TOWARDS A SPLITTER THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS IX: THE THEOREM

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ABSTRACT. Let M be a binary matroid that is internally 4-connected, that is, M is 3-connected, and one side of every 3-separation is a triangle or a triad. Let N be an internally 4-connected proper minor of M. In this paper, we show that M has a proper internally 4-connected minor with an N-minor that can be obtained from M either by removing at most three elements, or by removing some set of elements in an easily described way from one of a small collection of special substructures of M.

1. INTRODUCTION

Seymour's Splitter Theorem [15] shows that a 3-connected matroid M with a 3-connected proper minor N has a 3-connected proper minor M' with an N-minor such that $|E(M)|-|E(M')| \leq 2$. Moreover, such an M' can be found with $|E(M)|-|E(M')| = 1$ unless $r(M) \geq 3$ and M is a wheel or a whirl. This theorem has played an important role in inductive and constructive arguments for 3-connected matroids. In this paper, we prove a corresponding result for internally 4-connected binary matroids. Specifically, we show that if M and N are such matroids and M has a proper N-minor, then M has a proper minor M' such that M' is internally 4-connected with an N-minor, and M' can be obtained from M by a small number of simple operations. In a paper entitled "A splitter theorem for internally 4-connected binary matroids," Geelen and Zhou [9] proved a partial such result, observing that: "It is a shortcoming of [this theorem] that the intermediate matroids are only 4-connected up to separators of size 5; it would be preferable if this could be strengthened to internally 4-connected. There are, however, numerous obstacles to obtaining such a theorem, even for graphs". This paper settles this problem of Geelen and Zhou by explicitly revealing all of these obstacles.

Any unexplained matroid terminology used here will follow [14]. A graph is internally 4-connected exactly when its cycle matroid is internally 4-connected and it has no isolated vertices. In an internally 4-connected matroid, the only allowed 3-separations have a triangle or a triad on one side. A 3-connected matroid M is $(4, 4, S)$-*connected* if, for every 3-separation (X, Y) of M, one of X and Y is a triangle, a triad, or a 4-element fan, where the last structure is a 4-element set $\{x_1, x_2, x_3, x_4\}$ that can be ordered so that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad.

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To provide a context for our main theorem, we briefly describe our progress towards obtaining the desired splitter theorem. Johnson and Thomas [10] showed that, even for graphs, a splitter theorem in the internally 4-connected case must take account of some special examples. For $n \geq 3$, let G_{n+2} be the *biwheel* with $n+2$ vertices, that is, G_{n+2} consists of an n-cycle $v_1, v_2, \dots, v_n, v_1$, the *rim*, and two additional vertices, u and w, the *hubs*, both of which are adjacent to every v_i. Thus the dual of G_{n+2} is a cubic planar ladder. Let M be the cycle matroid of G_{2n+2} for some $n \geq 3$ and let N be the cycle matroid of the graph that is obtained by proceeding around the rim of G_{2n+2} and alternately deleting the edges from the rim vertex to u and to w. Both M and N are internally 4-connected but there is no internally 4-connected proper minor of M that has a proper N-minor. We can modify M slightly and still see the same phenomenon. Let G^+_{n+2} be obtained from G_{n+2} by adding a new edge z, the *axle*, joining the hubs u and w. Let Δ_{n+1} be the binary matroid that is obtained from $M(G^+_{n+2})$ by deleting the element $v_{n-1}v_n$ and adding the third element on the line spanned by wv_n and uv_{n-1}. This new element is also on the line spanned by uv_n and wv_{n-1}. For $r \geq 3$, Mayhew, Royle, and Whittle [13] call Δ_r the *rank-r triangular Möbius matroid* and note that $\Delta_r \backslash z$ is the dual of the cycle matroid of a cubic Möbius ladder. The following is the main result of [3, Theorem 1.2].

Theorem 1.1. *Let M be an internally 4-connected binary matroid with an internally 4-connected proper minor N such that $|E(M)| \geq 15$ and $|E(N)| \geq 6$. Then*

(i) *M has a proper minor M' such that $|E(M) - E(M')| \leq 3$ and M' is internally 4-connected with an N-minor; or*
(ii) *for some (M_0, N_0) in $\{(M,N),(M^*,N^*)\}$, the matroid M_0 has a triangle T that contains an element e such that $M_0 \backslash e$ is $(4,4,S)$-connected with an N_0-minor; or*
(iii) *M or M^* is isomorphic to $M(G^+_{r+1})$, $M(G_{r+1})$, Δ_r, or $\Delta_r \backslash z$ for some $r \geq 5$.*

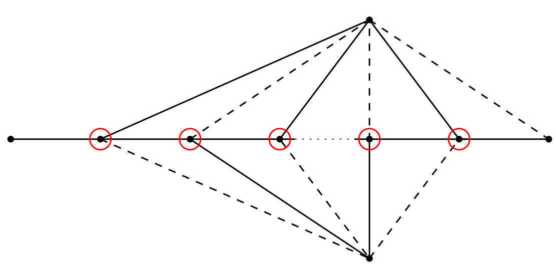

FIGURE 1. All the elements shown are distinct. There are at least three dashed elements, and all dashed elements are deleted.

That theorem prompted us to consider those matroids for which the second outcome in the theorem holds. In order to state the next result, we need to define some special structures. Let M be an internally 4-connected binary matroid and N be an internally 4-connected proper minor of M. Suppose M has disjoint triangles T_1 and T_2 and a 4-cocircuit D^* contained in their union. We call this structure a *bowtie* and denote it by (T_1, T_2, D^*). If D^* has an element d such that $M \backslash d$ has an N-minor and $M \backslash d$ is $(4,4,S)$-connected, then (T_1, T_2, D^*) is a *good bowtie*.

Motivated by outcome (ii) of the last theorem, we determined more about the structure of M when it has a triangle containing an element e such that $M\backslash e$ is $(4, 4, S)$-connected with an N-minor. One possibility here is that M has a good bowtie. Indeed, as the next result shows, if that outcome or its dual does not arise, we get a small number of easily described alternatives. We shall need two more definitions. A *terrahawk* is the graph that is obtained from a cube by adding a new vertex and adding edges from the new vertex to each of the four vertices that bound some fixed face of the cube. Although the matroid M that we are dealing with need not be graphic, we follow the convention begun in [2] of using a modified graph diagram to keep track of some of the circuits and cocircuits in M. Figure 1 shows such a modified graph diagram. Each of the cycles in such a graph diagram corresponds to a circuit of M while a circled vertex indicates a known cocircuit of M. We call the structure in Figure 1 an *open rotor chain* noting that all of the elements in the figure are distinct and, for some $n \geq 3$, there are n dashed edges. That figure may suggest that n must be even but we impose no such restriction. We refer to deleting the dashed elements from Figure 1 as *trimming an open rotor chain*.

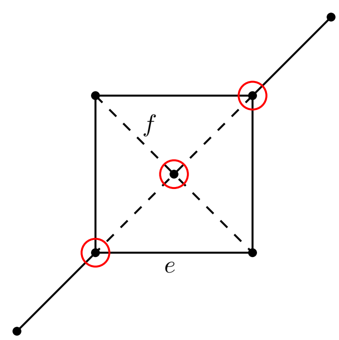

FIGURE 2. An augmented 4-wheel. All displayed elements are distinct.

An *augmented* 4-*wheel* is represented by the modified graph diagram in Figure 2, where the four dashed edges form the *central cocircuit*. We say that an augmented 4-wheel labelled in this way is *good* if $M\backslash e$ is $(4, 4, S)$-connected with an N-minor, while $M\backslash f$ also has an N-minor. The following theorem is [5, Corollary 1.4].

Theorem 1.2. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 6$. Suppose that M has a triangle T containing an element e for which $M\backslash e$ is $(4, 4, S)$-connected with an N-minor. Then one of the following holds.*

- (i) *M has an internally 4-connected minor M' that has an N-minor such that either $1 \leq |E(M) - E(M')| \leq 3$; or $|E(M) - E(M')| = 4$ and, for some (M_1, M_2) in $\{(M, M'), (M^*, (M')^*\}$, the matroid M_2 is obtained from M_1 by deleting the central cocircuit of a good augmented 4-wheel; or*
- (ii) *M or M^* has a good bowtie; or*
- (iii) *M is the cycle matroid of a terrahawk; or*
- (iv) *for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 contains an open rotor chain that can be trimmed to obtain an internally 4-connected matroid with an N_0-minor.*

This theorem led us to consider a good bowtie $(\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_4, x_5\})$ in an internally 4-connected binary matroid M where $M\backslash x_3$ is $(4, 4, S)$-connected with an N-minor. In $M\backslash x_3$, we see that $\{x_5, x_4, x_2\}$ is a triad and $\{x_6, x_5, x_4\}$ is a triangle, so $\{x_6, x_5, x_4, x_2\}$ is a 4-element fan. It follows, by [4, Lemma 2.5], that either

(i) $M\backslash x_3, x_6$ has an N-minor; or
(ii) $M\backslash x_3, x_6$ does not have an N-minor, but $M\backslash x_3/x_2$ is $(4, 4, S)$-connected with an N-minor.

In [7], we considered the case when (i) holds, but $M\backslash x_6$ is not $(4, 4, S)$-connected. In [8], we dealt with the case when (ii) holds. In this paper, we complete the work to obtain the splitter theorem by considering the case when $M\backslash x_3, x_6$ has an N-minor and $M\backslash x_6$ is $(4, 4, S)$-connected. Before stating the main results of [7] and [8], we define some structures that require special attention.

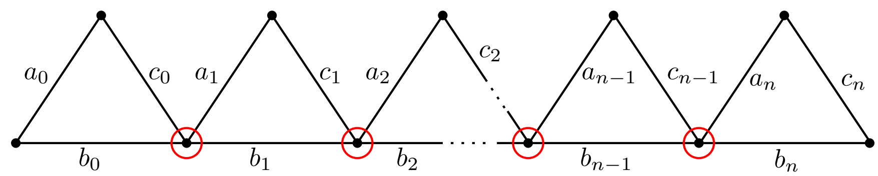

FIGURE 3. A string of bowties. All elements are distinct except that a_0 may be the same as c_n.

In a matroid M, a *string of bowties* is a sequence $\{a_0, b_0, c_0\}$, $\{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \{b_1, c_1, a_2, b_2\}, \dots, \{a_n, b_n, c_n\}$ with $n \geq 1$ such that

(i) $\{a_i, b_i, c_i\}$ is a triangle for all i in $\{0, 1, \dots, n\}$;
(ii) $\{b_j, c_j, a_{j+1}, b_{j+1}\}$ is a cocircuit for all j in $\{0, 1, \dots, n-1\}$; and
(iii) the elements $a_0, b_0, c_0, a_1, b_1, c_1, \dots, a_n, b_n$, and c_n are distinct except that a_0 and c_n may be equal.

The reader should note that this differs slightly from the definition we gave in [2] in that here we allow a_0 and c_n to be equal instead of requiring all of the elements to be distinct. Figure 3 illustrates a string of bowties, but this diagram may obscure the potential complexity of such a string. Evidently $M\backslash c_0$ has $\{c_1, b_1, a_1, b_0\}$ as a 4-fan. Indeed, $M\backslash c_0, c_1, \dots, c_i$ has a 4-fan for all i in $\{0, 1, \dots, n-1\}$. We say that $M\backslash c_0, c_1, \dots, c_n$ has been obtained from M by *trimming a string of bowties.* This operation plays a prominent role in our main theorem, and is the underlying operation in trimming an open rotor chain.

A string of bowties can attach to the rest of the matroid in a variety of ways. In most of these cases, the operation of trimming the string will produce an internally 4-connected minor of M with an N-minor. But, when the bowtie string is embedded in a modified quartic ladder in certain ways, we need to adjust the trimming process.

Consider the three configurations shown in Figures 4 and 5 where the elements in each configuration are distinct except that d_2 may equal w_k. We refer to each of these configurations as an *enhanced quartic ladder.* Indeed, in each configuration, we can see a portion of a quartic ladder, which can be thought of as two interlocking bowtie strings, one pointing up and one pointing down. In each case, we focus on $M\backslash c_2, c_1, c_0, v_0, v_1, \dots, v_k$ saying that this matroid has been obtained from M by an *enhanced-ladder move.* We will show that the structure in Figure 4(b) does not

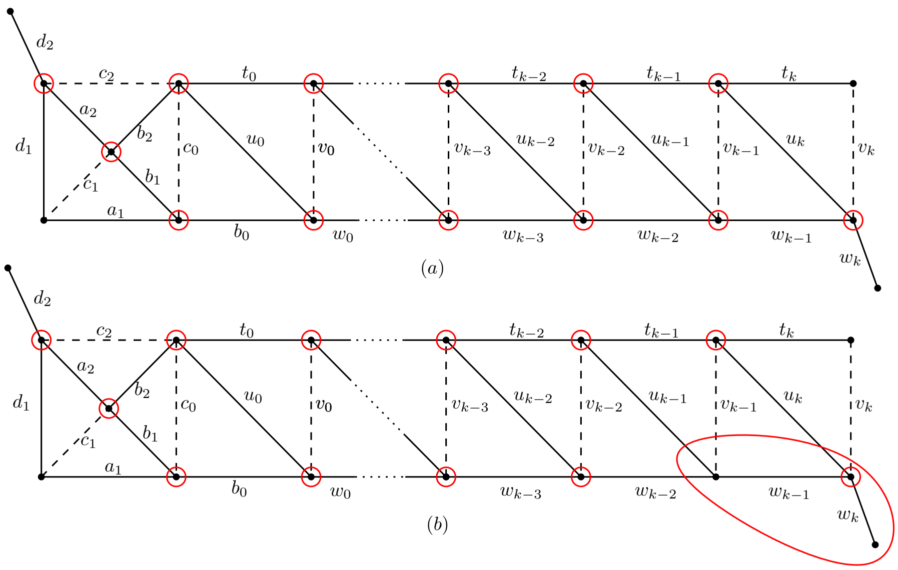

FIGURE 4. In both (a) and (b), the elements shown are distinct, except that d_2 may be w_k. Furthermore, in (a), $k \geq 0$; and, in (b), $k \geq 1$ and $\{w_{k-2}, u_{k-1}, v_{k-1}, u_k, v_k\}$ is a cocircuit.

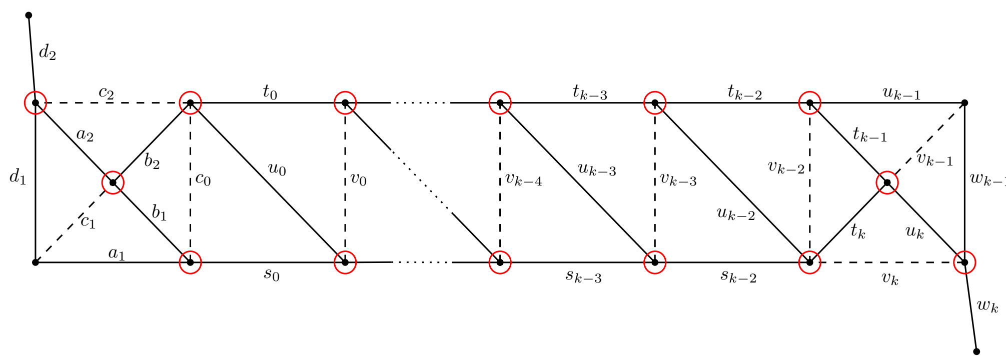

FIGURE 5. In this configuration, $k \geq 2$ and the elements are all distinct except that d_2 may be w_k.

arise in an internally 4-connected graphic matroid. Suppose M is graphic and M contains one of the structures in Figure 4(a) or Figure 5. Then d_2 and w_k must be distinct, and we say that $M\backslash c_2, c_1, c_0, v_0, v_1, \ldots, v_k$ has been obtained from M by a *graphic enhanced-ladder move*. In Figure 6, the configuration in Figure 5 has been redrawn omitting the triangles $\{c_0, b_1, b_2\}$ and $\{v_{k-2}, t_{k-1}, t_k\}$ as well as the cocircuits $\{c_2, b_2, c_0, u_0, t_0\}$ and $\{s_{k-2}, u_{k-2}, v_{k-2}, t_k, v_k\}$. The ladder structure is evident there and the enhanced-ladder move corresponds to deleting all of the dashed edges.

Suppose that $\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \ldots, \{a_n, b_n, c_n\}$ is a bowtie string for some $n \geq 2$. Assume, in addition, that $\{b_n, c_n, a_0, b_0\}$ is a cocircuit.

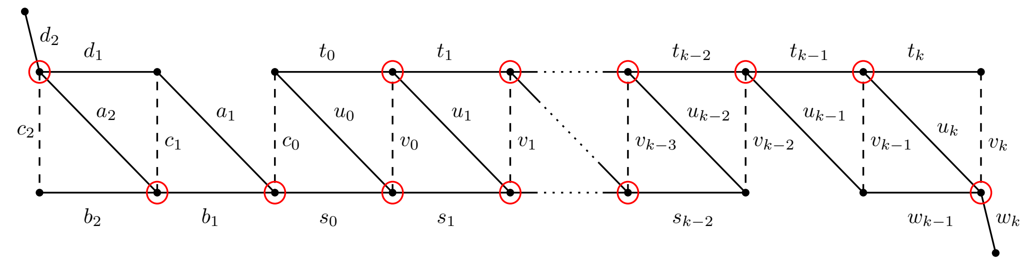

FIGURE 6. The configuration in Figure 5 redrawn omitting two triangles and two 5-cocircuits.

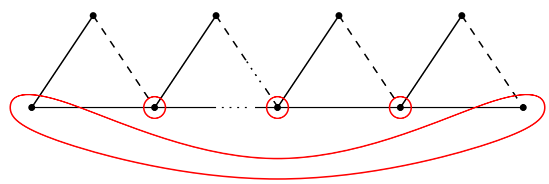

FIGURE 7. A bowtie ring. All elements are distinct. The ring contains at least three triangles.

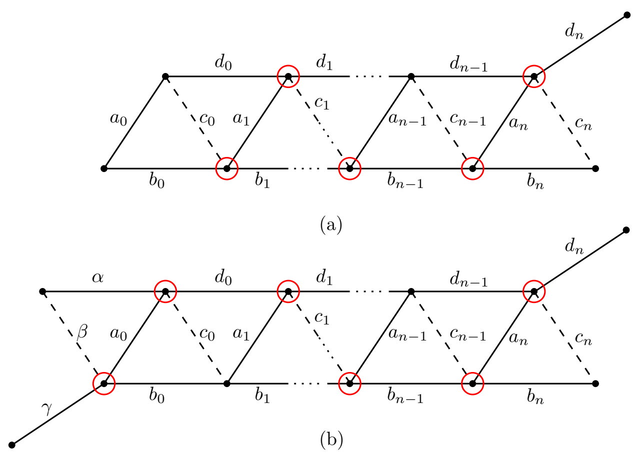

FIGURE 8. In (a) and (b), $n \geq 2$ and the elements shown are distinct, with the exception that d_n may be the same as γ in (b). Either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit in (a) and (b). Either $\{b_0, c_0, a_1, b_1\}$ or $\{\beta, a_0, c_0, a_1, b_1\}$ is also a cocircuit in (b).

Then the string of bowties has wrapped around on itself as in Figure 7 and we call the resulting structure a *ring of bowties*. Observe that, when such a ring occurs in an internally 4-connected binary matroid, $a_0 \neq c_n$ otherwise the matroid contains a 4-fan. We call each of the structures in Figure 8 a *ladder structure*, and we refer to removing the dashed elements in Figure 7 and Figure 8 as *trimming a ring of bowties* and *trimming a ladder structure*, respectively. A bowtie ring is *minimal* exactly when no proper subset of its set of triangles is the set of triangles of a bowtie ring. We showed in [7, Lemma 5.1] that if M is a binary internally 4-connected matroid having at least ten elements and trimming a bowtie ring in M produces an internally 4-connected matroid, then that bowtie ring is minimal.

It is straightforward to check that, in Figure 8(a) and (b), when $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ or $\{\beta, a_0, c_0, a_1, b_1\}$ is a cocircuit, the underlying matroid is not graphic. A *graphic ladder structure* is one of the structures in Figure 8 such that, in both (a) and (b), the set $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is a cocircuit and, in (b), $\{b_0, c_0, a_1, b_1\}$ is also a cocircuit. Removing the dashed elements from a graphic ladder structure will be referred to as *trimming a graphic ladder structure.* In the case that trimming a string of bowties in M yields an internally 4-connected matroid with an N-minor, we are able to ensure that the string of bowties belongs to one of the more highly structured objects we have described here. The following theorem is the main result of [7, Theorem 1.3]. Note that, although this theorem allows some moves that are not bounded in size, each such move is highly structured.

Theorem 1.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 13$ and $|E(N)| \geq 7$. Assume that M has a bowtie $(\{x_0, y_0, z_0\}, \{x_1, y_1, z_1\}, \{y_0, z_0, x_1, y_1\})$, where $M\backslash z_0$ is $(4, 4, S)$-connected, $M\backslash z_0, z_1$ has an N-minor, and $M\backslash z_1$ is not $(4, 4, S)$-connected. Then one of the following holds.*

(i) *M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$, and M' is internally 4-connected with an N-minor; or*
(ii) *M contains an open rotor chain, a ladder structure, or a ring of bowties, and this structure can be trimmed to obtain an internally 4-connected matroid with an N-minor; or*
(iii) *M contains an enhanced quartic ladder from which an internally 4-connected minor of M with an N-minor can be obtained by an enhanced-ladder move.*

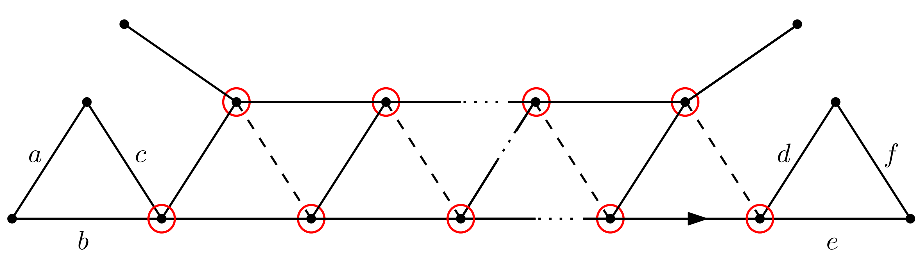

FIGURE 9. All of the elements are distinct except that a may be f or (a, b, c) may be (d, e, f).

When M contains the structure in Figure 9, where the elements are all distinct except that a may be f, or (a, b, c) may be (d, e, f), we say that M contains *an open quartic ladder*. Deleting the dashed elements and contracting the arrow edge is called *a mixed ladder move*. The following result is [8, Corollary 1.5].

Theorem 1.4. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 16$ and $|E(N)| \geq 7$. If M has a bowtie $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$, where $M\backslash 4$ is $(4, 4, S)$-connected with an N-minor, then either $M\backslash 1, 4$ has* *an N-minor and $M\backslash 1$ is $(4,4,S)$-connected, or, for some (M_0, N_0) in $\{(M,N),(M^*,N^*)\}$, there is an internally 4-connected minor M' of M_0 such that one of the following holds.*

(i) $|E(M)| - |E(M')| \leq 3$, *or* $|E(M)| - |E(M')| = 4$ *and M' is obtained from M_0 by deleting the central cocircuit of an augmented 4-wheel; or*

(ii) *M' is obtained from M_0 by trimming an open rotor chain, a ladder structure, or a ring of bowties; or*

(iii) *M_0 contains an open quartic ladder and M' is obtained from M_0 by a mixed ladder move; or*

(iv) *M_0 contains an enhanced quartic ladder and M' is obtained from M_0 by an enhanced-ladder move.*

Before stating our main theorem, we need to introduce one additional move along with some special families of matroids. Suppose M contains the structure in Figure 8(a) where $n = 2$ and $\{d_0, a_1, c_1, d_1\}$ is a cocircuit. Then we say that $M\backslash c_1, c_2/d_1, b_2$ has been obtained from M by a *ladder-compression move.* For $n \geq 3$, the *Möbius cubic ladder* consists of an even Hamiltonian cycle $\{u_1u_2, u_2u_3, \ldots, u_{2n-1}u_{2n}, u_{2n}u_1\}$ and a matching $\{u_iu_{i+n} : 1 \leq i \leq n\}$. The *Möbius quartic ladder* consists of an odd Hamiltonian cycle $\{v_1v_2, v_2v_3, \ldots, v_{2n-2}v_{2n-1}, v_{2n-1}v_1\}$ along with the set of edges $\{v_iv_{i+n-1}, v_iv_{i+n} : 1 \leq i \leq n\}$. In particular, when $n = 3$, the Möbius cubic ladder and the Möbius quartic ladder coincide with $K_{3,3}$ and K_5, respectively. It is straightforward to check that, for all $n \geq 3$, the cycle matroids of the Möbius cubic ladder and the Möbius quartic ladder are internally 4-connected. An illustration of the quartic Möbius ladder is shown in Figure 28 where $n \geq 2$ and the vertices v_1, v_2, and v_3 are identified with v_4, v_5, and v_6, respectively, so that $(c_n, d_n) = (a_0, b_0)$.

For each positive integer $n \geq 3$, let M_n be the binary matroid that is obtained from a wheel of rank n by adding a single element γ such that if B is the basis of $M(\mathcal{W}_n)$ consisting of the set of spokes of the wheel, then the fundamental circuit $C(\gamma, B)$ is $B \cup \gamma$. Observe that $M_3 \cong F_7$ and $M_4 \cong M^*(K_{3,3})$. Assume that the spokes of $M(\mathcal{W}_n)$, in cyclic order, are $x_1, x_2, \ldots, x_n$ and that $\{x_i, y_i, x_{i+1}\}$ is a triangle of $M(\mathcal{W}_n)$ for all i in $\{1, 2, \ldots, n\}$ where we interpret all subscripts modulo n. Then, for all i in $\{1, 2, \ldots, n\}$, the set $\{y_{i-1}, x_i, y_i\}$ is a triad of $M(\mathcal{W}_n)$ and $\{\gamma, y_{i-1}, x_i, y_i\}$ is a cocircuit of M_n. It is straightforward to check that M_n is internally 4-connected. Kingan and Lemos [12] denote M_n by F_{2n+1}. When n is odd, which is the case of most interest here, M_n^* is isomorphic to what Mayhew, Royle, and Whittle [13] call the *rank-$(n+1)$ triadic Möbius matroid*, Υ_{n+1}.

Each of the preliminary results stated above includes a lower bound on the size of the matroid M. In our main theorem, which we state next, no such bound is present. This is because small matroids were dealt with in [6], where most of the work was done using a computer. We denote the graph of a cube by Q_3.

Theorem 1.5. *Let M and N be internally 4-connected binary matroids such that N is a proper minor of M having at least six elements. Then, for some (M_0, N_0) in $\{(M,N),(M^*,N^*)\}$, the matroid M_0 has an internally 4-connected proper minor M' that has an N_0-minor such that*

(i) $|E(M)| - |E(M')| \leq 3$; *or*

(ii) $|E(M)| - |E(M')| = 4$, *and M' is obtained from M_0 by deleting the central cocircuit of a good augmented 4-wheel or by a ladder-compression move; or*

(iii) *M' is obtained from M_0 by removing at least four elements by*
 (a) *trimming an open rotor chain, a ladder structure, or a ring of bowties, or*
 (b) *a mixed ladder move, or*
 (c) *an enhanced-ladder move;*

 or

(iv) *M_0 is the cycle matroid of the quartic Möbius ladder of rank r for some $r \geq 4$, and N_0 is the cycle matroid of the cubic Möbius ladder of rank $r-1$; or*

(v) *M_0 is the triadic Möbius matroid of rank $2r$ for some $r \geq 3$, and N_0 is the triangular Möbius matroid of rank r; or*

(vi) $(M_0, N_0) \in \{(M(K_5), M(K_4)), (M(Q_3), M(K_4)), (\Upsilon_6^*, F_7)\}$.

By giving up successively more of the specificity offered by the last result, we immediately obtain the following two results.

Corollary 1.6. *Let M and N be internally 4-connected binary matroids such that N is a proper minor of M. Then, for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has an internally 4-connected proper minor M' that has an N_0-minor such that*

(i) $|E(M)| - |E(M')| \leq 4$; *or*

(ii) *M' is obtained from M_0 by removing at least five elements by*
 (a) *trimming an open rotor chain, a ladder structure, or a ring of bowties, or*
 (b) *a mixed ladder move, or*
 (c) *an enhanced-ladder move; or*

(iii) (M_0, N_0) *is* $(M(Q_3), M(K_4))$.

Corollary 1.7. *Let M and N be internally 4-connected binary matroids such that N is a proper minor of M. Then, for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has an internally 4-connected proper minor M' that has an N_0-minor such that*

(i) $|E(M)| - |E(M')| \leq 6$; *or*

(ii) *M' is obtained from M_0 by removing at least seven elements by*
 (a) *trimming an open rotor chain, a ladder structure, or a ring of bowties, or*
 (b) *a mixed ladder move, or*
 (c) *an enhanced-ladder move.*

We conclude this section with two further consequences of Theorem 1.5.

Corollary 1.8. *Let M and N be internally 4-connected binary matroids such that N is a proper minor of M having at least six elements. If neither M nor M^* contains a bowtie, then M has an internally 4-connected proper minor M' that has an N-minor such that $|E(M)| - |E(M')| \leq 3$.*

Corollary 1.9. *Let M and N be internally 4-connected binary matroids such that N is a proper minor of M having at least six elements. If neither M nor M^* has a string of bowties containing three triangles, then, for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has an internally 4-connected proper minor M' that has an N_0-minor such that one of the following holds.*

(i) $|E(M)| - |E(M')| \leq 3$; *or*
(ii) $|E(M)| - |E(M')| = 4$, *and*
  (a) M' *is obtained from* M_0 *by deleting the central cocircuit of an augmented* 4*-wheel or by a ladder-compression move; or*
  (b) (M_0, N_0) *is* $(M(K_5), M(K_4))$.

2. The theorem for graphs

In this section, we specialize the main theorem to graphs. To do that, we need to identify not only what the various moves look like for graphs but also what the duals of these moves look like in graphs.

We have used modified graph diagrams to describe various structures in matroids. We shall show in Section 8 that when such a structure arises in a graphic matroid, the graph diagram is an accurate representation of what occurs in the graph except that some vertices may be identified. Specifically, each of the cycles in the diagram represents a cycle in the graph. Moreover, each of the circled vertices corresponds to a vertex bond in the graph and no circled vertex equals any other vertex in the graph. While we do not insist that all of the uncircled vertices are distinct, the requirement that the graph G be internally 4-connected imposes constraints. In particular, most of the diagrams have at most four uncircled vertices. If two of these are equal, then there are at most three vertices to which the rest of G is attached, so G can have at most one vertex that is not part of the structure in the diagram. The only diagrams with more than four uncircled vertices are Figure 7, a bowtie ring which we shall discuss in more detail below, and Figure 9 in the case when $(a, b, c) \neq (d, e, f)$.

In (ii) and (iii) of Theorem 1.5, we allow various operations to be performed on either the matroid M or its dual. To specialize the splitter theorem to graphs, we now consider when $M = M(G)$ and we describe the effect on G of performing the following operations on M^*:

(a) deleting the central cocircuit of an augmented 4-wheel;
(b) a ladder-compression move;
(c) trimming an open rotor chain;
(d) trimming a ladder structure;
(e) a mixed ladder move;
(f) an enhanced-ladder move; and
(g) trimming a ring of bowties.

We have listed trimming a ring of bowties last in this list because it is this operation that allows for the most variation. We shall continue to use modified graph diagrams to describe the dual operations noting that these diagrams depict subgraphs of the graph G while also capturing the degrees of certain vertices in G. When G contains the structure in Figure 10 where all the vertices shown are distinct and $\{1, 2, 3, 4\}$ is a bond, the graph $G/5, 6, 7, 8$ is obtained from G by the dual of deleting the central cocircuit of an augmented 4-wheel.

Next suppose that G contains the structure in Figure 11 where all the edges shown are distinct. Then $G/c_1, c_2 \backslash d_1, b_2$ is obtained from G by the dual of a ladder-compression move. Evidently this operation corresponds to compressing a segment of an alternating biwheel.

The dual operation of trimming an open rotor chain is shown in Figure 12. This operation consists of contracting all of the edges in the figure that are marked with

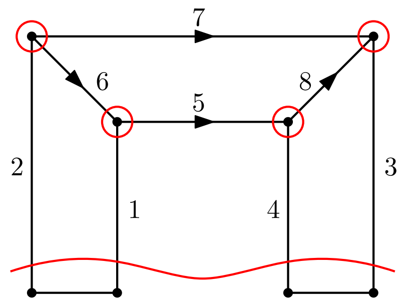

FIGURE 10. The dual of deleting the central cocircuit of an augmented 4-wheel: contract $5, 6, 7$, and 8.

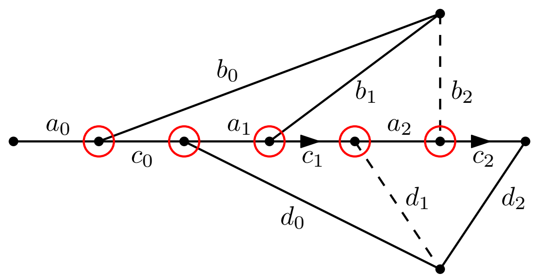

FIGURE 11. The dual of a ladder-compression move: contract c_1 and c_2 and delete d_1 and b_2.

an arrow. The number of such edges is at least three and may be even or odd. All of the edges in the figure are distinct. We see that this operation turns a cubic ladder segment into a quartic ladder segment.

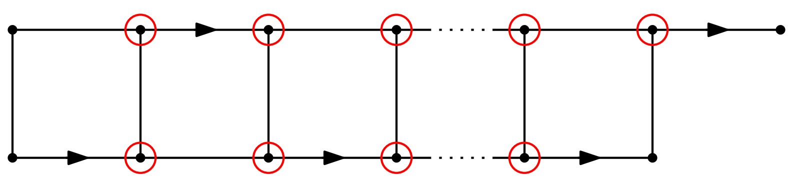

FIGURE 12. The dual of trimming an open-rotor chain: contract all of the edges marked with arrows.

The dual of trimming a ladder structure has several variants depending on whether the edges α, β, and γ are present in the ladder structure in Figure 8 and whether the ambiguous cocircuits in that figure have four or five elements. These variants are shown in Figure 13 where the dotted edges, which correspond to α, β, and γ, are either all present or all absent. In each of (a)–(c), the move consists of contracting all of the edges marked with an arrow, the number of such edges being at least three. All the edges in each part are distinct. We observe here that, although the operation of trimming a ladder structure has a restricted form in graphic matroids, the dual operation is less constrained.

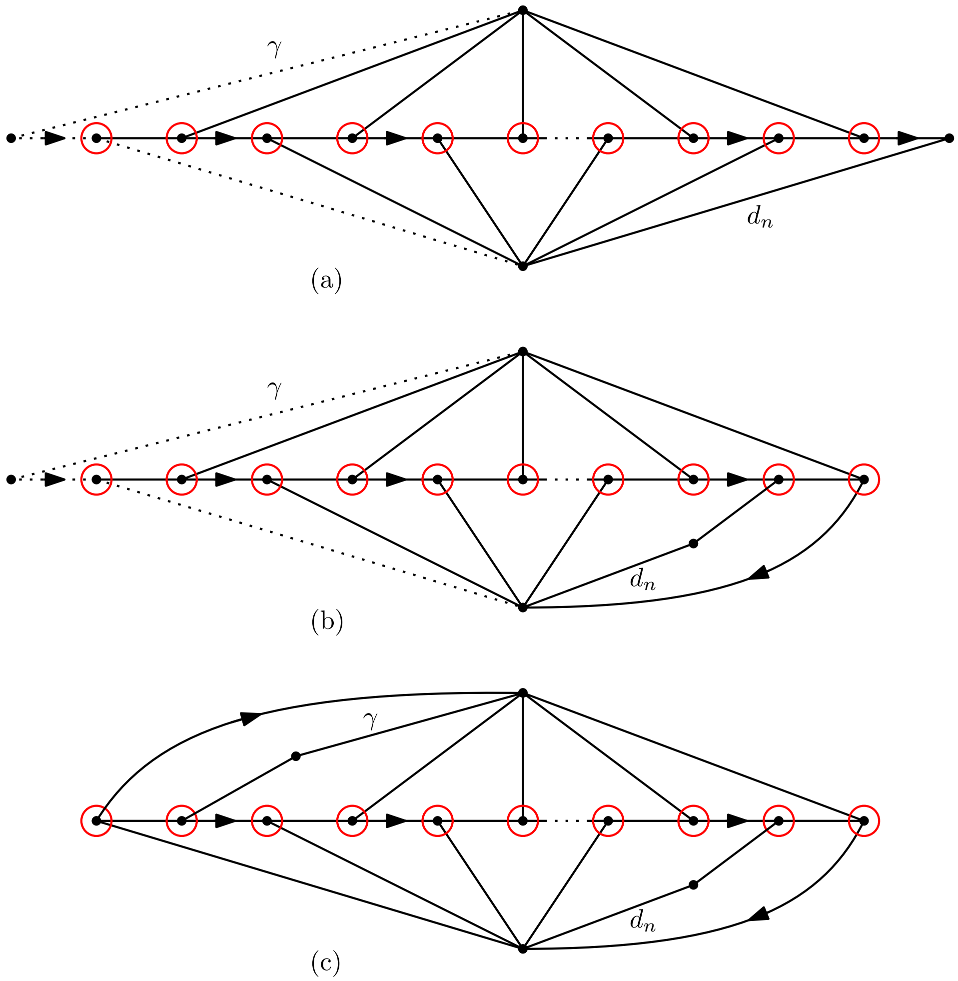

FIGURE 13. The dual of trimming a ladder structure: contract all of the edges marked with arrows.

The dual of a mixed ladder move is shown in Figure 14. All of the edges are distinct except that a may equal f, or $\{a,b,c\}$ may equal $\{d,e,f\}$ in such a way that $\{b,c\} \neq \{d,e\}$. The move contracts all of the edges marked with an arrow and deletes the dashed edge.

There are three variants of an enhanced-ladder move, which arise from the structures shown in Figures 4 and 5. As noted earlier, the structure shown in Figure 4(b) does not arise in a graphic matroid. However, the duals of all three variants of enhanced-ladder moves do arise in graphic matroids. The three possible structures are shown in Figure 15. In each, all of the edges are distinct and the move contracts all of the edges marked with arrows, the number of such edges being at least three, at least four, and at least six in (a), (b), and (c), respectively.

Let M be an internally 4-connected graphic matroid, say $M = M(G)$. When a bowtie ring $\{a_0,b_0,c_0\},\{b_0,c_0,a_1,b_1\},\{a_1,b_1,c_1\},\ldots,\{a_n,b_n,c_n\}$ arises in M, there are vertices $v_0,v_1,\ldots,v_n$ of G such that $\{b_i,c_i,a_{i+1},b_{i+1}\}$ is the

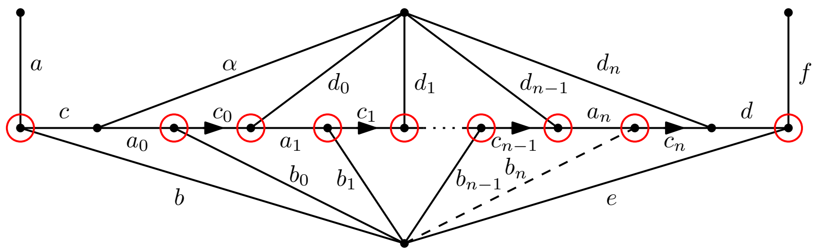

FIGURE 14. The dual of a mixed ladder move: contract all of the edges marked with arrows and delete the dashed edge.

set of edges of G meeting v_i. Hence $\{b_0, b_1, \dots, b_n\}$ is a cycle of G. By contrast, a bowtie ring in a more general matroid, even a cographic one, need not have $\{b_0, b_1, \dots, b_n\}$ as a circuit. If, in $M(G)$, the bowtie ring $\{a_0, b_0, c_0\}, \{b_0, c_0, a_1, b_1\}, \{a_1, b_1, c_1\}, \dots, \{a_n, b_n, c_n\}$ is trimmed to produce an internally 4-connected matroid, then the cycle $\{b_0, b_1, \dots, b_n\}$ in G must have at least four edges otherwise we get a 4-fan in the trimmed matroid. In G, let w_i be the vertex that meets a_i and c_i. The definition of a ring of bowties does not require $w_0, w_1, \dots, w_n$ to be distinct. Indeed, subject to some constraints to ensure that G stays internally 4-connected, they need not be. As an example [7], let $n = 99$ and take a copy of K_{10} with vertex set $\{u_0, u_1, \dots, u_9\}$. Suppose that, for all j in $\{0, 1, \dots, 99\}$, the vertex w_j is identified with u_t where $j \equiv t \mod 10$. Let G be the resulting 110-vertex graph. Then $M(G)$ is easily shown to be internally 4-connected. For $N = M(G)\backslash c_0, c_1, \dots, c_{99}$, we see that N is internally 4-connected but there is no internally 4-connected proper minor of M that has a proper N-minor.

The last example illustrates the considerable variance that can occur in bowtie rings even in graphic matroids. Unsurprisingly, bowtie rings can manifest themselves in a variety of ways in cographic matroids. The dual of a bowtie ring in a graph G consists of a sequence of 4-cycles, $\{b_0, c_0, a_1, b_1\}, \{b_1, c_1, a_2, b_2\}, \dots, \{b_n, c_n, a_0, b_0\}$ along with $n+1$ distinct degree-3 vertices $u_0, u_1, \dots, u_n$ where each u_i meets a_i, b_i, and c_i. We observe that this means that each of the distinguished 4-cycles meets exactly two of the vertices in $\{u_0, u_1, \dots, u_n\}$ and that these two degree-3 vertices are non-adjacent in G. Two possible variants of the dual of a bowtie-ring move are shown in Figure 16. In each of these, all of the edges are distinct but vertices with the same labels are to be identified. To illustrate the potential complexity of this move, a third variant is shown in Figure 17. In each of these figures, while all of the circled vertices must be distinct from all other vertices, some of the uncircled vertices may equal other uncircled vertices.

The following is our main result for graphs.

Theorem 2.1. *Let G and H be internally 4-connected graphs such that G has a proper H-minor and $|E(H)| \geq 6$. Then G has a proper minor G' such that G' is internally 4-connected with an H-minor and one of the following holds:*

(i) $|E(G)| - |E(G')| \leq 3$*; or*
(ii) $|E(G) - E(G')| = 4$ *and G' is obtained from G by deleting the central cocircuit of an augmented 4-wheel, by a ladder-compression move, or by the dual of one of these moves; or*

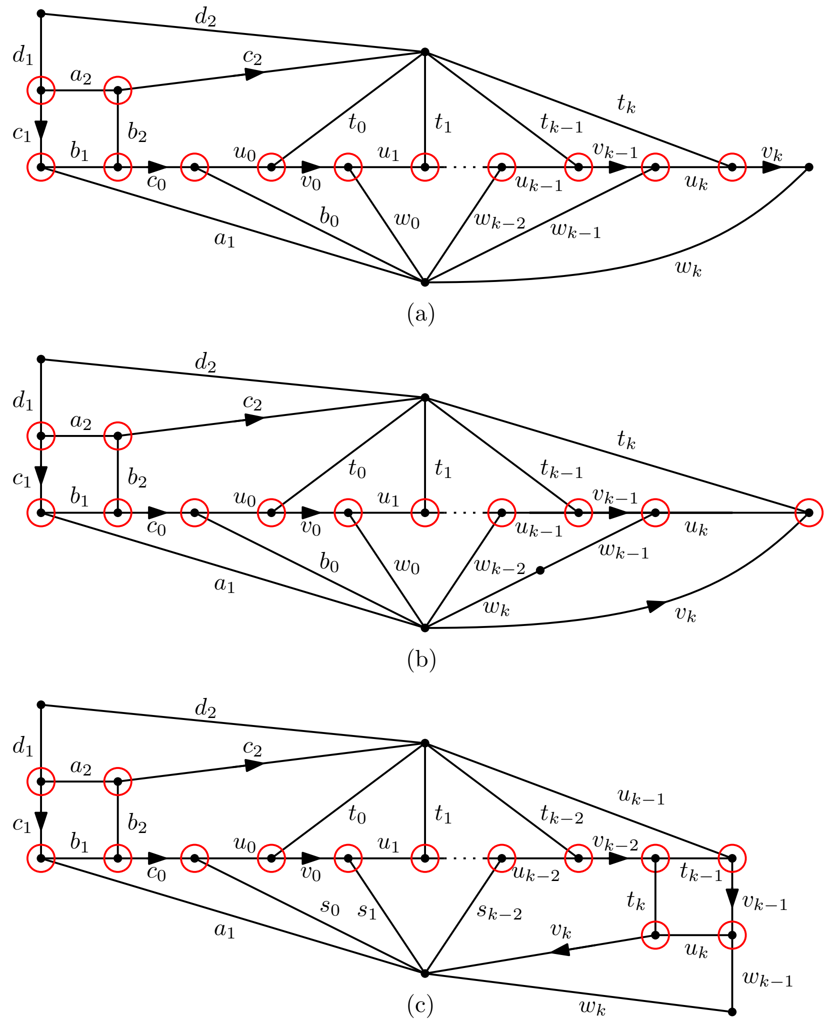

FIGURE 15. The dual of an enhanced-ladder move: contract all of the edges marked with arrows.

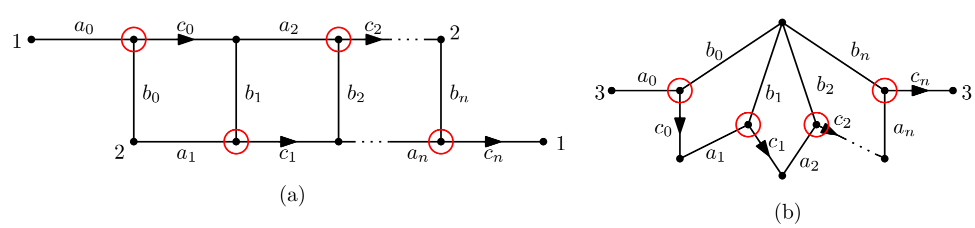

FIGURE 16. Two examples of the dual of trimming a bowtie ring: in each part, contract all the edges marked with arrows.

(iii) *G' is obtained from G by removing at least four edges by*

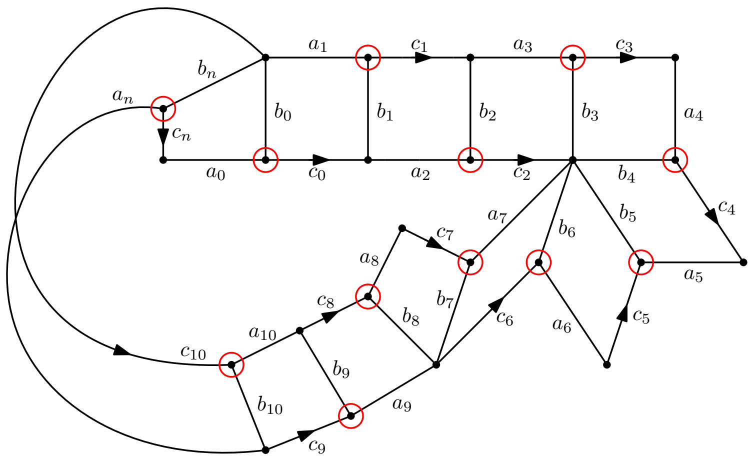

FIGURE 17. Another example of the dual of trimming a bowtie ring: contract all the edges marked with arrows.

(a) *trimming an open rotor chain, a graphic ladder structure, or a ring of bowties, or by the dual of trimming an open rotor chain, a ladder structure, or a ring of bowties; or*

(b) *a mixed ladder move or the dual of such a move; or*

(c) *a graphic enhanced-ladder move or the dual of an enhanced-ladder move;*

or

(iv) *G is a quartic Möbius ladder and H is a cubic Möbius ladder with $|V(G)|-1$ vertices; or*

(v) *G is K_5, Q_3, or $K_{2,2,2}$, and H is K_4.*

3. Preliminaries

In this section, we give some basic definitions mainly relating to matroid connectivity. Let M and N be matroids. We shall sometimes write $N \preceq M$ to indicate that M has an N-minor, that is, a minor isomorphic to N. Now let E be the ground set of M and r be its rank function. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E-X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate λ_M as λ. For a positive integer k, a subset X or a partition $(X, E-X)$ of E is *k-separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ of E is a *k-separation* if $|X|, |E-X| \geq k$. If n is an integer exceeding one, a matroid is *n-connected* if it has no k-separations for all $k < n$. This definition [16] has the attractive property that a matroid is n-connected if and only if its dual is. Moreover, this matroid definition of n-connectivity is relatively compatible with the graph notion of n-connectivity when n is 2 or 3. For example, when G is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if G is a 3-connected simple graph. But the link between n-connectivity for matroids and graphs breaks down for $n \geq 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geq 3$, neither $M(K_{r+1})$ nor

$PG(r-1,2)$ is 4-connected. This motivates the consideration of other types of 4-connectivity that allow certain 3-separations.

A matroid is *internally* 4-*connected* if it is 3-connected and, whenever (X, Y) is a 3-separation, either $|X| = 3$ or $|Y| = 3$. Equivalently, a 3-connected matroid M is internally 4-connected if and only if, for every 3-separation (X, Y) of M, either X or Y is a triangle or a triad of M. A graph G is *internally* 4-*connected* if $M(G)$ is internally 4-connected and G has no isolated vertices.

Let M be a matroid. A subset S of $E(M)$ is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of S. For convenience, we will often refer to the fan ordering as the fan. We will be mainly concerned with 4-element and 5-element fans. By convention, we shall always view a fan ordering of a 4-element fan as beginning with a triangle and we shall use the term 4-*fan* to refer to both the 4-element fan and such a fan ordering of it. Moreover, we shall use the terms 5-*fan* and 5-*cofan* to refer to the two different types of 5-element fan where the first contains two triangles and the second two triads. Let $(s_1, s_2, \ldots, s_n)$ be a fan ordering of a fan S. When M is 3-connected with at least five elements and $n \geq 4$, every fan ordering of S has its first and last elements in $\{s_1, s_n\}$. We call these elements the *ends* of the fan while the elements of $S - \{s_1, s_n\}$ are called the *internal elements* of the fan. When (s_1, s_2, s_3, s_4) is a 4-fan, our convention is that $\{s_1, s_2, s_3\}$ is a triangle, and we call s_1 the *guts element* of the fan and s_4 the *coguts element* of the fan since $s_1 \in \mathrm{cl}(\{s_2, s_3, s_4\})$ and $s_4 \in \mathrm{cl}^*(\{s_1, s_2, s_3\})$.

In a matroid M, a set U is *fully closed* if it is closed in both M and M^*. The *full closure* $\mathrm{fcl}(Z)$ of a set Z in M is the intersection of all fully closed sets containing Z. Let (X, Y) be a partition of $E(M)$. If (X, Y) is k-separating in M for some positive integer k, and y is an element of Y that is also in $\mathrm{cl}(X)$ or $\mathrm{cl}^*(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is k-separating, and we say that we have *moved* y into X. More generally, $(\mathrm{fcl}(X), Y - \mathrm{fcl}(X))$ is k-separating in M. Let n be an integer exceeding one. If M is n-connected, an n-separation (U, V) of M is *sequential* if $\mathrm{fcl}(U)$ or $\mathrm{fcl}(V)$ is $E(M)$. In particular, when $\mathrm{fcl}(U) = E(M)$, there is an ordering $(v_1, v_2, \ldots, v_m)$ of the elements of V such that $U \cup \{v_m, v_{m-1}, \ldots, v_i\}$ is n-separating for all i in $\{1, 2, \ldots, m\}$. When this occurs, the set V is called *sequential*. Moreover, if $n \leq m$, then $\{v_1, v_2, \ldots, v_n\}$ is a circuit or a cocircuit of M. A 3-connected matroid is *sequentially* 4-*connected* if all of its 3-separations are sequential. It is straightforward to check that, when M is binary, a sequential set with $3, 4$, or 5 elements is a fan. Let (X, Y) be a 3-separation of a 3-connected binary matroid M. We shall frequently be interested in 3-separations that indicate that M is, for example, not internally 4-connected. We call (X, Y) or X a $(4,3)$-*violator* if $|Y| \geq |X| \geq 4$. Similarly, (X, Y) is a $(4,4,S)$-*violator* if, for each Z in $\{X, Y\}$, either $|Z| \geq 5$, or Z is non-sequential. We also say that (X, Y) is a $(4,5,S,+)$-*violator* if, for each $Z \in \{X, Y\}$, either $|Z| \geq 6$, or Z is non-sequential, or Z is a 5-cofan. A binary matroid that has no $(4,4,S)$-violator is $(4,4,S)$-*connected*, as we defined in the introduction, and it is $(4,5,S,+)$-*connected* if it has no $(4,5,S,+)$-violator.

Next we note another special structure from [17], which has arisen frequently in our work towards the desired splitter theorem. In an internally 4-connected binary matroid M, we call $(\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{2,3,4,5\}, \{5,6,7,8\}, \{3,5,7\})$ a *quasi rotor* with *central triangle* $\{4,5,6\}$ and *central element* 5 if $\{1,2,3\}$, $\{4,5,6\}$, and $\{7,8,9\}$ are disjoint triangles in M such that $\{2,3,4,5\}$ and $\{5,6,7,8\}$ are cocircuits and $\{3,5,7\}$ is a triangle.

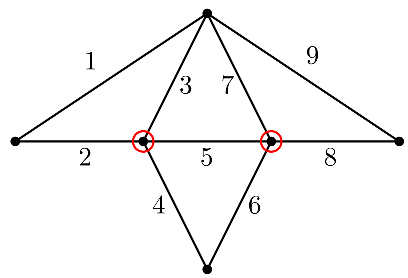

FIGURE 18. A quasi rotor, where $\{2,3,4,5\}$ and $\{5,6,7,8\}$ are cocircuits.

For all non-negative integers i, it will be convenient to adopt the convention of using T_i and D_{i+1} to denote, respectively, a triangle $\{a_i, b_i, c_i\}$ and a cocircuit $\{b_i, c_i, a_{i+1}, b_{i+1}\}$. Let M have $(T_0, T_1, T_2, D_0, D_1, \{c_0, b_1, a_2\})$ as a quasi rotor. Now T_2 may also be the central triangle of a quasi rotor. In fact, we may have a structure like one of the two depicted in Figure 19. If $T_0, D_0, T_1, D_1, \dots, T_n$ is a string of bowties in M, for some $n \geq 2$, and M has the additional structure that $\{c_i, b_{i+1}, a_{i+2}\}$ is a triangle for all i in $\{0, 1, \dots, n-2\}$, then we say that $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ is a *rotor chain*. Clearly, deleting a_0 from a rotor chain gives an open rotor chain. Observe that every three consecutive triangles within a rotor chain have the structure of a quasi rotor; that is, for all i in $\{0, 1, \dots, n-2\}$, the sequence $(T_i, T_{i+1}, T_{i+2}, D_i, D_{i+1}, \{c_i, b_{i+1}, a_{i+2}\})$ is a quasi rotor. Zhou [17] considered a similar structure that he called a *double fan of length* $n-1$; it consists of all of the elements in the rotor chain except for a_0, b_0, b_n, and c_n.

If a rotor chain $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n))$ cannot be extended to a rotor chain of the form $((a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_{n+1}, b_{n+1}, c_{n+1}))$, then we call it a *right-maximal rotor chain*. In the introduction, we defined a string of bowties. We say that such a string $T_0, D_0, T_1, D_1, \dots, T_n$ is a *right-maximal bowtie string* in M if M has no triangle $\{u, v, w\}$ such that $T_0, D_0, T_1, D_1, \dots, T_n, \{x, c_n, u, v\}, \{u, v, w\}$ is a bowtie string for some x in $\{a_n, b_n\}$.

When M and N are internally 4-connected and M has $T_0, D_0, T_1, D_1, \dots, T_n$ as a string of bowties, where $M\backslash c_0$ is $(4,4,S)$-connected with an N-minor, we call this string of bowties a *good bowtie string with n triangles*, or simply a *good bowtie string*. Furthermore, if $n \geq 2$ and $\{x, c_n, a_0, b_0\}$ is a cocircuit for some x in $\{a_n, b_n\}$, then call this ring of bowties a *good bowtie ring with n triangles*, or simply a *good bowtie ring*.

4. LADDERS AND MÖBIUS MATROIDS

In this section, we examine possible 3-connected minors of various special matroids that arise in our theorems. We begin with the cycle matroid of a quartic Möbius ladder.

Lemma 4.1. *Let M be the cycle matroid of a quartic Möbius ladder and let M' be an internally 4-connected proper minor of M having at least six elements. Then*

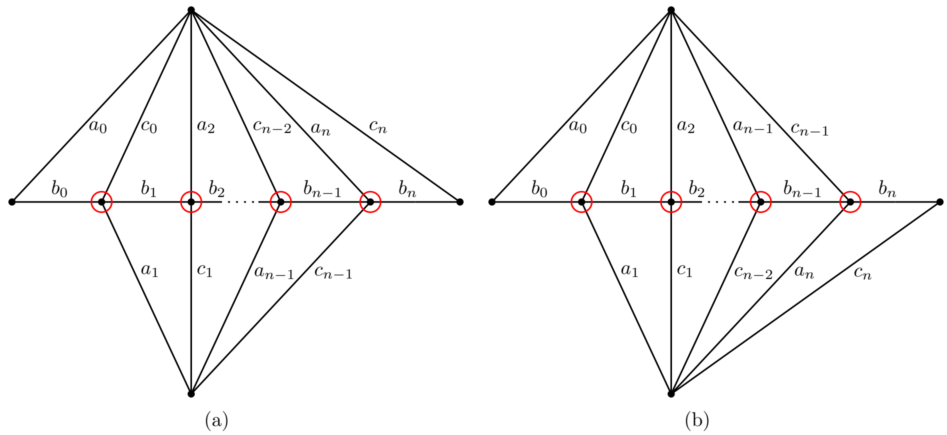

FIGURE 19. Right-maximal rotor chain configurations. In the case that n is even, the rotor chain is depicted on the left. If n is odd, then the rotor chain has the form on the right.

(i) *the matroid that is obtained from M by a ladder-compression move has an M'-minor; or*
(ii) *M' is the cycle matroid of a cubic Möbius ladder and $r(M') = r(M) - 1$.*

Proof. Let M be the cycle matroid of the quartic Möbius ladder that is obtained from the graph in Figure 28, where $n \geq 2$, by identifying the vertices v_1, v_2, and v_3 with v_4, v_5, and v_6, respectively, so that $(c_n, d_n) = (a_0, b_0)$. In addition, $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ is a cocircuit of M. Assume that (i) does not hold. Now $|E(M)| \geq 10$. Suppose $|E(M)| = 10$. Then $M \cong M(K_5)$. It is straightforward to check that the only internally 4-connected proper minor of $M(K_5)$ with at least six elements is $M(K_4)$. Since the last matroid is the cycle matroid of a cubic Möbius ladder of rank 3, it follows that (ii) holds. We may now assume that $|E(M)| \geq 14$. Since (i) does not hold, $|E(M')| \geq 7$. Moreover, for all i in $\{0, 1, \ldots, n\}$, neither $M/d_i, b_{i+1}$ nor $M/d_i, b_i$ has an M'-minor. We show next that

4.1.1. *$M\backslash c_i$ or $M\backslash a_i$ has an M'-minor for some i in $\{0, 1, \ldots, n\}$.*

Assume that this fails. Since every element of M is in a triangle, $M\backslash x$ has an M'-minor for some element x. By symmetry, we may assume that $x = d_1$. As $M\backslash d_1$ has $(a_2, b_2, c_2, d_2, a_3)$ as a 5-fan, [7, Lemma 4.1] implies that $M\backslash d_1, a_2$ or $M\backslash d_1\backslash a_3$ has an M'-minor; a contradiction. We conclude that 4.1.1 holds.

Now, by 4.1.1 and symmetry, we may assume that $M\backslash c_0$ has an M'-minor. Let t be maximal such that $M\backslash c_0, c_1, \ldots, c_t$ has an M'-minor. Assume first that $t \neq n$. Then, as $M\backslash c_0, c_1, \ldots, c_t$ has $(c_{t+1}, b_{t+1}, a_{t+1}, b_t)$ as a 4-fan, $M' \preceq M\backslash c_0, c_1, \ldots, c_t/b_t$. The last matroid has $(a_{t+1}, a_t, d_t, d_{t-1})$ as a 4-fan. Since $M' \not\preceq M/b_t, d_{t-1}$, we deduce that $M' \preceq M\backslash c_0, c_1, \ldots, c_t/b_t\backslash a_{t+1}$. But $M\backslash c_0, c_1, \ldots, c_t\backslash a_{t+1}/b_t \cong M\backslash c_0, c_1, \ldots, c_t\backslash a_{t+1}/b_{t+1} \cong M\backslash c_0, c_1, \ldots, c_t\backslash c_{t+1}/b_{t+1}$, so we have a contradiction.

We now know that $t = n$. Thus $M' \preceq M\backslash c_0, c_1, \ldots, c_n$. The last matroid has $\{b_n, d_0\}$ as a disjoint union of cocircuits. Let $L = M\backslash c_0, c_1, \ldots, c_n/b_n$. Then

$M' \preceq L$ and $r(L) = r(M) - 1$. Moreover, L is the cycle matroid of a cubic Möbius ladder.

To complete the proof, we now suppose that M' is a proper minor of L. Then, for some rung element s of L or some non-rung element u of L, one of $L\backslash s$, L/s, $L\backslash u$, or L/u has an M'-minor. We show next that

4.1.2. *L/u has an M'-minor.*

This follows using [7, Lemma 4.1] since L/s has a 5-cofan whose ends are non-rung elements while each of $L\backslash s$ and $L\backslash u$ has a 2-element cocircuit that contains a non-rung element.

With M' being a minor of L/u, the symmetry of L implies that the non-rung element u is in a 4-circuit of L with another non-rung element w such that w is in a 2-cocircuit in $M\backslash c_0, c_1, \dots, c_n$. It follows that M has an M'-minor that can be obtained by a ladder-compression move; a contradiction. □

For $n \geq 1$, let A_{2n+1} be the matrix over $GF(2)$ shown in Figure 20. Recall from the introduction that M_{2n+1} is the binary matroid for which A_{2n+1} is a reduced standard representative matrix.

$$
\begin{array}{c}
\\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \\ x_{2n} \\ x_{2n+1}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & \cdots & y_{2n} & y_{2n+1} & \gamma \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & \dots & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & \dots & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & \dots & 1 & 1 & 1
\end{bmatrix}
\end{array}.
$$

FIGURE 20. A_{2n+1}.

As noted earlier, $M_3 \cong F_7$. Moreover, M_5 is isomorphic to a single-element deletion of the matroid T_{12} that was introduced by Kingan [11] and that has a transitive automorphism group. In general, $M^*_{2n+1} \cong \Upsilon_{2n+2}$, the rank-$(2n+2)$ triadic Möbius matroid. It is straightforward to check that, for all i in $\{1, 2, \dots, 2n+1\}$, the matroid M_{2n+1} contains the configuration in Figure 21 where all subscripts are interpreted modulo $2n+1$. Moreover, M_{2n+1}/γ is the cycle matroid of a quartic Möbius ladder of rank $2n$. Hence M_{2n+1}/γ is internally 4-connected. It is not difficult to check that, for all $n \geq 2$, the matroid $M_{2n+1}\backslash x_1, x_3, \dots, x_{2n+1}/y_1$ is isomorphic to the dual of the rank-$(n+1)$ triangular Möbius matroid, Δ_{n+1}.

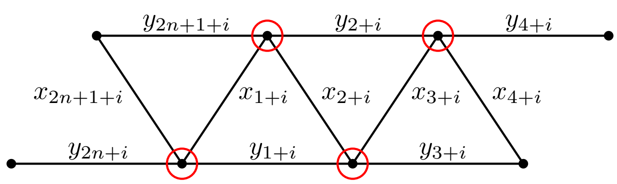

FIGURE 21. A configuration within M_{2n+1}.

Lemma 4.2. *Let M' be an internally 4-connected proper minor of M_{2n+1} having at least eight elements. Then one of the following holds.*

(i) *M' is a minor of M_{2n+1}/γ, that is, M' is a minor of the cycle matroid of a quartic Möbius ladder; or*
(ii) *the matroid that is obtained from M_{2n+1} by a ladder-compression move has an M'-minor; or*
(iii) *M' is the dual of Δ_{n+1}.*

Proof. Assume that the lemma fails. We show first that

4.2.1. $\gamma \in E(M')$.

As (i) does not hold, $M' \not\preceq M_{2n+1}/\gamma$. Suppose that $M' \preceq M_{2n+1}\backslash\gamma$. Then M' is an internally 4-connected minor of $M(\mathcal{W}_{2n+1})$. Thus $M' \cong M(K_4)$. But $|E(M')| \geq 8$. This contradiction implies that 4.2.1 holds.

4.2.2. *$M' \not\preceq M_{2n+1}/x_i$ for all i in $\{1, 2, \ldots, 2n+1\}$.*

By symmetry, it suffices to show that $M' \not\preceq M_{2n+1}/x_1$. Assume the contrary. As M_{2n+1}/x_1 has $\{x_{2n+1}, y_{2n+1}\}$ and $\{y_1, x_2\}$ as circuits, $M' \preceq M_{2n+1}/x_1\backslash y_{2n+1}, y_1$. But $M_{2n+1}\backslash y_{2n+1}, y_1/x_1 \cong M_{2n+1}\backslash y_{2n+1}, y_1/\gamma$, so $M' \preceq M_{2n+1}/\gamma$; a contradiction. Thus 4.2.2 holds.

4.2.3. *$M' \not\preceq M_{2n+1}\backslash y_i$ for all i in $\{1, 2, \ldots, 2n+1\}$.*

To see this, suppose that $M' \preceq M_{2n+1}\backslash y_1$. As $M_{2n+1}\backslash y_1$ has $(y_3, x_4, x_3, y_2, x_2)$ as a 5-fan, it follows by [7, Lemma 4.1] and 4.2.2 that $M' \preceq M_{2n+1}\backslash y_1, y_3$. But, using Figure 21, we see that $M_{2n+1}\backslash y_1, y_3$ has x_2 in a 2-cocircuit, so $M' \preceq M_{2n+1}/x_2$, a contradiction to 4.2.2. We conclude, using symmetry, that 4.2.3 holds.

Since M' is a proper minor of M_{2n+1}, we must have that, for some i, either $M' \preceq M_{2n+1}/y_i$, or $M' \preceq M_{2n+1}\backslash x_i$. But, in the former case, $M' \preceq M_{2n+1}/y_i\backslash x_i$. It follows that we may assume that $M' \preceq M_{2n+1}\backslash x_1$, say. The last matroid has $(x_3, x_2, y_2, y_{2n+1})$ as a 4-fan. Thus $M' \preceq M_{2n+1}\backslash x_1, x_3$, or $M' \preceq M_{2n+1}\backslash x_1/y_{2n+1}$. In the latter case, since $M_{2n+1}\backslash x_1/y_{2n+1}$ has $(y_{2n}, x_{2n}, x_{2n+1}, x_2, y_1)$ as a 5-cofan, by [7, Lemma 4.1], $M' \preceq M_{2n+1}\backslash x_1/y_{2n+1}, y_1$ or $M' \preceq M_{2n+1}\backslash x_1/y_{2n+1}, y_{2n}$. It follows by symmetry that $M' \preceq M_{2n+1}/y_{2n+1}, y_1\backslash x_1, x_2$, so M' is isomorphic to a minor of M_{2n+1} that can be obtained by a ladder-compression move; a contradiction.

We deduce that $M' \preceq M_{2n+1}\backslash x_1, x_3$. Let t be the maximal member of $\{1, 2, \ldots, n\}$ such that $M' \preceq M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}$. We show next that

4.2.4. $t = n$.

Assume $t < n$. Then, as $M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}$ has $(x_{2t+3}, y_{2t+2}, x_{2t+2}, y_{2t})$ as a 4-fan, it follows by the choice of t that $M' \preceq M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}/y_{2t}$. The last matroid has $(x_{2t+2}, x_{2t}, y_{2t+1}, y_{2t-1})$ as a 4-fan. Thus either $M' \preceq M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}/y_{2t}, y_{2t-1}$, or $M' \preceq M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}/y_{2t}\backslash x_{2t+2}$. The latter must hold since the former implies that M' is isomorphic to a minor of M_{2n+1} that is obtained by a ladder-compression move; a contradiction. But

$$\begin{aligned} M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}/y_{2t}\backslash x_{2t+2} &\cong M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}\backslash x_{2t+2}/y_{2t+2} \\ &\cong M_{2n+1}\backslash x_1, x_3, \ldots, x_{2t+1}/y_{2t+2}\backslash x_{2t+3}, \end{aligned}$$

so the choice of t is contradicted and 4.2.4 holds.

We now know that $M' \preceq M_{2n+1}\backslash x_1, x_3, \ldots, x_{2n+1}$. The last matroid has $\{y_{2n}, y_1\}$ as a union of cocircuits. Thus $M' \preceq M_{2n+1}\backslash x_1, x_3, \ldots, x_{2n+1}/y_1 \cong \Delta_{n+1}$. Suppose M' is isomorphic to a proper minor of $M_{2n+1}\backslash x_1, x_3, \ldots, x_{2n+1}/y_1$. Then this M'-minor must be obtained from the last matroid by deleting some member of $\{x_2, x_4, \ldots, x_{2n}\}$ or by contracting some member of $\{y_2, y_3, \ldots, y_{2n+1}\}$. It is straightforward to check, by exploiting the structure in Figure 21 in a familiar way, that $M' \preceq M_{2n+1}/y_1, y_2$; a contradiction. We conclude that the lemma holds. □

5. Bowties

In this section, we prove some results for bowties. We also prepare for the subsequent sections by defining some terminology concerning what we consider to be winning moves with respect to the splitter theorem. The following simple result from [7, Lemma 4.2] will be useful when dealing with bowtie structures.

Lemma 5.1. *Let M be an internally 4-connected matroid having at least ten elements. If $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ is a bowtie in M, then $\{2,3,4,5\}$ is the unique 4-cocircuit of M that meets both $\{1,2,3\}$ and $\{4,5,6\}$.*

When we contract b_1 in the good bowtie string T_0, D_0, T_1, D_1, T_2, we will apply the following result from [8, Lemma 3.9].

Lemma 5.2. *Let M be an internally 4-connected binary matroid having T_0, D_0, T_1, D_1, T_2 as a string of bowties and suppose that $M\backslash c_1$ is $(4,4,S)$-connected. Then*

(i) *$M\backslash c_1/b_1$ is internally 4-connected; or*
(ii) *$M\backslash c_1/b_1$ is $(4,5,S,+)$-connected and M has a triangle $\{1,2,3\}$ such that $\{2,3,a_1,c_1\}$ is a cocircuit; or*
(iii) *$M\backslash c_1/b_1$ is $(4,5,S,+)$-connected and M has elements d_0 and d_1 such that $\{d_0,d_1\}$ avoids $T_0 \cup T_1 \cup T_2$, and $\{d_0,a_1,c_1,d_1\}$ is a cocircuit, and $\{d_0,a_1,s\}$ or $\{d_1,c_1,t\}$ is a triangle for some s in $\{b_0,c_0\}$ or t in $\{a_2,b_2\}$.*

The following lemma is from [2, Lemma 8.3].

Lemma 5.3. *Let M be an internally 4-connected binary matroid with at least thirteen elements. Let M have $\{a,b,c\}$, $\{1,2,3\}$, $\{4,5,6\}$, and $\{7,8,9\}$ as triangles and $\{1,2,a,b\}$, $\{4,5,b,c\}$, and $\{7,8,a,c\}$ as cocircuits. Assume that $a,b,c,1,2,3,4,5,6,7,8$, and 9 are distinct except that, possibly, $3=9$, or $3=6$, or $6=9$. Then either*

(i) *$\{a,b,c\}$ is the central triangle of a quasi rotor; or*
(ii) *$M/a,b,c$ is internally 4-connected.*

We now prove a lemma regarding the structure that arises in the last lemma.

Lemma 5.4. *Let M be an internally 4-connected binary matroid with at least thirteen elements. Let M have $\{a,b,c\}$, $\{1,2,3\}$, $\{4,5,6\}$, and $\{7,8,9\}$ as triangles and $\{1,2,a,b\}$, $\{4,5,b,c\}$, and $\{7,8,a,c\}$ as cocircuits. Let N be an internally 4-connected matroid that is a minor of $M/a,b,c$. Then M has an internally 4-connected proper minor M' that has an N-minor such that $|E(M)-E(M')| \le 3$.*

Proof. Lemma 5.1 implies that $a,b,c,1,2,3,4,5,6,7,8$, and 9 are distinct except that, possibly, $3=9$, or $3=6$, or $6=9$. The lemma is immediate if $M/a,b,c$ is internally 4-connected. Thus, from the last lemma, we may assume that $\{a,b,c\}$ is the central triangle of a quasi rotor. As N is a minor of each of M/a, M/b, and M/c, and one of a, b, and c is in two triangles of the quasi rotor, the theorem follows by [7, Lemma 4.5]. □

The next lemma is helpful when dealing with strings of bowties.

Lemma 5.5. *Let M be a binary matroid with an internally 4-connected minor N with $|E(N)| \geq 6$. Let $T_0, D_0, T_1, D_1, \ldots, T_n$ be a string of bowties in M with $n \geq 2$, and suppose that $M\backslash c_0, c_1$ has an N-minor. Then*

(i) *$M\backslash c_0, c_1, \ldots, c_n$ has an N-minor; or $M\backslash c_0, c_1, \ldots, c_\ell/b_\ell$ has an N-minor for some ℓ in $\{1, 2, \ldots, n\}$, and $M\backslash c_0/b_0$ has an N-minor; and*

(ii) *for all j in $\{1, 2, \ldots, n\}$,*

$$\begin{aligned} M\backslash c_0, c_1, \ldots, c_n/b_n &\cong M\backslash c_0, c_1, \ldots, c_{j-1}, a_j, a_{j+1}, \ldots, a_n/b_j \\ &\cong M\backslash c_0, c_1, \ldots, c_{j-1}, a_j, a_{j+1}, \ldots, a_n/b_{j-1} \\ &\cong M\backslash a_0, a_1, \ldots, a_n/b_0; \end{aligned}$$

and

(iii) *if $M\backslash c_0, c_1, \ldots, c_n$ has an N-minor and has a 2-cocircuit containing a member x of $\{a_i, b_i\}$ for some i in $\{1, 2, \ldots, n\}$, then, for the element y of $\{a_i, b_i\} - x$ and for all k in $\{0, 1, \ldots, i-1\}$,*

$$N \preceq M\backslash c_0, c_1, \ldots, c_k, a_{k+1}, a_{k+2}, \ldots a_{i-1}, y, c_{i+1}, c_{i+2}, \ldots, c_n/b_k.$$

Proof. From [7, Lemma 5.2], we know that (ii) holds. To see that (i) holds, take ℓ as large as possible such that $M\backslash c_0, c_1, \ldots, c_\ell$ has an N-minor. If $\ell < n$, then $(c_{\ell+1}, b_{\ell+1}, a_{\ell+1}, b_\ell)$ is a 4-fan in $M\backslash c_0, c_1, \ldots, c_\ell$, so we can contract b_ℓ in this matroid keeping an N-minor since we have no N-minor when we delete $c_{\ell+1}$. By (ii), $M\backslash c_0, c_1, \ldots, c_\ell/b_\ell \cong M\backslash a_0, a_1, a_2, \ldots, a_\ell/b_0$. Thus M/b_0, and hence $M\backslash c_0/b_0$, has an N-minor.

Finally, we show that (iii) holds. Clearly $M\backslash c_0, c_1, \ldots, c_n/x$ has an N-minor, so $M\backslash c_0, c_1, \ldots, c_{i-1}, y, c_{i+1}, c_{i+2}, \ldots, c_n/x$ has an N-minor. By (ii), the last matroid is isomorphic to $M\backslash c_0, c_1, \ldots, c_k, a_{k+1}, a_{k+2}, \ldots a_{i-1}, y, c_{i+1}, c_{i+2}, \ldots, c_n/b_k$ for all k in $\{0, 1, \ldots, i-1\}$. Thus (iii) holds. □

When M contains a good bowtie, we know from Theorem 1.4 that Theorem 1.5 holds provided M satisfies certain requirements. Because of this, in our work towards the proof of Theorem 1.5, we shall frequently impose the following:

Hypothesis VIII. *If, for (M_1, N_1) in $\{(M, N), (M^*, N^*)\}$, the matroid M_1 has a bowtie $(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{b_0, c_0, a_1, b_1\})$, where $M_1\backslash c_0$ is $(4, 4, S)$-connected with an N_1-minor, then $M_1\backslash c_0, c_1$ has an N_1-minor, and $M_1\backslash c_1$ is $(4, 4, S)$-connected.*

The following lemma is an easy adaptation of [8, Lemma 3.5] in light of the preceding hypothesis. Indeed, if Hypothesis VIII holds, then so does Hypothesis VII of [8].

Lemma 5.6. *Let $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ be a bowtie in an internally 4-connected binary matroid M with $|E(M)| \geq 13$. Let N be an internally 4-connected minor of M having at least seven elements. Suppose that $M\backslash 4$ is $(4, 4, S)$-connected with an N-minor and that Hypothesis VIII holds. Then $N \preceq M\backslash 1, 4$ and $M\backslash 1$ is $(4, 4, S)$-connected with an N-minor. In addition,*

(i) *$M\backslash 1$ is internally 4-connected; or*

(ii) *M has a triangle $\{7,8,9\}$ such that $(\{1,2,3\},\{7,8,9\},\{7,8,1,s\})$ is a bowtie for some s in $\{2,3\}$, and $|\{1,2,\ldots,9\}| = 9$; or*
(iii) *every $(4,3)$-violator of $M\backslash 1$ is a 4-fan of the form $(4,t,7,8)$, for some t in $\{2,3\}$ where $\{1,2,3,4,5,6,7,8\}| = 8$; or*
(iv) *$M\backslash 6$ is internally 4-connected with an N-minor.*

Beginning with the next lemma and for the rest of the paper, we shall start abbreviating how we refer to the following five outcomes in the main theorem:

(i) M has a proper minor M' such that $|E(M)| - |E(M')| \leq 3$ and M' is internally 4-connected with an N-minor;
(ii) M contains an open rotor chain, a ladder structure, or a ring of bowties that can be trimmed to obtain an internally 4-connected matroid with an N-minor;
(iii) M contains an open quartic ladder where the deletion of all of the dashed elements followed by the contraction of the arrow element is internally 4-connected with an N-minor;
(iv) M contains an enhanced quartic ladder from which an internally 4-connected minor of M with an N-minor can be obtained by an enhanced-ladder move;
(v) a ladder-compression move in M yields an internally 4-connected minor with an N-minor.

When (i) or (iv) holds, we say, respectively, that M has a *quick win* or an *enhanced-ladder win*. When trimming an open rotor chain, a ladder structure, or a ring of bowties in M produces an internally 4-connected matroid with an N-minor, we say, respectively, that M has an *open-rotor-chain win*, a *ladder win*, or a *bowtie-ring win*. When (iii) or (v) holds, we say, respectively, that M has a *mixed-ladder win* or a *ladder-compression win*.

6. When a good bowtie is a right-maximal bowtie chain

Suppose that M has a bowtie $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$ where $M\backslash 4$ is $(4,4,S)$-connected with an N-minor. If Hypothesis VIII holds, then, in Lemma 5.6, we showed that we either get a quick win, or one of two possible configurations arises. Specifically, either M contains a good bowtie string with three triangles, or M contains the configuration shown in Figure 22, where we may have switched the labels on 2 and 3. In each of these, $M\backslash 1$ is $(4,4,S)$-connected.

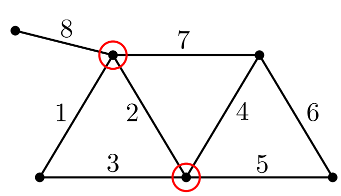

Figure 22. All the elements shown are distinct.

In this section, we give more details about the structure surrounding the configuration in Figure 22. We will show that if M contains this configuration, then either M contains a good bowtie string with three triangles, or, by removing at most three elements from M, we can find an internally 4-connected proper minor of M that has an N-minor. More precisely, we will prove the following result.

Theorem 6.1. *Let M and N be internally 4-connected binary matroids where $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose that Hypothesis VIII holds. If M contains the configuration in Figure 22 where $M\backslash 4$ is $(4,4,S)$-connected having an N-minor and $|\{1,2,\dots,8\}| = 8$, then either M has a quick win, or M contains a good bowtie string with three triangles.*

The proof of this theorem will use several lemmas.

Lemma 6.2. *Let M be an internally 4-connected binary matroid with at least thirteen elements. If M contains the configuration in Figure 22 where both $M\backslash 4$ and $M\backslash 1$ are $(4,4,S)$-connected, then $M\backslash 6$ is $(4,4,S)$-connected and $\{4,5,6\}$ is the only triangle in M containing 5.*

Proof. Assume that $M\backslash 6$ is not $(4,4,S)$-connected. Then, by [7, Lemma 4.3], M has a quasi rotor $(\{1,2,3\},\{4,5,6\},\{x,y,z\},\{2,3,4,5\},\{v,6,x,y\},\{u,v,x\})$ for some u in $\{2,3\}$ and v in $\{4,5\}$. Since $M\backslash 4$ is $(4,4,S)$-connected, $v = 4$. By orthogonality between $\{2,4,7\}$ and the cocircuit $\{4,6,x,y\}$, we deduce that $7 \in \{x,y\}$. Then orthogonality between $\{1,2,7,8\}$ and $\{x,y,z\}$ implies that $8 \in \{x,y,z\}$. Thus $\lambda(\{1,2,\dots,6,x,y,z\}) \leq 2$; a contradiction. Therefore $M\backslash 6$ is $(4,4,S)$-connected.

Suppose M has a triangle T other than $\{4,5,6\}$ containing 5. By orthogonality, T contains 2 or 3. Hence $T \cup \{1,2,3\}$ is a 5-fan in $M\backslash 4$; a contradiction. $\square$

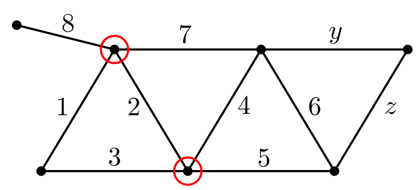

FIGURE 23. M has $\{4,6,7,y\}$ or $\{1,2,4,6,y\}$ as a cocircuit.

Lemma 6.3. *Let M and N be internally 4-connected binary matroids where $|E(M)| \geq 15$ and $|E(N)| \geq 7$. If M contains the configuration in Figure 22, where $M\backslash 4$ and $M\backslash 1$ are $(4,4,S)$-connected and $M\backslash 1,4$ has an N-minor, then $M\backslash 1,4$ is $(4,4,S)$-connected. Furthermore,*

(i) *$M\backslash 1,4$ is internally 4-connected; or*
(ii) *M contains the structure in Figure 23, where y and z are elements not in $\{1,2,\dots,8\}$, and $\{4,6,7,y\}$ or $\{1,2,4,6,y\}$ is a cocircuit of M; or*
(iii) *M contains a good bowtie string with three triangles.*

Proof. We apply [7, Lemma 6.1] and conclude that $M\backslash 1,4$ is sequentially 4-connected having no triangle containing 2. Moreover,

(a) $M\backslash 1,4$ is internally 4-connected; or
(b) $\{7,8\}$ is in a triangle of M; or
(c) $\{3,5\}$ is in a triangle of M; or
(d) $M\backslash 1,4$ is $(4,4,S)$-connected and M contains the structure shown in Figure 23, where y and z are not in $\{1,2,\dots,8\}$, and $\{4,6,7,y\}$ or $\{1,2,4,6,y\}$ is a cocircuit; or

(e) $M\backslash 1,4$ is $(4,4,S)$-connected having 3 as the coguts element of every 4-fan.

If (a) holds, then the lemma holds, so we assume not. As $M\backslash 1$ is $(4,4,S)$-connected, (b) does not hold. Now Lemma 6.2 implies that $\{4,5,6\}$ is the only triangle of M containing 5, so (c) does not hold. If (d) holds, then (ii) of the lemma holds.

We conclude that (e) holds. Hence we may assume that $M\backslash 1,4$ has a 4-fan of the form $(v_1,v_2,v_3,3)$. Then M contains a cocircuit C^* such that $\{v_2,v_3,3\} \subsetneqq C^* \subseteq \{v_2,v_3,3,1,4\}$. If $4 \in C^*$, then orthogonality implies that $\{v_2,v_3\}$ meets $\{2,7\}$ and $\{5,6\}$. As noted at the outset, 2 is in no triangles of $M\backslash 1,4$, so $7 \in \{v_1,v_2,v_3\}$. But $\{v_1,v_2,v_3\} \triangle \{2,5,6,7\}$ is a triangle of $M\backslash 1,4$ containing 2; a contradiction. Thus $C^* = \{v_2,v_3,3,1\}$, and M contains a good bowtie string with three triangles, so (iii) of the lemma holds. □

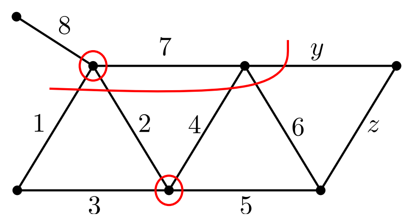

FIGURE 24. Here $\{1,2,4,6,y\}$ is a cocircuit.

In the ladder structures that arise, we occasionally need to consider a 5-cocircuit of the form shown in Figure 24. In the next lemma, we develop properties of this structure.

Lemma 6.4. *Suppose that M is an internally 4-connected binary matroid with at least thirteen elements, and that M contains the structure in Figure 24. If both $M\backslash 4$ and $M\backslash 1$ are $(4,4,S)$-connected, then the elements in Figure 24 are all distinct, $\{4,6\}$ is not contained in a 4-cocircuit of M, and $M\backslash 6$ or $M\backslash 6/5$ is internally 4-connected.*

Proof. First note that the elements in Figure 24 are all distinct otherwise $\lambda(\{1,2,\dots,8,y,z\}) = 2$; a contradiction. We show next that

6.4.1. $\{4,6\}$ *is not in a 4-cocircuit of M.*

Assume $\{4,6\}$ is in a 4-cocircuit C^* of M. Then the triangles $\{4,2,7\}$ and $\{6,y,z\}$ imply that $C^* \subseteq \{6,4,2,7,y,z\}$, so $\lambda(\{1,2,\dots,7,y,z\}) \le 2$; a contradiction. Hence 6.4.1 holds.

Now assume that neither $M\backslash 6$ nor $M\backslash 6/5$ is internally 4-connected. Lemma 6.2 implies that

6.4.2. $\{4,5,6\}$ *is the only triangle of M containing* 5.

We now apply [8, Lemma 3.4] noting that either $M\backslash 6$ is $(4,4,S)$-connected and one of (i)–(iv) of that lemma hold; or $M\backslash 6$ is not $(4,4,S)$-connected and M has $(\{1,2,3\},\{4,5,6\},\{7,8,9\},\{2,3,4,5\},\{y,6,7,8\},\{x,y,7\})$ as a quasi rotor for some x in $\{2,3\}$ and y in $\{4,5\}$. Neither the latter nor (iii) of [8, Lemma 3.4] holds because 6.4.1 and 6.4.2 hold and $M\backslash 4$ is $(4,4,S)$-connected. We deduce that $M\backslash 6$ is $(4,4,S)$-connected. It is immediate that neither (i) nor (iv) of [8, Lemma 3.4] holds. Thus (ii) of that lemma holds, so M has a 4-cocircuit $\{5,6,u,v\}$ and a triangle $\{u,v,w\}$ such that $|\{1,2,3,4,5,6,u,v,w\}| = 9$. Then M has $\{1,2,3\},\{2,3,4,5\},\{4,5,6\},\{5,6,u,v\},\{u,v,w\}$ as a string of bowties, and $M\backslash 6$ is $(4,4,S)$-connected. As M has no 4-cocircuit containing $\{4,6\}$, it follows by Lemma 5.2 that $M\backslash 6/5$ is internally 4-connected; a contradiction. $\square$

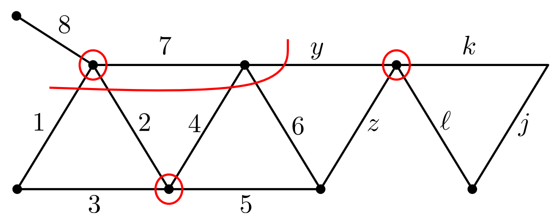

Figure 25. All of the elements shown in this figure are distinct.

Next we examine more closely what happens when M contains the structure in Figure 24.

Lemma 6.5. *Let M and N be internally 4-connected binary matroids where $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose M contains the structure in Figure 24 where all of the elements are distinct, and $\{4,5,6\}$ is the only triangle of M containing 5. Suppose that each of $M\backslash 4$, $M\backslash 1$, and $M\backslash 1,4$ is $(4,4,S)$-connected with an N-minor and that every 4-fan in $M\backslash 1$ has 4 as its guts element. Then every triangle that meets $\{1,2,\dots,8,y\}$ is shown in Figure 24 and either*

(i) *M has a quick win; or*
(ii) *M contains the structure in Figure 25, where all of the elements shown are distinct. Moreover, $M\backslash 1,4/2$ has no N-minor but $M\backslash 1,4,z,j$ does have an N-minor, and $M\backslash z$ is $(4,4,S)$-connected.*

Proof. First we show that

6.5.1. *every triangle that meets $\{1,2,\dots,8,y\}$ is shown in Figure 24.*

Suppose M has a triangle T that is not shown in Figure 24 such that T meets $\{1,2,\dots,8,y\}$. By assumption, $5 \notin T$. Moreover, by orthogonality, if T meets $\{y,6\}$, then T is $\{y,7,1\}$, $\{y,4,3\}$, or $\{6,7,1\}$. In each of these cases, $\{1,2,\dots,7,y\}$ has rank at most five and so is 3-separating; a contradiction. Thus T avoids $\{5,6,y\}$, so $T \subseteq \{1,2,3,4,7,8\}$. Hence $8 \in T$ and $\{1,2,3,4,7,8\}$ is 3-separating; a contradiction. Thus 6.5.1 holds.

Assume that (i) does not hold. We show next that

6.5.2. *$M\backslash 1,4/2$ has no N-minor but $M\backslash 1,4,z$ does have an N-minor.*

Suppose that $M\backslash 1,4/2$ has an N-minor. Then, since $M\backslash 1,4/2 \cong M\backslash 3,4/2 \cong M\backslash 3,4/5 \cong M\backslash 3,6/5$, we see that $M\backslash 6/5$ has an N-minor. Thus, by Lemma 6.4, we have a quick win; a contradiction. We deduce that $M\backslash 1,4/2$ has no N-minor. But $M\backslash 1,4$ has an N-minor and has $(z,y,6,2)$ as a 4-fan. Hence 6.5.2 holds.

6.5.3. *$M\backslash z$ is sequentially 4-connected.*

To see this, suppose that $M\backslash z$ has a non-sequential 3-separation (U, V). By [5, Lemma 3.3], we may assume that $\{1, 2, \dots, 7\} \subseteq U$. Hence we may also assume that $y \in U$. Then $(U \cup z, V)$ is a non-sequential 3-separation of M; a contradiction. Thus 6.5.3 holds.

Next we show the following.

6.5.4. *If (j, k, ℓ, m) is a 4-fan in $M\backslash z$, then either $m = y$; or $\{5, 6\} = \{k, \ell\}$.*

Clearly M has $\{j, k, \ell\}$ as a circuit and $\{k, \ell, m, z\}$ as a cocircuit. By orthogonality, $\{6, y\}$ meets $\{k, \ell, m\}$ in a single element. If $m = y$, then 6.5.4 holds. Suppose that $m = 6$. By orthogonality, $\{4, 5\}$ meets $\{k, \ell\}$. Since $\{4, 5, 6\}$ is the only triangle that contains 5, without loss of generality, $4 = \ell$. Thus $\{4, 6\}$ is contained in a 4-cocircuit, a contradiction to Lemma 6.4.

By symmetry between k and ℓ, it remains to consider when $\ell \in \{6, y\}$. By 6.5.1, y avoids $\{j, k, \ell\}$. Thus $6 = \ell$, and 6.5.1 also implies that $\{j, k\} = \{4, 5\}$. Lemma 6.4 implies that $4 = j$, so $5 = k$, and 6.5.4 holds.

We now show that

6.5.5. *y is the coguts element of some 4-fan of $M\backslash z$.*

By 6.5.3, since $M\backslash z$ is not internally 4-connected, it has a 4-fan. Suppose 6.5.5 fails. Then 6.5.4 implies that $\{5, 6, z, x\}$ is a cocircuit of M for some element x. Then, by orthogonality, either $x = 8$, or x avoids the elements in Figure 24. In the first case, $\lambda(\{1, 2, 3, 4, 5, 6, 7, 8, y, z\}) \leq 2$; a contradiction. In the second case, we observe that M is not the cycle matroid of a quartic Möbius ladder since M has an odd circuit but every vertex degree, and hence every bond, in a quartic Möbius ladder is even. Thus we can apply [8, Lemma 4.1] to get that $M\backslash 1, 4, z$ is $(4, 4, S)$-connected and every 4-fan of this matroid is a 4-fan of $M\backslash z$ or $M\backslash 1$. Since, by hypothesis, 4 is the guts element of every 4-fan of $M\backslash 1$, and we have assumed that M has no quick win, we deduce that $M\backslash 1, 4, z$ has a 4-fan F that is a 4-fan of $M\backslash z$. Then 6.5.4 implies that either 6.5.5 holds, or 5 and 6 are the interior elements of F. Hence 4 is in F; a contradiction. Thus 6.5.5 holds.

By 6.5.5, M contains the structure in Figure 25. The elements in this figure are distinct since $\{j, k, \ell\}$ certainly avoids z and, by 6.5.1, $\{j, k, \ell\}$ also avoids $\{1, 2, \dots, 8, y\}$. Since $M\backslash 1, 4, z$ has (j, k, ℓ, y) as a 4-fan, $M\backslash 1, 4, z, j$ or $M\backslash 1, 4, z/y$ has an N-minor. As $M\backslash 1, 4, z/y \cong M\backslash 1, 4, 6/y \cong M\backslash 1, 4, 6/2$, and $M\backslash 1, 4/2$ has no N-minor, we deduce that $M\backslash 1, 4, z, j$ has an N-minor.

To complete the proof that (ii) holds, it remains only to show that $M\backslash z$ is $(4, 4, S)$-connected. Suppose that $M\backslash z$ has a 5-fan $(\alpha, \beta, \gamma, \delta, \varepsilon)$. By 6.5.4, since β and δ cannot both equal y, it follows that $\{5, 6\}$ is $\{\beta, \gamma\}$ or $\{\gamma, \delta\}$. Thus 5 or 6 is in two triangles of $M\backslash z$; a contradiction to 6.5.1. Now suppose that $M\backslash z$ has a 5-cofan $(\zeta, \alpha, \beta, \gamma, \delta)$. Then 6.5.4 implies, after possibly reversing the order of the fan elements, that $\zeta = y$ and $\{5, 6\} = \{\beta, \gamma\}$, so $\alpha = 4$. Thus $\{4, y, z, 5\}$ or $\{4, y, z, 6\}$ is a cocircuit in M; a contradiction to orthogonality with $\{2, 4, 7\}$. We conclude that $M\backslash z$ is $(4, 4, S)$-connected. Thus (ii) holds. □

When combined with earlier results, the next result establishes that when M contains the structure in Figure 24, either M contains a good bowtie string with three triangles, or M has an internally 4-connected minor M' that has an N-minor such that $1 \leq |E(M) - E(M')| \leq 2$.

Lemma 6.6. *Let M and N be internally 4-connected binary matroids with $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose that Hypothesis VIII holds, that M contains the configuration in Figure 25, where all of the elements are distinct, that $N \preceq M\backslash 1,4,z,j$, and that each of $M\backslash 1$, $M\backslash 4$, $M\backslash 1,4$, and $M\backslash z$ is $(4,4,S)$-connected. Suppose also that every 4-fan in $M\backslash 1$ has 4 as its guts element and that every triangle that meets $\{1,2,\dots,8,y\}$ is displayed in Figure 25. Then either*

- (i) *M has an internally 4-connected minor M' that has an N-minor such that $1 \leq |E(M) - E(M')| \leq 2$; or*
- (ii) *M contains a good bowtie string with three triangles.*

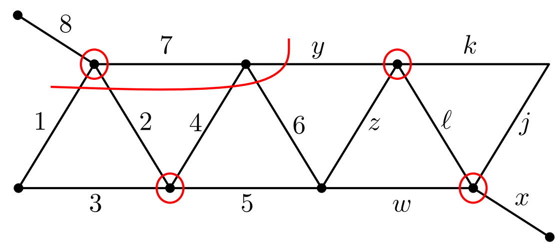

FIGURE 26. This structure arises in the proof of Lemma 6.6.

Proof. Assume that the lemma fails. Then $M\backslash j$ is not internally 4-connected. We apply Lemma 5.6 to the good bowtie $(\{j,k,\ell\},\{6,y,z\},\{y,z,k,\ell\})$. Neither (i) nor (iv) from that lemma holds. Moreover, (ii) does not hold otherwise M has a good bowtie string with three triangles. Thus (iii) from Lemma 5.6 holds, that is, $M\backslash j$ is $(4,4,S)$-connected and every $(4,3)$-violator of it is a 4-fan of the form (z,v,w,x) for some v in $\{k,\ell\}$. Without loss of generality, we may assume that $v = \ell$ and that M contains the structure in Figure 26. By orthogonality with the cocircuits displayed in Figure 25, we deduce that w is not among the elements in that figure. Using orthogonality again, this time with the circuits displayed in Figure 25, we see that all of the elements in Figure 26 are distinct except that x may equal 8.

Now we apply Lemma 6.3 to the right-hand end of the structure in Figure 26 and get that $M\backslash z,j$ is $(4,4,S)$-connected. Moreover, since, by assumption, $M\backslash z,j$ is not internally 4-connected, M has $\{6,w,z\}$ in a 4-cocircuit or has $\{6,z,\ell,j\}$ in a 5-cocircuit. Thus, by orthogonality with the circuits in Figure 26, $\{5,6,w,z\}$ or $\{5,6,z,\ell,j\}$ is a cocircuit of M. The former implies that (ii) holds; a contradiction. Hence $\{5,6,z,\ell,j\}$ is a cocircuit. The symmetric difference of the cocircuits $\{1,2,4,6,y\},\{5,6,z,\ell,j\},\{y,z,\ell,k\}$, and $\{2,3,4,5\}$ is $\{1,3,j,k\}$, which must be a cocircuit of M. Since $N \preceq M\backslash 1,4,z,j$, it follows that $N \preceq M\backslash 1,4,z,j/3$. But $M\backslash 1,4/3 \cong M\backslash 2,4/3 \cong M\backslash 2,4/5 \cong M\backslash 2,6/5$. Hence $N \preceq M/5\backslash 6$. Then Lemma 6.4 implies that (i) holds; a contradiction. □

We now consider the case when M contains the structure in Figure 23 where $\{4,6,7,y\}$ is a cocircuit and $M\backslash 1,4$ is $(4,4,S)$-connected having an N-minor. Then $M\backslash 1,4$ has $(z,y,6,7)$ as a 4-fan. Thus $M\backslash 1,4\backslash z$ or $M\backslash 1,4/7$ has an N-minor. The next two lemmas treat these two cases.

Lemma 6.7. *Let M and N be internally 4-connected binary matroids with $|E(M)| \geq 15$ and $|E(N)| \geq 7$ and suppose that Hypothesis VIII holds. Let M*

contain the configuration in Figure 23 where $\{4,6,7,y\}$ *is a cocircuit. Suppose that each of* $M\backslash 4$ *and* $M\backslash 1,4$ *is* $(4,4,S)$*-connected, and that* $\{4,5,6\}$ *is the only triangle that contains* 5*. Suppose also that* M *has no quick win. If* $N \preceq M\backslash 1,4,z$*, then* M *contains a good bowtie string with three triangles.*

Proof. By Hypothesis VIII, $M\backslash 1$ is $(4,4,S)$-connected. We show first that

6.7.1. *if* M *has* $\{\alpha,\beta,\gamma\}$ *as a triangle and* $\{z,\beta,\gamma,\delta\}$ *as a cocircuit, for some* $\delta \in \{6,y\}$*, then the lemma holds.*

By Lemma 5.1, $\{\beta,\gamma\}$ avoids $\{2,4,7\}$. Hence $|\{2,4,7,6,y,z,\beta,\gamma\}| = 8$, and orthogonality implies that α avoids $\{2,4,7,6,y,z,\beta,\gamma\}$ unless $\alpha = 2$. Consider the exceptional case. Then $\{\alpha,\beta,\gamma\} = \{2,3,1\}$ and $\lambda(\{1,2,3,4,5,6,7,y,z\}) \le 2$; a contradiction. We deduce that $\alpha \notin \{2,4,7,6,y,z,\beta,\gamma\}$. Since $M\backslash 4$ is $(4,4,S)$-connected with an N-minor, it follows that $\{2,4,7\}$, $\{6,y,z\}$, and $\{\alpha,\beta,\gamma\}$ are three triangles in a good bowtie string in M, so 6.7.1 holds.

We now apply Lemma 5.6 to the bowtie $(\{6,y,z\},\{2,4,7\},\{4,6,7,y\})$. Since $M\backslash z$ has an N-minor but M has no quick win, $M\backslash z$ is not internally 4-connected. Clearly $M\backslash 2$ is not internally 4-connected. Using 6.7.1, it follows that we may assume that (iii) of Lemma 5.6 holds. Thus

6.7.2. *every* $(4,3)$*-violator of* $M\backslash z$ *is a* 4*-fan of the form* $(4,v,w,x)$ *where* $v \in \{6,y\}$*.*

If $v = y$, then orthogonality between $\{4,v,w\}$ and $\{2,3,4,5\}$ implies that $w = 3$. Hence $\lambda(\{1,2,\dots,7,y\}) \le 2$; a contradiction. Thus $v = 6$, and it follows that $w = 5$, so $\{5,6,x,z\}$ is a cocircuit of M. By orthogonality with the circuits in Figure 23, we know that x avoids the elements in that figure unless $x = 8$.

We may assume that M is not the cycle matroid of a quartic Möbius ladder otherwise the lemma holds. Thus [8, Lemma 4.1] implies that $M\backslash 1,4,z$ is $(4,4,S)$-connected and every $(4,3)$-violator of this matroid is a 4-fan in either $M\backslash z$ or $M\backslash 1$. Since M has no quick win, $M\backslash 1,4,z$ has a 4-fan F. By 6.7.2, F cannot be a 4-fan of $M\backslash z$ otherwise it has 4 as its guts element. Thus F is a 4-fan, $(\alpha,\beta,\gamma,\delta)$, of $M\backslash 1$. Then M has $\{\alpha,\beta,\gamma\}$ as a triangle and has $\{\beta,\gamma,\delta,1\}$ as a cocircuit. By orthogonality, $\{2,3\}$ meets $\{\beta,\gamma,\delta\}$. By Lemma 5.1, $\{\beta,\gamma,\delta\}$ avoids $\{4,5,6\}$. Suppose $\{2,3\}$ meets $\{\beta,\gamma\}$. Then, by orthogonality, $\{\alpha,\beta,\gamma\}$ contains two elements of $\{2,3,4,5\}$. But $\{\alpha,\beta,\gamma\}$ avoids $\{4,5\}$, so $\{\alpha,\beta,\gamma\}$ contains $\{2,3\}$ and hence is $\{1,2,3\}$; a contradiction since $\{\alpha,\beta,\gamma\}$ avoids 1. We deduce that $\{2,3\}$ avoids $\{\beta,\gamma\}$. Hence $\delta \in \{2,3\}$.

Now observe that

6.7.3. $\{\alpha,\beta,\gamma\}$ *avoids* $\{7,y\}$*.*

Suppose $\{\alpha,\beta,\gamma\}$ meets $\{7,y\}$. Then, by orthogonality, $\{\alpha,\beta,\gamma\}$ is a subset of $\{1,2,3,4,5,6,7,8,y\}$. Since $\{\alpha,\beta,\gamma\}$ avoids $\{1,4,z\}$, it is a triangle not shown in Figure 23, so $\lambda(\{1,2,3,4,5,6,7,8,y\}) \le 2$; a contradiction. Thus 6.7.3 holds.

If $\delta = 2$, then orthogonality implies that $\{4,7\}$ meets $\{\beta,\gamma\}$; a contradiction. We deduce that $\delta = 3$. Then M has $\{6,5,4\},\{5,4,2,3\},\{2,3,1\},\{3,1,\gamma,\beta\},\{\gamma,\beta,\alpha\}$ as a good bowtie string of three triangles unless $\{\alpha,\beta,\gamma\}$ meets $\{4,5,6\}$. In the exceptional case, as $\{\alpha,\beta,\gamma\}$ avoids $\{4,5\}$, we must have that $6 \in \{\alpha,\beta,\gamma\}$. Then, by orthogonality, $\{y,7\}$ meets $\{\alpha,\beta,\gamma\}$; a contradiction to 6.7.3. □

When M contains the structure in Figure 23 where $\{4, 6, 7, y\}$ is a cocircuit, we have completed our analysis of the case when $M\backslash 1, 4, z$ has an N-minor; we now turn our attention to the case when $M\backslash 1, 4/7$ has an N-minor.

Lemma 6.8. *Let M and N be internally 4-connected binary matroids with $|E(M)| \geq 15$ and $|E(N)| \geq 7$ and suppose that Hypothesis VIII holds. Let M contain the configuration in Figure 23 where $\{4, 6, 7, y\}$ is a cocircuit. Suppose that each of $M\backslash 4$ and $M\backslash 1, 4$ is $(4, 4, S)$-connected, that $M\backslash 4$ has an N-minor, and that $\{4, 5, 6\}$ is the only triangle of M containing 5. Then either M has a quick win, or M contains a good bowtie string with three triangles.*

Proof. Assume that the lemma fails. By Hypothesis VIII, $M\backslash 1$ is $(4, 4, S)$-connected and $M\backslash 1, 4$ has an N-minor. If $M\backslash 1, 4, z$ has an N-minor, then Lemma 6.7 implies that the lemma holds, so we assume not.

We show next that

6.8.1. *$M\backslash 2, 6$ has an N-minor.*

As $M\backslash 1, 4$ has an N-minor and has $(z, y, 6, 7)$ as a 4-fan, it follows that $M\backslash 1, 4/7$ has an N-minor. As the last matroid has $(6, 5, 2, 3)$ as a 4-fan, either $M\backslash 1, 4/7\backslash 6$ or $M\backslash 1, 4/7/3$ has an N-minor. The former implies, since $M\backslash 1, 4/7\backslash 6 \cong M\backslash 1, 4/y\backslash 6 \cong M\backslash 1, 4/y\backslash z$, that $M\backslash 1, 4, z$ has an N-minor; a contradiction. Thus $M\backslash 1, 4/7, 3$ has an N-minor. Since $M\backslash 1, 4/7, 3 \cong M\backslash 2, 4/7, 3 \cong M\backslash 2, 4/7, 5 \cong M\backslash 2, 6/7, 5$, we deduce that 6.8.1 holds.

By Lemma 6.2, $M\backslash 6$ is $(4, 4, S)$-connected. Thus, by applying Hypothesis VIII to the bowtie $(\{z, y, 6\}, \{4, 7, 2\}, \{y, 6, 4, 7\})$, we deduce that

6.8.2. *$M\backslash 2$ is $(4, 4, S)$-connected.*

All the elements in Figure 23 are distinct otherwise $\lambda(\{1, 2, \dots, 8, y, z\}) \leq 2$. We may now apply [7, Lemma 6.1] and deduce that

(i) $M\backslash 2, 6$ is internally 4-connected; or
(ii) $\{7, y\}$ is in a triangle of M; or
(iii) $\{3, 5\}$ is in a triangle of M; or
(iv) M has a triangle containing 1 and avoiding $\{2, 3\}$; or
(v) $M\backslash 2, 6$ is $(4, 4, S)$-connected, and 5 is the coguts element of all of its 4-fans.

By assumption, neither (i) nor (iii) holds. If $\{7, y\}$ is in a triangle, then orthogonality implies that the third element of this triangle is in $\{1, 2, 8\}$, so $\lambda(\{1, 2, \dots, 8, y\}) \leq 2$; a contradiction. Thus (ii) does not hold. Suppose 1 is in a triangle T avoiding $\{2, 3\}$. Then orthogonality implies that 7 or 8 is in T. In the first case, orthogonality implies that the third element of T is in $\{4, 6, y\}$, so $\lambda(\{1, 2, 3, 4, 5, 6, 7, y\}) \leq 2$; a contradiction. Thus $8 \in T$ and, by orthogonality, the third element of T avoids $\{4, 6, 7, y\}$. Thus M has a bowtie string containing T, $\{2, 4, 7\}$, and $\{6, y, z\}$, and, as $M\backslash 1$ is $(4, 4, S)$-connected having an N-minor, this bowtie string is good; a contradiction. We deduce that (iv) does not hold. It follows that (v) holds, so $M\backslash 2, 6$ has a 4-fan of the form $(\alpha, \beta, \gamma, 5)$. Thus M has a cocircuit C^* such that $\{\beta, \gamma, 5\} \subsetneq C^* \subseteq \{\beta, \gamma, 5, 2, 6\}$.

Suppose that $C^* = \{\beta, \gamma, 5, 6\}$. Then Lemma 5.1 implies that $\{\beta, \gamma\}$ avoids $\{1, 2, 3\}$. Thus, by orthogonality, α does not meet $\{2, 3, 4\}$. As (iv) does not hold, $\alpha \neq 1$. Hence M has $\{1, 2, 3\}, \{2, 3, 4, 5\}, \{4, 5, 6\}, \{5, 6, \beta, \gamma\}, \{\alpha, \beta, \gamma\}$ as a bowtie string, and 6.8.2 implies that it is a good bowtie string; a contradiction.

We deduce that C^* is $\{\beta, \gamma, 5, 2\}$ or $\{\beta, \gamma, 5, 2, 6\}$. If the former holds, then Lemma 5.1 implies that $\{\beta, \gamma\} = \{3, 4\}$, and orthogonality between $\{\alpha, \beta, \gamma\}$ and $\{4, 6, 7, y\}$ implies that $\alpha = y$, so $\lambda(\{1, 2, \dots, 7, y\}) \leq 2$; a contradiction. It remains to consider when M has $\{\beta, \gamma, 5, 2, 6\}$ as a cocircuit. By orthogonality with $\{1, 2, 3\}$ and $\{2, 4, 7\}$, it follows that $\{\beta, \gamma\}$ meets $\{1, 3\}$ and $\{4, 7\}$, so $\lambda(\{1, 2, \dots, 7\}) \leq 2$; a contradiction. □

We are now ready to prove the main result of this section.

Proof of Theorem 6.1. Assume that the theorem fails. Lemma 5.6 implies that $M\backslash 1$ is $(4, 4, S)$-connected and that every 4-fan of it has the form $(4, \beta, \gamma, \delta)$, where $\beta \in \{2, 3\}$.

Now Lemma 6.2 implies that $\{4, 5, 6\}$ is the only triangle containing 5, and $M\backslash 6$ is $(4, 4, S)$-connected. By Hypothesis VIII, $M\backslash 1, 4$ has an N-minor. Furthermore, by Lemma 6.3, $M\backslash 1, 4$ is $(4, 4, S)$-connected, and M contains the structure in Figure 23 where $\{4, 6, 7, y\}$ or $\{1, 2, 4, 6, y\}$ is a cocircuit.

By Lemma 6.8, M does not have $\{4, 6, 7, y\}$ as a cocircuit. Therefore M has $\{1, 2, 4, 6, y\}$ as a cocircuit. Then M contains the structure in Figure 24. By Lemma 6.4, all the elements in this figure are distinct. By Lemma 6.5, every triangle that meets $\{1, 2, \dots, 8, y\}$ is displayed in Figure 24, and M contains the structure in Figure 25 where all of the elements shown there are also distinct. Furthermore, $M\backslash 1, 4, z, j$ has an N-minor, and $M\backslash z$ is $(4, 4, S)$-connected. By Lemma 6.6, we deduce that M has a good bowtie string with three triangles; a contradiction. □

7. Good bowtie strings containing three triangles

One outcome of the preceding section is that M contains a good bowtie string with three triangles. In this section, we analyze this case.

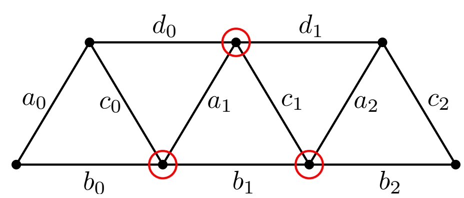

FIGURE 27. This structure is analyzed in the proof of Lemma 7.1.

Lemma 7.1. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose that Hypothesis VIII holds and that M contains the structure in Figure 27 and that $N \preceq M\backslash c_0, c_1/d_0, b_1$. Then*

(i) *M has a quick win; or*
(ii) *M has a ladder-compression win; or*
(iii) *$M\backslash c_0, c_1/d_0, b_1$ is $(4, 4, S)$-connected and every 4-fan in this matroid is of the form (d_1, v, w, x), where $\{v, w, x\}$ is a triad of M and $\{d_0, d_1, v, w\}$ is a circuit of M.*

Proof. We note that all of the elements in Figure 27 are distinct, as otherwise this set of elements is 3-separating; a contradiction. Assume that the lemma does not hold. We show first that

7.1.1. *$M\backslash a_1, c_0$ and $M\backslash a_0/b_0, d_0$ have N-minors.*

This follows from the fact that $N \preceq M\backslash c_0, c_1/d_0, b_1$ by observing that $M\backslash c_0, c_1/d_0, b_1 \cong M\backslash c_0, a_1/d_0, b_1 \cong M\backslash c_0, a_1/d_0, b_0 \cong M\backslash c_0, a_0/d_0, b_0$.

We show next that

7.1.2. *$M\backslash c_0$ and $M\backslash c_1$ are $(4, 4, S)$-connected.*

By Hypothesis VIII, this holds provided $M\backslash c_0$ is $(4, 4, S)$-connected. Assume it is not. Applying [7, Lemma 4.3] to the bowtie $(\{a_2, c_1, d_1\}, \{a_1, d_0, c_0\}, \{c_1, d_1, a_1, d_0\})$, we see that M has a quasi rotor $(\{a_2, c_1, d_1\}, \{a_1, d_0, c_0\}, \{7, 8, 9\}, \{c_1, d_1, a_1, d_0\}, \{y, c_0, 7, 8\}, \{x, y, 7\})$ for some x in $\{c_1, d_1\}$ and some y in $\{a_1, d_0\}$. As $N \preceq M/d_0$, it follows by [7, Lemma 4.5] that $y = a_1$. Now $M\backslash a_1, c_0$ has an N-minor and has $\{7, 8\}$ as a cocircuit. Thus $N \preceq M\backslash a_1, c_0/7$. Therefore, by [7, Lemma 4.5] again, M has a quick win; a contradiction. We conclude that 7.1.2 holds.

Next we show that

7.1.3. *$\{a_0, c_0\}$ is contained in a 4-cocircuit of M.*

Suppose not. By 7.1.1, M/b_0 and $M/b_0\backslash a_0$ have N-minors. Then $M\backslash a_0$ is not internally 4-connected, otherwise M has a quick win; a contradiction.

As the next step towards proving 7.1.3, we now show that

7.1.4. *$M\backslash a_0$ is $(4, 4, S)$-connected.*

Suppose not. Then [7, Lemma 4.3] implies that M has a quasi rotor of the form $(T_1, T_0, \{7, 8, 9\}, D_0, \{y, a_0, 7, 8\}, \{x, y, 7\})$ for some x in $\{a_1, b_1\}$ and some y in $\{b_0, c_0\}$. If $x = b_1$, then orthogonality implies that $7 \in \{a_2, b_2\}$, so the set of elements in Figure 27 is 3-separating; a contradiction. Thus $x = a_1$. If $y = b_0$, then $7 = d_1$, so $\lambda(T_0 \cup T_1 \cup \{d_0, d_1\}) \leq 2$; a contradiction. Hence $y = c_0$ and 7.1.3 holds; a contradiction. We conclude that 7.1.4 holds.

Continuing with the proof of 7.1.3, we note that [7, Lemma 4.3] implies, since $\{a_0, c_0\}$ is not in a 4-cocircuit, that either M has a triangle $\{1, 2, 3\}$ and a cocircuit $\{2, 3, a_0, b_0\}$ where $\{1, 2, 3\}$ avoids $T_0 \cup T_1$; or M has a cocircuit $\{a_0, b_0, 7, 8\}$ and a triangle $\{b_0, 7, x\}$ for some $x \in \{a_1, b_1\}$. If the latter holds, then orthogonality using the two possibilities for x gives that $7 \in \{d_0, c_1, d_1\}$ or $7 \in \{c_1, a_2, b_2\}$. In each case, we see that the set of elements in Figure 27 is 3-separating; a contradiction. Thus the former holds. Clearly, M has $T_1, D_0, T_0, \{a_0, b_0, 2, 3\}, \{1, 2, 3\}$ as a bowtie string. Since, by 7.1.4, $M\backslash a_0$ is $(4, 4, S)$-connected but $\{a_0, c_0\}$ is not in a 4-cocircuit, [8, Lemma 3.8] implies that $M\backslash a_0/b_0$ is internally 4-connected, so (i) holds; a contradiction. We conclude that 7.1.3 holds.

Let C^* be a 4-cocircuit of M containing $\{a_0, c_0\}$. We will show that M has a ladder containing the structure in Figure 27. By orthogonality between C^* and $\{c_0, d_0, a_1\}$, we deduce, using Lemma 5.1, that $d_0 \in C^*$. Now orthogonality between C^* and the triangles in Figure 27 implies that the fourth element of C^* is a new element, γ, that is not shown in the figure. By reflecting the structure in Figure 27 about a vertical line and adding γ, we obtain a structure that has the form of Figure 8(a) where the ambiguous extra cocircuit in this structure is a 4-cocircuit rather than a 5-cocircuit.

We deduce that if $M\backslash c_0, c_1/d_0, b_1$ is internally 4-connected, then M has a ladder-compression win; a contradiction. Thus

7.1.5. *$M\backslash c_0, c_1/d_0, b_1$ is not internally 4-connected.*

We show next that

7.1.6. *$M\backslash c_0, c_1/d_0, b_1$ is 3-connected.*

Assume not. By 7.1.2 and Lemma 6.3, $M\backslash c_0, c_1$ is $(4, 4, S)$-connected. As this matroid has (c_2, b_2, a_2, b_1) as a 4-fan, we deduce that $M\backslash c_0, c_1/b_1$ is 3-connected. The last matroid has (a_2, d_1, a_1, d_0) as a 4-fan. Since $M\backslash c_0, c_1/d_0, b_1$ is not 3-connected, it follows by Bixby's Lemma [1] (or see [14, Lemma 8.7.3]) that $M\backslash c_0, c_1/b_1$ has a triangle T containing d_0. Suppose T is a triangle of M. Then, as $T \neq \{d_0, a_1, c_0\}$, orthogonality implies that $d_1 \in T$ so $T = \{d_0, d_1, e\}$, say. Then orthogonality between T and $\{\gamma, a_0, c_0, d_0\}$ implies that $e \in \{\gamma, a_0, c_0\}$, and $\lambda(T_0 \cup T_1 \cup T) \leq 2$; a contradiction. It follows that T is not a circuit of M, so $T \cup b_1$ is a circuit of $M\backslash c_0, c_1$. Then orthogonality in $M\backslash c_0, c_1$ with the cocircuits $\{b_0, a_1, b_1\}, \{d_0, a_1, d_1\}$, and $\{b_1, a_2, b_2\}$ implies that $T \cup b_1$ contains a_1 and meets $\{a_2, b_2\}$. Since $\{a_1, b_1, d_1, a_2\}$ is a circuit, we deduce that $T \cup b_1 = \{d_0, a_1, b_1, b_2\}$. Hence $\lambda(T_1 \cup T_2 \cup \{d_0, d_1\}) \leq 2$; a contradiction. Thus 7.1.6 holds.

7.1.7. *$M\backslash c_0, c_1/d_0, b_1$ is sequentially 4-connected.*

To show this, suppose that $M\backslash c_0, c_1/d_0, b_1$ has a non-sequential 3-separation (U, V). By [5, Lemma 3.3], we may assume that $\{a_0, b_0, a_1, d_1, a_2, b_2, c_2\} \subseteq U$. Then $(U \cup \{d_0, b_1, c_0, c_1\}, V)$ is a non-sequential 3-separation of M; a contradiction. Thus 7.1.7 holds.

By 7.1.6, $M\backslash c_0, c_1/d_0, b_1$ is not internally 4-connected. Let $(1, 2, 3, 4)$ be a 4-fan in it. Then $M/d_0, b_1$ has a cocircuit D^* such that $\{2, 3, 4\} \subseteq D^* \subseteq \{2, 3, 4, c_0, c_1\}$. Suppose $|D^*| = 4$. Then we get a contradiction to orthogonality between D^* and the circuit $\{c_0, c_1\}$ in $M/d_0, b_1$. Suppose next that $D^* = \{2, 3, 4, c_0, c_1\}$. Then orthogonality with the circuits $T_0, \{c_0, a_1\}$, and $\{c_1, d_1, a_2\}$ in $M/d_0, b_1$ implies that $\{2, 3, 4\}$ meets $\{a_0, b_0\}, \{a_1\}$, and $\{d_1, a_2\}$, so the eleven-element set in Figure 27 is 3-separating in M; a contradiction. We conclude that $D^* = \{2, 3, 4\}$. Now M has a circuit C such that $\{1, 2, 3\} \subsetneqq C \subseteq \{1, 2, 3, d_0, b_1\}$. Suppose $b_1 \in C$. Then orthogonality implies that C meets $\{b_0, a_1\}$ and $\{a_2, b_2\}$, so the triad $\{2, 3, 4\}$ meets one of these sets; a contradiction. It follows that $C = \{1, 2, 3, d_0\}$. Then orthogonality implies that C meets $\{a_1, d_1\}$. If $a_1 \in C$, then orthogonality with D_0 implies that $b_0 \in C$, so the triad $\{2, 3, 4\}$ meets $\{a_1, b_0\}$; a contradiction. We deduce that $d_1 \in C$, so $d_1 = 1$. Thus (iii) holds; a contradiction. $\square$

In our next lemma, we consider $M\backslash c_{\ell-1}, c_\ell/d_{\ell-1}, b_\ell$, for $\ell \in \{1, 2, \ldots, n\}$, in the structures in Figure 8 when $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit.

Lemma 7.2. *Let M and N be internally 4-connected binary matroids where M contains structure (a) or (b) in Figure 8 where $n \geq 2$ and $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit, and all of the elements are distinct except that d_n may be the same as γ in (b). If $M\backslash c_{\ell-1}, c_\ell/d_{\ell-1}, b_\ell$ has an N-minor for some ℓ in $\{1, 2, \ldots, n\}$, then $M\backslash a_{n-1}/b_{n-1}$ has an N-minor.*

Proof. If $\ell = n$, then $M\backslash c_{n-1}, c_n/d_{n-1}, b_n \cong M\backslash c_{n-1}, a_n/d_{n-1}, b_n \cong M\backslash c_{n-1}, a_n/d_{n-1}, b_{n-1} \cong M\backslash c_{n-1}, a_{n-1}/d_{n-1}, b_{n-1}$. Hence $M\backslash a_{n-1}/b_{n-1}$, has an N-minor.

Now suppose that $\ell \le n-1$. We have that

$$\begin{aligned} M\backslash c_{\ell-1}, c_\ell/d_{\ell-1}, b_\ell &\cong M\backslash a_\ell, c_\ell/d_{\ell-1}, b_\ell \\ &\cong M\backslash a_\ell, c_\ell/d_\ell, b_\ell \\ &\cong M\backslash c_\ell, a_{\ell+1}/d_\ell, b_\ell \\ &\cong M\backslash c_\ell, a_{\ell+1}/d_\ell, b_{\ell+1} \\ &\cong M\backslash c_\ell, c_{\ell+1}/d_\ell, b_{\ell+1} \\ &\ \ \vdots \\ &\cong M\backslash c_{n-2}, c_{n-1}/d_{n-2}, b_{n-1} \\ &\cong M\backslash a_{n-1}, c_{n-1}/d_{n-2}, b_{n-1}. \end{aligned}$$

Thus $M\backslash a_{n-1}/b_{n-1}$ has an N-minor. □

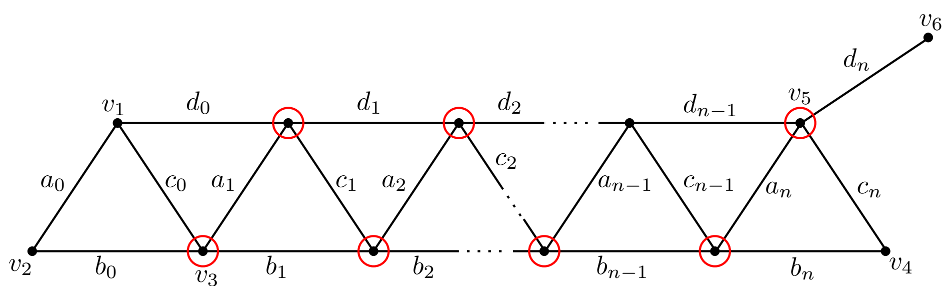

FIGURE 28. M has $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit.

Lemma 7.3. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \ge 15$ and $|E(N)| \ge 7$. Suppose that M contains the structure in Figure 28, where $n \ge 1$ and, when $n \ge 2$, either $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ or $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ is a cocircuit. Suppose also that $N \preceq M\backslash c_0, c_1$ and $M\backslash c_i$ is $(4,4,S)$-connected for all i in $\{0,1,\dots,n\}$. Then*

(i) *$M\backslash c_0, c_1, \dots, c_n$ has an N-minor; or*
(ii) *M has a quick win; or*
(iii) *M has a ladder-compression win.*

Proof. Since $M\backslash c_n$ is $(4,4,S)$-connected, we know that $\{d_{n-1}, d_n\}$ is not contained in a triangle. Therefore, by [7, Lemma 6.4], either all the elements in Figure 28 are distinct, or (a_0, b_0) is (c_n, d_n) or (d_{n-1}, d_n) but all the other elements in the figure are distinct. Suppose that the lemma fails. Then $n \ge 2$ and, for some ℓ in $\{1,2,\dots,n-1\}$, the matroid $M\backslash c_0, c_1, \dots, c_\ell$ has an N-minor but $M\backslash c_0, c_1, \dots, c_{\ell+1}$ does not. We show next that

7.3.1. $(\ell, n) = (1,2)$ *and $N \preceq M\backslash c_0, c_1/b_1, d_0$. Moreover, M has $\{d_0, a_1, c_1, d_1\}$ as a cocircuit and has a triad $\{2,3,4\}$ such that $\{d_0, d_1, 2, 3\}$ is a circuit and $\{2, d_1, a_2, c_2\}$ is a cocircuit.*

As $M\backslash c_0, c_1, \dots, c_\ell$ has $(c_{\ell+1}, b_{\ell+1}, a_{\ell+1}, b_\ell)$ as a 4-fan, we deduce that $M\backslash c_0, c_1, \dots, c_\ell/b_\ell$ has an N-minor. Either the last matroid has $(a_{\ell+1}, a_\ell, d_\ell, d_{\ell-1})$

as a 4-fan, or $\ell = n-1$ and M has $\{d_{n-2}, a_{n-1}, c_{n-1}, a_n, c_n\}$ as a cocircuit. Suppose the latter occurs. Then Lemma 6.4 implies, since $n \geq 2$, that $M\backslash a_{n-1}/b_{n-1}$ is internally 4-connected. As $M\backslash c_0, c_1, \ldots, c_{n-1}/b_{n-1} \cong M\backslash c_0, c_1, \ldots, c_{n-2}/b_{n-1}\backslash a_{n-1}$, it follows that $N \preceq M\backslash a_{n-1}/b_{n-1}$, so (ii) holds; a contradiction. We deduce that $M\backslash c_0, c_1, \ldots, c_\ell/b_\ell$ has $(a_{\ell+1}, a_\ell, d_\ell, d_{\ell-1})$ as a 4-fan and that if $\ell = n-1$, then M has $\{d_{n-2}, a_{n-1}, c_{n-1}, d_{n-1}\}$ as a cocircuit.

If $M\backslash c_0, c_1, \ldots, c_\ell/b_\ell\backslash a_{\ell+1}$ has an N-minor, then, as $M\backslash c_0, c_1, \ldots, c_\ell/b_\ell\backslash a_{\ell+1} \cong M\backslash c_0, c_1, \ldots, c_\ell, a_{\ell+1}/b_{\ell+1} \cong M\backslash c_0, c_1, \ldots, c_\ell, c_{\ell+1}/b_{\ell+1}$, we see that $M\backslash c_0, c_1, \ldots, c_{\ell+1}$ has an N-minor; a contradiction. We deduce that $M\backslash c_0, c_1, \ldots, c_\ell/b_\ell/d_{\ell-1}$ has an N-minor. Since $n \geq 2$, we can apply Lemma 7.1 to our structure, deducing that M has a triad $\{2, 3, 4\}$ such that $\{d_\ell, d_{\ell-1}, 2, 3\}$ is a circuit. By orthogonality between $\{d_\ell, d_{\ell-1}, 2, 3\}$ and the cocircuit $\{d_\ell, a_{\ell+1}, c_{\ell+1}, d_{\ell+1}\}$, it follows, since neither 2 nor 3 is in a triangle of M, that $d_{\ell+1} \in \{2, 3\}$ and $\ell + 1 = n$. By symmetry, we may assume that $d_{\ell+1} = 2$. Moreover, if $\ell > 1$, then orthogonality between $\{d_\ell, d_{\ell-1}, 2, 3\}$ and $\{d_{\ell-2}, a_{\ell-1}, c_{\ell-1}, d_{\ell-1}\}$ implies that 3 is in a triangle of M; a contradiction. We conclude that $\ell = 1$ and it follows that 7.3.1 holds where we note that the cocircuit $\{2, d_1, a_2, c_2\}$ is $\{d_2, d_1, a_2, c_2\}$.

Next we observe the following.

7.3.2. *All the elements in Figure 28 are distinct.*

If this fails, then, as noted above, (a_0, b_0) is (c_n, d_n) or (d_{n-1}, d_n). But $n = 2$, so $\lambda(T_0 \cup T_1 \cup T_2 \cup \{d_0, d_1\}) \leq 2$; a contradiction. Hence 7.3.2 holds.

We now show that

7.3.3. $\{a_0, c_0\}$ *is contained in a* 4*-cocircuit,* C^**, of* M*.*

Suppose not. Since $M\backslash c_0, c_1/b_1, d_0$ has an N-minor and $M\backslash c_0, c_1/b_1, d_0 \cong M\backslash c_0, a_1/b_1, d_0 \cong M\backslash c_0, a_1/b_0, d_0 \cong M\backslash a_0, a_1/b_0, d_0$, we deduce that $M\backslash a_0$ and $M\backslash c_0/b_0$ have N-minors. It now follows, by [7, Lemma 4.3], that M has $\{a_0, b_0, 7, 8\}$ as a cocircuit and has $\{b_0, 7, x\}$ or $\{7, 8, 9\}$ as a triangle where $x \in \{a_1, b_1\}$ in the former case, and $\{7, 8, 9\}$ avoids $T_0 \cup T_1$ in the latter case. If $\{b_0, 7, x\}$ is a triangle, then $M\backslash c_0$ has a 5-fan; a contradiction. Thus $\{7, 8, 9\}$ is a triangle, so $\{7, 8, 9\}, \{7, 8, a_0, b_0\}, T_0, D_0, T_1$ is a string of bowties and $M\backslash c_0$ is $(4, 4, S)$-connected. Therefore, as $\{a_0, c_0\}$ is not in a 4-cocircuit, Lemma 5.2 implies that $M\backslash c_0/b_0$ is internally 4-connected. As the last matroid has an N-minor, we obtain the contradiction that (ii) holds thereby completing the proof of 7.3.3.

By orthogonality between C^* and $\{c_0, d_0, a_1\}$, it follows by Lemma 5.1 that $d_0 \in C^*$. Moreover, orthogonality between C^* and the circuits $\{3, d_0, d_1, 2\}$ and $\{c_1, d_1, a_2\}$ implies that C^* meets $\{2, 3\}$. If $\{a_0, c_0, d_0, 2\}$ is a cocircuit, then its symmetric difference with $\{2, d_1, a_2, c_2\}$ is $\{a_0, c_0, d_0, d_1, a_2, c_2\}$, which must be a cocircuit. Hence $\lambda(T_0 \cup T_1 \cup T_2 \cup \{d_0, d_1\}) \leq 2$; a contradiction. We deduce that $\{a_0, c_0, d_0, 3\}$ is a cocircuit, and M contains the structure in Figure 29. Furthermore, all of the elements in that figure are distinct since the triad $\{2, 3, 4\}$ cannot meet any triangles. Let S be this set of 14 elements. Observe that $\lambda(S - 4) \leq 2$, so $|E(M)| \leq 16$.

We show next that $S-4$ does not span M. Assume the contrary. Since $r(S-4) \leq 7$, and $S - 4$ contains five cocircuits, none of which is a symmetric difference of some of the others, we deduce that $r(S - 4) = 7$ otherwise $\lambda(S - 4) \leq 1$. Thus $\{2, 3, a_0, c_0, a_1, a_2, c_2\}$ is a basis, B, of M. Then orthogonality with the cocircuits

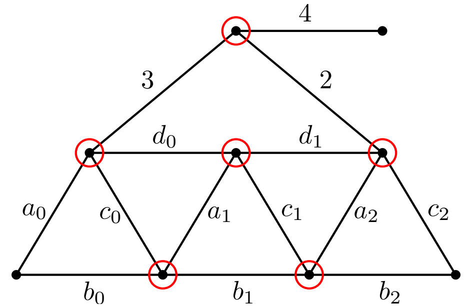

FIGURE 29. All elements shown are distinct.

in Figure 29 implies that the fundamental circuit $C(4,B)$ is $\{4,3,a_0\}$ or $\{4,2,c_2\}$, so 4 is in a triangle of M; a contradiction. We conclude that $S-4$ does not span M. Thus $|E(M)|=16$, and $\{4,5,6\}$ is a triad of M, where $E(M)-S=\{5,6\}$. Therefore $\{2,3,a_0,c_0,a_1,a_2,c_2,4\}$ is a basis, B', of M. After possibly switching the labels on 5 and 6, we see that $C(5,B')$ and $C(6,B')$ are $\{5,4,3,a_0\}$ and $\{6,4,2,c_2\}$, respectively. Now $E(M)-B'$ is a cobasis, $\{5,6,b_0,d_0,b_1,c_1,d_1,b_2\}$. By orthogonality with the circuits in Figure 29 and with $\{5,4,3,a_0\}$ and $\{6,4,2,c_2\}$, we see that the fundamental cocircuit $C^*_M(a_0,E(M)-B')$ is $\{a_0,b_0,5\}$. Thus M has a triad meeting a triangle; a contradiction. □

Next we consider the case when M contains the structure in Figure 8(b) where the two ambiguous cocircuits are both 5-cocircuits. We relabel this ladder structure as shown in Figure 30.

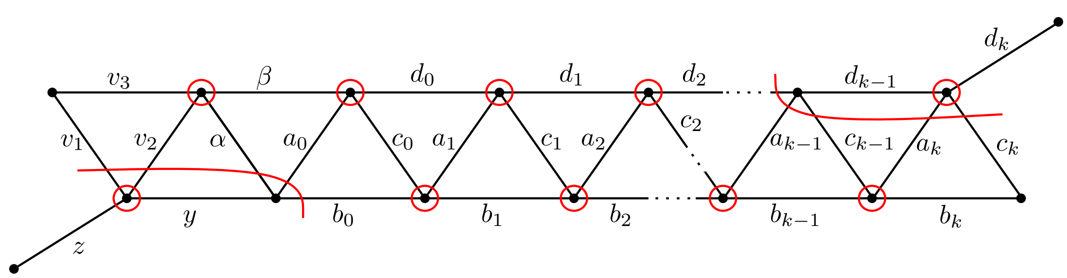

FIGURE 30. Both $\{d_{k-2},a_{k-1},c_{k-1},a_k,c_k\}$ and $\{v_1,v_2,\alpha,a_0,b_0\}$ are cocircuits.

Lemma 7.4. *Let M be an internally 4-connected binary matroid such that $|E(M)|\geq 15$. Suppose that M contains the structure in Figure 30, where $k\geq 1$ and the elements are all distinct except that z may equal d_k. Let $S=\{v_1,\alpha,c_0,c_1,\dots,c_k\}$. Suppose that $M\backslash e$ is $(4,4,S)$-connected for every e in S. Suppose $\{v_1,v_3,b_k,c_k\}$ is not a cocircuit. Then $M\backslash S$ is sequentially 4-connected, and every 4-fan of $M\backslash S$ is either a 4-fan of $M\backslash c_k$ that has b_k as its coguts element or is a 4-fan of $M\backslash v_1$ that has v_3 as its coguts element.*

Proof. First observe that

7.4.1. *neither $\{d_{k-1},d_k\}$ nor $\{b_0,b_1\}$ is contained in a triangle.*

The first of these follows since $M\backslash c_k$ is $(4,4,S)$-connected; the second follows by orthogonality.

Next observe that M is not the cycle matroid of a quartic Möbius ladder, since all of the cocircuits of the latter have even cardinality. Then, by 7.4.1 and [7, Lemmas 6.1 and 6.5], $M\backslash c_0, c_1, \dots, c_k$ is $(4,4,S)$-connected. The last matroid has α as the guts element of a 4-fan, so $M\backslash c_0, c_1, \dots, c_k, \alpha$ is 3-connected.

We show next that

7.4.2. *$M\backslash S$ is 3-connected.*

Suppose not. Since $M\backslash(S - v_1)$ is 3-connected having $\{v_1, v_2, v_3\}$ as a triangle such that $M\backslash(S - v_1)\backslash v_i$ is not 3-connected for all i in $\{1,2,3\}$, Tutte's Triangle Lemma [16] (see also [14, Lemma 8.7.7]) implies that $M\backslash(S - v_1)$ has a triad, $\{v_1, f, g\}$, with f in $\{v_2, v_3\}$. Then M has a cocircuit C^* such that $\{v_1, f, g\} \subsetneqq C^* \subseteq \{v_1, f, g\} \cup S$. Suppose $f = v_2$. Then, since $\{v_2, \beta, a_0, y\}$ is a circuit of $M\backslash(S - v_1)$, orthogonality implies that $g \in \{y, a_0, \beta\}$. Then orthogonality with the triangles in Figure 30 implies that $\{c_0, c_1, \dots, c_k\}$ avoids C^*, so $C^* = \{v_1, f, g, \alpha\}$; a contradiction to Lemma 5.1. Thus $f = v_3$. Every element in $S - v_1 - c_k$ is in two triangles of M that avoid all the other elements in this set as well as $\{v_1, v_3\}$. Thus orthogonality implies that C^* avoids $S - v_1 - c_k$, so $C^* = \{v_1, v_3, g, c_k\}$. Now orthogonality between C^* and the triangles $\{c_{k-1}, d_{k-1}, a_k\}$ and T_k implies that $g = b_k$, a contradiction to a hypothesis of the lemma. We conclude that 7.4.2 holds.

7.4.3. *$M\backslash S$ is sequentially 4-connected.*

To see this, suppose that $M\backslash S$ has a non-sequential 3-separation, (U, V). Without loss of generality, the triad $\{\beta, v_2, v_3\} \subseteq U$. If $\{a_0, d_0\}$ meets U, then we may assume that $\{a_0, d_0\} \subseteq U$, and then $(U \cup \{v_1, \alpha\}, V)$ is a non-sequential 3-separation of $M\backslash c_0, c_1, \dots, c_k$; a contradiction. Thus $\{a_0, d_0\} \subseteq V$, and we can move β into V, and then $((U - \beta) \cup v_1, (V \cup \beta) \cup \alpha)$ is a non-sequential 3-separation of $M\backslash c_0, c_1, \dots, c_k$; a contradiction. Hence 7.4.3 holds.

Now suppose that $M\backslash S$ has a 4-fan $F = (x_1, x_2, x_3, x_4)$ such that F is not a 4-fan of $M\backslash c_k$ having b_k as its coguts element, and F is not a 4-fan of $M\backslash v_1$ having v_3 as its coguts element. Then M has a cocircuit C^* such that $\{x_2, x_3, x_4\} \subsetneqq C^* \subseteq \{x_2, x_3, x_4\} \cup S$. Next we show that

7.4.4. *C^* meets both $\{v_1, \alpha\}$ and $\{c_{k-1}, c_k\}$.*

By the rotational symmetry of Figure 30, it suffices to prove that C^* meets $\{v_1, \alpha\}$. Assume C^* avoids this set.

Suppose first that $k = 1$. Then F is a 4-fan of $M\backslash c_0, c_1$. Thus, by 7.4.1 and [7, Lemma 6.1], as $M\backslash c_0$ and $M\backslash c_1$ are both $(4,4,S)$-connected, it follows that $M\backslash c_0, c_1$ is $(4,4,S)$-connected and every 4-fan of it has α as its guts element or b_1 as its coguts element. Since F is a 4-fan of $M\backslash v_1, \alpha, c_0, c_1$, we deduce that $x_1 \neq \alpha$, so $x_4 = b_1$. By assumption, (x_1, x_2, x_3, b_1) is not a 4-fan of $M\backslash c_1$. Thus $\{x_2, x_3, b_1, c_0\}$ is a cocircuit of $M\backslash c_1$. By orthogonality, $\{x_2, x_3\}$ meets both $\{d_0, a_1\}$ and $\{a_0, b_0\}$. Now $\{x_2, x_3\}$ avoids a_0, otherwise orthogonality implies that $\{x_2, x_3\}$ also meets $\{\alpha, \beta\}$, so $\{x_1, x_2, x_3\} = \{\alpha, \beta, a_0\}$; a contradiction. Thus $b_0 \in \{x_2, x_3\}$. Moreover, $d_0 \notin \{x_2, x_3\}$ otherwise $\{d_0, d_1\} \subseteq \{x_1, x_2, x_3\}$, a contradiction to 7.4.1. Hence $\{x_2, x_3\} = \{a_1, b_0\}$. Then $M\backslash c_0$ has a 5-fan and so is not $(4,4,S)$-connected; a contradiction. We conclude that 7.4.4 holds when $k = 1$.

Now suppose that $k \geq 2$. Then F is a 4-fan of $M\backslash c_0, c_1, \dots, c_k$. By [7, Lemma 6.5], since F is not a 4-fan of $M\backslash c_k$ having b_k as its coguts element, it follows that $x_4 = d_0$, that $a_0 \in \{x_2, x_3\}$, and that F is a 4-fan of $M\backslash c_0$. Then $\{x_2, x_3, x_4\} = \{d_0, a_0, \beta\}$, so $x_1 = \alpha$; a contradiction since F is a fan in $M\backslash S$. We conclude that 7.4.4 holds when $k \geq 2$ and so it holds in general.

We now show that

7.4.5. $\{x_1, x_2, x_3\}$ *avoids* $\{a_{k-1}, b_{k-1}, d_{k-1}, a_k\}$ *and* $\{y, \beta, v_2, a_0\}$.

By symmetry, it suffices to prove that $\{x_1, x_2, x_3\}$ avoids the first of these sets. Assume the contrary. As $\{x_1, x_2, x_3\}$ is a triangle of $M\backslash \alpha, c_0, c_1, \dots, c_k$, [7, Lemma 6.3] implies that $\{x_1, x_2, x_3\}$ meets $\{v_2, y, \beta, a_0, b_0, d_0, \dots, a_k, b_k, d_k\}$ in $\{v_2\}$, $\{d_{k-1}, d_k\}$, or $\{v_2, d_{k-1}, d_k\}$. By 7.4.1, the last two possibilities are excluded. By orthogonality, $\{x_1, x_2, x_3\}$ must contain v_3 as well as v_2. Thus $\{x_1, x_2, x_3\} = \{v_1, v_2, v_3\}$; a contradiction since $v_1 \notin \{x_1, x_2, x_3\}$. We conclude that 7.4.5 holds.

We now apply 7.4.4. If $c_{k-1} \in C^*$, then orthogonality implies that $\{x_2, x_3, x_4\}$ meets $\{a_{k-1}, b_{k-1}\}$ and $\{d_{k-1}, a_k\}$. Then $\{x_1, x_2, x_3\}$ meets at least one of these sets; a contradiction to 7.4.5. Thus $c_{k-1} \notin C^*$ and, by symmetry, $\alpha \notin C^*$. We deduce, by 7.4.4, that $\{x_2, x_3, x_4, v_1, c_k\} \subseteq C^* \subseteq \{x_2, x_3, x_4\} \cup S$.

By orthogonality, both $\{v_2, v_3\}$ and $\{a_k, b_k\}$ meet $\{x_2, x_3, x_4\}$. Thus, by symmetry, we may assume that $\{x_1, x_2, x_3\}$ meets $\{a_k, b_k\}$. Since $c_k \notin \{x_1, x_2, x_3\}$, it follows by orthogonality that $\{x_1, x_2, x_3\}$ contains b_{k-1}; a contradiction to 7.4.5. $\square$

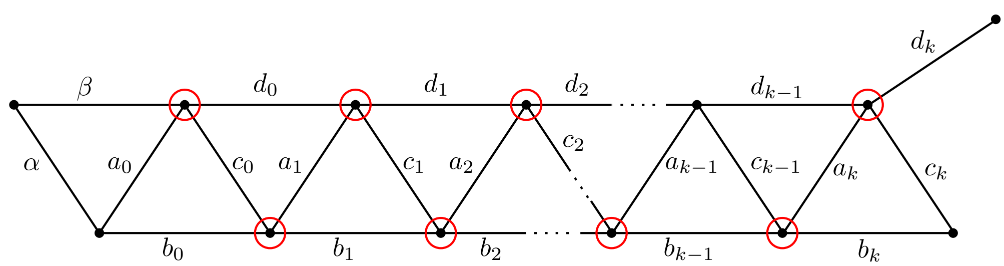

FIGURE 31. M has either $\{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$ or $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\}$ as a cocircuit.

We are now in a position to address the case when M has a good bowtie string with three triangles. Although the proof of the next lemma is long, it is broken into a number of pieces.

Lemma 7.5. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose Hypothesis VIII holds. If M has T_0, D_0, T_1, D_1, T_2 as a string of bowties, and $M\backslash c_0$ is $(4,4,S)$-connected with an N-minor, then*

(i) *M has a quick win; or*
(ii) *M has a ladder win; or*
(iii) *M has a good bowtie ring; or*
(iv) *M has a ladder-compression win; or*
(v) *M is the cycle matroid of a quartic Möbius ladder, $r(N) = r(M) - 1$, and N is the cycle matroid of a cubic Möbius ladder; or*

(vi) *M is the dual of a triadic Möbius matroid of rank $2r$ for some $r \geq 4$ and N is the dual of a triangular Möbius matroid of rank r.*

Proof. Suppose the lemma does not hold. Then, by Lemma 4.1, M is not the cycle matroid of a quartic Möbius ladder otherwise (iv) or (v) holds. Take a right-maximal bowtie string $T_0, D_0, T_1, D_1, \dots, T_k$ in M. This is a good bowtie string. We assume that it is a maximum-length good bowtie string in M. By repeated application of Hypothesis VIII, we deduce that

7.5.1. *$M\backslash c_i, c_{i+1}$ has an N-minor for all i in $\{0, 1, \dots, k-1\}$ and $M\backslash c_j$ is $(4, 4, S)$-connected for all j in $\{0, 1, \dots k\}$.*

Next we show that

7.5.2. *M has a maximum-length good bowtie string $T'_0, D'_0, T'_1, D'_1, \dots, T'_k$ such that M has no bowtie (T'_k, T'_{k+1}, D'_k) where D'_k contains the element of $T'_k - D'_{k-1}$.*

Assume that this assertion fails. Then M has a bowtie of the form $(T_k, T_{k+1}, \{x, c_k, a_{k+1}, b_{k+1}\})$ for some x in $\{a_k, b_k\}$. By possibly interchanging the labels on a_k and b_k, we may assume that $x = b_k$.

Suppose $a_0 \neq c_k$. As our string of bowties is right-maximal, by [7, Lemma 5.4], $T_{k+1} = T_\ell$, for some ℓ with $0 \leq \ell \leq k-2$. Now $a_\ell \notin \{a_{k+1}, b_{k+1}\}$, otherwise M has a good bowtie ring with triangles $T_\ell, T_{\ell+1}, \dots, T_k$; a contradiction. Thus $\{b_\ell, c_\ell\} = \{a_{k+1}, b_{k+1}\}$, and the symmetric difference of $\{b_k, c_k, b_\ell, c_\ell\}$ and $\{b_\ell, c_\ell, a_{\ell+1}, b_{\ell+1}\}$ is $\{b_k, c_k, a_{\ell+1}, b_{\ell+1}\}$, which must be a cocircuit. It follows that M has a good bowtie ring with triangles $T_{\ell+1}, T_{\ell+2}, \dots, T_k$; a contradiction.

We now know that $a_0 = c_k$. Orthogonality implies that $\{b_0, c_0\}$ meets $\{a_{k+1}, b_{k+1}\}$ and hence that T_{k+1} contains two elements in D_0. As $M\backslash c_0$ has no 5-fans, neither $\{b_0, a_1\}$ nor $\{b_0, b_1\}$ is contained in a triangle, so $\{c_0, a_1\}$ or $\{c_0, b_1\}$ is contained in T_{k+1}. The latter implies, by orthogonality with D_1, that T_{k+1} meets $\{a_2, b_2\}$, so the cocircuit $\{b_k, c_k, a_{k+1}, b_{k+1}\}$ meets T_0, T_1, or T_2 in a single element; a contradiction. We conclude that $\{c_0, a_1\} \subseteq T_{k+1}$. Label the third element of T_{k+1} by d_0.

Using orthogonality, it is straightforward to check that $d_0 \notin T_0 \cup T_1 \cup \dots \cup T_k$. By orthogonality between T_0 and $\{b_k, c_k, a_{k+1}, b_{k+1}\}$, we deduce, since $T_{k+1} = \{c_0, a_1, d_0\}$, that $c_0 \in \{a_{k+1}, b_{k+1}\}$. Hence we may assume that $c_0 = a_{k+1}$. Using orthogonality again, we deduce that $a_1 = c_{k+1}$, so $d_0 = b_{k+1}$ and $(a_{k+1}, b_{k+1}, c_{c+1}) = (c_0, d_0, a_1)$.

Now $T_1, D_1, T_2, D_2, \dots, T_{k+1}$ is a maximum-length good bowtie string, and Hypothesis VIII implies that $M\backslash c_k, c_{k+1}$ has an N-minor and $M\backslash c_{k+1}$ is $(4, 4, S)$-connected. By assumption, M has a bowtie $(T_{k+1}, T_{k+2}, \{x', c_{k+1}, a_{k+2}, b_{k+2}\})$ for some x' in $\{a_{k+1}, b_{k+1}\}$. Moreover, the argument above implies that T_{k+2} contains $\{c_1, a_2\}$ and some element d_1 that is not in $T_1 \cup T_2 \cup \dots \cup T_{k+1}$. In addition, orthogonality implies that $d_1 \notin T_0$. As in the last paragraph, we may assume that $(a_{k+2}, b_{k+2}, c_{k+2}) = (c_1, d_1, a_2)$. By orthogonality between $\{x', c_{k+1}, a_{k+2}, b_{k+2}\}$ and $\{a_0, b_0, c_0\}$ recalling that $c_0 = a_{k+1}$, we deduce that $x' = b_{k+1}$. Thus M has $\{b_k, a_0, c_0, d_0\}$ and $\{d_0, a_1, c_1, d_1\}$ as cocircuits where we recall that $d_0 = b_{k+1}$. Hence $(a_{k+2}, b_{k+2}, c_{k+2}) = (c_1, d_1, a_2)$. Furthermore, the elements in $T_0 \cup T_1 \cup \dots \cup T_k \cup \{d_0, d_1\}$ are distinct.

By applying the argument above to the maximum-length good bowtie string $T_2, D_2, T_3, D_3, \dots, T_{k+2}$, we deduce that M has a triangle of the form $\{c_2, a_3, d_2\}$

and a cocircuit of the form $\{d_1, a_2, c_2, d_2\}$. Repeating this process, we eventually find that M has a triangle T_{2k+1} where $(a_{2k+1}, b_{2k+1}, c_{2k+1}) = (c_k, d_k, a_{k+1})$. Indeed, M has distinct elements $d_0, d_1, \ldots, d_k$ where $d_0, d_1, \ldots, d_{k-1}$ are not in $T_0 \cup T_1 \cup \cdots \cup T_k$, and M has all of the triangles shown in Figure 28 where $k = n$. Moreover, M has $\{b_k, a_0, c_0, d_0\}$ and $\{d_0, a_1, c_1, d_1\}, \{d_1, a_2, c_2, d_2\}, \ldots, \{d_{k-1}, a_k, c_k, d_k\}$ as cocircuits. Since $(a_{2k+1}, b_{2k+1}, c_{2k+1}) = (c_k, d_k, a_{k+1})$ and $(c_k, a_{k+1}) = (a_0, c_0)$, we deduce that $d_k = b_0$, so $(a_0, b_0) = (c_k, d_k)$.

By [7, Lemma 6.4] and Section 4, M is the cycle matroid of a quartic Möbius ladder, or M is the dual of a triadic Möbius matroid of even rank. In the first case, it follows by Lemma 4.1 that (iv) or (v) holds; a contradiction. In the second case, using the notation of Section 4, $M \cong M_{2n+1}$. Since neither (v) nor (vi) holds, we may assume that N is a minor of M_{2n+1}/γ. Since the last matroid is internally 4-connected, (i) holds; a contradiction. We conclude that 7.5.2 holds.

We now relabel so that $T_0, D_0, T_1, D_1, \ldots, T_k$ is the maximum-length good bowtie string whose existence is guaranteed by 7.5.2. By 7.5.1, $M\backslash c_{k-1}$ and $M\backslash c_k$ are $(4,4,S)$-connected having an N-minor. Applying Theorem 5.6 to the bowtie (T_k, T_{k-1}, D_{k-1}) gives that every $(4,3)$-violator of $M\backslash c_k$ is a 4-fan of the form $(c_{k-1}, t, d_{k-1}, d_k)$, where $t \in \{a_k, b_k\}$ and $|T_{k-1} \cup T_k \cup \{d_{k-1}, d_k\}| = 8$. After possibly switching the labels on a_k and b_k, we may assume that $t = a_k$. Thus $\{c_{k-1}, d_{k-1}, a_k\}$ is a triangle and $\{d_{k-1}, a_k, c_k, d_k\}$ is a cocircuit. Take m minimal such that, for all i in $\{m, m+1, \ldots, k\}$, there is an element d_i for which $\{c_i, d_i, a_{i+1}\}$ is a triangle and $\{d_i, a_{i+1}, c_{i+1}, d_{i+1}\}$ or $\{d_i, a_{i+1}, c_{i+1}, a_{i+2}, c_{i+2}\}$ is a cocircuit, where we only allow the 5-cocircuit in the case that $i = n-2$.

Since the lemma fails, Lemma 7.3 implies that

7.5.3. *$M\backslash c_m, c_{m+1}, \ldots, c_k$ has an N-minor but is not internally 4-connected.*

In order to apply [7, Lemma 6.1 or Lemma 6.5] to our structure, we have to show that

7.5.4. *the elements in $T_m \cup T_{m+1} \cup \ldots T_k \cup \{d_m, d_{m+1}, \ldots, d_k\}$ are all distinct.*

This is certainly true if $m = k-1$, so assume that $m \le k-2$. Then the elements in $T_{k-1} \cup T_k \cup \{d_{k-2}, d_{k-1}, d_k\}$ are all distinct, otherwise $d_{k-2} = d_k$ and $\lambda(T_{k-1} \cup T_k \cup \{d_{k-1}, d_k\}) \le 2$; a contradiction. We know that $\{d_{k-1}, d_k\}$ is not contained in a triangle, since $M\backslash c_k$ contains no 5-fans. Now, 7.5.2 implies that M is not the cycle matroid of a quartic Möbius ladder and is not the dual of a triadic Möbius matroid. Then [7, Lemma 6.4] implies that 7.5.4 holds.

By Hypothesis VIII and 7.5.1, both $M\backslash c_{k-1}$ and $M\backslash c_k$ are $(4,4,S)$-connected. Hence

7.5.5. *neither $\{d_{k-1}, d_k\}$ nor $\{b_{k-1}, b_k\}$ is contained in a triangle of M.*

Suppose $m = k-1$ and apply [7, Lemma 6.1] noting that (iv) or (v) of that lemma must occur. In the first case, M has a triangle $\{\alpha, \beta, a_{k-1}\}$ where $\{\beta, a_{k-1}, d_{k-1}, c_{k-1}\}$ or $\{\beta, a_{k-1}, c_{k-1}, a_k, c_k\}$ is a cocircuit of M. As $M\backslash c_{k-2}$ is $(4,4,S)$-connected, it follows that $\alpha = c_{k-2}$ and so the minimality of m is contradicted. In the second case, b_k is the coguts element of every 4-fan of $M\backslash c_{k-1}, c_k$. As M has no quick win, there is a 4-fan $(1,2,3,b_k)$ in $M\backslash c_{k-1}, c_k$. Thus M has a cocircuit C^* such that $\{2,3,b_k\} \subsetneqq C^* \subseteq \{2,3,b_k,c_{k-1},c_k\}$. Also [7, Lemma 6.1] gives that $a_k \notin \{1,2,3\}$. Hence, by orthogonality, $c_k \in C^*$. If $c_{k-1} \notin C^*$, then we contradict the choice of the bowtie string. Thus $c_{k-1} \in C^*$. By orthogonality,

$d_{k-1} \in \{2,3\}$, so M has a triangle containing $\{d_{k-1}, d_k\}$; a contradiction to 7.5.5. We conclude that $m < k-1$.

We can now apply [7, Lemma 6.5] to get that $M\backslash c_m, c_{m+1}, \dots, c_k$ is $(4,4,S)$-connected, and has a 4-fan $F = (\alpha, \beta, \gamma, \delta)$ where either F is a 4-fan in $M\backslash c_m$ with $\delta = d_m$ and a_m in $\{\beta, \gamma\}$; or F is a 4-fan of $M\backslash c_k$ and $\delta = b_k$. If $(T_k, \{\alpha, \beta, \gamma\}, \{b_k, c_k, \beta, \gamma\})$ is a bowtie, then we have a contradiction to 7.5.2, so we assume not. Thus, without loss of generality, M has $\{\alpha, \beta, a_m\}$ as a triangle and $\{\beta, a_m, c_m, d_m\}$ as a cocircuit.

Suppose $m > 0$. Then $c_{m-1} \neq \alpha$ by the minimality of m, and $c_{m-1} \neq \beta$ by Lemma 5.1. Orthogonality between $\{\alpha, \beta, a_m\}$ and the cocircuit $\{b_{m-1}, c_{m-1}, a_m, b_m\}$ implies that $b_{m-1} \in \{\alpha, \beta\}$. Then $\{\alpha, \beta, a_m, b_m, c_m\}$ is a 5-fan in $M\backslash c_{m-1}$; a contradiction to 7.5.1. Thus $m = 0$.

Suppose $\{\alpha, \beta\}$ meets the elements in Figure 28. Then [7, Lemma 6.3] implies that $\{\alpha, \beta, a_0\} = \{a_0, d_{k-1}, d_k\}$; a contradiction to 7.5.5. Thus

7.5.6. *M contains the structure in Figure 31, and all of the elements in $\{\alpha, \beta, a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, \dots, a_k, b_k, c_k, d_k\}$ are distinct.*

We show next that

7.5.7. *$M\backslash \alpha, c_0, c_1, \dots, c_k$ has an N-minor.*

Suppose not. Since $m = 0$, it follows by 7.5.3 that $M\backslash c_0, c_1, \dots, c_k$ has an N-minor. Moreover, as $M\backslash c_0, c_1, \dots, c_k$ has $(\alpha, \beta, a_0, d_0)$ as a 4-fan, we deduce that $M\backslash c_0, c_1, \dots, c_k/d_0$ has an N-minor. The last matroid has (a_0, b_0, a_1, b_1) as a 4-fan. Since $M\backslash c_0, c_1, \dots, c_k/d_0\backslash a_0 \cong M\backslash c_0, c_1, \dots, c_k/\beta\backslash a_0 \cong M\backslash c_0, c_1, \dots, c_k/\beta\backslash \alpha$, we know that $M\backslash c_0, c_1, \dots, c_k/d_0\backslash a_0$ has no N-minor. Thus $M\backslash c_0, c_1, \dots, c_k/d_0/b_1$ has an N-minor. Lemma 7.1 implies that $\{d_0, d_1, 2, 3\}$ is a circuit of M for some triad $\{2,3,4\}$ of M. As M has $\{\beta, a_0, c_0, d_0\}$ as a cocircuit, it follows by orthogonality that $\{2,3\}$ meets $\{\beta, a_0, c_0\}$; a contradiction. Thus 7.5.7 holds.

Next we show the following.

7.5.8. *Let T be a triangle of M that contains c_i for some i in $\{0, 1, \dots, k-1\}$. Then T is T_i or $\{c_i, d_i, a_{i+1}\}$. Moreover, if M has a 4-cocircuit D such that $|D \cap T| = 2$ and $D - T$ avoids $T_0 \cup T_1 \cup \dots \cup T_k \cup \{d_0, d_1, \dots, d_k\}$, then $D \cap T = \{a_0, b_0\}$.*

Assume the first assertion fails. By orthogonality, T has two common elements with $\{b_i, c_i, a_{i+1}, b_{i+1}\}$ and has two common elements with whichever of $\{d_{i-1}, a_i, c_i, d_i\}$ and $\{d_{i-1}, a_i, c_i, a_{i+1}, c_{i+1}\}$ is a cocircuit, where $d_{-1} = \beta$. Thus $b_{i+1} \in T$. Moreover, either d_{i-1} or c_{i+1} is in T. But $c_{i+1} \notin T$ otherwise $T = T_{i+1}$ yet $c_i \in T$. Hence $T = \{c_i, b_{i+1}, d_{i-1}\}$ and $\lambda(T_i \cup T_{i+1} \cup \{d_{i-1}, d_i\}) \leq 2$; a contradiction. We deduce that the first assertion holds.

Now let D be a 4-cocircuit satisfying the hypothesis. Suppose $c_i \in D$. Then, by orthogonality, D meets both $\{a_i, b_i\}$ and $\{d_i, a_{i+1}\}$. Hence $D - T$ meets $\{a_i, b_i, d_i, a_{i+1}\}$; a contradiction. We deduce that $c_i \notin D$. Thus $D \cap T$ is $\{a_i, b_i\}$ or $\{d_i, a_{i+1}\}$. In the latter case, we get a contradiction to orthogonality between D and T_{i+1}. Thus $D \cap T = \{a_i, b_i\}$. Hence $i = 0$ otherwise we get the contradiction that D meets the circuit $\{c_{i-1}, d_{i-1}, b_i, c_i\}$ in a single element. We conclude that 7.5.8 holds.

We now show that

7.5.9. *M has no bowtie $(\{a_0, \alpha, \beta\}, \{v_1, v_2, v_3\}, \{\alpha, \beta, v_2, v_3\})$.*

Suppose otherwise. As a first step towards establishing 7.5.9, we show that

7.5.10. *$\{v_1, v_2, v_3\}$ avoids the elements in Figure 31.*

Assume that 7.5.10 fails. Clearly $\{v_1, v_2, v_3\}$ avoids $\{\alpha, \beta, a_0\}$. Since $\{d_{k-1}, d_k\}$ is not contained in a triangle, we know from [7, Lemma 6.3] that $c_i \in \{v_1, v_2, v_3\}$ for some i in $\{0, 1, \dots, k\}$. Suppose $i < k$. Taking $T = \{v_1, v_2, v_3\}$ and $D = \{\alpha, \beta, v_2, v_3\}$ in 7.5.8, we deduce that $\{v_2, v_3\} = \{a_0, b_0\}$. Thus $\{\alpha, \beta, v_2, v_3\}$ contains the triangle $\{a_0, \alpha, \beta\}$; a contradiction.

We now consider the case when $i = k$, so $c_k \in \{v_1, v_2, v_3\}$. By 7.5.2, $v_1 \neq a_k$ otherwise the bowtie $(\{a_0, \alpha, \beta\}, T_k, \{\alpha, \beta, v_2, v_3\})$ gives a contradiction. Thus $T_k \neq \{v_1, v_2, v_3\}$ otherwise, by orthogonality between $\{c_{k-1}, d_{k-1}, a_k\}$ and $\{\alpha, \beta, v_2, v_3\}$, we get the contradiction that $v_1 = a_k$. It follows that $c_k \notin \{v_2, v_3\}$ otherwise, by orthogonality between T_k and $\{\alpha, \beta, v_2, v_3\}$, we find that $T_k = \{v_1, v_2, v_3\}$; a contradiction. Hence $c_k = v_1$. By orthogonality between $\{v_1, v_2, v_3\}$ and $\{d_{k-1}, a_k, c_k, d_k\}$, we deduce that $\{v_2, v_3\}$ meets $\{d_{k-1}, d_k\}$. If $\{v_2, v_3\}$ meets $\{c_{k-1}, d_{k-1}, a_k\}$, then, by orthogonality between $\{\alpha, \beta, v_2, v_3\}$ and $\{c_{k-1}, d_{k-1}, a_k\}$, we deduce that $\{v_1, v_2, v_3\} = \{c_{k-1}, d_{k-1}, a_k\}$; a contradiction. Thus $d_k \in \{v_2, v_3\}$ so, without loss of generality, $d_k = v_2$. Moreover, orthogonality implies that $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\}$ is a not cocircuit, so $\{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$ is a cocircuit. Now we note that $\{v_3, d_k, c_k\}$, $\{d_k, c_k, a_k, d_{k-1}\}, \{a_k, d_{k-1}, c_{k-1}\}, \{d_{k-1}, c_{k-1}, a_{k-1}, d_{k-2}\}, \dots, \{a_1, d_0, c_0\}, \{d_0, c_0, a_0, \alpha\}, \{a_0, \alpha, \beta\}$ is a good bowtie string that is longer than our original good bowtie string; a contradiction. Thus 7.5.10 holds.

Since we chose a good bowtie string of maximum length, we know now that $\{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$ is not a cocircuit otherwise $\{a_k, d_{k-1}, c_{k-1}\}, \{d_{k-1}, c_{k-1}, a_{k-1}, d_{k-2}\}, \{a_{k-1}, d_{k-2}, c_{k-2}\}, \dots, \{a_0, \beta, \alpha\}, \{\beta, \alpha, v_2, v_3\}, \{v_2, v_3, v_1\}$ is a good bowtie string violating our original choice of such a string. Hence

7.5.11. *M has $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\}$ as a cocircuit.*

We continue the proof of 7.5.9 by showing that

7.5.12. *M has no triangle disjoint from $\{v_1, v_2, v_3\}$ that meets a 4-cocircuit containing v_1.*

Assume the contrary. After possibly relabelling v_2 and v_3, we may suppose M has a triangle $\{w_1, w_2, w_3\}$ disjoint from $\{v_1, v_2, v_3\}$ and a cocircuit $\{w_2, w_3, v_1, v_2\}$. Then $\{w_1, w_2, w_3\}$ meets the elements in Figure 31, otherwise we have a good bowtie string that is longer than our original good bowtie string. Now $\{w_1, w_2, w_3\}$ avoids $\{\alpha, \beta\}$ otherwise, by orthogonality, $\{w_1, w_2, w_3\} = \{\alpha, \beta, a_0\}$ and we get a contradiction to Lemma 5.1. By 7.5.5 $\{d_{k-1}, d_k\}$ is not in a triangle, so [7, Lemma 6.3] implies that either $\{w_1, w_2, w_3\}$ meets the set of elements in Figure 31 in $\{a_0\}$; or $c_i \in \{w_1, w_2, w_3\}$ for some i in $\{0, 1, \dots, k\}$. The former gives a contradiction to orthogonality with $\{\beta, a_0, c_0, d_0\}$, so the latter holds.

Suppose $i < k$. Then, taking $T = \{w_1, w_2, w_3\}$ and $D = \{w_2, w_3, v_1, v_2\}$ in 7.5.8, we deduce that $\{w_2, w_3\} = \{a_0, b_0\}$. Then $D = \{a_0, b_0, v_1, v_2\}$ and we have a contradiction to Lemma 5.1 in the bowtie $(\{a_0, \alpha, \beta\}, \{v_1, v_2, v_3\}, \{\alpha, \beta, v_2, v_3\})$. We deduce that $i = k$. Suppose $a_k \in \{w_1, w_2, w_3\}$. Then the triangle is T_k, and $\{a_{k-1}, d_{k-2}, c_{k-2}\}, \{d_{k-2}, c_{k-2}, a_{k-2}, d_{k-3}\}, \{a_{k-2}, d_{k-3}, c_{k-3}\}, \{d_{k-3}, c_{k-3}, a_{k-3}, d_{k-4}\}, \dots, \{a_1, d_0, c_0\}, \{d_0, c_0, a_0, \beta\}, \{a_0, \beta, \alpha\}, \{\beta, \alpha, v_3, v_2\}, \{v_3, v_2, v_1\}, \{v_2, v_1, w_3,$

$w_2\}, T_k$ is a longer good bowtie chain than our original good bowtie chain; a contradiction. Thus $a_k \notin \{w_1, w_2, w_3\}$. By orthogonality with the cocircuits $\{d_{k-1}, a_k, c_k, d_k\}$ and $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\}$, we deduce that $\{w_1, w_2, w_3\}$ is contained in $T_{k-1} \cup T_k \cup \{d_{k-2}, d_{k-1}, d_k\}$, so the last set is 3-separating; a contradiction. We deduce that 7.5.12 holds.

We continue the proof of 7.5.9. Since $M\backslash c_0$ is $(4,4,S)$-connected with an N-minor, Hypothesis VIII implies that $M\backslash \alpha$ and $M\backslash v_1$ are each $(4,4,S)$-connected, and $M\backslash \alpha, v_1$ has an N-minor. From Lemma 5.6, we also know, by 7.5.12, that every $(4,3)$-violator of $M\backslash v_1$ is a 4-fan of the form (α, x, y, z), where $x \in \{v_2, v_3\}$ and $|\{v_1, v_2, v_3, y, z, \alpha, \beta, a_0\}| = 8$. Without loss of generality, we may assume that $x = v_2$. By orthogonality between $\{\alpha, v_2, y\}$ and the cocircuits in Figure 31 as well as the cocircuit $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\}$, we deduce that y differs from all of the elements in Figure 31. Now 7.5.12 implies that $\{y, z\}$ is not contained in a triangle of M. By [7, Lemma 6.1],

(a) $\{v_3, \beta\}$ is contained in a triangle; or
(b) M has a triangle $\{x_1, x_2, a_0\}$ such that $\{x_2, a_0, \alpha, y\}$ or $\{x_2, a_0, \alpha, v_2, v_1\}$ is a cocircuit; or
(c) $M\backslash \alpha, v_1$ is $(4,4,S)$-connected and has v_3 in the coguts of every 4-fan.

If $\{v_3, \beta\}$ is in a triangle, then this triangle together with $\{v_1, v_2\}$ forms a 5-fan in $M\backslash \alpha$; a contradiction. Thus (a) does not hold. Suppose (c) holds, and let $(1, 2, 3, v_3)$ be a 4-fan in $M\backslash \alpha, v_1$. We know that $\{2, 3, v_3, v_1\}$ is not a cocircuit of M by 7.5.12, so either $\{2, 3, v_3, \alpha\}$ or $\{2, 3, v_3, \alpha, v_1\}$ is a cocircuit C^* of M. By orthogonality, C^* meets $\{y, v_2\}$ and $\{\beta, a_0\}$, so $\{2, 3\} \subseteq \{y, v_2, \beta, a_0\}$. If $C^* \neq \{v_2, \beta, v_3, \alpha\}$, then $\lambda(\{v_1, v_2, v_3, y, \alpha, \beta, a_0\}) \leq 2$; a contradiction. Thus $\{2, 3\} = \{v_2, \beta\}$ and $M\backslash \alpha$ has a 5-fan; a contradiction. We conclude that (c) does not hold. Thus (b) holds. By orthogonality between the triangle $\{x_1, x_2, a_0\}$ and the cocircuit $\{\beta, a_0, c_0, d_0\}$, we know that $\{x_1, x_2\}$ meets $\{c_0, d_0\}$. If $\{a_0, d_0\}$ is in a triangle, then this triangle together with $\{\alpha, \beta\}$ is a 5-fan in $M\backslash c_0$; a contradiction. Thus $c_0 \in \{x_1, x_2\}$, so $\{x_1, x_2\} = \{b_0, c_0\}$. By orthogonality between $\{c_0, d_0, a_1\}$ and whichever of $\{x_2, a_0, \alpha, y\}$ and $\{x_2, a_0, \alpha, v_2, v_1\}$ is a cocircuit, we know that $x_2 \notin \{c_0, d_0, a_1\}$, so $x_2 = b_0$ and $x_1 = c_0$. If $\{b_0, a_0, \alpha, y\}$ is a cocircuit, then M has a longer good bowtie chain than the original good bowtie chain; a contradiction. Therefore $\{b_0, a_0, \alpha, v_2, v_1\}$ is a cocircuit, and we have the structure in Figure 30.

We complete the proof of 7.5.9 by obtaining the contradiction that (i) holds. Recall from 7.5.7 that $M\backslash \alpha, c_0, c_1, \dots, c_k$ has an N-minor. We also know that the elements in Figure 30 are all distinct except that z may equal one of the other elements. By orthogonality between the cocircuit $\{v_1, v_2, y, z\}$ and the triangles in Figure 30, we deduce that the elements in the figure are all distinct except that z may be d_k. By 7.5.12, Lemma 7.4 implies that $M\backslash v_1, \alpha, c_0, c_1, \dots, c_k$ is sequentially 4-connected and every 4-fan of this matroid is either a 4-fan of $M\backslash c_k$ having b_k as its coguts element or is a 4-fan of $M\backslash v_1$ that has v_3 as its coguts element. By 7.5.2 and 7.5.12, we deduce that $M\backslash v_1, \alpha, c_0, c_1, \dots, c_k$ has no 4-fans. Hence this matroid is internally 4-connected. As (ii) does not hold, we deduce that $M\backslash v_1, \alpha, c_0, c_1, \dots, c_k$ does not have an N-minor. By 7.5.7, $M\backslash \alpha, c_0, c_1, \dots, c_k$ has an N-minor. Since it also has (v_1, v_2, v_3, β) as a 4-fan, $M\backslash \alpha, c_0, c_1, \dots, c_k/\beta$ has an N-minor. As $M/\beta\backslash \alpha, c_0 \cong M/\beta\backslash a_0, c_0$, we deduce that $M\backslash a_0/\beta$ has an N-minor. But Lemma 6.4 implies that $M\backslash a_0/\beta$ or $M\backslash a_0$ is internally 4-connected, so (i) holds; a contradiction. This completes the proof of 7.5.9.

We now know, by 7.5.6 and 7.5.9, that M contains the structure in Figure 31 and M has no triangle that meets a 4-cocircuit that contains $\{\alpha, \beta\}$. As $M\backslash\alpha$ has an N-minor, $M\backslash\alpha$ is not internally 4-connected. Suppose M has a triangle $\{7, 8, 9\}$ such that $(\{\alpha, \beta, a_0\}, \{7, 8, 9\}, \{\alpha, t, 7, 8\})$ is a bowtie. Then 7.5.9 implies that $t = a_0$, and orthogonality implies that $\{7, 8\}$ meets $\{b_0, c_0\}$. Lemma 5.1 implies that $c_0 \notin \{7, 8\}$, so, without loss of generality, $7 = b_0$. Thus $\{7, 8, 9, a_0, c_0\}$ is a 5-fan in $M\backslash\alpha$; a contradiction to Hypothesis VIII. Hence M has no such triangle $\{7, 8, 9\}$. This combined with 7.5.9 implies that (ii) of Lemma 5.6 does not hold. Since $M\backslash a_1$ is not internally 4-connected, Lemma 5.6 implies that every $(4, 3)$-violator of $M\backslash\alpha$ is a 4-fan of the form (c_0, v, γ, δ) for some v in $\{a_0, \beta\}$. If $v = \beta$, then orthogonality between $\{c_0, \beta, \gamma\}$ and D_0 implies that $\gamma = b_1$ and $\lambda(T_0 \cup T_1 \cup \{\beta, d_0\}) \leq 2$; a contradiction. Thus $v = a_0$, so $\gamma = b_0$ and $\{\alpha, a_0, b_0, \delta\}$ is a cocircuit of M. By orthogonality with the triangles in Figure 31, we know that δ avoids all of these elements except possibly d_k.

If $M\backslash\alpha, c_0, c_1, \ldots, c_k$ is internally 4-connected, then (ii) holds; a contradiction. If M is the cycle matroid of a quartic Möbius ladder, then we get a contradiction to 7.5.9. Thus [8, Lemma 4.1] implies that $M\backslash\alpha, c_0, c_1, \ldots, c_k$ is $(4, 4, S)$-connected and every $(4, 3)$-violator of this matroid is a 4-fan of $M\backslash c_k$ or a 4-fan of $M\backslash\alpha$. Since c_0 is the guts element of every 4-fan in $M\backslash\alpha$, we know that $M\backslash\alpha, c_0, c_1, \ldots, c_k$ has a 4-fan, (u_1, u_2, u_3, u_4), that is also a 4-fan of $M\backslash c_k$. Thus $\{u_2, u_3, u_4, c_k\}$ is a cocircuit of M, so orthogonality implies that $\{a_k, b_k\}$ meets this cocircuit. By [7, Lemma 6.3], $u_4 \in \{a_k, b_k\}$, so $(T_k, \{u_1, u_2, u_3\}, \{u_2, u_3, u_4, c_k\})$ is a bowtie in M for some u_4 in $\{a_k, b_k\}$; a contradiction to 7.5.2 as $T_0, D_0, T_1, D_1, \ldots, T_k$ is a maximum-length good bowtie string for which there is no bowtie (T_k, T_{k+1}, D_k). $\square$

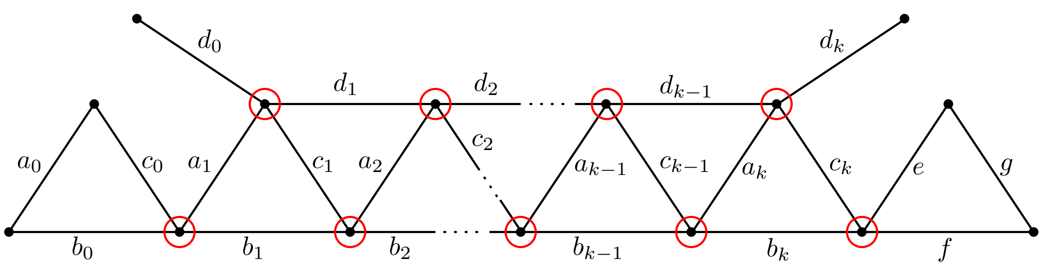

FIGURE 32. All of the elements are distinct except a_0 may be g, or T_0 may be $\{e, f, g\}$.

We will use the following consequence of [8, Lemma 6.3] to deal with the structure in Figure 32.

Lemma 7.6. *Let M be an internally 4-connected binary matroid having at least fifteen elements. Suppose that M contains the structure in Figure 32, where $k \geq 2$ and all of the elements are distinct except that a_0 may be g, or T_0 may be $\{e, f, g\}$. If $M\backslash c_i$ is $(4, 4, S)$-connected for all i in $\{1, 2, \ldots, k\}$, then*

(i) *$\{d_0, a_1\}$ is contained in a triangle; or*
(ii) *$\{c_k, d_k\}$ is contained in a triangle; or*
(iii) *$M\backslash c_1, c_2, \ldots, c_k/b_k$ is internally 4-connected.*

The only outcome in Lemma 7.5 that is not an outcome of Theorem 1.5 is that M has a good bowtie ring.

Lemma 7.7. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose M has $T_0, D_0, T_1, D_1, \ldots, T_k, \{b_k, c_k, a_0, b_0\}$ as a ring of bowties, and that $M\backslash c_0$ is $(4,4,S)$-connected with an N-minor, and that Hypothesis VIII holds. Suppose also that M has $T_0, D_0, T_1, D_1, \ldots, T_k, \{a_k, c_k, e_1, f_1\}, \{e_1, f_1, g_1\}, \{f_1, g_1, e_2, f_2\}, \ldots, \{e_m, f_m, g_m\}$ as a right-maximal bowtie string for some $m \geq 1$. If $(\{e_m, f_m, g_m\}, \{e_{m+1}, f_{m+1}, g_{m+1}\}, \{z, g_m, e_{m+1}, f_{m+1}\})$ is a bowtie for some z in $\{e_m, f_m\}$, then*

(i) *M has a quick win; or*
(ii) *M has a ladder win or a mixed-ladder win; or*
(iii) *M has a ladder-compression win.*

Proof. Let $S_i = \{e_i, f_i, g_i\}$ and $C_i = \{f_i, g_i, e_{i+1}, f_{i+1}\}$ for all $i \geq 1$. Let $\{b_k, c_k, a_0, b_0\} = D_k$. By orthogonality between D_k and S_m, we know that $g_m \neq a_0$. Thus all of the elements in the right-maximal bowtie string are distinct. It will be convenient sometimes to think of T_k as also being S_0, where $(a_k, b_k, c_k) = (f_0, e_0, g_0)$. Also we take C_0 to be $\{a_k, c_k, e_1, f_1\}$.

By Hypothesis VIII, we have that

7.7.1. *$M\backslash c_i$ is $(4,4,S)$-connected and $M\backslash c_i, c_{i+1}$ has an N-minor for all i in $\{0, 1, \ldots, k\}$, where indices are calculated modulo $k+1$; in addition, $M\backslash g_j$ is $(4,4,S)$-connected and $M\backslash g_{j-1}, g_j$ has an N-minor for all j in $\{1, 2, \ldots, m+1\}$.*

Suppose the lemma does not hold. By the symmetry between e_m and f_m, we may restrict our attention to the case when $z = f_m$. Since M has (S_m, S_{m+1}, C_m) as a bowtie, we know that S_{m+1} meets the bowtie string. Then [7, Lemma 5.4] implies that S_{m+1} meets the bowtie string in $\{a_0\}$, or S_{m+1} is a triangle in the bowtie string other than S_{m-1} or S_m. The former gives a contradiction to orthogonality with D_k, so the latter holds.

We show next that

7.7.2. *M has a bowtie $(\{e_m, f_m, g_m\}, \{e'_{m+1}, f'_{m+1}, g'_{m+1}\}, \{f_m, g_m, e'_{m+1}, f'_{m+1}\})$ where either $\{e'_{m+1}, f'_{m+1}, g'_{m+1}\} = T_i$ for some i in $\{0, 1, \ldots, k\}$ and $g'_{m+1} \in \{b_i, c_i\}$; or $\{e'_{m+1}, f'_{m+1}, g'_{m+1}\} = S_j$ for some j in $\{1, 2, \ldots, m-2\}$ and $g'_{m+1} \in \{e_j, f_j\}$.*

We know that S_{m+1} is a triangle in the right-maximal bowtie string. Suppose first that $S_{m+1} = T_i$ for some i in $\{0, 1, \ldots, k\}$. If $g_{m+1} \in \{b_i, c_i\}$, then 7.7.2 holds with $(e'_{m+1}, f'_{m+1}, g'_{m+1}) = (e_{m+1}, f_{m+1}, g_{m+1})$. Thus, we may assume that $g_{m+1} = a_i$. Then $C_m \triangle \{b_i, c_i, a_{i+1}, b_{i+1}\} = \{f_m, g_m, a_{i+1}, b_{i+1}\}$, which must be a cocircuit. Hence $(S_m, T_{i+1}, \{f_m, g_m, a_{i+1}, b_{i+1}\})$ is a bowtie and 7.7.2 holds with $(e'_{m+1}, f'_{m+1}, g'_{m+1}) = (a_{i+1}, b_{i+1}, c_{i+1})$.

We may now assume that $S_{m+1} = S_j$ for some j in $\{1, 2, \ldots, m-2\}$. Then 7.7.2 holds when $g_{m+1} \in \{e_j, f_j\}$ by taking $(e'_{m+1}, f'_{m+1}, g'_{m+1}) = (e_{m+1}, f_{m+1}, g_{m+1})$. On the other hand, if $g_{m+1} = g_j$, then $\{f_m, g_m, e_{m+1}, f_{m+1}\} \triangle \{f_{j-1}, g_{j-1}, e_j, f_j\} = \{f_m, g_m, f_{j-1}, g_{j-1}\}$, which must be a cocircuit of M and so $(S_m, S_{j-1}, \{f_m, g_m, f_{j-1}, g_{j-1}\})$ is a bowtie. In this case, 7.7.2 holds with $\{e'_{m+1}, f'_{m+1}, g'_{m+1}\} = S_{j-1}$ and $g'_{m+1} = e_{j-1}$. We conclude that 7.7.2 holds.

For convenience of notation, we now relabel so that $(e'_{m+1}, f'_{m+1}, g'_{m+1}) = (e_{m+1}, f_{m+1}, g_{m+1})$.

Next we show the following.

7.7.3. *If $M\backslash g_u, g_{u+1}, \dots, g_w$ has an N-minor for some u and w with $0 \le u \le w \le m$, then $M\backslash g_u, g_{u+1}, \dots, g_w/f_w$ has an N-minor.*

Assume that 7.7.3 fails. By 7.7.2 either $S_{m+1} = S_j$ for some j in $\{1, 2, \dots, m-2\}$ and g_{m+1} in $\{e_j, f_j\}$; or $S_{m+1} = T_i$ for some i in $\{0, 1, \dots, k\}$ and g_{m+1} in $\{b_i, c_i\}$. In the first case, consider the bowtie string $S_u, C_u, \dots, C_{m-1}, S_m, \{f_m, g_m, e_{m+1}, f_{m+1}\}, S_j, C_{j-1}, S_{j-1}, \dots, C_1, S_1, C_0, T_k,$ $D_{k-1}, T_{k-1}, D_{k-2}, \dots, T_0$. In the second case, consider the bowtie string $S_u, C_u, \dots, C_{m-1}, S_m, \{f_m, g_m\} \cup (T_i - g_{m+1}), T_i, D_i, T_{i+1}, D_{i+1}, \dots, T_{i-1}$.

Now $M\backslash g_u, g_{u+1}, \dots, g_w$ either has $(g_{w+1}, f_{w+1}, e_{w+1}, f_w)$ as a 4-fan, or has f_w in a cocircuit of size at most two. As 7.7.3 fails, $M\backslash g_u, g_{u+1}, \dots, g_{w+1}$ has an N-minor. The last matroid either has a 4-fan F_1 with f_{w+1} as its coguts element, or has f_{w+1} in a cocircuit of size at most two. Thus either $M\backslash g_u, g_{u+1}, \dots, g_{w+1}/f_{w+1}$ has an N-minor, or $M\backslash g_u, g_{u+1}, \dots, g_{w+1}\backslash g'_1$ has an N-minor where g'_1 is the guts element of F_1. By Lemma 5.5(ii), the first possibility gives the contradiction that 7.7.3 holds. Hence $M\backslash g_u, g_{u+1}, \dots, g_{w+1}\backslash g'_1$ has an N-minor. Continuing in this way, we see that if, at any stage, we can keep an N-minor by contraction of an element that is either a coguts element of a 4-fan or is in a cocircuit of size at most two, we get a contradiction. Hence we can continue to the end of the bowtie string keeping the N-minor by repeatedly deleting the guts element of each consecutive 4-fan we encounter.

Suppose we are dealing with the first bowtie string noted in the second-last paragraph. Then $M\backslash g_u, g_{u+1}, \dots, g_m, g_{m+1}, e_{j-1}, \dots, e_1, b_k, c_{k-1}, c_{k-2}, \dots, c_0$ has an N-minor. The last matroid has $\{a_k, b_{k-1}\}$ as a disjoint union of cocircuits, so $M\backslash g_u, g_{u+1}, \dots, g_m, g_{m+1}, e_{j-1}, \dots, e_1, b_k/a_k$ has an N-minor. Hence so does $M\backslash g_u, g_{u+1}, \dots, g_m, g_{m+1}, e_{j-1}, \dots, e_1, c_k/f_1$. But, in the 4-fan (b_k, c_k, a_k, f_1) that arises in $M\backslash g_u, g_{u+1}, \dots, g_m, g_{m+1}, e_{j-1}, \dots, e_1$, we see that we can contract the coguts element. This contradicts our assumption. We deduce that we must be dealing with the second bowtie string from the second-last paragraph. Then $M\backslash g_u, g_{u+1}, \dots, g_m, g_{m+1}, c_{i+1}, c_{i+2} \dots, c_{i-1}$ has an N-minor. The last matroid has $\{b_{i-1}, a_i\}$ as a disjoint union of cocircuits. Thus $M\backslash g_u, g_{u+1}, \dots, g_m, g_{m+1}/a_i$ has an N-minor. Hence, so does $M\backslash g_u, g_{u+1}, \dots, g_m/f_m$; a contradiction. We conclude that 7.7.3 holds.

Next we show that

7.7.4. *M has no bowtie of the form $(S_m, \{1, 2, 3\}, \{e_m, g_m, 2, 3\})$.*

Suppose M has such a bowtie. By [7, Lemma 5.4], $\{1, 2, 3\}$ is a triangle in the bowtie string $T_0, D_0, T_1, D_1, \dots, T_k, C_0, S_1, C_1, \dots, S_m$, so $\{1, 2, 3\}$ is T_ℓ or S_ℓ for some ℓ. We show first that we can choose $\{1, 2, 3\}$ so that $1 \notin \{a_\ell, g_\ell\}$. Suppose 1 is a_ℓ or g_ℓ. Then, taking the symmetric difference of $\{e_m, g_m, 2, 3\}$ with D_ℓ or $C_{\ell-1}$, respectively, we get that M has $\{e_m, g_m, a_{\ell+1}, b_{\ell+1}\}$ or $\{e_m, g_m, f_{\ell-1}, g_{\ell-1}\}$, respectively, as a cocircuit. Thus M has $(S_m, T_{\ell+1}, \{e_m, g_m, a_{\ell+1}, b_{\ell+1}\})$ or $(S_m, S_{\ell-1}, \{e_m, g_m, f_{\ell-1}, g_{\ell-1}\})$ as a bowtie. By taking $\{1, 2, 3\}$ to be $T_{\ell+1}$ or $S_{\ell-1}$, respectively and relabelling these to be T_ℓ and S_ℓ, we have a bowtie $(S_m, \{1, 2, 3\}, \{e_m, g_m, 2, 3\})$ where $\{1, 2, 3\}$ is T_ℓ or S_ℓ and $1 \notin \{a_\ell, g_\ell\}$.

Continuing with the proof of 7.7.4, we note that, by 7.7.3, $M\backslash g_m/f_m$ has an N-minor, and this matroid also has $(1, 2, 3, e_m)$ as a 4-fan. Now

7.7.5. *$M\backslash g_m/f_m/e_m$ does not have an N-minor.*

Since (i) does not hold, this follows by Lemma 5.4 as $M\backslash g_m/f_m/e_m = M/g_m, f_m, e_m$.

By 7.7.5, $M\backslash g_m/f_m\backslash 1$ has an N-minor. Now, depending on whether $\{1,2,3\}$ is T_ℓ or S_ℓ, the matroid M has, as a bowtie string, one of $T_\ell, D_\ell, T_{\ell+1}, D_{\ell+1}, \dots, T_{\ell-1}$ or $S_\ell, C_{\ell-1}, S_{\ell-1}, C_{\ell-2}, \dots, S_1, C_0, T_k, D_{k-1}, \dots, T_0$. As M is internally 4-connected, one easily checks that this bowtie string remains a bowtie string in $M\backslash g_m/f_m$. We want to be able to apply Lemma 5.5(i) to this bowtie string. We know that $M\backslash g_m/f_m\backslash 1$ has an N-minor. To be able to apply the lemma, we need to show that

7.7.6. *$M\backslash g_m/f_m\backslash 1, c_{\ell+1}$ or $M\backslash g_m/f_m\backslash 1, e_{\ell-1}$ has an N-minor depending on whether $\{1,2,3\}$ is T_ℓ or S_ℓ.*

The matroid $M\backslash g_m/f_m\backslash 1$ has $(c_{\ell+1}, b_{\ell+1}, a_{\ell+1}, w)$ or $(e_{\ell-1}, f_{\ell-1}, g_{\ell-1}, w)$ as a 4-fan where w is the element of $\{b_\ell, c_\ell\}-1$ or $\{e_\ell, f_\ell\}-1$, respectively. Thus 7.7.6 holds unless $M\backslash g_m/f_m\backslash 1/w$ has an N-minor. But $M\backslash g_m/f_m\backslash 1/w \cong M\backslash g_m/f_m/w\backslash v$ where $\{w,v\} = \{2,3\}$. Now $M\backslash g_m/f_m\backslash v$ has e_m in a 1- or 2-cocircuit. Thus $M\backslash g_m/f_m/e_m$ has an N-minor; a contradiction to 7.7.5. Hence 7.7.6 holds.

Now 7.7.6 implies, using Lemma 5.5(i), that $M\backslash g_m/f_m\backslash 1, c_{\ell+1}, c_{\ell+2}, \dots, c_{\ell-1}$ or $M\backslash g_m/f_m\backslash 1, e_{\ell-1}, e_{\ell-2}, \dots, e_1, b_k, a_{k-1}, a_{k-2}, \dots, a_0$ has an N-minor. Suppose the second matroid has an N-minor. Since this matroid has $\{c_k, b_0\}$ as a disjoint union of cocircuits, $M\backslash g_m/f_m\backslash 1, e_{\ell-1}, e_{\ell-2}, \dots, e_1, b_k, a_{k-1}, a_{k-2}, \dots, a_0/c_k$ has an N-minor, so $M\backslash g_m/f_m\backslash 1, e_{\ell-1}, e_{\ell-2}, \dots, e_1, b_k/c_k$ has an N-minor. By Lemma 5.5(ii), it follows that $M\backslash g_m/f_m\backslash 1, g_{\ell-1}/f_{\ell-1}$ has an N-minor. But $M\backslash g_m/f_m\backslash 1, g_{\ell-1}/f_{\ell-1} \cong M\backslash g_m/f_m\backslash x, g_{\ell-1}/y$ for some $\{x,y\} = \{2,3\}$. The last matroid is isomorphic to $M\backslash g_m/f_m\backslash x, g_{\ell-1}/e_m$, so we get a contradiction to 7.7.5. We deduce that $T_\ell = \{1,2,3\}$ and $M\backslash g_m/f_m\backslash 1, c_{\ell+1}, c_{\ell+2}, \dots, c_{\ell-1}$ has an N-minor. Now either $1 = c_\ell$ and $\{e_m, g_m, b_{\ell-1}, c_{\ell-1}\}$, which is $D_{\ell-1} \triangle \{e_m, g_m, a_\ell, b_\ell\}$, is a cocircuit; or $1 = b_\ell$ and $\{e_m, g_m, a_\ell, b_\ell\}$ is a cocircuit. In both cases, $M\backslash g_m/f_m\backslash 1, c_{\ell+1}, c_{\ell+2}, \dots, c_{\ell-1}$ has e_m contained in a 1- or 2-cocircuit. Thus $M\backslash g_m/f_m/e_m$ has an N-minor; a contradiction to 7.7.5. We conclude that 7.7.4 holds.

By 7.7.1 and 7.7.3, $M\backslash g_m/f_m$ has an N-minor. By Lemma 5.2, $M\backslash g_m/f_m$ is $(4,5,S,+)$-connected and M has elements h_{m-1} and h_m such that $\{h_{m-1}, h_m\}$ avoids $S_{m-1} \cup S_m \cup S_{m+1}$, and $\{h_{m-1}, e_m, g_m, h_m\}$ is a cocircuit and either $\{h_{m-1}, e_m, s\}$ or $\{h_m, g_m, t\}$ is a triangle, for some s in $\{f_{m-1}, g_{m-1}\}$ or t in $\{e_{m+1}, f_{m+1}\}$. By 7.7.4,

7.7.7. *M has no triangle containing $\{h_{m-1}, h_m\}$.*

Next we show that

7.7.8. *M has no triangle containing $\{h_m, g_m\}$.*

Assume that M has such a triangle. Then, by orthogonality, this triangle contains an element t of $\{e_{m+1}, f_{m+1}\}$. Now recall from 7.7.2 and the remarks following it that either $\{e_{m+1}, f_{m+1}\}$ is $\{a_i, b_i\}$ or $\{a_i, c_i\}$; or $\{e_{m+1}, f_{m+1}\}$ is $\{g_j, e_j\}$ or $\{g_j, f_j\}$. First we distinguish the cases when $t = c_i$ and when $t \in \{a_i, b_i\}$. Then orthogonality between the triangle $\{h_m, g_m, t\}$ and either $\{b_i, c_i, a_{i+1}, b_{i+1}\}$ or $\{b_{i-1}, c_{i-1}, a_i, b_i\}$, respectively, implies that $h_m \in T_{i+1}$ or $h_m \in T_{i-1}$. Then, by orthogonality between T_{i+1} or T_{i-1} and $\{h_{m-1}, e_m, g_m, h_m\}$, it follows that $\{h_{m-1}, h_m\}$ is contained in T_{i+1} or T_{i-1}, respectively; a contradiction to 7.7.7.

Similarly, in the cases when $t = g_i$ and when $t \in \{e_i, f_i\}$, orthogonality implies that $h_m \in S_{i+1}$ or $h_m \in S_{i-1}$, so $\{h_{m-1}, h_m\}$ is contained in S_{i+1} or S_{i-1}; a contradiction to 7.7.7. Thus 7.7.8 holds.

We now know that $\{h_{m-1}, e_m, s\}$ is a triangle for some s in $\{f_{m-1}, g_{m-1}\}$. Take n to be the smallest positive integer such that M has elements $h_{n-1}, h_n, \dots, h_{m-1}$, and h_m where, for all i in $\{n, n+1, \dots, m\}$, the set $\{h_{i-1}, e_i, g_i, h_i\}$ is a cocircuit and $\{h_{i-1}, e_i\}$ is in a triangle that meets $\{f_{i-1}, g_{i-1}\}$.

Suppose $n = 1$. Then M has $\{h_0, e_1\}$ in a triangle with a_k or c_k, and orthogonality implies that h_0 is in $\{b_{k-1}, c_{k-1}\}$ or $\{a_0, b_0\}$, respectively. Then orthogonality with the cocircuit $\{h_0, e_1, g_1, h_1\}$ implies that h_1 is in T_{k-1} or T_0, respectively. Thus $M\backslash g_1$ has a 5-fan; a contradiction. We conclude that $n > 1$. As $m \geq n$, it follows that $m > 1$.

If $\{h_{i-1}, e_i, f_{i-1}\}$ is a triangle for any i in $\{n, n+1, \dots, m\}$, then $M\backslash g_{i-1}$ has a 5-fan; a contradiction to 7.7.1. Thus $\{h_{i-1}, e_i, g_{i-1}\}$ is a triangle for all i in $\{n, n+1, \dots, m\}$. By 7.7.3, $M/f_{n-1}\backslash g_{n-1}$ has an N-minor. As M has no quick win, Lemma 5.2 implies that M has a 4-cocircuit C^* containing $\{e_{n-1}, g_{n-1}\}$. By orthogonality and Lemma 5.1, C^* contains h_{n-1}. Let the fourth element of C^* be h_{n-2}.

7.7.9. *The elements $h_{n-2}, h_{n-1}, \dots, h_m$ are distinct and none is in $T_0 \cup T_1 \cup \dots \cup T_k \cup S_1 \cup S_2 \cup \dots \cup S_m$.*

Assume $\{h_{n-2}, h_{n-1}, \dots, h_m\}$ meets $T_0 \cup T_1 \cup \dots \cup T_k \cup S_1 \cup S_2 \cup \dots \cup S_m$. Let p be the largest member of $\{n-2, n-1, \dots, m\}$ such that $h_p \in T_0 \cup T_1 \cup \dots \cup T_k \cup S_1 \cup S_2 \cup \dots \cup S_m$. Then, by orthogonality, $p \neq n-2$ so $\{h_{p-1}, e_p, g_p, h_p\}$ is a cocircuit and, using orthogonality again, we deduce that $\{h_{p-1}, h_p\}$ is contained in a triangle of M. Thus $M\backslash g_p$ has a 5-fan; a contradiction to 7.7.1. Hence $\{h_{n-2}, h_{n-1}, \dots, h_m\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_k \cup S_1 \cup S_2 \cup \dots \cup S_m$.

Now let q be the least member of $\{n-2, n-1, \dots, m\}$ such that $h_q = h_i$ for some $i \neq q$. Then $i \geq q+2$ and the triangle containing $\{h_q, e_{q+1}\}$ has a single common element with the cocircuit $\{h_{i-1}, e_{i-1}, g_{i-1}, h_i\}$; a contradiction to orthogonality. Thus 7.7.9 holds.

By Lemma 7.3, $M\backslash g_{n-1}, g_n, \dots, g_m$ has an N-minor. By Lemma 7.6, 7.7.3, and 7.7.8, since M does not have a mixed-ladder win, M has a triangle containing $\{h_{n-2}, e_{n-1}\}$. Since $\{h_{n-2}, e_{n-1}, g_{n-1}, h_{n-1}\}$ is a cocircuit of M, we have a contradiction to the choice of n that completes the proof of the lemma. $\square$

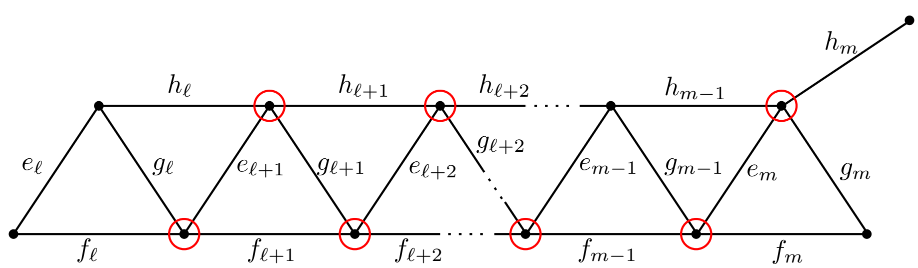

FIGURE 33. Either $\{h_{m-2}, e_{m-1}, g_{m-1}, h_{m-1}\}$ or $\{h_{m-2}, e_{m-1}, g_{m-1}, e_m, g_m\}$ is a cocircuit, and $1 \leq \ell < m$.

The next lemma strengthens the preceding lemma.

Lemma 7.8. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 15$ and $|E(N)| \geq 7$. Suppose that M has $T_0, D_0, T_1, D_1, \ldots, T_k, \{b_k, c_k, a_0, b_0\}$ as a ring of bowties, that $M\backslash c_0$ is $(4, 4, S)$-connected with an N-minor, and that Hypothesis VIII holds. Suppose also that M has $T_0, D_0, T_1, D_1, \ldots, T_k, \{a_k, c_k, e_1, f_1\}, \{e_1, f_1, g_1\}$ as a bowtie string. Then*

(i) *M has a quick win; or*
(ii) *M has a ladder win or a mixed-ladder win; or*
(iii) *M has a ladder-compression win.*

Proof. Let $S_i = \{e_i, f_i, g_i\}$ and $C_i = \{f_i, g_i, e_{i+1}, f_{i+1}\}$ for all $i \geq 1$. Suppose $T_0, D_0, T_1, D_1, \ldots, T_k, \{a_k, c_k, e_1, f_1\}, S_1, C_1, S_2, C_2, \ldots, S_m$ is a right-maximal string of bowties. Let $\{b_k, c_k, a_0, b_0\} = D_k$. By orthogonality between D_k and S_m, we know that $g_m \neq a_0$. Thus all of the elements in the bowtie string are distinct. It will be convenient sometimes to think of T_k as also being S_0, where $(a_k, b_k, c_k) = (f_0, e_0, g_0)$. Also we take C_0 to be $\{a_k, c_k, e_1, f_1\}$.

By Hypothesis VIII, we know that $M\backslash c_i$ is $(4, 4, S)$-connected with an N-minor for all i in $\{0, 1, \ldots, k\}$, and $M\backslash g_j$ is $(4, 4, S)$-connected with an N-minor for all j in $\{1, 2, \ldots, m\}$.

Suppose the lemma does not hold. By Lemma 5.6, since neither $M\backslash e_{m-1}$ nor $M\backslash g_m$ is internally 4-connected, either M has a triangle S_{m+1} such that $\{w, g_m, e_{m+1}, f_{m+1}\}$ is a cocircuit for some w in $\{e_m, f_m\}$; or every $(4, 3)$-violator of $M\backslash g_m$ is a 4-fan of the form $(g_{m-1}, v, h_{m-1}, h_m)$, where $v \in \{e_m, f_m\}$ and $|S_{m-1} \cup S_m \cup \{h_{m-1}, h_m\}| = 8$. By Lemma 7.7, the first option does not hold. Thus, as $M\backslash g_m$ is not internally 4-connected, it has a 4-fan as specified in the second option.

Now $\{h_{m-1}, h_m\}$ avoids $T_0, T_1, \ldots, T_k, S_1, S_2, \ldots, S_{m-3}$, and S_{m-2}, otherwise, by orthogonality, $\{h_{m-1}, h_m\}$ is contained in this triangle, and Lemma 7.7 gives a contradiction. Furthermore, if $m = 1$, then $g_{m-1} = c_k$, and orthogonality implies that h_{m-1} is in $\{b_k, a_0, b_0\}$; a contradiction. Thus $m \geq 2$. Without loss of generality, we may assume that $v = e_m$. Take ℓ to be the least non-negative integer such that, for all i in $\{\ell, \ell+1, \ldots, m-1\}$, there is an element h_i such that $\{g_i, h_i, e_{i+1}\}$ is a triangle and $\{h_i, e_{i+1}, g_{i+1}, h_{i+1}\}$ or $\{h_i, e_{i+1}, g_{i+1}, e_{i+2}, g_{i+2}\}$ is a cocircuit. Since h_{m-1} and h_m avoid the triangles in our right-maximal bowtie string, orthogonality implies that h_{m-2} is a new element, and so h_{m-3} is a new element. Continuing in this way, we see that all of the elements in $T_0 \cup T_1 \cup \cdots \cup T_k \cup S_1 \cup S_2 \cup \cdots \cup S_m \cup \{h_\ell, h_{\ell+1}, \ldots, h_m\}$ are distinct. By orthogonality, $\{h_i, e_{i+1}, g_{i+1}, e_{i+2}, g_{i+2}\}$ can only be a cocircuit in the case that $i = m-2$.

Suppose that $\ell = 0$. Then orthogonality between $\{c_, h_0, e_1\}$ and D_k implies that $h_0 \in \{b_k, a_0, b_0\}$; a contradiction. Thus $\ell \geq 1$. Since $\ell < m$, we have that M contains the structure shown in Figure 33. By Lemma 7.3, $M\backslash g_\ell, g_{\ell+1}, \ldots, g_m$ has an N-minor.

We will apply [7, Lemma 6.1] and [7, Lemma 6.5], depending on whether $\ell = m-1$ or $\ell < m-1$, but first we eliminate a case that is common to both. Specifically, we show that

7.8.1. *M has no triangle $\{\alpha, \beta, e_\ell\}$, where $\{\beta, e_\ell, g_\ell, h_\ell\}$ or $\{\beta, e_\ell, g_\ell, e_{\ell+1}, g_{\ell+1}\}$ is a cocircuit.*

Suppose otherwise. Observe that $\{\alpha, \beta\}$ avoids $\{f_\ell, g_\ell\}$ otherwise $\{\alpha, \beta\} = \{f_\ell, g_\ell\}$ and we contradict the fact that M is binary and internally 4-connected. By orthogonality between $\{\alpha, \beta, e_\ell\}$ and the cocircuit $C_{\ell-1}$, we know that $\{\alpha, \beta\}$ meets $\{f_{\ell-1}, g_{\ell-1}\}$. Moreover, orthogonality between $\{e_{\ell-1}, f_{\ell-1}, g_{\ell-1}\}$ and whichever of $\{\beta, e_\ell, g_\ell, h_\ell\}$ and $\{\beta, e_\ell, g_\ell, e_{\ell+1}, g_{\ell+1}\}$ is a cocircuit implies that $\beta \notin \{f_{\ell-1}, g_{\ell-1}\}$. By the choice of ℓ, we deduce that $\alpha = f_{\ell-1}$. Then orthogonality with the cocircuit $C_{\ell-2}$, or D_{k-1} in the case that $f_{\ell-1} = f_0 = a_k$, implies that β is in $S_{\ell-2}$ or $T_{k-1} \cup T_k$; a contradiction to orthogonality with $\{\beta, e_\ell, g_\ell, h_\ell\}$ or $\{\beta, e_\ell, g_\ell, e_{\ell+1}, g_{\ell+1}\}$. Thus 7.8.1 holds.

Next, we show that

7.8.2. *$M\backslash g_\ell, g_{\ell+1}, \dots, g_m$ is $(4,4,S)$-connected and f_m is the coguts element of every 4-fan of this matroid. Furthermore, either every $(4,3)$-violator of $M\backslash g_\ell, g_{\ell+1}, \dots, g_m$ is a 4-fan in $M\backslash g_m$, or $\ell = m-1$.*

Assume that 7.8.2 does not hold. Suppose $\ell = m-1$. Then [7, Lemma 6.1] and 7.8.1 imply that M has $\{h_{m-1}, h_m\}$ or $\{f_{m-1}, f_m\}$ in a triangle. Thus $M\backslash g_m$ or $M\backslash g_{m-1}$ has a 5-fan; a contradiction.

We now know that $\ell < m-1$. By 7.8.1, M is not the cycle matroid of a quartic Möbius ladder. Now, by [7, Lemma 6.5], either $\{h_{m-1}, h_m\}$ is in a triangle; or $M\backslash g_\ell, g_{\ell+1}, \dots, g_m$ is $(4,4,S)$-connected and every $(4,3)$-violator of this matroid is a 4-fan that is also a 4-fan of $M\backslash g_\ell$ with e_ℓ as an interior element and h_ℓ as its coguts element, or is a 4-fan of $M\backslash g_m$ with f_m as its coguts element. The first option implies that $M\backslash g_m$ has a 5-fan; a contradiction. If $M\backslash g_\ell$ has a 4-fan with e_ℓ as an interior element and h_ℓ as its coguts element, then M has a triangle $\{\alpha, \beta, e_\ell\}$ where $\{\beta, e_\ell, g_\ell, h_\ell\}$ is a cocircuit; a contradiction to 7.8.1. We deduce that 7.8.2 holds.

Since we have assumed that (ii) does not hold, we know that $M\backslash g_\ell, g_{\ell+1}, \dots, g_m$ has a 4-fan, $(1,2,3,f_m)$. If $\{2,3,f_m,g_m\}$ is a cocircuit, then Lemma 7.7 implies that the lemma holds; a contradiction. Thus $(1,2,3,f_m)$ is not a 4-fan in $M\backslash g_m$ and it follows using 7.8.2 that $\ell = m-1$, and $\{2,3,f_m,g_{m-1}\}$ or $\{2,3,f_m,g_{m-1},g_m\}$ is a cocircuit of M. If $\{2,3,f_m,g_{m-1}\}$ is a cocircuit, then Lemma 5.1 implies that $\{2,3\} = \{f_{m-1}, e_m\}$, so $\{1,2,3,f_m,g_m\}$ is a 5-fan in $M\backslash g_{m-1}$; a contradiction. Thus $\{2,3,f_m,g_{m-1},g_m\}$ is a cocircuit, and orthogonality implies that $\{2,3\}$ meets $\{e_{m-1}, f_{m-1}\}$ and $\{h_{m-1}, e_m\}$, so $\lambda(S_{m-1} \cup S_m \cup h_{m-1}) \leq 2$; a contradiction. $\square$

We continue our consideration of good bowtie rings with the next lemma.

Lemma 7.9. *Let M and N be internally 4-connected binary matroids such that $|E(M)| \geq 15$ and $|E(N)| \geq 7$ and Hypothesis VIII holds. Suppose M has $T_0, D_0, T_1, D_1, \dots, T_k, \{b_k, c_k, a_0, b_0\}$ as a ring of bowties, and that $M\backslash c_0$ is $(4,4,S)$-connected with an N-minor. Then*

- (i) *M has a quick win; or*
- (ii) *M has a ladder win, bowtie-ring win, or an open-rotor-chain win; or*
- (iii) *M has an enhanced-ladder win; or*
- (iv) *M has a mixed-ladder win; or*
- (v) *M has a ladder-compression win.*

Proof. Throughout the proof, all indices related to the bowtie ring will be interpreted modulo $k+1$. Suppose the lemma does not hold. Assume we have selected a good bowtie ring with the least number of triangles. Let $D_k = \{b_k, c_k, a_0, b_0\}$ and

$S = \{c_0, c_1, \dots, c_k\}$. Since $M\backslash c_0$ is $(4,4,S)$-connected with an N-minor, Hypothesis VIII implies that $M\backslash c_i$ is $(4,4,S)$-connected, and $M\backslash c_i, c_{i+1}$ has an N-minor for all i in $\{0, 1, \dots, k\}$.

We now show that

7.9.1. *after possibly shifting the indices, $M\backslash c_i, c_{i+1}/b_{i+1}$ has an N-minor for all i in $\{0, 1, \dots, k-1\}$.*

Suppose that $M\backslash S$ has an N-minor. Since M has no bowtie-ring win, we know that $M\backslash S$ is not internally 4-connected. Suppose $M\backslash S$ has a 1- or 2-element cocircuit that meets $\{a_0, b_0, a_1, b_1, \dots, a_k, b_k\}$. Then, since M is internally 4-connected, orthogonality with the triangles $T_0, T_1, \dots, T_{k-1}$, and T_k implies that M has a 4-cocircuit C^* that contains $\{c_i, c_j\}$ for some i and j in $\{0, 1, \dots, k\}$, where $i < j$. By possibly shifting the indices in the bowtie ring, we may assume that $j = k$. By Lemma 5.1, we know that $i \notin \{k-1, 0\}$. By orthogonality with T_i and T_k, we know that $C^* = \{p, c_i, q, c_k\}$, for some p in $\{a_i, b_i\}$ and some q in $\{a_k, b_k\}$, so $(T_i, T_k, \{p, c_i, q, c_k\})$ is a bowtie. If $p = a_i$, then M has $T_{i+1}, D_{i+1}, \dots, T_k, \{p, c_i, q, c_k\}, T_i, D_i$ as a good bowtie ring with at most k triangles, a contradiction to our selection of the original good bowtie ring. Thus $p = b_i$, and $\{p, c_i, q, c_k\} \triangle D_i$ is $\{a_{i+1}, b_{i+1}, q, c_k\}$, which must be a cocircuit, and again M has a smaller good bowtie ring; a contradiction. We conclude that $M\backslash S$ has no 1- or 2-element cocircuits meeting $\{a_0, b_0, a_1, b_1, \dots, a_k, b_k\}$.

Now [7, Lemma 5.5] implies that $M\backslash S$ is sequentially 4-connected and every 4-fan of it has the form $(1,2,3,4)$, where $\{1,2,3\}$ is a triangle disjoint from the bowtie ring, and $\{2,3,4,c_j\}$ is a cocircuit of M for some j in $\{0, 1, \dots, k\}$ where $4 \in \{a_j, b_j\}$. By shifting the indices in the bowtie ring, we may assume that $M\backslash c_k$ has $(1,2,3,4)$ as a 4-fan where $4 \in \{a_k, b_k\}$. Thus M has $(T_k, \{1,2,3\}, \{4, c_k, 2, 3\})$ as a bowtie. By Lemma 7.8, $4 \neq a_k$. Then it follows by [7, Lemma 10.4] that $M\backslash c_i, c_{i+1}/b_{i+1}$ has an N-minor for all i in $\{0, 1, \dots, k-1\}$. Thus 7.9.1 holds if $M\backslash S$ has an N-minor.

We may now assume that $M\backslash S$ has no N-minor. Then, for each i in $\{0, 1, \dots, k-1\}$, there is some ℓ in $\{0, 1, \dots, k\}$, such that $M\backslash c_i, c_{i+1}, \dots, c_\ell$ has an N-minor, but $M\backslash c_i, c_{i+1}, \dots, c_{\ell+1}$ has no N-minor. Then, by Hypothesis VIII, $\ell \neq i$. Moreover, $\ell \neq i-1$ as $M\backslash S$ has no N-minor. Since $M\backslash c_i, c_{i+1}, \dots, c_\ell$ has $(c_{\ell+1}, b_{\ell+1}, a_{\ell+1}, b_\ell)$ as a 4-fan, we know that $M\backslash c_i, c_{i+1}, \dots, c_\ell/b_\ell$ has an N-minor, and Lemma 5.5(ii) implies that $M\backslash c_i, c_{i+1}, a_{i+2}, a_{i+3}, \dots, a_\ell/b_{i+1}$ has an N-minor. Thus 7.9.1 holds.

By 7.9.1, $M\backslash c_1/b_1$ has an N-minor. Moreover, by Lemmas 5.2 and 7.8, $M\backslash c_1/b_1$ is $(4,5,S,+)$-connected and M has elements d_0' and e_1' such that $\{d_0', a_1, c_1, e_1'\}$ is a cocircuit and $\{d_0', a_1, s\}$ or $\{e_1', c_1, t\}$ is a triangle, for some s in $\{b_0, c_0\}$ or some t in $\{a_2, b_2\}$.

We show next that

7.9.2. *for some i in $\{0, 1\}$, there are elements d_i and e_{i+1} of M such that $\{c_i, d_i, a_{i+1}\}$ is a triangle, $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ is a cocircuit, and $M\backslash c_i, c_{i+1}/b_{i+1}$ has an N-minor.*

Suppose M has $\{d_0', a_1, s\}$ as a triangle for some s in $\{b_0, c_0\}$. If $s = b_0$, then, by orthogonality between $\{d_0', a_1, s\}$ and D_k, we deduce that $d_0' \in \{b_k, c_k\}$. By orthogonality again, $d_0' = c_k$ and $e_1 \in T_k$. Then $\lambda(T_k \cup T_0 \cup T_1 \cup \{d_0', e_1'\}) \leq 2$; a contradiction. We deduce that $s = c_0$. Thus, by taking $(d_0, e_1) = (d_0', e_1')$ and using 7.9.1, we see that 7.9.2 holds when $\{d_0', a_1, s\}$ is a triangle.

We may now assume that M has $\{e'_1, c_1, t\}$ as a triangle for some t in $\{a_2, b_2\}$. By orthogonality with D_2, either $t = a_2$; or $t = b_2$ and $e'_1 \in T_3$. The latter implies that $M\backslash c_2$ has a 5-fan; a contradiction. Thus $t = a_2$, so M has $\{e'_1, c_1, a_2\}$ as a triangle.

By 7.9.1, $M\backslash c_1, c_2/b_2$ has an N-minor. Hence, so does $M\backslash c_2/b_2$. By Lemma 5.2, M has a 4-cocircuit C^* containing $\{a_2, c_2\}$. Orthogonality with $\{e'_1, c_1, a_2\}$ using Lemma 5.1 implies that $e'_1 \in C^*$. We let the fourth element of C^* be e_2 and relabel e'_1 as d_1 to get that 7.9.2 holds.

Next we show the following.

7.9.3. *Suppose $\{c_i, d_i, a_{i+1}\}$ is a triangle and $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ is a cocircuit for some i in $\{0, 1, \dots, k\}$. Then $\{d_i, e_{i+1}\}$ avoids $T_0 \cup T_1 \cup \dots \cup T_k$. If, in addition, $\{c_{i+1}, d_{i+1}, a_{i+2}\}$ is a triangle and $\{d_{i+1}, a_{i+2}, c_{i+2}, e_{i+2}\}$ is a cocircuit, then $e_{i+1} = d_{i+1}$.*

If $\{d_i, e_{i+1}\}$ meets T_j for some j in $\{0, 1, \dots, k\}$, then orthogonality implies that $\{d_i, e_{i+1}\} \subseteq T_j$. Hence $T_j \cup \{c_i, a_{i+1}\}$ is a 5-fan in $M\backslash c_{i+1}$; a contradiction. We deduce that $\{d_i, e_{i+1}\}$ avoids the bowtie ring. Likewise, $\{d_{i+1}, e_{i+2}\}$ avoids the bowtie ring. Now, orthogonality between the triangle $\{c_{i+1}, d_{i+1}, a_{i+2}\}$ and the cocircuit $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ implies that $d_{i+1} \in \{d_i, e_{i+1}\}$. If $d_{i+1} = d_i$, then orthogonality between $\{c_i, d_{i+1}, a_{i+1}\}$ and $\{d_{i+1}, a_{i+2}, c_{i+2}, e_{i+2}\}$ implies that $e_{i+2} \in \{c_i, a_{i+1}\}$; a contradiction. Thus $d_{i+1} = e_{i+1}$, and 7.9.3 holds.

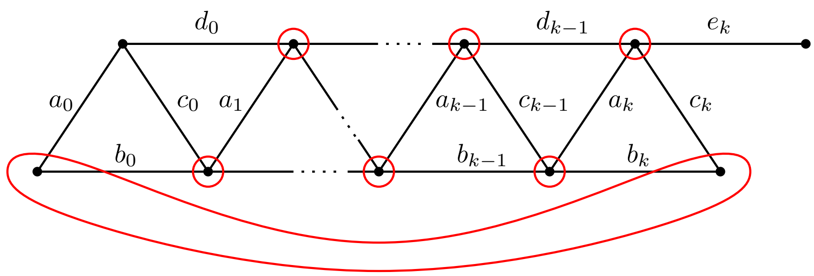

FIGURE 34. This structure arises in the proof of 7.9.4.

Next, we show that

7.9.4. *there are distinct elements j_1 and j_2 of $\{0, 1, \dots, k\}$ such that, for each i in $\{j_1, j_2\}$, either $\{c_i, d_i, a_{i+1}\}$ is not a triangle, or $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ is not a cocircuit.*

Assume that this fails. We adjusted indices to show that 7.9.1 holds. Here we will suppose that, for all i in $\{0, 1, \dots, k-1\}$, we have $\{c_i, d_i, a_{i+1}\}$ as a triangle and $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ as a cocircuit. This assumption is confined to the proof of 7.9.4, and this proof does not use either 7.9.1 or 7.9.2, which depends on 7.9.1. It does use 7.9.3 but that does not depend on 7.9.1 or 7.9.2. By 7.9.3, $d_{i+1} = e_{i+1}$ for all i in $\{0, 1, \dots, k-2\}$, and M contains the structure in Figure 34. By [7, Lemma 6.4], the elements in the figure are all distinct, or $a_0 = c_k$, or $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\}$ is a cocircuit. The second option gives an immediate contradiction; the third implies that M has $\{d_{k-2}, a_{k-1}, c_{k-1}, a_k, c_k\} \triangle \{d_{k-2}, a_{k-1}, c_{k-1}, d_{k-1}\}$, that is, $\{d_{k-1}, a_k, c_k\}$ as a cocircuit; a contradiction. We conclude that the elements in Figure 34 are all distinct. Let X be the set of elements in Figure 34 excluding e_k. Since $\lambda(X) \leq 2$, we know that $E(M)$ contains at most three elements that are not in X. By Lemma 7.3, $M\backslash S$ has an N-minor. By [7, Lemma 6.5], $M\backslash S$ is $(4,4,S)$-connected, or $M\backslash c_k$ is not $(4,4,S)$-connected, or $\{b_k, c_k, p, q\}$ is a cocircuit, where $\{p, q, a_0\}$ is a triangle other than T_0. The second option gives a contradiction, and the third option implies that $\{b_k, c_k, p, q\} \triangle D_k$, which equals $\{p, q, a_0, b_0\}$, is a cocircuit. As this cocircuit contains the triangle $\{p, q, a_0\}$, we have a contradiction to M being binary. We conclude that $M\backslash S$ is $(4,4,S)$-connected. Since M does not have a ladder win, $M\backslash S$ has a 4-fan, $(1,2,3,4)$. As $X \cup e_k$ contains at least $|E(M)| - 2$ elements, $\{1,2,3\}$ meets the elements in Figure 34, and, by [7, Lemma 6.3], these two sets meet in $\{a_0\}$, $\{d_{k-1}, e_k\}$, or $\{a_0, d_{k-1}, e_k\}$. We know that $\{d_{k-1}, e_k\}$ is not contained in a triangle, since $M\backslash c_k$ has no 5-fan. Thus $\{1,2,3\}$ meets the elements in Figure 34 in $\{a_0\}$; a contradiction to orthogonality with D_k. Hence 7.9.4 holds.

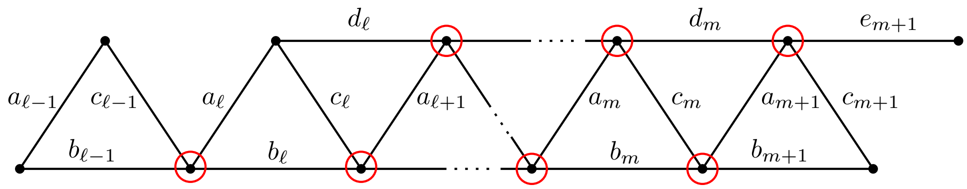

FIGURE 35. Possibly $\ell = m$.

By 7.9.2, for some i in $\{0,1\}$,

7.9.5. *$\{c_i, d_i, a_{i+1}\}$ is a triangle, $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ is a cocircuit, and $M\backslash c_i, c_{i+1}/b_{i+1}$ has an N-minor.*

If 7.9.5 holds when $i = 0$, let $\zeta = 0$; otherwise it holds for $i = 1$ and we take $\zeta = 1$. Let ℓ and m be such that $\{c_i, d_i, a_{i+1}\}$ is a triangle and $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ is a cocircuit for all i in the cyclically consecutive set $\{\ell, \ell+1, \dots, k, \dots, \zeta, \dots, m\}$ but, for each i in $\{\ell-1, m+1\}$, either $\{c_i, d_i, a_{i+1}\}$ is not a triangle or $\{d_i, a_{i+1}, c_{i+1}, e_{i+1}\}$ is not a cocircuit. By 7.9.4, $\ell \notin \{m+1, m+2\}$. When $\ell \neq m$, 7.9.3 implies that $d_{i+1} = e_{i+1}$ for all i in $\{\ell, \ell+1, \dots, m-1\}$. We deduce that M contains the structure in Figure 35 where ℓ and m may be equal, in which case their common value is ζ. By 7.9.3, we know that $T_{\ell-1}$ avoids $\{d_\ell, d_{\ell+1}, \dots, d_m, e_{m+1}\}$. We show next that

7.9.6. *the elements in Figure 35 are distinct.*

By 7.9.3, this certainly holds if $\ell = m$. If $\ell \neq m$, it holds by [7, Lemma 6.4] unless $a_\ell = c_{m+1}$, or $\{d_{m-1}, a_m, c_m, a_{m+1}, c_{m+1}\}$ is a cocircuit. The first option gives a contradiction, and the second implies that $\{d_{m-1}, a_m, c_m, a_{m+1}, c_{m+1}\} \triangle \{d_{m-1}, a_m, c_m, d_m\}$, which is $\{d_m, a_{m+1}, c_{m+1}\}$, is a triad; a contradiction. Thus 7.9.6 holds.

7.9.7. *$M\backslash c_\ell, c_{\ell+1}, \dots, c_{m+1}/b_{m+1}$ has no N-minor.*

To see this, suppose that $M\backslash c_\ell, c_{\ell+1}, \dots, c_{m+1}/b_{m+1}$ has an N-minor. Then Lemma 5.5(ii) implies that $M\backslash c_\ell, a_{\ell+1}, \dots, a_{m+1}/b_\ell$ has an N-minor, so Lemma 5.2 implies that $\{a_\ell, c_\ell\}$ is in a 4-cocircuit C^*. By orthogonality between C^* and

$\{c_\ell, d_\ell, a_{\ell+1}\}$, Lemma 5.1 implies that $d_\ell \in C^*$. Let $d_{\ell-1}$ be the fourth element of C^*. Suppose $d_{\ell-1} = e_{m+1}$. Then $\{d_{\ell-1}, a_\ell, c_\ell, d_\ell\} \triangle \{d_m, a_{m+1}, c_{m+1}, e_{m+1}\}$, which is $\{a_\ell, c_\ell, d_\ell, d_m, a_{m+1}, c_{m+1}\}$ or $\{a_\ell, c_\ell, a_{m+1}, c_{m+1}\}$, must be a cocircuit. The latter possibility arises if and only if $\ell = m$ and it violates Lemma 5.1. Hence $\ell \neq m$ and the former possibility arises. In that case, $T_\ell \cup T_{\ell+1} \cup \cdots \cup T_{m+1} \cup \{d_\ell, d_{\ell+1}, \ldots, d_m\}$ is 3-separating, so its complement in $E(M)$ contains at most three elements. This complement contains $T_{\ell-1}$ and e_{m+1}, so $e_{m+1} \in T_{\ell-1}$; a contradiction. We deduce that $d_{\ell-1} \neq e_{m+1}$. By orthogonality, $d_{\ell-1}$ avoids each of the triangles in the bowtie ring and in Figure 35.

We apply Lemma 7.6 to the structure obtained from that in Figure 35 by adjoining $d_{\ell-1}$ and the triangle T_{m+2}. Since $M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}$ is not internally 4-connected, we deduce that $\{d_{\ell-1}, a_\ell\}$ or $\{c_{m+1}, e_{m+1}\}$ is in a triangle. The first option implies, by orthogonality with $D_{\ell-2}$ and $D_{\ell-1}$, that $\{c_{\ell-1}, d_{\ell-1}, a_\ell\}$ is a triangle; a contradiction to our choice of ℓ. Thus $\{c_{m+1}, e_{m+1}\}$ is in a triangle. By orthogonality with D_{m+1} and D_{m+2}, we know that $\{c_{m+1}, e_{m+1}, a_{m+2}\}$ is a triangle. Since $M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}$ has $(a_{m+2}, e_{m+1}, a_{m+1}, d_m)$ as a 4-fan, either $M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}\backslash a_{m+2}$ or $M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}, d_m$ has an N-minor. Suppose the former holds. Then, since $M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}\backslash a_{m+2} \cong \backslash c_\ell, c_{\ell+1}, \ldots, c_{m+2}/b_{m+2}$ by Lemma 5.5(ii), we see that $M\backslash c_{m+2}/b_{m+2}$ has an N-minor. Then Lemma 5.2 implies that $\{a_{m+2}, c_{m+2}\}$ is in a 4-cocircuit. Orthogonality with $\{c_{m+1}, e_{m+1}, a_{m+2}\}$ together with Lemma 5.1 imply that e_{m+1} is in this cocircuit, and we obtain a contradiction to the choice of m. We deduce that $M\backslash c_{m+2}/b_{m+2}$ has no N-minor. Thus $M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}, d_m$ has an N-minor. But

$$\begin{aligned}
M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m+1}/b_{m+1}, d_m &\cong M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m-1}, a_{m+1}, c_{m+1}/b_{m+1}, d_m \\
&\cong M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m-1}, a_{m+1}, c_{m+1}/b_{m+1}, e_{m+1} \\
&\cong M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m-1}, a_{m+2}, c_{m+1}/b_{m+1}, e_{m+1} \\
&\cong M\backslash c_\ell, c_{\ell+1}, \ldots, c_{m-1}, a_{m+2}, c_{m+2}/b_{m+2}, e_{m+1}.
\end{aligned}$$

Thus $M\backslash c_{m+2}/b_{m+2}$, has an N-minor; a contradiction. We conclude that 7.9.7 holds.

Recall the ζ was defined following 7.9.5. We show next that

7.9.8. *$\ell \neq m$.*

Suppose $\ell = m$. Then $m = \zeta$. Since $M\backslash c_\zeta, c_{\zeta+1}/b_{\zeta+1}$ has an N-minor, we have a contradiction to 7.9.7 that establishes 7.9.8.

Next we show that

7.9.9. *$M\backslash c_\ell, c_{\ell+1}/b_{\ell+1}$ has an N-minor.*

We may assume that $\ell = k$ otherwise the assertion holds by 7.9.1. Now $M\backslash c_0, c_1/b_1$ has an N-minor and has (c_k, d_k, a_0, d_0) as a 4-fan. Thus $M\backslash c_0, c_1/b_1, d_0$ or $M\backslash c_0, c_1/b_1\backslash c_k$ has an N-minor. Now

$$\begin{aligned}
M\backslash c_0, c_1/b_1, d_0 &\cong M\backslash c_0, a_1/b_1, d_0 \\
&\cong M\backslash c_0, a_1/b_0, d_0 \\
&\cong M\backslash c_0, a_0/b_0, d_k \\
&\cong M\backslash c_k, c_0/b_0, d_k.
\end{aligned}$$

Moreover, $M\backslash c_k, c_0, c_1/b_1 \cong M\backslash c_k, c_0, a_1/b_1 \cong M\backslash c_k, c_0, a_1/b_0$. We conclude that 7.9.9 holds.

Choose j such that $M\backslash c_\ell, c_{\ell+1}, \dots, c_i/b_i$ has an N-minor for all i in $\{\ell+1, \ell+2, \dots, j\}$, but $M\backslash c_\ell, c_{\ell+1}, \dots, c_{j+1}/b_{j+1}$ does not have an N-minor. By 7.9.7, $j \le m$. Now $M\backslash c_\ell, c_{\ell+1}, \dots, c_j/b_j$ has $(a_{j+1}, d_j, a_j, d_{j-1})$ as a 4-fan, so $M\backslash c_\ell, c_{\ell+1}, \dots, c_j/b_j\backslash a_{j+1}$ or $M\backslash c_\ell, c_{\ell+1}, \dots, c_j/b_j, d_{j-1}$ has an N-minor. As $M\backslash c_\ell, c_{\ell+1}, \dots, c_j/b_j\backslash a_{j+1} \cong M\backslash c_\ell, c_{\ell+1}, \dots, c_j/b_{j+1}\backslash c_{j+1}$, by Lemma 5.5(ii), the choice of j implies that the former does not hold. Thus $M\backslash c_\ell, c_{\ell+1}, \dots, c_j/b_j, d_{j-1}$ has an N-minor. Lemma 7.1 implies that $\{d_{j-1}, d_j, v, w\}$ is a circuit and $\{v, w, x\}$ is a cocircuit. We know that $\{v, w\}$ meets no triangle of M. By orthogonality with the vertex cocircuits in Figure 35, we see that $j = m$, that $\ell = m-1$, and that $\{v, w\}$ meets $\{a_{\ell+2}, c_{\ell+2}, e_{\ell+2}\}$. Thus, without loss of generality, $v = e_{\ell+2}$. Then, using the argument in 7.9.9 with the indices shifted, we see that $M\backslash c_{\ell+1}, c_{\ell+2}/b_{\ell+2}, d_{\ell+1} \cong M\backslash c_\ell, c_{\ell+1}/b_{\ell+1}, d_\ell$. But $M\backslash c_\ell, c_{\ell+1}/b_{\ell+1}, d_\ell \cong M\backslash c_\ell, a_{\ell+1}/b_{\ell+1}, d_\ell \cong M\backslash c_\ell, a_{\ell+1}/b_\ell, d_\ell$. Thus $M\backslash c_\ell/b_\ell$ has an N-minor. By Lemma 5.2, we know that $\{a_\ell, c_\ell\}$ is in a 4-cocircuit C^* of M. By orthogonality with $\{c_\ell, d_\ell, a_{\ell+1}\}$, it follows by Lemma 5.1 that $d_\ell \in C^*$. Let $d_{\ell-1}$ be the fourth element in C^*. By orthogonality between C^* and the circuit $\{d_\ell, d_{\ell+1}, e_{\ell+2}, w\}$, we know that $d_{\ell-1}$ is in this circuit. Hence $T_\ell \cup T_{\ell+1} \cup T_{\ell+2} \cup \{d_\ell, d_{\ell+1}, e_{\ell+2}, w\}$ is 3-separating, so the complement of this set in $E(M)$ contains at most three elements. Thus $x \in T_{\ell-1}$; a contradiction. $\square$

We now prove the main result of this sequence of papers.

Proof of Theorem 1.5. We assume that the theorem does not hold. Suppose $|E(M)| \le 15$. Note that $M(K_5)$ and $M(K_4)$ are a rank-four quartic Möbius ladder and a rank-three cubic Möbius ladder, respectively. Then [6, Theorem 3.1] gives a contradiction. We assume therefore that $|E(M)| \ge 16$.

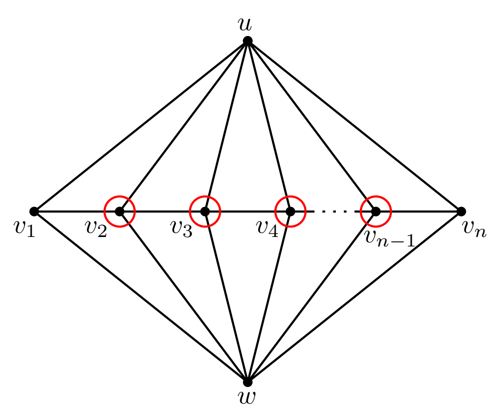

FIGURE 36. The elements are all distinct and $n \ge 5$.

We show first that

7.10.1. *for all $r \ge 6$, the matroid M is not isomorphic to $M(G_{r+1})$, $M(G^+_{r+1})$, Δ_r, or $\Delta_r\backslash z$.*

Assume that M is isomorphic to one of these matroids. Then M contains the structure in Figure 36 where $r = n+1$. Note that G_{r+1} and G^+_{r+1} are obtained from the graph in Figure 36 by, respectively, adding the edge v_1v_n, and adding the edges v_1v_n and z, where $z = uw$. Furthermore, we recall that Δ_r is the binary non-graphic matroid obtained from the cycle matroid of G^+_{r+1} by deleting v_1v_n and adding a new element e in a triangle with uv_n and wv_1. Thus M has a restriction isomorphic to the structure in Figure 36 and, for ease of notation, we will refer to the elements in M by the graph edges in the figure and by the elements defined above.

As the next step towards proving 7.10.1, we show that

7.10.2. *v_iv_{i+1} is in every N-minor of M for all i in $\{1, 2, \dots, n-1\}$ and, if v_1v_n is an element of M, then it is also in every N-minor of M. Moreover, if M/f has an N-minor, then $f \in \{z, e\}$.*

If contracting the element v_iv_{i+1} from M keeps an N-minor for any i in $\{1, 2, \dots n-1\}$, or if v_1v_n is an element of M and it can be contracted keeping an N-minor, then [8, Lemma 3.11] gives a contradiction.

Suppose deleting v_1v_2 keeps an N-minor. Then, since $M\backslash v_1v_2$ has $(uv_3, uv_2, v_2v_3, wv_2, wv_3)$ as a 5-fan, we can retain an N-minor either by deleting both uv_3 and wv_3, or by contracting uv_2 or wv_2. The latter gives a contradiction to [8, Lemma 3.11], so the former holds. But $M\backslash uv_3, wv_3$ has v_2v_3 in a 2-cocircuit. Hence M/v_2v_3 has an N-minor; a contradiction. A similar argument shows that we destroy all N-minors of M by deleting v_iv_{i+1} for any i in $\{1, 2, \dots, n-1\}$, or by deleting v_1v_n in the case that it exists. Thus the first part of 7.10.2 holds. Now each uv_i and each wv_i is in a triangle with a member of $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. It follows that 7.10.2 holds.

Next we show the following.

7.10.3. *If M has z as an element, then z is in every N-minor of M.*

Since $M(G_{r+1})$ and $\Delta_r\backslash z$ are internally 4-connected, and M has no quick win, if M is isomorphic to $M(G^+_{r+1})$ or Δ_r, then deleting z destroys all N-minors. Suppose M/z has an N-minor. As M/z has $\{uv_2, wv_2\}$ and $\{uv_3, wv_3\}$ as circuits, $M\backslash uv_2, uv_3$ has an N-minor. But this matroid has v_1v_2 as a coguts element of a 5-cofan. Hence this element can be contracted keeping an N-minor; a contradiction to 7.10.2. We conclude that 7.10.3 holds.

Continuing with the proof of 7.10.1, we now show that

7.10.4. *M is not isomorphic to $M(G_{r+1})$ or $M(G^+_{r+1})$.*

Suppose that M is isomorphic to one of these matroids. As every element of M is in a triangle, to obtain an N-minor of M, we must delete some element other than z. By symmetry and 7.10.2, we may assume that this element is uv_1, that is, $M\backslash uv_1$ has an N-minor. The last matroid has $(wv_2, v_2v_1, wv_1, v_1v_n)$ as a 4-fan. By 7.10.2, it follows that $M\backslash uv_1, wv_2$ has an N-minor. Again the last matroid has a 4-fan, $(uv_3, v_3v_2, uv_2, v_2v_1)$, so $M\backslash uv_1, wv_2, uv_3$ has an N-minor. Continuing in this way, we see that $M\backslash uv_1, wv_2, uv_3, wv_4, \dots, uv_{2k-1}, wv_{2k}$ has an N-minor for all k such that $2 \le 2k \le n+1$. It follows that n is even otherwise the last matroid has v_1v_2 in a 2-cocircuit; a contradiction to 7.10.2. Now $M\backslash uv_1, wv_2, uv_3, wv_4, \dots, wv_n$ is the cycle matroid of an alternating biwheel or an alternating biwheel with axle, and so is internally 4-connected. Since the last matroid has been obtained from M by trimming a bowtie ring, we deduce that (iii)(a) holds; a contradiction.

We now know that M is isomorphic to Δ_r or $\Delta_r\backslash z$ and that, in addition to the 4-cocircuits displayed in Figure 36, the edges incident with v_1 are in a 4-cocircuit with e, and the edges incident with v_n are in a 4-cocircuit with e. Furthermore, $\{e, uv_n, wv_1\}$ and $\{e, uv_1, wv_n\}$ are triangles of M.

Next we show that

7.10.5. *e is in every N-minor of M.*

Suppose first that $M\backslash e$ has an N-minor. Now $M\backslash e$ has a 5-fan with uv_2 and wv_2 as its guts elements. By 7.10.2 and 7.10.3, no element of $M\backslash e$ can be contracted keeping an N-minor. Thus $M\backslash e, uv_2, wv_2$ has an N-minor. But the last matroid has $\{v_1v_2, v_2v_3\}$ as a cocircuit. Hence $M\backslash e, uv_2, wv_2/v_1v_2$ has an N-minor; a contradiction to 7.10.2. We conclude that $M\backslash e$ does not have an N-minor. Now suppose that M/e has an N-minor. Since M/e has $\{uv_n, wv_1\}$ and $\{uv_1, wv_n\}$ as circuits, $M\backslash uv_n, wv_n$ has an N-minor. As the last matroid has $\{v_{n-1}v_n, e\}$ as a cocircuit, $M/v_{n-1}v_n$ has an N-minor; a contradiction to 7.10.2. Thus 7.10.5 holds.

By 7.10.2, 7.10.3, and 7.10.5, it follows that $r(N) = r(M)$. Moreover, by symmetry, we may assume that $M\backslash uv_i$ has an N-minor for some i. By deleting the guts elements of successively exposed 4-fans, we see that $M\backslash uv_1$ or $M\backslash wv_1$ has an N-minor so, by symmetry, we may assume the former. Then, again deleting the guts elements of successively exposed 4-fans, we see that $M\backslash uv_1, wv_2, uv_3, wv_4, \dots, uv_{2k-1}, wv_{2k}$ has an N-minor for all k with $2 \leq 2k \leq n$. Next we note that n must be odd for, if $n = 2k$, then $M\backslash uv_1, wv_2, uv_3, wv_4, \dots, uv_{n-1}, wv_n$ has $(wv_1, e, uv_n, v_nv_{n-1})$ as a 4-fan, so $M\backslash uv_1, wv_2, uv_3, wv_4, \dots, uv_{n-1}, wv_n\backslash wv_1$, and hence $M\backslash uv_1, wv_1$ has an N-minor. This gives a contradiction to 7.10.2 since the last matroid has v_1v_2 in a 2-cocircuit.

Now, when n is odd, let $M' = M\backslash uv_1, wv_2, uv_3, wv_4, \dots, uv_{n-1}, wv_n$ observing that this matroid has been obtained from M by trimming an open rotor chain. If z is not an element of M, then one easily checks that M' is the rank-$(n+1)$ triadic Möbius matroid, which is internally 4-connected. Hence (iii)(a) holds; a contradiction. We may now assume that z is an element of M and hence of M'. Then M' is a single-element extension of a rank-$(n+1)$ triadic Möbius matroid, so $M'\backslash z$ is internally 4-connected. We prove that M' is internally 4-connected by showing that it has no 4-element 3-separating set Z that contains z. Suppose M' contains such a set Z. Then Z is a 4-fan of M' or is both a circuit and a cocircuit of M'. But z is not in a triangle of M' so Z is a not a 4-fan of M'. If Z is both a circuit and a cocircuit of M', then orthogonality gives us a contradiction. We conclude that M contains no such set Z, so M' is internally 4-connected. Thus (iii)(a) holds; a contradiction. We conclude that 7.10.1 holds.

By Theorem 1.1, for some (M_0, N_0) in $\{(M, N), (M^*, N^*)\}$, the matroid M_0 has a triangle T that contains an element e such that $M_0\backslash e$ is $(4, 4, S)$-connected with an N_0-minor. Then Theorem 1.2 implies that M or M^* has a good bowtie otherwise M is the cycle matroid of a terrahawk and it is not difficult to check that N must be a minor of the cycle matroid of a cube or an octahedron, so the theorem holds; a contradiction. Then Theorems 1.3 and 1.4 imply that Hypothesis VIII holds. Lemma 5.6 implies that either M contains the structure in Figure 22, where $M\backslash 4$ is $(4, 4, S)$-connected with an N-minor, or M has a good bowtie string with three triangles. Thus, by Theorem 6.1, M has a good bowtie string with three triangles. Now Lemma 7.5 implies that M has a good bowtie ring, and Lemma 7.9 gives a contradiction. $\square$

8. Graphic matroids

In Section 2, we interpreted our main theorem for graphic matroids to state a splitter theorem for internally 4-connected graphs. To prove this graph theorem, we need to do two things: first, to justify our observation that, when we are in an internally 4-connected graphic matroid $M(G)$, the modified graph diagrams we have been using provide an accurate representation of what occurs in G except that some of the uncircled vertices may be identified; and, second, to check that, when the duals of the various moves are applied to a graphic matroid, they look as shown in the various diagrams in Section 2. Clearly the only triads that occur in $M(G)$ arise from vertex bonds in G. The next result shows that, provided $G \not\cong K_4$, every 4-cocircuit of $M(G)$ that meets a triangle is a vertex bond in G.

Lemma 8.1. *In an internally 4-connected graph G, let T be a triangle and C^* be a 4-edge bond that meets T. Then either $G \cong K_4$, or C^* is a vertex bond of G.*

Proof. Let $C^* = \{a, b, c, d\}$ and $T = \{a, b, e\}$. Assume that the lemma fails. Then $|E(G)| \geq 8$. Let u be the vertex of G that meets a and b, and let H_u be the component of $G \backslash C^*$ containing u. Let u_c and u_d be the endpoints of c and d that are in H_u. Now $|E(H_u)| \leq 3$ otherwise $(E(H_u), E(G) - E(H_u))$ is a 3-separation of $M(G)$ that contradicts the fact that G is internally 4-connected. In G, each of u_c and u_d has degree at least three while u has degree at least four. Now $|\{u, u_c, u_d\}| = 3$ otherwise $G \backslash \{u, u_c, u_d\}$ is disconnected, so G is not 3-connected; a contradiction. Since $|E(H_u)| \leq 3$, it follows that H_u is a triangle and each of u_c and u_d has degree three in G. Thus G has a 4-fan; a contradiction. $\square$

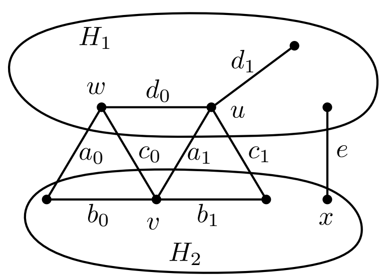

FIGURE 37. This structure arises in the proof of Lemma 8.2.

In the ladder structures in Figures 4(b) and 8, a specific 4-cocircuit can be replaced by a certain 5-cocircuit. We asserted in the introduction that this 5-cocircuit does not arise when the underlying matroid is graphic. The next lemma proves this assertion.

Lemma 8.2. *Let $M(G)$ be an internally 4-connected graphic matroid having (T_0, T_1, D_0) as a bowtie, $\{c_0, a_1, d_0\}$ as a triangle, and $\{d_0, a_1, c_1, d_1\}$ as a cocircuit, where $|T_0 \cup T_1 \cup \{d_0, d_1\}| = 8$. Then $\{a_0, c_0, a_1, c_1\}$ is not contained in a 5-cocircuit.*

Proof. Suppose that $\{a_0, c_0, a_1, c_1, e\}$ is a cocircuit C^* in $M(G)$. Lemma 8.1 implies that G has vertices v and u such that D_0 and $\{d_0, a_1, c_1, d_1\}$ are the sets of edges meeting these vertices. Since u and v are the endpoints of a_1, they are in different components, say H_1 and H_2, of $G\backslash C^*$. Now d_0 and d_1 are in $E(H_1)$, since they are incident with u and are not in C^*. Likewise, $\{b_0, b_1\} \subseteq E(H_2)$. Thus G contains the structure in Figure 37, where we have labelled the endpoint of e that is in H_2 as x, and the vertex incident with $\{a_0, c_0, d_0\}$ as w. The vertices shown in the figure need not all be distinct, although neither u nor v coincides with another vertex. Now $\{w, u, x\}$ is a vertex cut in G, so $\lambda_{M(G)}(E(H_1) \cup e) \leq 2$. But the complement of this set in $E(G)$ contains $T_0 \cup T_1$, so $E(H_1) \cup e$, which is $\{e, d_0, d_1\}$, is a triangle or a triad. It follows by orthogonality with C^* that $\{e, d_0, d_1\}$ is not a triangle, so it must be a triad. But this gives a contradiction to orthogonality with $\{c_0, d_0, a_1\}$. □

Among all of the diagrams in Sections 1 and 2, only Figures 4 and 5 contain circled vertices of degree exceeding four, these having degree five. In each case, when $M(G)$ contains such a 5-element cocircuit, this 5-cocircuit must be a vertex bond in G. To see this, it suffices to show that $\{c_2, b_2, c_0, u_0, t_0\}$ is a vertex bond in Figure 4(a). Observe that, by using the last lemma to treat 4-cocircuits that meet triangles, it follows that $\{c_2, b_2, c_0\}$ must meet at a common vertex of G, and u_0 and t_0 must also meet this vertex. Hence $\{c_2, b_2, c_0, u_0, t_0\}$ is indeed a vertex bond in G.

Most of the diagrams in Sections 1 and 2 have four uncircled vertices. We argued above that, when $M = M(G)$, each of the circled vertices in the diagram corresponds to a vertex bond in G. Using this, along with the numerous triangles in these diagrams, one can easily check that edges that meet at an uncircled vertex in the diagram must meet at the same vertex of G. However, unlike with the circled vertices, we do not know that the uncircled vertices correspond to distinct vertices of G. Indeed, as noted in Section 2, when a bowtie ring occurs in a graphic matroid, numerous uncircled vertices can be identified.

Finally, we need to check the accuracy of our depictions in Section 2 of how the duals of the various moves look in graphic matroids. These checks are straightforward and rely heavily on the fact that a triad in an internally 4-connected graphic matroid must correspond to a vertex bond in the graph. We omit the details.

In all of the figures in Section 2 except Figure 14, all of the edges are distinct. This requires justification since this is not true in Section 1. The structure in Figure 8(b) allows for various options including the possibility that d_n equals γ. But, in the dual structures in Figure 13, if $d_n = \gamma$, each of the two possible ways of identifying the endpoints of these edges results in $M(G)$ having a 4-fan or a 2-cycle; a contradiction. Similarly, in the structures, in Figures 4 and 5, d_2 may equal w_k. But, in the dual structures in Figures 15, if $d_2 = w_k$, we obtain the contradiction that $M(G)$ has a 4-fan.

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