \cup v \cup x, C_2, \ldots, C_{i-1}, C_i \cup w, C_{i+1}, \ldots, C_{t-1}\}$$

is a collection of $t-1$ circuits of M whose union is $E(M)$. Hence (4) holds.

To complete case II(ii), suppose that for all i in $\{1, 2, \ldots, t-1\}$, C_i is a circuit of M and hence of $M\backslash x$. In this case, we have, on taking $N = M\backslash x$ in Lemma 5, that there is a covering $\mathscr{K}$ of $E(M\backslash x)$ with $t-1$ circuits of $M\backslash x$ such that w is in exactly two of these circuits, say K_1 and K_2. Now $\mathscr{C}(M\backslash x) \subseteq \mathscr{C}(M)$ and $(K_2\backslash w) \cup x \in \mathscr{C}(M)$. Thus replacing K_2 by $(K_2\backslash w) \cup x$ in $\mathscr{K}$ gives a covering of $E(M)$ with $t-1$ circuits of M; that is (4) holds.

Case III. If every cocircuit of M contains at least three points and $\bar{M}$ is the simple matroid associated with M, then every cocircuit of $\bar{M}$ contains at least three elements. Moreover, $\bar{M}$ is connected.

The next lemma is an analogue for binary matroids of a graph-theoretic result of Kaugars (see (2), p. 31).

LEMMA 6. (P. D. Seymour, private communication.) *Let N be a simple connected binary matroid having no cocircuits of size less than three. Then N has a connected hyperplane.*

Proof. Suppose that every circuit of N has size $\rho(N)+1$. Then, since N has no co-circuits of size two, N has at least two circuits. Let y and z be distinct elements of a circuit C_1 of N and suppose that $x \in E(N)\backslash C_1$. Then $(C_1\backslash y) \cup x$ and $(C_1\backslash z) \cup x$ are circuits of N and hence by Lemma 3, $((C_1\backslash y) \cup x) \triangle ((C_1\backslash z) \cup x) = \{y, z\}$ is a disjoint union of circuits of N, contradicting the simplicity of N.

We may therefore assume that N has a circuit of size less than $\rho(N)+1$ and hence that $E(N)$ has a non-empty subset A which is maximal with respect to being both connected and non-spanning. Clearly A is a flat of N.

As N is connected there is a circuit intersecting both A and its complement. Choose such a circuit C_1 so that $|C_1 \cap (E(N)\backslash A)| = m$ is minimal. We show that $m = 2$ from which it follows that A is a hyperplane and hence that A is the required connected hyperplane of N.

If $C_1 \supseteq E(N)\backslash A$ and c and d are distinct elements of $E(N)\backslash A$, then, by the choice of C_1, every circuit of N containing one of c and d also contains the other. Thus, by ((5), theorem 2.1.6), $\{c, d\}$ is a cocircuit of N, a contradiction. Therefore $E(N)\backslash(A \cup C_1)$ is non-empty so let x be an element of this set. As $C_1 \cup A$ is connected, we have, by the choice of A, that $C_1 \cup A$ is spanning. Thus $C_1 \cup A$ contains a base B of N. Let C_2 be the fundamental circuit of x with respect to B. Then either $C_2 \cap A = \varnothing$ or not. In the first case, by Lemma 3, $C_1 \triangle C_2$ contains a circuit C_4 containing x. Then, as $C_2\backslash x \subseteq C_1$, we have, by Lemma 2(*a*), that $C_4 \cap A \neq \varnothing$. Hence $|C_4 \cap C_1 \cap (E(N)\backslash A)| \geqslant m-1$ and so $|C_2| \leqslant 2$, a contradiction. Thus we may assume that $C_2 \cap A \neq \varnothing$. Then, since $|C_2 \cap (E(N)\backslash A)| \geqslant m$, by Lemma 3, C_2 contains exactly $m-1$ elements of $C_1 \cap (E(N)\backslash A)$. But $C_1 \triangle C_2$ contains a circuit C_3 containing x which, since M is simple, intersects A. Thus $m = 2$ as required. ∎

By this lemma we have, in case III, that $\bar{M}$ has a connected hyperplane. Therefore M has a connected hyperplane, H say. Let $E(M)\backslash H = C^*$ and let $\{C_1, C_2, \ldots, C_{\beta^*}\}$ be a maximal set of pairwise disjoint circuits of M.

If there is an element x of

$$C^*\backslash\left(\bigcup_{i=1}^{\beta^*} C_i\right),$$

then since $M\backslash C^*$ is connected and C^* contains at least three points of M we have, by Lemma 1, that $M\backslash x$ is connected. The result now follows by applying the induction assumption to $M\backslash x$.

Now suppose that $\bigcup_{i=1}^{\beta^*} C_i \supseteq C^*$. Then since, by Lemma 2 (b), $|C_i \cap C^*|$ is even for all i, $|C^*|$ is even and so $|C^*| \geqslant 4$. We choose two elements x and y from C^* as follows. If, for some i in $\{1, 2, \ldots, \beta^*\}$, $|C_i| = 2 = |C_i \cap C^*|$, then let $C_i = \{x, y\}$. Otherwise choose C_j such that $C_j \cap C^* \neq \varnothing$ and let x and y be any two elements of this intersection. In either case, it follows by Lemma 1 that $M\backslash x, y$ is connected. To see this, recall that C^* contains at least three points of M, $|C^*| \geqslant 4$, each element of M is parallel to at most one other element and $M\backslash C^*$ is connected. The result follows by applying the induction assumption to $M\backslash x, y$.

This completes the proof of case III and thereby finishes the proof of Theorem 1.∎

The above proof makes frequent use of the fact that M is binary and the method does not seem to generalize to arbitrary non-binary matroids. In particular, Lemma 6 fails for $M \cong U_{3,5}$. However the method may be adapted to prove Conjecture 1 for bicircular matroids, such matroids having been studied in detail by Matthews (3).

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