UNAVOIDABLE CONNECTED MATROIDS RETAINING A SPECIFIED MINOR

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Abstract. A sufficiently large connected matroid $M$ contains a big circuit or a big cocircuit. Pou-Lin Wu showed that we can ensure that $M$ has a big circuit or a big cocircuit containing any chosen element of $M$. In this paper, we prove that, for a fixed connected matroid $N$, if $M$ is a sufficiently large connected matroid having $N$ as a minor, then, up to duality, either $M$ has a big connected minor in which $N$ is a spanning restriction and the deletion of $E(N)$ is a large connected uniform matroid, or $M$ has, as a minor, the 2-sum of a big circuit and a connected single-element extension or coextension of $N$. In addition, we find a set of unavoidable minors for the class of graphs that have a cycle and a bond with a big intersection.

1. Introduction

In this paper, we consider a fixed connected matroid $N$ and look at a sufficiently large connected matroid $M$ that has $N$ as a minor. By a result of Lovász, Schrijver, and Seymour (in [5, Exercise 4.3.11]), $M$ has a big circuit or a big cocircuit. Hence $M$ itself is a connected matroid that has $N$ as a minor and also has a big uniform minor. But $M$ may have many other elements that are not in either of these minors. Our goal is to pack these two minors compactly into some connected minor of $M$. The first of our two main theorems proves that this can be done. Two elements, $e$ and $f$, of $M$ are clones if the map that interchanges $e$ and $f$ but fixes every other element of $E(M)$ is an automorphism of $M$.

Theorem 1.1. Let $N$ be a connected matroid with $n$ elements and let $k$ be a positive integer. There is a positive integer $f(n,k)$ such that, whenever $M$ is a connected matroid that has at least $f(n,k)$ elements and has $N$ as a minor, one of the following holds for some $(M_0, N_0) \in \{(M, N), (M^*, N^*)\}$.

(i) $M_0$ has a connected minor $M'_0$ with $r(M'_0) = r(N_0)$ and $|E(M'_0) - E(N_0)| \geq k$ where $M'_0$ has $N_0$ as a restriction, and all the elements of $E(M'_0) - E(N_0)$ are clones in $M'_0$; in particular, $M'_0\backslash E(N_0)$ is a connected uniform matroid having at least $k$ elements; or
(ii) for some connected single-element extension or coextension $N'_0$ of $N_0$ by an element $p$, the matroid $M_0$ has, as a minor, the 2-sum, with basepoint $p$, of $N'_0$ and a circuit that contains $p$ and has at least $k$ other elements.

One way to view the last theorem is that if we have a connected matroid $M$ with a fixed connected minor $N$ and a huge connected uniform minor $U$, then, up to duality, we can find a connected minor $M_0$ of $M$ such that either $E(M_0)$ has a partition $(X,Y)$ such that $M_0|X = N$ and $M_0|Y$ is a big connected uniform matroid; or $M_0$ is the 2-sum of a single-element extension or coextension of $N$ and a big circuit. Thus every element of $M_0$ is in $E(N)$ or the ground set of a big connected uniform minor of $M_0$. This has some similarity to intertwining two fixed matroids $M_1$ and $M_2$ where one seeks a minor-minimal matroid having minors isomorphic to $M_1$ and $M_2$. In our theorem, we seek to keep one matroid $N$ as a minor but we allow ourselves to change the other matroid, in this case, so that it remains a big connected uniform matroid.

We continue with the same theme in Section 7 where we suppose that a connected matroid $M$ has both a huge circuit and a huge cocircuit and we try to efficiently pack a big circuit and a big cocircuit into a minor of $M$. We show that either $M$ has a big set that is the intersection of a circuit and a cocircuit, or $M$ has, as a minor, the 2-sum of a big circuit and a big cocircuit. This leads us to consider what more we can say about $M$ in the former case. In Section 8, we prove our second main result, the following theorem for graphs, and we pose a conjecture about the corresponding result for binary matroids.

Theorem 1.2. Let $n$ be an integer exceeding two. There is a positive $g(n)$ such that if a graph $G$ has a set $Z$ of at least $g(n)$ edges such that $Z$ is the intersection of a cycle and a bond, then $G$ has, as a minor, one of the graphs $\Gamma_1(n), \Gamma_2(n), \Gamma_3(n), \text{ or } \Gamma_4(n)$ shown in Figure 1.

Observe that each of $\Gamma_2(n)$ and $\Gamma_3(n)$ can be formed from $n$ copies of $K_4$ by a sequence of 2-sums. But, although the cycle matroids of $\Gamma_2(2)$ and $\Gamma_3(2)$ are isomorphic, when $n \geq 3$, the cycle matroids of $\Gamma_2(n)$ and $\Gamma_3(n)$ are not isomorphic. It is also worth noting that $\Gamma_1(n)$ and $\Gamma_4(n)$ are dual graphs.

Sections 2–4 of the paper present some preliminary results that will be used in the proof of Theorem 1.1. In Section 5, we show that if $M$ is a sufficiently large connected matroid having a connected matroid $N$ as a minor, then either $M$ has a large connected minor that has $N$ as a spanning restriction, or, up to duality, $M$ has a minor that is the 2-sum of a big circuit and a single-element circuit or cocircuit of $N$. The proof of Theorem 1.1 is given in Section 6.
Figure 1. The graphs in Theorem 1.2.

2. Preliminaries

The matroid terminology used here will follow Oxley [6]. Let $M$ be a matroid. For subsets $X$ and $Y$ of $E(M)$, the local connectivity $\cap(X,Y)$ between $X$ and $Y$ is defined by $\cap(X,Y) = r(X) + r(Y) - r(X \cup Y)$.

The following elementary property of matroids (see, for example, [6, Exercise 1.1.5]) will be used repeatedly throughout the paper.

**Lemma 2.1.** In a matroid $M$, let $C$ be a circuit and $e$ be a non-loop element of $E(M) - C$. If $M$ has a circuit $D$ that contains $e$ and is contained in $C \cup e$, 

then $M$ has a circuit that contains $(C - D) \cup e$ and is contained in $C \cup e$. In particular, if $e$ is in the closure of $C$, then $M$ has distinct circuits $C_1$ and $C_2$ both of which contain $e$ such that $C_1 \cup C_2 = C \cup e$.

For a set $I$, let $\mathcal{A}$ be a family $(A_i : i \in I)$ of subsets of a set $S$. If there is a subset $K$ of $S$ such that $A_j \cap A_k = K$ for all distinct $j$ and $k$ in $I$, then $\mathcal{A}$ is a sunflower with kernel $K$. The following result, which is sometimes called the sunflower lemma, was proved by Erdős and Rado [2].

**Lemma 2.2.** Let $h$ and $k$ be positive integers and $I$ be a set. Let $\mathcal{A}$ be a family $(A_i : i \in I)$ of subsets of a set $S$. If $|I| \geq k!h^{k+1}$ and $|A_i| \leq k$ for all $i$ in $I$, then $\mathcal{A}$ contains a subset $J$ with more than $h$ members such that $(A_i : i \in J)$ is a sunflower.

Let $(X, Y)$ be a partition of the ground set of a matroid $M$. An $X$-arc is a minimal non-empty subset $A$ of $Y$ such that $M$ has a circuit $C$ with $C - X = A$ and $C \cap X \neq \emptyset$. This terminology is due to Seymour [8]. It should be noted that Geelen and Whittle [3] call an $X$-arc a $Y$-strand. Clearly all $X$-arcs are non-empty independent sets in $M$. Moreover, no $X$-arc is a proper subset of another $X$-arc.

Seymour [8, (3.1), (3.3)] proved the first two parts of the following result. The third part is a straightforward consequence of the second.

**Lemma 2.3.** Let $X$ be a set in a matroid $M$.

(i) If $C$ is a circuit of $M$ that meets $X$, then $C - X$ is expressible as a union of $X$-arcs.

(ii) If $x$ and $y$ are distinct elements of an $X$-arc $A$, then $\{x, y\}$ is a cocircuit of $M|(X \cup A)$.

(iii) If $A$ is an $X$-arc, then $M|(X \cup A)$ is the 2-sum, with basepoint $p$, of an extension $M_1$ of $M|X$ by $p$ and of a circuit with ground set $A \cup p$. Moreover, $r(M_1) = r(M|X)$.

**Proof.** Let $A$ be an $X$-arc. An immediate consequence of (ii) is that

$$r(X \cup A) = r(X) + |A| - 1.$$ 

As $A$ is an independent set in $M$, it follows that $\cap(X, A) = 1$. Using this, the proof of (iii) can be completed by following the proof of Theorem 8.3.1 of [6]. \qed

The next two lemmas establish certain properties of $X$-arcs that will be used in the proof of Theorem 1.1.

**Lemma 2.4.** Let $X$ be a set in a matroid $M$ and $C$ be a circuit of $M$ that meets $X$. Let $A_1$ and $A_2$ be disjoint $X$-arcs contained in $C - X$. If $a_1 \in A_1$, then $A_2$ contains an $X$-arc in $M/(A_1 - a_1)$.

**Proof.** We argue by induction on $|A_1|$ noting that the result is immediate if $|A_1| = 1$. Assume the result is true if $|A_1| = k$ and let $|A_1| = k + 1 \geq 2$. Take $b_1$ in $A_1 - a_1$. Evidently $C - b_1$ is a circuit of $M/b_1$, and $A_1 - b_1$ is an $X$-arc in $M/b_1$. It suffices to show that $A_2$ contains an $X$-arc in $M/b_1$. 


Lemma 2.5. Let $\text{X}$ be a matroid $M$ has distinct circuits $C$ is an independent in $X$. Let $\text{A}$ be the kernel of this sunflower. Then $\text{A}$ contains a set $\text{C}$ of disjoint $X$-arcs in $M/K$, each of which is contained in $C - K$. We complete the proof of the lemma by establishing the contradiction that $2.5.1. \ r_M(X) \geq n_1 + 1$. Choose an element $a_1$ of $A_1$ and consider $M/K/(A_1 - a_1)$. In this matroid, $\{a_1\}$ is an X-arc and $C - K - (A_1 - a_1)$ is a circuit. Thus $a_1$ is in the closure of $X$ in $M/K/(A_1 - a_1)$. Also, by Lemma 2.4, for each $i$ in $\{2, 3, \ldots, n_1 + 1\}$, the set $A_1 - (A_1 - a_1)$ contains an X-arc $A'_i$ in $M/K/(A_1 - a_1)$. Thus $A'_i$ contains an element $a_2$ such that $\{a_2\}$ is an X-arc of $M/K/(A_1 - a_1)/(A'_i - a_2)$. Moreover, for each $i$ in $\{3, 4, \ldots, n_1 + 1\}$, the set $A'_i - (A'_i - a_2)$ contains an X-arc in the last matroid. By repeating this process, we see that $C - X$ contains disjoint subsets $\{a_1, a_2, \ldots, a_{n_1 + 1}\}$ and $Z$ such that, in $M/Z$, the set $\{a_1, a_2, \ldots, a_{n_1 + 1}\}$ is contained in the closure of $X$. As $C - X$ is independent in $M$, it follows that $\{a_1, a_2, \ldots, a_{n_1 + 1}\}$ is independent in $M/Z$. Since $\{a_1, a_2, \ldots, a_{n_1 + 1}\} \subseteq c_{M/Z}(X)$, we deduce that $r_{M/Z}(X) \geq n_1 + 1$. Thus 2.5.1 holds and the lemma follows.

3. Ramsey’s theorem

The main theorem of this paper should be seen in the context of Ramsey theory, which, loosely speaking, asserts that, within a sufficiently large object, some structure must emerge. In this section, we first state Ramsey’s original theorem [7] and then give a straightforward consequence of it that we shall need. Let $X$ be a finite set and $k$ be a positive integer. We denote the set of all subsets of $X$ by $2^X$ and write $\binom{X}{k}$ for the set of all $k$-element subsets of $X$. The set of all non-empty subsets of $X$ with at most $k$ elements
will be denoted by \( (\mathcal{X}_k) \). A mapping from a set \( Y \) into a set \( Z \) is called a coloring. The members of \( Z \) are called colors. When \( |Z| = t \), such a mapping is also called a \( t \)-coloring.

**Theorem 3.1.** Let \( k \), \( t \), and \( n_1, n_2, \ldots, n_t \) be positive integers. There is a function \( f_{3,1}(k; n_1, n_2, \ldots, n_t) \) with the property that, for every set \( X \) with at least \( f_{3,1}(k; n_1, n_2, \ldots, n_t) \) elements and every coloring of \( (\mathcal{X}_k) \) by elements of \( \{1, 2, \ldots, t\} \), there is an element \( j \) of \( \{1, 2, \ldots, t\} \) and an \( n_j \)-element subset \( Y \) of \( X \) such that every member of \( (\mathcal{Y}_k) \) receives the color \( j \).

The following is a well-known corollary of this theorem.

**Corollary 3.2.** Let \( k \), \( t \), and \( n \) be positive integers. There is a function \( f_{3,2}(k, t, n) \) with the property that, for every set \( X \) with at least \( f_{3,2}(k, t, n) \) elements and every \( t \)-coloring of \( (\mathcal{X}_k) \), there is a \( n \)-element subset \( Y \) of \( X \) such that, for all \( i \) in \( \{1, 2, \ldots, k\} \), all members of the set \( (\mathcal{Y}_i) \) receive the same color.

4. **Unavoidable minors of big connected matroids**

As noted in the introduction, Lovász, Schrijver, and Seymour were the first to show that, in a connected matroid, by bounding the size of a largest circuit and a largest cocircuit, we are also bounding the size of the ground set of the matroid. The following result of Lemos and Oxley [4] gives a best-possible such bound.

**Theorem 4.1.** Let \( M \) be a connected matroid with at least two elements having largest circuit with \( c \) elements and largest cocircuit with \( c^* \) elements. Then

\[ |E(M)| \leq \frac{1}{2} cc^*. \]

In the proof of our main theorem, we shall use the following result of Pou-Lin Wu [11], which shows that if \( e \) is an element of a connected matroid \( M \), and \( M \) has a big circuit, then \( M \) has a big circuit containing \( e \).

**Theorem 4.2.** Let \( M \) be a connected matroid with at least two elements having largest circuit with \( c \) elements. Then every element of \( M \) is in a circuit of size at least \( \frac{c}{2} + 1 \).

5. **A big spanning restriction or a 2-sum**

Given a sufficiently large connected matroid \( M \) that has \( N \) as a connected minor, one possibility is that \( M \) has a large connected minor of which \( N \) is a spanning restriction. When this possibility does not arise, we show in this section that, up to duality, \( M \) has, as a minor, the 2-sum of a big circuit and a single-element extension or coextension of \( N \).

**Theorem 5.1.** Let \( N \) be a non-empty connected matroid with \( n \) elements and let \( k \) be a positive integer. There is a positive integer \( f_{5,1}(n, k) \) such
that, whenever $M$ is a connected matroid that has at least $f_{5,1}(n,k)$ elements and has an $N$-minor, one of the following holds for some $(M_0,N_0)$ in $\{(M,N),(M^*,N^*)\}$.

(i) $M_0$ has a connected minor $M'_0$ having at least $n + k$ elements such that $r(M'_0) = r(N_0)$, and $M'_0$ has $N_0$ as a restriction; or

(ii) for some connected single-element extension or coextension $N'_0$ of $N_0$ by an element $p$, the matroid $M_0$ has, as a minor, the 2-sum, with basepoint $p$, of $N'_0$ and a circuit that contains $p$ and has at least $k$ other elements.

The next lemma contains the core of the proof of this theorem.

**Lemma 5.2.** Let $N$ be a non-empty connected matroid with $n$ elements and let $k$ be a positive integer. There is a positive integer $f_{5,2}(n,k)$ such that, whenever $M$ is a connected matroid that has at least $f_{5,2}(n,k)$ elements and has an $N$-minor, one of the following holds.

(i) $M^*$ has a connected minor $M'$ having at least $n + k$ elements such that $r(M') = r(N^*)$, and $M'$ has $N^*$ as a restriction; or

(ii) for some connected single-element extension or coextension $N'$ of $N$ by an element $p$, the matroid $M$ has, as a minor, the 2-sum, with basepoint $p$, of $N'$ and a circuit that contains $p$ and has at least $k$ other elements.

**Proof.** Let $M$ be a connected matroid having a circuit with at least $2(3^k k! n^k + 3^k n + k)$ elements and having an $N$-minor. We shall show that (i) or (ii) holds for $M$. Let $X$ and $Y$ be subsets of $E(M)$ such that $N = M/X \setminus Y$ and $Y$ is maximal. Then $r^*(M \setminus Y) = r^*(N)$. Moreover, since $N$ is connected and $Y$ is maximal, $M \setminus Y$ is connected. If $|X| \geq k$, then (i) of the lemma holds. Thus we may assume that $|X| < k$.

As $M$ has a circuit with at least $2(3^k k! n^k + 3^k n + k)$ elements, it follows by Theorem 4.2 that $M$ has a circuit $C$ that meets $E(N)$ and has at least $3^k k! n^k + 3^k n + k$ elements. Let $M_1 = M \setminus (Y - C)/(X \cap C)$. Clearly $C - X$ is a circuit of $M_1$ meeting $E(N)$ and, since $|X| < k$, it follows that $|C - X| > 3^k k! n^k + 3^k n$.

In proving the next assertion, the key features of $M_1$ that we need are that it has a circuit $C_1$ that meets $E(N)$ and has more than $3^k k! n^k + 3^k n$ elements and that the connected matroid $N$ is $M_1 \setminus D/X_1$ where $D$ is maximal.

**5.2.1.** Either

(i) $M_1/X_1$ has a circuit that meets $E(N)$ and has at least $|C_1| 3^{-|X_1|}$ elements; or

(ii) for some connected single-element coextension $N_1$ of $N$ by an element $p$ of $X_1$, the matroid $M$ has, as a minor, the 2-sum, with basepoint $p$, of $N_1$ and a circuit that contains $p$ and has at least $2|C_1| 3^{-|X_1|}$ other elements.
We prove this by induction on \(|X_1|\). The result is immediate if \(|X_1| = 0\). Suppose \(|X_1| \geq 1\) and consider \(x_1\) in \(X_1\). Either \(C_1\) is a circuit of \(M_1/x_1\), or \(x_1 \in \text{cl}_{M_1}(C_1)\). In the first case, the result follows by induction as \(M_1/X_1 = (M_1/x_1)/(X_1 - x_1)\). Thus we may assume that \(x_1 \in \text{cl}_{M_1}(C_1)\). Then Lemma 2.1 implies that \(M_1\) has a circuit that contains \(x_1\), meets \(E(N)\), and is contained in \(C_1 \cup x_1\). Clearly either

(a) \(M_1/x_1\) has a circuit of size at least \(\frac{|C_1|}{3}\) that is contained in \(C_1\) and meets \(E(N)\); or

(b) every circuit of \(M_1/x_1\) that is contained in \(C_1\) and that meets \(E(N)\)

has size less than \(\frac{|C_1|}{3}\).

In the first case, let \(M'_1 = M_1/x_1\) and \(X'_1 = X_1 - x_1\). Then \(N = M'_1/D/X'_1\) and the result follows by applying the induction assumption to \(M'_1\). Thus we may assume that (a) does not hold. Then (b) holds and, by Lemma 2.1, \(M_1\) has a circuit \(D_1\) that contains \(x_1\), that avoids \(E(N)\), that is contained in \(C_1 \cup x_1\), and that has more than \(\frac{2|C_1|}{3} + 1\) elements.

As the next step towards proving 5.2.1, we now show the following.

5.2.2. There is a partition \((F_1, G_1)\) of \(C_1\) such that \(F_1 \cup x_1\) and \(G_1 \cup x_1\) are circuits of \(M_1\) where \(C_1 \cap E(N) \subseteq G_1\) and \(|G_1| < \frac{|C_1|}{3}\) elements. Moreover, \(F_1\) and \(G_1\) are contained in distinct components of \(M_1/x_1\).

Suppose that \(M_1/x_1\) has a component \(H\) containing \(C_1\). Since \(H\) contains \(E(N)\) and has \(D_1 - x_1\) as a circuit, it follows by Theorem 4.2 that \(H\) has a circuit that meets \(E(N)\) and has at least \(\frac{|C_1|}{3}\) elements. This contradicts the fact that (a) does not hold. We deduce that \(C_1\) is not contained in a component of \(M_1/x_1\), and 5.2.2 follows.

Suppose \(X_1 = \{x_1\}\). By the maximality of \(D\), the matroid \(M_1/D\) is a connected single-element coextension \(N_1\) of \(N\) by the element \(x_1\), and \(M_1/(D - F_1)\) is the parallel connection, with basepoint \(x_1\), of \(N_1\) and a circuit with ground set \(F_1 \cup x_1\). Thus \(M_1/(D - F_1) \cup x_1\) is the 2-sum, with basepoint \(x_1\), of \(N_1\) and the circuit \(F_1 \cup x_1\). Since \(|F_1| \geq \frac{2|C_1|}{3}\), the result follows in this case.

We may now assume that \(|X_1| \geq 2\). Let \(x_1\) and \(x'_1\) be distinct elements of \(X_1\). Let \((F_1, G_1)\) and \((F'_1, G'_1)\) be corresponding partitions of \(C_1\) given by 5.2.2, where \(|F_1|, |F'_1| \geq \frac{2|C_1|}{3}\).

Suppose \(F'_1 \subseteq F_1\). In \(M_1/x_1\), we have \(F_1\) and \(G_1\) contained in separate components. But \(x'_1 \in \text{cl}_{M_1}(F'_1) \subseteq \text{cl}_{M_1}(F_1)\). Hence we can move \(x'_1\) from \(X_1\) into \(D\) contradicting the maximality of the latter. Thus \(F'_1 \not\subseteq F_1\) and, by symmetry, \(F_1 \not\subseteq F'_1\). Hence \(F'_1 \cap G_1\) and \(F_1 \cap G'_1\) are both non-empty. Moreover, \(G_1 \cap G'_1 \not= \emptyset\) as \(G_1 \cap G'_1 \supseteq C_1 \cap E(N)\). Finally, as \(|G_1|, |G'_1| \leq \frac{|C_1|}{3}\), we see that \(|G_1 \cup G'_1| \leq \frac{2|C_1|}{3}\), so \(|F_1 \cap F'_1| \geq \frac{|G_1|}{3}\).

Let \(P_1\) be the dual of \(M_1/(C_1 \cup x_1 \cup x'_1)\). Then \(r(P_1) = 3\), and \(P_1\) has \(\{x_1, x'_1\}\) as a line, that is, a rank-2 flat. Moreover, \(P_1/x_1\) has \(\{x'_1\}, F'_1\), and \(G'_1\) as parallel classes, while \(P_1/x'_1\) has \(\{x_1\}, F_1\), and \(G_1\) as parallel classes.
In \( P_1 \), there are exactly three lines through \( x_1 \), namely, \( \{x_1, x_1'\}, F_1' \cup x_1, \) and \( G_1' \cup x_1 \). Likewise, the three lines through \( x_1' \) are \( \{x_1, x_1'\}, F_1 \cup x_1', \) and \( G_1 \cup x_1' \). Thus \( si(P_1) \) is isomorphic to \( M(K_4) \), and \( P_1 \) has \( F_1 \cap F_1', F_1 \cap G_1', G_1 \cap F_1', G_1 \cap G_1' \), \( \{x_1\} \), and \( \{x_1'\} \) as parallel classes. Hence \( P_1 \) has \( \{x_1, x_1'\} \cup (F_1 \cap F_1') \cup (G_1 \cap G_1') \) as a cocircuit that meets \( E(N) \). Thus \( M_1/x_1 \geq x_1' \) has \( (F_1 \cap F_1') \cup (G_1 \cap G_1') \) as a circuit that meets \( E(N) \) and that has at least \( |C_1| \) elements. It now follows, by induction, that 5.2.1 holds.

Now assume that 5.2.1(i) occurs and let \( C_2 \) be a circuit of \( M_1/X_1 \) that meets \( E(N) \) and has at least \( |C_1|3^{-|X_1|} \) elements. Since \( |X_1| \leq |X| < k \) and \( |C_1| > 3^k k! n^k + 3^k n \), we see that

\[
|C_2| > \frac{3^k k! n^k + 3^k n}{3^{|X_1|}} > k! n^k + n.
\]

Then \( |C_2 - E(N)| > k! n^k \). Since \( r(N) \leq |E(N)| = n \), it follows by Lemma 2.5 that \( C_2 - E(N) \) contains an \( E(N) \)-arc with at least \( k \) elements. Hence, by Lemma 2.3(iii), part (ii) of the current lemma holds.

Finally, assume that 5.2.1(ii) holds. Then, as \( \frac{2|C_1|}{3} \geq k \), we again get that (ii) of the lemma holds. \( \square \)

**Proof of Theorem 5.1.** Let \( M \) be a connected matroid having at least \( [f_{3.2}(n, k)]^2 \) elements and having an \( N \)-minor. Then, by Theorem 4.1, \( M \) has a circuit or a cocircuit with at least \( f_{3.2}(n, k) \) elements. By switching to the dual if necessary, we may assume the former. The theorem is now an immediate consequence of Lemma 5.2. \( \square \)

### 6. Large Matroids of Bounded Rank

For a positive integer \( k \) and a sufficiently large connected matroid \( M \) having some fixed connected matroid \( N \) as a minor, Theorem 5.1 tells us that one possibility is that \( M \) has, as a minor, a connected extension \( M' \) of \( N \), such that \( r(M') = r(N) \) and \( |E(M')| - |E(N)| \geq k \). In this section, we show that, when \( k \) is sufficiently large, \( M' \) has a connected restriction \( M'' \) such that all the elements of \( E(M'') - E(N) \) are clones. This result gives us the final piece we need to prove the first main theorem of the paper, and that proof appears at the end of the section.

**Theorem 6.1.** Let \( N \) be a matroid with \( n \) elements and let \( k \) be a positive integer. There is a positive integer \( f_{6.1}(n, k) \) such that, whenever \( M \) is a matroid with at least \( f_{6.1}(n, k) \) elements such that \( M \) has \( N \) as a spanning restriction, \( M \) has a restriction \( M' \) with at least \( n+k \) elements such that \( N \) is a spanning restriction of \( M' \) and all the elements of \( E(M') - E(N) \) are clones in \( M' \). In particular, \( M' \setminus E(N) \) is uniform.

**Proof.** Let \( M \) be a matroid with at least \( f_{3.2}(1+r(N), 2^{1+2^a}, k) \) elements and suppose that \( M \) has \( N \) as a spanning restriction. Let \( X = E(M) - E(N) \)
and let \( d = 1 + r(N) \). For every \( A \in \binom{X}{d} \), let
\[
c_1(A) = \begin{cases} 
1 & \text{if } A \text{ is a circuit of } M; \\
0 & \text{if } A \text{ is not a circuit of } M;
\end{cases}
\]
and let
\[
c_2(A) = \{ D \in 2^{E(N)} : D \cup A \text{ is a circuit of } M \}.
\]
Finally, let \( c_0 = c_1 \times c_2 \). Then \( c_0 \) is a \( 2^{1+2^n} \)-coloring of \( \binom{X}{d} \). By Corollary 3.2, \( X \) contains a \( k \)-element subset \( Y \) such that, for all \( i \) in \( \{1, 2, \ldots, 1 + r(N)\} \), all members of \( \binom{Y}{i} \) receive the same color. Let \( M' = M|(Y \cup E(N)) \). Then all the elements of \( Y \) are clones in \( M' \) and the theorem holds. \( \square \)

In the last theorem, there are potentially many different ways for all of the elements of \( E(M') - E(N) \) to be clones. For example, all these elements could be parallel, or they could all be added freely to \( N \). In addition, for any non-loop element \( p \) of \( N \), we could take the parallel connection, with basepoint \( p \), of \( N \) and a line with ground set \( [E(M') - E(N)] \cup p \) and then truncate this matroid. In general, if \( r(M' - E(N)) = t \), then \( M' \) can be obtained by extending \( N \) by some independent set \( Z \) of \( t \) elements such that the resulting matroid \( M'' \) has rank \( r(N) \) and has all the elements of \( Z \) as clones. We then freely add the elements of \( E(M') - E(M'') \) to the flat of \( M'' \) that is spanned by \( Z \). It is straightforward to check that, in the resulting matroid, all the elements of \( E(M') - E(N) \) are clones.

Applying the last theorem to connected matroids, we immediately obtain the following result.

**Corollary 6.2.** Suppose \( N \) is a connected matroid with \( n \) elements and non-zero rank, and let \( k \) be a positive integer. There is a positive integer \( f_{6.2}(n, k) \) such that, whenever \( M \) is a connected matroid with at least \( f_{6.2}(n, k) \) elements such that \( M \) has \( N \) as a spanning restriction, \( M \) has a connected restriction \( M' \) with at least \( n + k \) elements such that \( N \) is a spanning restriction of \( M' \) and all the elements of \( E(M') - E(N) \) are clones in \( M' \). In particular, \( M' \setminus E(N) \) is a connected uniform matroid.

We are now ready to prove the first of our two main theorems.

**Proof of Theorem 1.1.** Let \( M \) be a connected matroid having at least \( f_{5.1}(n, f_{6.2}(n, k)) \) elements and having an \( N \)-minor. It is immediate from Theorem 5.1 and Corollary 6.2 that
\[
f_{5.1}(n, k) \geq n + k \text{ and } f_{6.2}(n, k) \geq n + k.
\]
Suppose that \((M_0, N_0)\) is a member of \( \{(M, N), (M^*, N^*)\} \) for which (i) or (ii) of Theorem 5.1 holds. In the latter case, for some connected single-element extension or coextension \( N'_0 \) of \( N_0 \) by an element \( p \), the matroid \( M_0 \) has, as a minor, the 2-sum, with basepoint \( p \), of \( N'_0 \) and a circuit that contains \( p \) and has at least \( f_{6.2}(n, k) \) other elements. Since \( f_{6.2}(n, k) \geq n + k \), it follows that part (ii) of the theorem holds.
We may now assume that (i) of Theorem 5.1 holds. Then $M_0$ has a connected minor $M'_0$ having at least $n + f_{6,2}(n, k)$ elements such that $r(M'_0) = r(N_0)$, and $M'_0$ has $N_0$ as a restriction. Thus, by Corollary 6.2, $M'_0$ has a connected restriction $M''_0$ with at least $n + k$ elements such that $N_0$ is a spanning restriction of $M''_0$, and $M''_0 \setminus E(N_0)$ is a connected uniform matroid with at least $k$ elements, so part (i) of the theorem holds.

7. A big circuit and a big cocircuit

By Theorem 4.1, a sufficiently large connected matroid has a big circuit or a big cocircuit. In this section, we consider what can be said about a connected matroid that has both a big circuit and a big cocircuit. The next lemma is well known (see, for example, [6, Exercise 3.3.11]). It and the subsequent lemma will be used in the proof of the main result of this section.

**Lemma 7.1.** Suppose that, in a matroid $M$, a non-empty set $Z$ is the intersection of a circuit and a cocircuit. Then $M$ has a minor $M'_0$ in which $Z$ is a spanning circuit of both $M'_0$ and $M'_0^*$.  

**Lemma 7.2.** Let $M$ be a connected matroid and $D$ be a set of clones in $M$ with $r(D) = t \geq 1$ and $|D| = t + d$. Assume that $M$ has a circuit $C$ with $|C| \geq 2^t(s + 2) - 2$ for some positive integer $s$. Then $M$ has, as a minor, the parallel connection of a circuit of size at least $s + 1$ and a cocircuit of size at least $d + 1$.

**Proof.** Since $C$ is a circuit, it follows by Theorem 4.2 that $M$ has a circuit $C_1$ of size at least $2^{t-1}(s + 2) - 1$ that meets $D$. We argue by induction on $t$. If $t = 1$, then the result follows immediately. Now assume the result holds when $t < m$ and let $t = m \geq 2$. Take $e$ in $D \cap C_1$. Then $C_1 - e$ is a circuit of $M/e$ and $D - e$ is a set of clones in $M/e$. Evidently $|D - e| = (t - 1) + d$ and $|C_1 - e| \geq 2^{t-1}(s + 2) - 2$. We may assume that $D - e$ and $C_1 - e$ are in different components of $M/e$, otherwise the result follows by induction. Thus $D \cap C_1 = \{e\}$. Hence $M$ is the parallel connection, with basepoint $e$, of a connected matroid that contains $D$ and a connected matroid that contains $C_1$. It follows that the elements of $D - e$ are all parallel to $e$ in $M$. We conclude, by induction, that the required result holds. 

**Theorem 7.3.** Let $n$ be a positive integer. There is a positive integer $f_{7,3}(n)$ such that, whenever $M$ is a connected matroid having both a circuit and a cocircuit with at least $f_{7,3}(n)$ elements, $M$ has as a minor either

(i) the 2-sum of an $(n + 1)$-element circuit and an $(n + 1)$-element cocircuit; or
(ii) a matroid $M_0$ that contains a set with at least $n$ elements that is a spanning circuit in both $M_0$ and $M_0^*$.  

**Proof.** It is immediate from Corollary 6.2 that $$f_{6,2}(n, k) \geq n + k.$$
Let \( m = 2^n(n + 2) \) and let \( M \) be a connected matroid having a circuit \( C \) and a cocircuit each of which has at least \( 2[f_{5.2}(m, f_{6.2}(m, 2m)) + m + n] \) elements. Then, by Theorem 4.2, \( M \) has a cocircuit \( C^* \) that meets \( C \) and has at least \( f_{5.2}(m, f_{6.2}(m, 2m)) + m + n \) elements.

We may assume that \(|C \cap C^*| < n\) otherwise it follows, by Lemma 7.1, that (ii) holds. Let \( J \) be a subset of \( C^* - C \) such that \(|C^* - J| = m \). Let \( C_1^* \) be \( C^* - J \). Then \( M \setminus J \) has a component \( M_1 \) containing both \( C_1^* \) and \( C \). Let \( N = M_1.C_1^* \). Then \( N \) is an \( m \)-element cocircuit. Since \( M_1 \) has \( C \) as a circuit with at least \( f_{5.2}(m, f_{6.2}(m, 2m)) \) elements, it follows by Lemma 5.2 that either

(i) \( M^*_1 \) has a connected minor \( M' \) with at least \( m + f_{6.2}(m, 2m) \) elements such that \( r(M') = r(N^*) \) and \( M' \) has \( N^* \) as a restriction; or

(ii) for some connected single-element extension or coextension \( N_1 \) of \( N \) by an element \( p \), the matroid \( M_1 \) has, as a minor, the 2-sum, with basepoint \( p \), of \( N_1 \) and a circuit that contains \( p \) and has at least \( f_{6.2}(m, 2m) \) other elements.

Suppose that (ii) holds. If \( N_1 \) is an extension of \( N \), then \( N_1 \) is also a cocircuit and (i) of the theorem holds. If \( N_1 \) is a coextension of \( N \), then, by Theorem 4.2, \( p \) is in a cocircuit of \( N_1 \) with at least \( \frac{1}{2} f_{6.2}(m, 2m) \) other elements and again (i) of the theorem holds.

We may now assume that (i) holds. Then, since \(|E(N)| = m\), it follows, by Corollary 6.2, that \( M' \) has a connected restriction \( M'' \) with at least \( m + 2m \) elements such that \( N^* \) is a spanning restriction of \( M'' \), and the set \( D \) of elements of \( E(M'') - E(N^*) \) is a set of clones in \( M'' \). Clearly \(|D| \geq 2m \) and \( M'' \setminus D \) is uniform. As \( M'' \) is spanned by \( E(N^*) \), it follows that \( r(D) \leq r(N^*) \leq m \). If \( r(D) \geq n \), then, since \(|D| \geq 2m \) and \( n < m \), we see that \( M'' \) has a \( U_{n,2n} \)-minor, so (ii) of the theorem holds. We may now assume that \( r(D) = t < n \). Clearly \(|D| \geq t + m \), and \( M'' \) has \( E(N^*) \) as an \( m \)-element circuit. But \( m = 2^n(n + 2) \). Thus, by Lemma 7.2, \( M'' \) has, as a minor, the parallel connection of a circuit of size at least \( n + 1 \) and a cocircuit of size at least \( m + 1 \). Since \( m \geq n \), we conclude that (i) of the theorem holds in this case. Thus the proof is complete.

8. Graphs with a cycle and bond having a big intersection

By Theorem 7.3, one of the possibilities for a connected matroid \( M \) that has both a big circuit and a big cocircuit is that \( M \) has a minor that contains a big set that is the intersection of a circuit and a cocircuit. We had hoped to be able to identify a family of unavoidable minors for the class of matroids having a big such intersection. But we were unable to solve this problem even when we restrict to the binary case. In the latter case, we have a potential list of unavoidable minors, but we are unsure that this list is complete. The list is given at the end of the section. The main result of this section, Theorem 1.2, solves the problem when we restrict to the class of graphic matroids.
Let $G$ be a graph with vertex set $V$. For any subset $X$ of $V$, let $G[X] = G - (V - X)$, and let $\delta_G(X)$ denote the set of edges of $G$ that have one end in each of $X$ and $V - X$. When $X \notin \{V, \emptyset\}$ and both $G[X]$ and $G[V - X]$ are connected, $\delta_G(X)$ is a bond in $G$ or, equivalently, it is a cocircuit in $M(G)$.

Our proof of Theorem 1.2 will require some preliminary results.

**Lemma 8.1.** Let $d$ and $l$ be non-negative integers. There is an integer $f_{8.1}(d, l)$ such that if $G$ is a connected graph with at least $f_{8.1}(d, l)$ vertices and $v$ is a vertex of $G$, then either $G$ has a vertex of degree exceeding $d$, or $G$ has a path that has $v$ as an end and has length exceeding $l$.

**Proof.** Let $n = 1 + d + d(d - 1) + d(d - 1)^2 + \cdots + d(d - 1)^{l-1}$. Suppose $G$ is a connected graph with $|V(G)| \geq \max\{3, 1 + n\}$, and let $v$ be a vertex of $G$. Since $|V(G)| \geq 3$, $G$ has a path of length two. It follows that the lemma holds if $d \leq 1$ or $l \leq 1$. Hence we may assume that $d > 1$ and $l > 1$. We may also assume that every vertex of $G$ has degree at most $d$. It follows that $G$ has at most $n$ vertices that are distance at most $l$ away from $v$. Since $|V(G)| \geq 1 + n$, we deduce that $G$ has a path that has $v$ as an end and has length exceeding $l$. We conclude that the lemma holds if we take $f_{8.1}(d, l) = \max\{3, 1 + n\}$. \hfill $\square$

For a positive integer $m$, we shall denote by $mK_2$ the disjoint union of $m$ copies of $K_2$, that is, $mK_2$ is a matching with $m$ edges.

**Lemma 8.2.** Let $m, s$, and $t$ be positive integers. There exists an integer $f_{8.2}(m, s, t)$ such that every simple graph with at least $f_{8.2}(m, s, t)$ edges has one of $mK_2$, $K_{1,s}$, or $K_t$ as an induced subgraph.

**Proof.** Let $G$ be a simple graph with vertex set $\{1, 2, \ldots, n\}$ having at least $f_{3.1}(2; s, t, t, 2s, m)$ edges. For any two distinct edges $uv$ and $xy$ of $G$, where $u < v$ and $x < y$, we let $Z = \{u, v, x, y\}$ and define

$$c(uv, xy) = \begin{cases} 1 & \text{if } |Z| = 3 \text{ and } |E(G[Z])| = 2; \\ 2 & \text{if } |Z| = 3 \text{ and } |E(G[Z])| = 3; \\ 3 & \text{if } |Z| = 4 \text{ and } vy \in E(G); \\ 4 & \text{if } |Z| = 4 \text{ and } vy \notin E, \text{ but } ux \in E; \\ 5 & \text{if } |Z| = 4 \text{ and } \{vy, ux\} \cap E = \emptyset, \text{ and } |E(G[Z])| \geq 3; \\ 6 & \text{if } |Z| = 4 \text{ and } \{vy, ux\} \cap E = \emptyset, \text{ and } |E(G[Z])| = 2. \end{cases}$$

The six situations arising above are illustrated in Figure 2, where $\{e, f\} = \{uv, xy\}$ and, in the fifth case, $vx$ may be present instead of $uy$.

Clearly $c$ is a 6-coloring of $\binom{S}{2}$. By Theorem 3.1, for some $j$ in \{1, 2, 3, 4, 5, 6\} and some $n_j$-element subset $F$ of $E$, all pairs in $\binom{F}{2}$ have the same color $j$, where $(n_1, n_2, n_3, n_4, n_5, n_6) = (s, t, t, 2s, m)$. Let $F = \{x_1y_1, x_2y_2, \ldots, x_{n_j}y_{n_j}\}$, where $x_i < y_i$ for all $i$. Let $X = \{x_1, x_2, \ldots, x_{n_j}\}$ and $Y = \{y_1, y_2, \ldots, y_{n_j}\}$.

If $j$ is 1 or 6, then $F$ forms an induced subgraph of $G$ isomorphic to $K_{1,s}$ or $mK_2$, respectively. If $j = 2$, then $G[X \cup Y]$ is $K_{t+1}$, unless $t = 3$ when
Let $G[X \cup Y] \in \{K_3, K_4\}$. If $j$ is 3 or 4, then $G[Y]$ or $G[X]$, respectively, is $K_j$. Finally, if $j = 5$, then, for all $i$ in $\{2, 3, \ldots, n_3\}$, at least one of $x_1y_i$ and $y_1x_i$ is an edge of $G$. It follows that at least one of $G[(x_1 \cup Y]$ and $G[(y_1 \cup X]$ contains an induced subgraph isomorphic to $K_{1, n}$. We conclude that the lemma holds if we take $f_{8,2}(m,s,t) = f_{3,1}(2; s, t, t, 2s, m)$.

**Lemma 8.3.** Let $t_1, t_2, t_3$, and $t_4$ be positive integers. There is an integer $f_{8,3}(t_1, t_2, t_3, t_4)$ such that if $G$ is a simple graph with vertex set $\{1, 2, \ldots, n\}$ and $G$ has at least $f_{8,3}(t_1, t_2, t_3, t_4)$ edges, then $G$ has distinct edges $x_1y_1, x_2y_2, \ldots, x_4y_4$ such that at least one of the following holds:

(i) $t = t_1$ and either $x_1 = x_2 = \cdots = x_t$ or $y_1 = y_2 = \cdots = y_t$;
(ii) $t = t_2$ and $x_1 < y_1 < x_2 < y_2 < \cdots < x_t < y_t$;
(iii) $t = t_3$ and $x_1 < x_2 < \cdots < x_t < y_1 < y_2 < \cdots < y_t$;
(iv) $t = t_4$ and $x_1 < x_2 < \cdots < x_t < y_t < y_{t-1} < \cdots < y_1$.

**Proof.** Let $G$ be a simple graph with vertex set $\{1, 2, \ldots, n\}$ and suppose that $G$ has at least $f_{3,1}(2; t_1t_4, 2t_2 - 1, t_3)$ edges. Let $uw$ and $xy$ be an arbitrary pair of distinct edges of $G$ with $u < v$ and $x < y$, where $u \leq x$. Define

$$c(uv, xy) = \begin{cases} 
1 & \text{if } u = x \text{ or } y \leq v; \\
2 & \text{if } v \leq x; \\
3 & \text{otherwise, that is, if } u < x < v < y.
\end{cases}$$

Then $c$ is a 3-coloring of $\binom{E}{2}$. By Theorem 3.1, there is an element $j$ in $\{1, 2, 3\}$ and an $n_j$-element subset $F$ of $E$ such that all pairs in $\binom{F}{2}$ have the same color $j$, where $(n_1, n_2, n_3) = (t_1, t_4, 2t_2 - 1, t_3)$. Let $F = \{x_1y_1, x_2y_2, \ldots, x_n_1y_n_1\}$. If $j = 1$, then we may assume $x_1 \leq x_2 \leq \cdots \leq x_{n_j} \leq y_{n_j} \leq y_{n_j-1} \leq \cdots \leq y_1$. In this case, either a subset of $F$ satisfies (i), or $\{x_1+i_1y_1+i_1 : i \in \{0, 1, \ldots, t_4 - 1\}\}$ satisfies (iv). If $j = 2$, then we may assume $x_1 < y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_{n_j} < y_{n_j}$. In this case, $\{x_2i-1y_2i-1 : i \in \{1, 2, \ldots, t_2\}\}$ satisfies (ii). Finally, if $j = 3$, then $F$ satisfies (iii).

From now on, we will call a vertex in a graph *universal* if it is adjacent to all other vertices of the graph. In addition, when $x$ and $y$ are vertices of a path $P$, we denote by $P[x,y]$ the subpath of $P$ between $x$ and $y$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Dashed lines represent non-edges of $G$; unlinked vertices may or may not be adjacent in $G$.}
\end{figure}
Lemma 8.4. Let $n$ be a positive integer. There exists an integer $f_{8,4}(n)$ with the following property. If a graph $G$ has a bond $\delta_G(V_1)$ that is contained in a cycle $C$ such that $G[V_1]$ has a universal vertex, and $|\delta_G(V_1)| \geq f_{8,4}(n)$, then $G$ has $\Gamma_1(n)$, $\Gamma_3(n)$, or $\Gamma_4(n)$ as a minor.

Proof. Let $l = f_{8,3}(3, n+2, n+1, n+1)$. We shall show that the lemma holds with $f_{8,4}(n) = 2f_{8,1}(2n-1, l-1)$. Let $G$ be a graph satisfying the specified conditions, and let $D = \delta_G(V_1)$ and $u_1$ be a universal vertex of $G[V_1]$. Let $V_2 = V(G) - V_1$. By contracting edges of $C$ in $G[V_2]$, we may assume that no edge of $C$ is contained in $G[V_2]$. Also, by repeatedly contracting edges of $G[V_2]$ that have at most one end on $C$, we may assume that $V_2 \subseteq V(C)$. After these reductions, every vertex of $V_2$ meets exactly two edges of $C$ and so meets exactly two edges of $D$.

Since $|V_2| = |D|/2 \geq f_{8,1}(2n-1, l-1)$, it follows by Lemma 8.1 that $G[V_2]$ has subgraph $T$ such that $T$ is a star or a path on $l+1$ vertices. Let us delete all the edges of $G[V_2]$ that are not in $T$. Then each vertex $v$ in $V_2 - V(T)$ has degree two as it meets two edges in $D$. For all such $v$, contract exactly one of the two edges of $D$ that are incident with $v$. Let $G'$ be the resulting graph, and let $C' = E(G') \cap C$. Then $C'$ is a cycle of $G'$. Let $V_2' = V(T)$. Then $(V_1, V_2')$ is a partition of $V(G')$ and $G'[V_2'] = T$. Moreover, $u_1$ is a universal vertex of $G'[V_1]$, and $\delta_{G'}(V_1)$ is a bond $D'$ of $G'$ that is contained in the cycle $C'$. Now we apply the same reductions used in the first paragraph to $G'[V_1]$. First we contract edges if necessary to get that no edge of $C'$ is contained in $G'[V_1]$. Then further contractions, this time of edges with at most one end in $V(C')$, mean that we may assume that $V_1 \subseteq V(C')$. Note that $u_1$ remains a universal vertex of $G'[V_1]$ where we follow the convention that if an edge $e$ that is incident with $u_1$ is contracted, the composite vertex that results from identifying the ends of $e$ will retain the label $u_1$.

Suppose that $T$ is a star with center vertex $u_2$. Let $P$ be a $u_1u_2$-path in $C'$ with $|V(P)| \geq \left( |V(C')| + 2 \right)/2$. Then $|V(P)| \geq |V_2'| + 1 = 2n + 2$. Now the vertices of $P$ alternate between $V_1$ and $V_2'$. By contracting all of the even-numbered edges in $P$, we get a path with at least $n$ edges in which every interior vertex is adjacent to both $u_1$ and $u_2$. Thus the union of $P$, $T$, and the edges between $u_1$ and $P$ contains $\Gamma_4(n)$ as a minor.

We may now suppose that $T$ is an $l$-edge path $P$ and that its vertices are $1, 2, \ldots, l+1$, listed as they appear on the path. Note that $C'$ is divided by these vertices into $l+1$ two-edge paths $P_1, P_2, \ldots, P_{l+1}$, each having both ends in $P$, and each vertex in $P$ is an end of exactly two such paths. Assume that $P_{l+1}$ contains the universal vertex $u_1$. For each $i$ in $\{1, 2, \ldots, l\}$, let $P_i$ have ends $x_i$ and $y_i$. Apply Lemma 8.3 to the graph with vertex set $V(P)$ and edge set $\{x_iy_i : 1 \leq i \leq l\}$. Recall that $l = f_{8,3}(3, n+2, n+1, n+1)$. Without loss of generality, we may assume that $x_1y_1, x_2y_2, \ldots, x_ly_l$ satisfy one of the outcomes (ii)–(iv) in Lemma 8.3. If (ii) holds, then, as $t = t_2 = n+2$, the union of $P_i$, $P_i$, $P_i$, $P_i$, and edges from $u_1$ to each $P_i$ with $i$ in
{1, 2, \ldots, t} \text{ has } \Gamma_3(n) \text{ as a minor. If (iii) or (iv) of Lemma 8.3 holds, then, as } t_3 = t_4 = n + 1, \text{ the union of } P[x_1, x_t], P[y_1, y_t], P_1, P_2, \ldots, P_t, \text{ and an edge from } u_1 \text{ to each of } P_1 \text{ and } P_t \text{ has } \Gamma_1(n) \text{ as a minor.} \qedhere

We are now ready to prove the main result of this section.

Proof of Theorem 1.2. Let } d = f_{8.4}(n)/2 \text{ and } m = f_{8.3}(2, n + 2, 2, 3n + 3). \text{ In addition, let } l_2 = f_{8.2}(m, 2n, n + 1) \text{ and } l_1 = f_{8.1}(d - 2, l_2 + 2). \text{ We shall prove that the theorem holds when we take } f_{1.2}(n) = 2f_{8.1}(d - 2, l_1). \text{ Specifically, we show that, for a graph } G \text{ having a set } Z \text{ with at least } 2f_{8.1}(d - 2, l_1) \text{ edges such that } Z \text{ is the intersection of a cycle and a bond, } G \text{ has, as a minor, one of } \Gamma_1(n), \Gamma_2(n), \Gamma_3(n), \text{ or } \Gamma_4(n).

Suppose } Z \text{ is the intersection of a cycle } C \text{ and a bond } C^*. \text{ By deleting the edges of } C^* - C, \text{ we obtain a minor of } G \text{ in which } C \text{ is a cycle and } C \cap C^* \text{ is a bond } D. \text{ For notational convenience, we shall relabel this minor of } G \text{ as } G. \text{ Let } (V_1, V_2) \text{ be a partition of } V(G) \text{ such that } D = \delta_C(V_1). \text{ As in the proof of the preceding lemma, we also assume that } V_1 \subseteq V(C) \text{ and that no edge of } C \text{ is contained in } G[V_1]. \text{ It follows that every vertex of } V_1 \text{ meets exactly two edges of } C \text{ and so meets exactly two edges of } D. \text{ Thus } |V_1| = |D|/2 \geq f_{8.4}(d - 2, l_1).

Suppose } G[V_1] \text{ has a vertex } u \text{ of degree at least } d - 1. \text{ Delete all the edges of } G[V_1] \text{ that are not incident with } u \text{ letting } G' \text{ be the resulting graph. Let } V_0 \text{ be } u \text{ together with its neighbors in } V_1 \text{ and let } D_0 = \delta_{G'}(V_0). \text{ Then } |V_0| \geq d, \text{ so } |D_0| \geq 2d. \text{ We can now apply Lemma 8.4 to the bond } D_0 \text{ of } G'. \text{ As } G'[V_0] \text{ has } u \text{ as a universal vertex and } |D_0| \geq f_{8.4}(n), \text{ the theorem follows in this case. We may now assume that the maximum degree of } G[V_1] \text{ is at most } d - 2. \text{ Thus, by Lemma 8.1, } G[V_1] \text{ has a path } P_1 \text{ of length } l_1. \text{ By deleting all the edges of } G[V_1] \text{ that are not in } P_1, \text{ we obtain a subgraph } G_0 \text{ of } G \text{ in which } \delta_{G_0}(V(P_1)) \text{ is a bond } D' \text{ contained in } C. \text{ Note that } V(P_1) \subseteq V(C), \text{ that } E(C) \cap E(P_1) = \emptyset, \text{ and that } P_1 \text{ is an induced path in } G_0. \text{ By contracting edges in } G_0 - V(P_1) \text{ as in the first paragraph of the last proof, we can obtain a minor } G' \text{ such that the edges of } D' \text{ form a cycle that spans } M(G') \text{ and if } (V'_1, V'_2) \text{ is the partition of } G' \text{ with } V'_1 = V(P_1), \text{ then } V'_2 \subseteq V(D') \text{ and } E(D') \cap G'[V'_2] = \emptyset. \text{ As every vertex of each of } V'_1 \text{ and } V'_2 \text{ meets exactly two edges of } D', \text{ it follows that } |V'_1| = |V'_2|. \text{ Let } u_1, v_1 \text{ be the ends of } P_1.

If } G'[V'_2] \text{ has a vertex of degree at least } d - 1, \text{ then, as in the previous paragraph, the theorem holds by Lemma 8.4. Thus we may assume that the maximum degree of } G'[V'_2] \text{ is at most } d - 2. \text{ Let } p \text{ label } u_1u_2, \text{ one of the two edges in } D' \text{ that are incident with } u_1. \text{ Since } |V'_2| = |V'_1| = l_1 + 1 > f_{8.1}(d - 2, l_2 + 2), \text{ we deduce from Lemma 8.1 that } G'[V'_2] \text{ has a path } P_2 \text{ of length } l_2 + 3 \text{ having } u_2 \text{ as an end. Let } v_2 \text{ be the other end of this path. Let } G'' \text{ be obtained by deleting all the edges of } G'[V'_2] \text{ that are not in } P_2, \text{ and, for each } w \in V'_2 - V(P_2), \text{ contracting exactly one of the two edges of } D' \text{ that are incident with } w. \text{ Note that, after these contractions, the remaining edges of } D' \text{ form a cycle } C'' \text{ that spans } M(G''). \text{ Let } P \text{ be the union of } P_1, P_2, \text{ and } p. \text{ Then } P \text{ is a path with vertex set } V(G'') \text{ and ends } v_1 \text{ and } v_2.
Moreover, \((E(C'' \setminus p), E(P))\) is a partition of \(E(G'')\). We also observe that no edge of \(C''\) has both ends in \(P_2\) but \(C''\) may have some edges with both ends in \(P_1\).

We shall say that two edges \(x_1x_2\) and \(y_1y_2\) of \(C'' \setminus p\) cross if \(P[x_1, x_2]\) and \(P[y_1, y_2]\) have at least one common edge yet neither is a subpath of the other.

**8.6.1.** If \(w_1w_2\) is an edge \(e\) having one end in \(V(P_1) - \{u_1, v_1\}\) and the other in \(V(P_2) - \{u_2, v_2\}\), then \(e\) crosses some edge of \(C'' \setminus p\).

Let \(Q\) be the component of \(C'' \setminus \{e, p\}\) that contains \(v_1\). Then \(Q\) is a path with ends \(u_i\) and \(w_j\) for some \(i\) and \(j\) in \(\{1, 2\}\). Note that \(Q[u_i, v_1]\) is a path from the interior of \(P[w_1, w_2]\) to \(v_1\), and it contains neither \(w_1\) nor \(w_2\). Thus its last edge leaving \(P[w_1, w_2]\) crosses \(e\). Hence 8.6.1 holds.

Let \(F\) be the set of edges joining a vertex in \(V(P_1) - \{u_1, v_1\}\) to a vertex in \(V(P_2) - \{u_2, v_2\}\), and let \(F'\) be the set of edges of \(C'' - p\) that cross at least one edge in \(F\). We now construct an auxiliary simple graph \(H\) that has \(F \cup F'\) as its vertex set. Two vertices in \(H\) are adjacent if and only if the corresponding edges of \(G''\) cross. By 8.6.1, \(H\) has no isolated vertices. Now each vertex of \(V(P_2) - \{u_2, v_2\}\) meets exactly two edges of \(C'' \setminus p\). Since each of these edges has its other end in \(V(P_1)\) and at most four of these edges have \(u_1\) or \(v_1\) as an end, \(|F| \geq 2(|V(P_2)| - 2) - 4 = 2|V(P_2)| - 8\). Thus \(|E(H)| \geq |V(H)|/2 = |F \cup F'|/2 \geq |F'|/2 \geq |V(P_2)| - 4 = l_2 = f_8.2(m, 2n, n + 1). By Lemma 8.2, \(F \cup F'\) has a subset \(F_0\) such that \(H[F_0]\) is \(mK_2, K_{1, 2n}, \) or \(K_{n + 1}\). In the third case, \(F_0\) corresponds to a set of \(n + 1\) edges of \(C'' \setminus p\) every two of which cross. Beginning at \(v_1\), order the vertices of \(P\) as they occur along the path. Then the edges of \(F_0\) can be ordered \(x_1y_1, x_2y_2, \ldots, x_{n+1}y_{n+1}\) so that \(x_1 < x_2 < \cdots < x_{n+1} < y_1 < y_2 < \cdots < y_{n+1}\). Then the union of \(P[x_1, x_{n+1}], P[y_{n+1}, y_1]\), and the edges of \(F_0\) form a subdivided ladder. Since the path \(P[x_{n+1}, y_1]\) is internally disjoint from this ladder, when we add this path to the ladder and suppress the degree-two vertices, we get \(\Gamma_1(n)\).

Next suppose that \(H[F_0]\) is \(K_{1, 2n}\). Then, in \(G''\), the edges of \(F_0\) consist of a \(2n\)-edge matching together with a single edge \(e\) that crosses all of the edges in the matching. If \(xy\) and \(x'y'\) are in the matching, then \(\{x, y\} \cap \{x', y'\} = \emptyset\). Moreover, either \(P[x, y]\) and \(P[x', y']\) are disjoint, or one of \(P[x, y]\) and \(P[x', y']\) is a subpath of the other. As \(e\) crosses all the edges in the matching, it follows that the matching contains \(n\) edges, \(x_1y_1, x_2y_2, \ldots, x_{n}y_{n}\), such that, after interchanging \(x_i\) and \(y_i\) where necessary, \(x_1 < x_2 < \cdots < x_n < y_n < y_{n-1} < \cdots < y_1\). Moreover, we may assume that \(e = x_0y_0\) where \(x_0 < x_1\). Then the union of \(P[x_1, x_n], P[y_1, y_n]\), and \(\{x_1y_1, x_2y_2, \ldots, x_{n}y_{n}\}\) is a subdivided ladder. Taking the union of it with \(x_0y_0, P[x_n, y_0], P[y_0, y_n]\), and \(P[x_0, x_1]\) gives a graph that has \(\Gamma_1(n)\) as a minor.

It remains to consider the case when \(H[F_0]\) is \(mK_2\). In that case, the \(m\) edges of \(H[F_0]\) correspond to \(m\) disjoint pairs \(\{e_i, e'_i\}\) of edges of \(C'' \setminus p\) in \(G''\) where the edges in each pair cross each other but edges from different pairs...
do not cross. Now we apply Lemma 8.3 to the subgraph of $G''$ with edge set $\{e_1, e_2, \ldots, e_m\}$, where their end vertices are ordered as they occur on $P$. We know that the edges in $\{e_1, e_2, \ldots, e_m\}$ form a matching and no two of them cross. Since $m = f_{8,3}(2, n + 2, 2, 3n + 3)$, it follows by Lemma 8.3 that either (ii) of the lemma holds with $t = n + 2$, or (iv) of the lemma holds with $t = 3n + 3$.

Consider the first possibility. Then at most one $e_i$ has one end in $P_1$ and the other end in $P_2$. Thus, since no $e_i$ has both ends in $P_2$, it follows that $n + 1$ of the edges, say $e_1, e_2, \ldots, e_{n+1}$, have both ends in $P_1$. Then each of $e_1, e_2, \ldots, e_{n+1}$ is in $F'$. The definition of $F'$ guarantees that, for each $i$ in $\{1, 2, \ldots, n + 1\}$, there is an edge $g_i$ of $F$ that crosses $e_i$. Then, by contracting $P_2$ to a single vertex, we see that the union of $P$ and $\{e_1, e_2, \ldots, e_{n+1}, g_1, g_2, \ldots, g_{n+1}\}$ contains $\Gamma_3(n)$ as a minor.

Now suppose (iv) of Lemma 8.3 occurs. Let $e_i = x_i y_i$ for all $i$ in $\{1, 2, \ldots, 3n + 3\}$ and suppose that $x_1 < x_2 < \cdots < x_{3n+3} < y_{3n+3} < \cdots < y_2 < y_1$. For each $i$ in $\{2, 3, \ldots, 3n + 2\}$, there are three possibilities for the position of $e_i'$:

(a) both ends of $e_i'$ are in $P[x_{i-1}, x_{i+1}]$;
(b) both ends of $e_i'$ are in $P[y_{i+1}, y_{i-1}]$;
(c) $e_i'$ joins a vertex in $P[x_{i-1}, x_{i+1}]$ to a vertex in $P[y_{i+1}, y_{i-1}]$.

Put $e_i'$ into $I_a, I_b$, or $I_c$ depending on which of (a), (b), or (c) holds. Then $I_a, I_b$, and $I_c$ are disjoint sets whose union is $\{2, 3, \ldots, 3n + 2\}$. Hence one of $I_a, I_b$, or $I_c$ has at least $n + 1$ members. If $|I_a| \geq n + 1$, then the union of $P$ and the edges $e_i e_i'$ with $i$ in $I_a$ contains $\Gamma_3(n)$ as a minor. By symmetry, when $|I_b| \geq n + 1$, we also get $\Gamma_3(n)$ as a minor. Thus we may assume that $|I_c| \geq n + 1$. By relabelling, we may assume that $\{2, 3, \ldots, n+2\} \subseteq I_c$. Then, by taking the union of $P[x_1, x_{n+3}], P[y_{n+3}, y_1]$, and $\{e_i, e_i': i \in \{2, 3, \ldots, n+2\}\}$, we get a graph that contains $\Gamma_2(n)$ as a minor.

A list of unavoidable minors for the class of binary matroids that contain a big set that is the intersection of a circuit and a cocircuit must include the cycle matroids of $\Gamma_1(n), \Gamma_2(n), \Gamma_3(n)$, and $\Gamma_4(n)$. In addition, the list should include the tipless binary spike of rank $n$, that is, the vector matroid of the binary matrix $[I_n | J_n - I_n]$, where $J_n$ is the $n \times n$ matrix of all ones. We were unable to prove that this list is complete, nor could we find a counterexample.

**Conjecture 8.7.** Let $n$ be an integer exceeding two. There is an integer $f_{8,7}(n)$ such that if a binary matroid $M$ contains a set that is the intersection of a circuit and a cocircuit and has at least $f_{8,7}(n)$ elements, then $M$ has, as a minor, one of $M(\Gamma_1(n)), M(\Gamma_2(n)), M(\Gamma_3(n)), M(\Gamma_4(n))$, or the vector matroid of the binary matrix $[I_n | J_n - I_n]$.

The next result shows that if the list in the conjecture is incomplete, any other matroids on the list can be assumed to be 3-connected.
Theorem 8.8. Let $M$ be a binary matroid that contains a set $Z$ that is the intersection of a circuit and a cocircuit. For each integer $n$ exceeding two, there is a positive integer $f_{8,8}(n)$ such that if $|Z| \geq f_{8,8}(n)$, then

(i) $M$ has a 3-connected minor $M'$ that contains a set $C'$ that has at least $n$ elements and is both a spanning circuit and a cospanning cocircuit; or

(ii) $M$ has $M(\Gamma_2(n))$ or $M(\Gamma_3(n))$ as a minor.

Proof. Assume that $|Z| \geq n f_{8,1}(2^n, n^2)$. By Lemma 7.1, $M$ has a minor $M_0$ such that $Z$ is a spanning circuit of both $M_0$ and $M_0^*$. For notational convenience, we relabel $M_0$ as $M$. We may assume that $M$ is not 3-connected, otherwise (i) certainly holds. We now apply a result of Cunningham and Edmonds (see Cunningham [1]) following the treatment given in [6, Section 8.3]. By that result, $M$ has a canonical tree decomposition. This consists of a tree $T$ whose vertex set is labelled by a set $\{M_1, M_2, \ldots, M_k\}$ of matroids and whose edge set is $\{e_1, e_2, \ldots, e_{k-1}\}$, say, such that

(i) each $M_i$ is a circuit, a cocircuit, or a 3-connected matroid; and no two adjacent vertices of $T$ are both labelled by circuits, or are both labelled by cocircuits;

(ii) if $M_{j_1}$ and $M_{j_2}$ are joined by an edge $e_i$ of $T$, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$, and $\{e_i\}$ is not a component of $M_{j_1}$ or $M_{j_2}$;

(iii) $E(M_{j_1}) \cap E(M_{j_2}) = \emptyset$;

(iv) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k)) \cup \{e_1, e_2, \ldots, e_{k-1}\}$;

(v) $|E(M_i)| \geq 3$ for all $i$; and

(vi) $M$ is the matroid that labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$ where, when an edge is contracted, the resulting composite vertex is labelled by the 2-sum of the two vertices that had labelled its ends.

As $M$ has $Z$ as a spanning circuit, whenever $M$ is written as a 2-sum of $N_1$ and $N_2$ with respect to the basepoint $p$, either

(a) $Z = (C_1 - p) \cup (C_2 - p)$ where $C_i$ is a spanning circuit of $N_i$ containing $p$; or

(b) $Z$ is a spanning circuit of one of $N_1$ and $N_2$ avoiding $p$ while the other $N_i$ has rank one.

Because $Z$ is also a spanning circuit of $M^*$, the latter cannot occur. Thus (a) holds. Hence no vertex of the tree $T$ labels a cocircuit otherwise $M$ has a circuit that is properly contained in $Z$. Because $T$ is also the canonical tree decomposition for $M^*$, where each vertex label $M_i$ is replaced by $M_i^*$, we deduce that no vertex of $T$ labels a circuit. Hence every vertex of $T$ labels a 3-connected binary matroid, which must have at least six elements. If some $M_i$ has rank at least $n - 1$, then the theorem holds. Thus we may assume that $r(M_i) < n - 1$ for all $i$. But $r(M) = \sum_{i=1}^{k} r(M_i) - (k - 1)$. Hence $r(M) < k(n - 1)$.
We also know that \( r(M) \geq |Z| - 1 \geq nf_{8.1}(2^n, n^2) - 1 \geq (n-1)f_{8.1}(2^n, n^2) \). Thus
\[
 k(n-1) > r(M) > (n-1)f_{8.1}(2^n, n^2).
\]
Hence \( k > f_{8.1}(2^n, n^2) \), that is, \( T \) has at least \( f_{8.1}(2^n, n^2) \) vertices. Therefore, by Lemma 8.1, either \( T \) has a vertex of degree exceeding \( 2^n \), or \( T \) contains a path with at least \( n^2 \) vertices. The first possibility does not arise since a simple binary matroid with more than \( 2^n \) elements has rank more than \( n \), and we know \( r(M_i) < n - 1 \) for all \( i \). Thus we may assume that \( T \) contains a path \( P \) with vertex set \( \{N_1, N_2, \ldots, N_{n^2}\} \) where \( E(N_i) \cap E(N_{i+1}) = \{f_i\} \) for all \( i \in \{1, 2, \ldots, n^2 - 1\} \). Let \( f_0 \) be an element of \( E(N_1) - f_1 \), and let \( f_{n^2} \) be an element of \( E(N_{n^2}) - f_{n^2-1} \).

Now \( P \) is the canonical tree decomposition for a minor \( N \) of \( M \). For each \( i \) in \( \{1, 2, \ldots, n^2\} \), since \( N_i \) is 3-connected and binary, it follows by Tutte’s Wheels-and-Whirls Theorem [10] that \( N_i \) has an \( M(K_4) \)-minor. Moreover, by a result of Seymour [9], \( N_i \) has an \( M(K_4) \)-minor \( N_i' \) whose ground set contains \( \{f_{i-1}, f_i\} \). Thus \( N \) has a minor \( N' \) for which \( P \) is the canonical tree decomposition where we replace the vertex label \( N_i \) by \( N_i' \). We color \( N_i' \) black if it has a triangle containing \( \{f_{i-1}, f_i\} \) and color \( N_i' \) white otherwise. Since \( P \) has at least \( n^2 \) vertices, it certainly has at least \( n \) black vertices or at least \( n \) white vertices. We can eliminate a vertex \( N_i' \) of \( P \) by contracting elements of it until \( f_{i-1} \) and \( f_i \) are parallel. We then delete the remaining elements of \( N_i' \) except \( f_i \) and relabel \( f_{i-1} \) in \( N_i' \) by \( f_i \). By this process, we get a minor of \( M \) for which the canonical tree decomposition is an \( n \)-vertex path in which all vertices are the same color. We conclude that \( M \) has \( M(\Gamma_2(n)) \) or \( M(\Gamma_3(n)) \) as a minor. \( \square \)

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