INTERNALLY 4-CONNECTED BINARY MATROIDS WITH EVERY ELEMENT IN THREE TRIANGLES

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ABSTRACT. Let M be an internally 4-connected binary matroid with every element in exactly three triangles. Then M has at least four elements e such that $\mathrm{si}(M/e)$ is internally 4-connected. This technical result is a crucial ingredient in Abdi and Guenin's theorem determining the minimally non-ideal binary clutters that have a triangle.

1. INTRODUCTION

The terminology in this note will follow [4]. A matroid is *internally* 4*-connected* if it is 3-connected and, for every 3-separation (X, Y) of M, either X or Y is a triangle or a triad of M.

The purpose of this note is to prove the following technical result.

Theorem 1.1. *Let M be a binary internally 4-connected matroid in which every element is in exactly three triangles. Then M has at least four elements e such that $si(M/e)$ is internally 4-connected.*

We were motivated to prove this result because Abdi and Guenin [1] needed it to prove that the only minimally non-ideal binary clutters that have a triangle consist of the lines of the Fano matroid and the odd circuits of K_5. Indeed, the above result appears as Theorem 15 in [1].

Following [1, 2, 6, 7, 8], we now give the background relating to clutters needed to understand Abdi and Guenin's theorem. A *clutter* $\mathcal{A}$ on a finite set $E(\mathcal{A})$ is a family of subsets of $E(\mathcal{A})$ none of which is a proper subset of another. A clutter $\mathcal{A}$ is *trivial* if $\mathcal{A} = \emptyset$ or $\mathcal{A} = \{\emptyset\}$. The *blocker* $b(\mathcal{A})$ of $\mathcal{A}$ is the family of minimal subsets of $E(\mathcal{A})$ that have non-empty intersection with every member of $\mathcal{A}$. Edmonds and Fulkerson [3] showed that $b(b(\mathcal{A})) = \mathcal{A}$ for all clutters $\mathcal{A}$. A clutter $\mathcal{A}$ is *binary* if there is a binary matroid M having an element e such that $E(\mathcal{A}) = E(M) - \{e\}$ and $\mathcal{A}$ is the collection of sets of the form $C - e$ where C is a circuit of M containing e. It is well known that a clutter $\mathcal{A}$ is binary if and only if $|S \cap R|$ is odd for all S in $\mathcal{A}$ and all R in $b(\mathcal{A})$.

For a clutter $\mathcal{A}$, if C and D are disjoint subsets of $E(\mathcal{A})$, then the minimal members of $\{S - C : S \in \mathcal{A},\ S \cap D = \emptyset\}$ forms a clutter $\mathcal{A}/C\backslash D$ on $E(\mathcal{A}) - (C \cup D)$. Such a clutter is a *minor* of $\mathcal{A}$. This minor is *proper* if $C \cup D \neq \emptyset$. Seymour [6] showed that if a clutter is binary, then so are its blocker and all of its minors. Two important binary clutters are $\mathbb{L}_7$ and $\mathbb{O}_5$. The first consists of the seven lines in the Fano matroid while the second consists of the odd circuits in $M(K_5)$.

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Let $\mathcal{A}$ be a non-trivial clutter and $A(\mathcal{A})$ be the matrix whose columns are indexed by the elements of $E(\mathcal{A})$ and whose rows are the characteristic vectors of the members S of $\mathcal{A}$. If all of the extreme points of the real polyhedron $\{\mathbf{x} \geq \mathbf{0} : A(\mathcal{A})\mathbf{x} \geq \mathbf{1}\}$ have all of their coordinates in $\{0,1\}$, then the clutter $\mathcal{A}$ is *ideal*.

Seymour [7, 8] proposed the following.

Conjecture 1.2. *A binary clutter is ideal if and only if it has none of* $\mathbb{L}_7$*,* $\mathbb{O}_5$*, or* $b(\mathbb{O}_5)$ *as a minor.*

A member of a clutter with exactly three elements is called a *triangle*. Abdi and Guenin [1] proved the following partial result towards this conjecture. Theorem 1.1 is essential in their proof of this result.

Theorem 1.3. *The only binary non-ideal clutters that have a triangle and have all of their proper minors ideal are* $\mathbb{L}_7$ *and* $\mathbb{O}_5$*.*

The next section introduces some preliminaries. In Section 3, we prove the main theorem when M has at most thirteen elements, while Section 4 deals with when M has small cocircuits. Section 5 completes the proof of the main theorem. In Section 6, we show that the main theorem cannot be extended to ensure that $\mathrm{si}(M/e)$ is internally 4-connected for every element e of M.

2. Preliminaries

This section introduces some basic material relating to matroid connectivity. For a matroid M having ground set E and rank function r, its *connectivity function* λ_M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E-X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate λ_M as λ. For a positive integer k, a subset X or a partition $(X, E-X)$ of E is *k-separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ of E is a *k-separation* if $|X|, |E-X| \geq k$. If n is an integer exceeding one, a matroid is *n-connected* if it has no k-separations for all $k < n$. Let (X, Y) be a 3-separation in a matroid M. If $|X|, |Y| \geq 4$, then we call X, Y, or (X, Y) a $(4,3)$-*violator* since it certifies that M is not internally 4-connected. For example, if X is a 4-*fan*, that is, a 4-element set containing a triangle and a triad, then X is a $(4,3)$-violator provided $|Y| \geq 4$.

In a matroid M, a set U is *fully closed* if it is closed in both M and M^*. The *full closure* $\mathrm{fcl}(Z)$ of a set Z in M is the intersection of all fully closed sets containing Z. The full closure of Z may be obtained by alternating between taking the closure and the coclosure until both operations leave the set unchanged. Let (X, Y) be a partition of $E(M)$. If (X, Y) is k-separating in M for some positive integer k, and y is an element of Y that is also in $\mathrm{cl}(X)$ or $\mathrm{cl}^*(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is k-separating, and we say that we have *moved y* into X. More generally, $(\mathrm{fcl}(X), Y - \mathrm{fcl}(X))$ is k-separating in M.

The following elementary result will be used repeatedly.

Lemma 2.1. *Let e be an element of an internally 4-connected binary matroid M. Then $\mathrm{si}(M/e)$ is 3-connected.*

Proof. The result is easily checked if $|E(M)| < 4$, so we may assume that $|E(M)| \geq 4$. Since M is 3-connected and binary, $|E(M)| \geq 6$ and both M/e and $\mathrm{si}(M/e)$ are 2-connected. If $|E(M)| \in \{6,7\}$, then M is isomorphic to $M(K_4), F_7$, or F_7^* and again the result is easily checked. Thus we may assume that $|E(M)| \geq 8$.

Now let $M' = \mathrm{si}(M/e)$ and suppose that M' has a 2-separation (X, Y). We may assume that $|X| \geq |Y|$. Suppose $|Y| = 2$. Then Y is a 2-cocircuit $\{y_1, y_2\}$ of M'. As $\{y_1, y_2\}$ is not a 2-cocircuit of M/e and M is binary, we see that, in M/e, either one or both of y_1 and y_2 is in a 2-element parallel class. Thus we may assume that M/e has $\{y_1, y_1'\}$ as a circuit and $\{y_1, y_1', y_2\}$ as a cocircuit, or M/e has $\{y_1, y_1'\}$ and $\{y_2, y_2'\}$ as circuits and has $\{y_1, y_1', y_2, y_2'\}$ as a cocircuit. Hence M has $\{e, y_1, y_1', y_2\}$ as a 4-fan or has $\{y_1, y_1', y_2, y_2'\}$ as both a circuit and a cocircuit. Since $|E(M)| \geq 8$, each possibility contradicts the fact that M is internally 4-connected. We conclude that $|Y| \geq 3$.

Let (X', Y') be obtained from (X, Y) by adjoining each element of $E(M/e) - E(M')$ to the side of (X, Y) that contains an element parallel to it. Then $r_{M/e}(X') = r_{M'}(X)$ and $r_{M/e}(Y') = r_{M'}(Y)$, so (X', Y') is a 2-separation of M/e. Hence $(X', Y' \cup e)$ and $(X' \cup e, Y')$ are 3-separations of M. As $|Y' \cup e| \geq 4$ and $|E(M) \geq 8$, this gives a contradiction. □

Let n be an integer exceeding one. If M is n-connected, an n-separation (U, V) of M is *sequential* if $\mathrm{fcl}(U)$ or $\mathrm{fcl}(V)$ is $E(M)$. In particular, when $\mathrm{fcl}(U) = E(M)$, there is an ordering $(v_1, v_2, \ldots, v_m)$ of the elements of V such that $U \cup \{v_m, v_{m-1}, \ldots, v_i\}$ is n-separating for all i in $\{1, 2, \ldots, m\}$. When this occurs, the set V is called *sequential*.

We conclude this section by noting two useful results.

Lemma 2.2. *Let M be a matroid in which every element is in exactly three triangles. Then M has exactly $|E(M)|$ triangles.*

Proof. Consider the set of ordered pairs (e, T) where $e \in E(M)$ and T is a triangle of M containing e. The number of such pairs is $3|E(M)|$ since each element is in exactly three triangles. As each triangle contains exactly three elements, this number is also three times the number of triangles of M. □

Lemma 2.3. *Let M be an internally 4-connected binary matroid in which every element is in exactly three triangles. Then M has no cocircuits of odd size.*

Proof. For a cocircuit C^* of M, we construct an auxiliary graph G as follows. Let $V(G) = C^*$, and let c_1c_2 be an edge of G exactly when c_1 and c_2 are members of C^* that are contained in a triangle of M. Since every element in is three triangles of M, every vertex in G has degree three by orthogonality and the fact that M is binary. Hence $|C^*|$, which equals the number of vertices of G of odd degree, is even. □

3. Small matroids

In this section we prove the main theorem for matroids with at most thirteen elements. To prove this, we shall use the following theorem of Qin and Zhou [5].

Theorem 3.1. *Let M be an internally 4-connected binary matroid with no minor isomorphic to any of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$. Then either M is isomorphic to the cycle matroid of a planar graph, or M is isomorphic to F_7 or F_7^*.*

Lemma 3.2. *Let M be an internally 4-connected binary matroid in which every element is in exactly three triangles and $|E(M)| \leq 13$. Then M is isomorphic to F_7 or $M(K_5)$. Hence $\mathrm{si}(M/e)$ is internally 4-connected for all elements e of M.*

Proof. Assume that M is not isomorphic to F_7 or $M(K_5)$. Suppose first that M has none of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$ as a minor. As F_7^* has no triangles, it follows that M is isomorphic to the cycle matroid of a planar graph G. As every edge of G is in exactly three triangles, but $M(G)$ is internally 4-connected, every vertex has degree at least four. Hence $|E(G)| \geq 2|V(G)|$. Moreover, by Lemma 2.3, every vertex of G has even degree. Clearly $|V(G)| \neq 4$. Moreover, $|V(G)| \neq 5$, otherwise $M \cong M(K_5)$; a contradiction. As $|E(G)| \leq 13$, it follows that $|V(G)| = 6$ and $|E(G)| = 12$. Then G is obtained from K_6 by deleting the edges of a perfect matching. But no edge of this graph is in exactly three triangles.

We may now assume that M has an N-minor for some N in $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$. By the Splitter Theorem for 3-connected matroids, there is a sequence $M_0, M_1, \dots, M_k$ of 3-connected matroids such that $M_0 \cong N$ and $M_k \cong M$, while $|E(M_{i+1}) - E(M_i)| = 1$ for all i in $\{0, 1, \dots, k-1\}$. Since $|E(M) \geq 9$ and $|E(M)| \leq 13$, it follows that $k \in \{0, 1, 2, 3, 4\}$.

Suppose that some M_i is obtained from its successor by contracting an element e. Then M/e has an N-minor. But $\text{si}(M/e)$ has at most nine elements. Thus $|E(M)| = 13$ and N is $M(K_{3,3})$ or $M^*(K_{3,3})$. Since $\text{si}(M/e)$ must contain triangles, N is $M^*(K_{3,3})$. Now, by Lemma 2.3, every cocircuit of M/e is even. Moreover, M/e has exactly three 2-circuits. The union of these three 2-circuits cannot have rank two in M/e otherwise M has F_7 as a restriction and then the remaining six elements of M cannot all be in exactly three triangles of M. Let a, b and c be the three elements of $M^*(K_{3,3})$ that are in 2-circuits in M/e. Then one easily checks that there are two intersecting triangles of $M^*(K_{3,3})$ whose union contains exactly two elements of $\{a, b, c\}$. The cocircuit of M/e whose complement is the union of the closure of these two triangles is odd; a contradiction.

We now know that M is an extension of N by at most four elements. Let $N = M \backslash D$. Then $|D| \geq 1$ so $|E(M)| \geq 10$. Moreover, N has at least $|E(M)| - 3|D|$ triangles. It is straightforward to check that the last number is positive, so N cannot be $M(K_{3,3})$ or $M^*(K_5)$. Thus N is $M^*(K_{3,3})$ or $M(K_5)$. Each element of $M(K_5)$ is in three triangles, so $N \neq M(K_5)$ since each element of $E(M) - E(N)$ must be in a triangle with some element of $M(K_5)$; a contradiction. We deduce that $N = M^*(K_{3,3})$. Now $M^*(K_{3,3})$ has exactly six triangles with each element being in precisely two triangles. Thus, in M, there are six triangles each containing a single element of $M^*(K_{3,3})$ and two elements of $E(M) - E(N)$. As $|E(M)| - E(N)| \leq 4$, there are at most six triangles containing exactly two elements of $E(M) - E(N)$. We deduce that $|E(M)| = 13$ so M can be obtained from $PG(3, 2)$ by deleting exactly two elements. As $PG(3, 2)$ has exactly seven triangles containing each element, deleting two elements leaves each element in at least five triangles; a contradiction. □

4. Small cocircuits

In this section, we move towards proving the main result by dealing with 4-cocircuits and certain special 6-cocircuits in M. Throughout the section, we will assume that M is an internally 4-connected binary matroid in which every element is in exactly three triangles, and $|E(M)| \geq 14$.

Lemma 4.1. *If C^* is a 4-element cocircuit of M, then, for all e in C^*, the matroid* $\text{si}(M/e)$ *is internally 4-connected having no triads.*

Proof. Suppose that $C^* = \{e, f_1, f_2, f_3\}$ and $\mathrm{si}(M/e)$ is not internally 4-connected. As M is internally 4-connected, $r(C^*) = 4$. As e is in three triangles of M, there are elements $\{g_1, g_2, g_3\}$ such that $\{e, f_i, g_i\}$ is a triangle for all i. As f_i is in three triangles for all i, by orthogonality and the fact that M is binary, there are elements $\{h_1, h_2, h_3\}$ such that $\{f_1, f_2, h_1\}, \{f_1, f_3, h_3\}$, and $\{f_2, f_3, h_2\}$ are triangles. This forces $\{g_1, g_2, h_1\}, \{g_1, g_3, h_3\}$, and $\{g_2, g_3, h_2\}$ to be triangles, so g_i is in no other triangle of M for all i.

Let $M' = \mathrm{si}(M/e) = M/e\backslash f_1, f_2, f_3$. Lemma 2.1 implies that M' is 3-connected. The set $\{g_1, g_2, g_3, h_1, h_2, h_3\}$ forms an $M(K_4)$-restriction in M'. Suppose M' has a non-sequential 3-separation. Then we may assume that $\{g_1, g_2, g_3, h_1, h_2, h_3\}$ is contained in one side of the 3-separation. Since $\{f_i, g_i\}$ is a circuit in M/e, we may add f_1, f_2, and f_3 to the side containing the $M(K_4)$-restriction, and then add e to get a $(4, 3)$-violator of M; a contradiction. We deduce that a $(4, 3)$-violator of $\mathrm{si}(M/e)$ is a sequential 3-separation.

We show next that

4.1.1. *$M/e\backslash f_1, f_2, f_3$ has no triads.*

Suppose $M/e\backslash f_1, f_2, f_3$ has a triad $\{\beta, \gamma, \delta\}$. Then $M\backslash f_1, f_2, f_3$ has $\{\beta, \gamma, \delta\}$ as a cocircuit. By Lemma 2.3, we may assume that $\{\beta, \gamma, \delta, f_1, f_2, f_3\}$ or $\{\beta, \gamma, \delta, f_1\}$ is a cocircuit of M. By orthogonality, in the first case, $\{\beta, \gamma, \delta\} = \{g_1, g_2, g_3\}$ while, in the second case, $g_1 \in \{\beta, \gamma, \delta\}$. In the first case, let $Z = \{e, f_1, f_2, f_3, g_1, g_2, g_3\}$. Then $r(Z) \leq 4$ while $|Z| - r^*(Z) \geq 2$, so $\lambda(Z) \leq 2$, a contradiction as $|E(M)| \geq 14$.

In the second case, M has a 4-cocircuit D^* such that $C^* \cap D^* = \{f_1\}$ and $g_1 \in D^*$. Apart from $\{f_1, e, g_1\}$, the other triangles containing f_1 must meet $C^* - \{f_1, e\}$ in distinct elements and must meet $D^* - \{f_1, g_1\}$ in distinct elements. Thus $r(C^* \cup D^*) \leq 4$ and $|C^* \cup D^*| - r^*(C^* \cup D^*) \geq 2$, so $\lambda(C^* \cup D^*) \leq 2$; a contradiction since $|E(M)| \geq 14$. Thus 4.1.1 holds.

By 4.1.1, $M/e\backslash f_1, f_2, f_3$ has no 4-fans and so has no sequential 3-separation that is a $(4, 3)$-violator. This contradiction completes the proof. □

Lemma 4.2. *Take $e \in E(M)$ and the three triangles T_1, T_2, and T_3 containing e. If $(T_1 \cup T_2 \cup T_3) - e$ is a cocircuit C^*, then $\mathrm{si}(M/x)$ is internally 4-connected for every element x of C^*.*

Proof. Let $T_i = \{e, f_i, g_i\}$ for each $i \in \{1, 2, 3\}$. Note that T_1, T_2, and T_3 are not coplanar, otherwise their union forms an F_7-restriction, and C^* contains a triangle; a contradiction to the fact that M is binary. Suppose the lemma fails. Then we may assume that $\mathrm{si}(M/f_3)$ is not internally 4-connected.

As f_1 is in two triangles other than T_1, orthogonality and the fact that M is binary imply that each of these triangles contains an element of $\{f_2, g_2, f_3, g_3\}$. If $\{f_1, f_2\}$ and $\{f_1, g_2\}$ are each contained in a triangle, then the plane containing T_1 and T_2 is an F_7-restriction, so e is in a fourth triangle; a contradiction. Hence f_1 is in a single triangle with an element of $\{f_2, g_2\}$ and a single triangle with an element of $\{f_3, g_3\}$. Without loss of generality, $\{f_1, g_2, x_1\}$ and $\{f_1, g_3, x_2\}$ are triangles. By taking the symmetric difference of these triangles with the circuits $\{f_1, g_1, f_2, g_2\}$ and $\{f_1, g_1, f_3, g_3\}$, respectively, we see that $\{g_1, f_2, x_1\}$ and $\{g_1, f_3, x_2\}$ are also triangles. We have now identified all three of the triangles containing each element in $\{f_1, g_1\}$. But, for each element in $\{f_2, g_2, f_3, g_3\}$, one of the triangles containing the element remains undetermined.

Either $\{f_2, g_3, x_3\}$ and $\{g_2, f_3, x_3\}$ are triangles, or $\{f_2, f_3, y_3\}$ and $\{g_2, g_3, y_3\}$ are triangles. In each of these cases, we will obtain the contradiction that $\mathrm{si}(M/f_3)$ is internally 4-connected. By Lemma 2.1, $M' = \mathrm{si}(M/f_3)$ is 3-connected. Take (U, V) to be a $(4, 3)$-violator in M'.

Let $X = \{e, f_1, f_2, g_1, g_2, x_1\}$. Clearly the restriction of M/f_3 to X is isomorphic to $M(K_4)$. We may assume that $M' = M/f_3\backslash Y$ where Y is $\{g_3, x_2, x_3\}$ or $\{g_3, x_2, y_3\}$ depending on whether $\{f_3, g_2, x_3\}$ or $\{f_3, f_2, y_3\}$ is a triangle of M. Without loss of generality, we may also assume that U spans X in M'. Then $(U \cup X, V - X)$ is 3-separating in M' and it follows that $(U \cup X \cup Y \cup f_3, V - X)$ is 3-separating in M. Since M has no $(4, 3)$-violator, we deduce that V is a sequential 3-separating set in M'. Thus M' has a triad $\{\beta, \gamma, \delta\}$. By Lemma 2.3, M has a cocircuit D^* where D^* is $\{\beta, \gamma, \delta\} \cup Y$ or $\{\beta, \gamma, \delta\} \cup y$ for some y in Y. In the first case, by orthogonality, $\{\beta, \gamma, \delta\} \subseteq X$. The last inclusion also follows by orthogonality in the second case since $\{\beta, \gamma, \delta\}$ must meet X and $M|X \cong M(K_4)$. Hence $X \cup Y \cup f_3$ contains at least two cocircuits. Since $r(X \cup Y \cup f_3) = 4$, it follows that $\lambda(X \cup Y \cup f_3) \leq 2$; a contradiction as $|E(M)| \geq 14$. □

Lemma 4.3. *Let (X, Y) be an exact 4-separation in M with $X \subseteq \mathrm{fcl}(Y)$. If M has no 4-cocircuits, then X is coindependent, $r(X) = 3$, and $X \subseteq \mathrm{cl}(Y)$.*

Proof. If $X \subseteq \mathrm{cl}(Y)$, then Y contains a basis of M, and X is coindependent. As $r(X) + r^*(X) - |X| \leq 3$, the rank of X is at most three, and the result holds. If $X \subseteq \mathrm{cl}^*(Y)$, then X is independent, so $r^*(X) = 3$. As $|X| \geq 4$, it follows that X is a 4-cocircuit; a contradiction.

Beginning with Y, look at $\mathrm{cl}(Y), \mathrm{cl}^*(\mathrm{cl}(Y)), \mathrm{cl}(\mathrm{cl}^*(\mathrm{cl}(Y))), \ldots$ until the first time we get $E(M)$. Consider the set Y' that occurs before $E(M)$ in this sequence, let $X' = E(M) - Y'$, and let e be the last element that was added in taking the closure or coclosure that equals Y'. Then either Y' is a hyperplane and X' is a cocircuit, or Y' is a cohyperplane and X' is a circuit.

Suppose X' is a circuit. As $r(X') + r^*(X') - |X'| \leq 3$, we see that $r^*(X') \leq 4$. Thus, as X' does not contain a 4-cocircuit, it is coindependent, so it has size at most four. We may assume that $X' \subsetneqq X$, otherwise the lemma holds. Suppose $|X'| = 4$. Then both $(X' \cup e, Y' - e)$ and (X', Y') are exact 4-separations. Thus $e \in \mathrm{cl}^*(X') \cap \mathrm{cl}^*(Y' - e)$ or $e \in \mathrm{cl}(X') \cap \mathrm{cl}(Y' - e)$. The latter holds otherwise M has a 4-cocircuit; a contradiction. But Y' is coclosed, so e was added by coclosure; that is, $e \in \mathrm{cl}^*(Y - e)$ and we have a contradiction to orthogonality since $e \in \mathrm{cl}(X)$. It remains to consider the case when $|X'| = 3$. Then $|X' \cup e| = 4$. The lemma holds if $X' \cup e = X$, so there is an element f of $Y' - e$ that was added immediately before e in the construction of Y'. Now if f is added via closure, then we can also add e and X' via closure, so we violate our choice of Y'. Thus f is added via coclosure so $f \in \mathrm{cl}^*(Y' - e - f) \cap \mathrm{cl}^*(X' \cup e)$. Hence M has a 4-cocircuit; a contradiction.

We may now assume that X' is a cocircuit. Then X' has at least six elements. As X' is 4-separating, $3 = r(X') + r^*(X') - |X'| = r(X') - 1$. Hence $r(X') = 4$, so $M|X'$ is a restriction of $PG(3, 2)$. As M is binary, X' contains no triangle and no 5-circuits, so $M|X'$ is a restriction of $AG(3, 2)$. As X' has six or eight elements, it follows that X' is a union of 4-circuits so $\mathrm{fcl}(Y')$ cannot contain X'; a contradiction. □

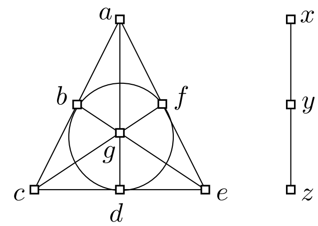

FIGURE 1. A skew plane and line in a binary matroid. Squares indicate positions that may be occupied by elements of M.

Lemma 4.4. *Assume M has no 4-cocircuits. If every exact 4-separation in M is sequential, then, for every element $e \in E(M)$, the matroid $\mathrm{si}(M/e)$ is internally 4-connected with no triads.*

Proof. Let $\{e, f_i, g_i\}$ be a triangle for all $i \in \{1,2,3\}$. The matroid $M' = \mathrm{si}(M/e) = M/e\backslash f_1, f_2, f_3$ is 3-connected by Lemma 2.1. Let (U,V) be a $(4,3)$-violator in M'. Then $|U|, |V| \geq 4$. Add f_i to the side of the 3-separation containing g_i for all $i \in \{1,2,3\}$ to obtain (U', V'), a 3-separation in M/e. Neither $(U' \cup e, V')$ nor $(U', V' \cup e)$ is a 3-separation in M. Hence both are 4-separations in M. Thus, by hypothesis, each is a sequential 4-separation in M. Lemma 4.3 implies that, without loss of generality, either $U' \cup e$ is coindependent and has rank at most three in M; or both U' and V' have rank at most three and are contained in $\mathrm{cl}(V' \cup e)$ and $\mathrm{cl}(U' \cup e)$, respectively. In the first case, as $U' \cup e$ is contained in a plane, U is contained in a triangle in $\mathrm{si}(M/e)$; a contradiction. In the second case, $r(M) = 4$, so U' and V' span planes in $PG(3,2)$. These planes meet in a line, so $|U' \cup V'| \leq 7 + 7 - 3 = 11$. Hence $E(M) \leq 12$; a contradiction.

Suppose $M/e\backslash f_1, f_2, f_3$ has a triad $\{a,b,c\}$. Then, by Lemma 4.1, M has $\{a,b,c,f_1,f_2,f_3\}$ as a cocircuit, so we may assume that $(a,b,c) = (g_1, g_2, g_3)$. Now M has a triangle containing f_1 and exactly one of f_2, g_2, f_3, or g_3. It follows that $\mathrm{si}(M/e)$ has a triangle meeting $\{g_1, g_2, g_3\}$, so $\mathrm{si}(M/e)$ has a 4-fan; a contradiction. □

The next three lemmas deal with a plane and a line in M.

Lemma 4.5. *Suppose M contains a plane P and a line L that are skew and are labelled as in Figure 1 where not every element in the figure must be in M. If $a,b,c,d,e,f,x,y,$ and z are in M, and $\{x,y,a,b,d,e\}$ and $\{y,z,b,c,e,f\}$ are cocircuits in M, then $\mathrm{si}(M/w)$ is internally 4-connected for all w in $\{a,b,c,d,e,f\}$.*

Proof. By symmetric difference, $\{x,z,a,c,d,f\}$ is a cocircuit. As z is in three triangles of M, orthogonality implies that z is in a triangle with c, say $\{z,c,c'\}$, and a triangle with f, say $\{z,f,f'\}$. Likewise, x is in triangles $\{x,a,a'\}$ and $\{x,d,d'\}$, while y is in triangles $\{y,b,b'\}$ and $\{y,e,e'\}$, for some elements a', d', b', e'. As P and L are skew, all of a', b', c', d', e', f' are distinct and none is in P or L.

By symmetry, it suffices to show that $\mathrm{si}(M/a)$ is internally 4-connected. Let $M' = \mathrm{si}(M/a) = M/a\backslash a', b, f$. Let $Z = \{c,d,e,x,y,z,d',b',f'\}$. The restriction of M' to Z is isomorphic to $M^*(K_{3,3})$. Suppose (U,V) is a $(4,3)$-violator of M'. Without loss of generality, U spans Z in M'. Thus U spans $\{c', e'\}$. Hence $(U \cup Z \cup \{c',e'\} \cup \{a',b,f\}, V - Z - \{c',e'\})$ is 3-separating in M/a, so $(U \cup Z \cup \{c',e'\} \cup \{a',b,f\} \cup a, V - Z - \{c',e'\})$ is 3-separating in M.

Thus V is a sequential 3-separating set in M', so V contains a triad $\{\beta,\gamma,\delta\}$. Thus either $\{x,c,e,a',b,f\}$ or $\{\beta,\gamma,\delta\}\cup t$ is a cocircuit of M for some t in $\{a',b,f\}$. The first possibility gives a contradiction to orthogonality with $\{y,b,b'\}$. Thus $\{\beta,\gamma,\delta,b\}$, $\{\beta,\gamma,\delta,f\}$, or $\{\beta,\gamma,\delta,a'\}$ is a cocircuit. Suppose $\{\beta,\gamma,\delta,b\}$ or $\{\beta,\gamma,\delta,f\}$ is a cocircuit. Then orthogonality implies that $\{\beta,\gamma,\delta\}$ contains $\{b,c,d\}$ or $\{f,e,d\}$ and so we get a contradiction to orthogonality with at least one of $\{x,d,d'\},\{z,c,c'\},\{z,f,f'\},\{y,b,b'\}$ and $\{y,e,e'\}$. Thus $\{\beta,\gamma,\delta,a'\}$ is a cocircuit. This cocircuit also contains x so either contains y and elements from each of $\{b,b'\}$ and $\{e,e'\}$, or contains z and elements from each of $\{f,f'\}$ and $\{c,c'\}$. Each case gives a contradiction to orthogonality. We conclude that $\mathrm{si}(M/a)$ is internally 4-connected, so the lemma holds. □

Lemma 4.6. *Assume M has no 4-cocircuits. Let (U,V) be a non-sequential 4-separation of M where U is closed and V is contained in the union of a plane P and a line L of M. Then either V is 6-cocircuit, or $|V|=9$ and $|P|=6$. Moreover, $\mathrm{si}(M/v)$ is internally 4-connected for at least six elements v of V.*

Proof. By Lemma 2.3, each cocircuit contained in V has exactly six elements otherwise it contains a triangle. Suppose $r(V)=3$. As $r(V)+r^*(V)-|V|=3$, we know that V is coindependent. Hence it is contained in $\mathrm{cl}(U)$; a contradiction. Evidently $r(V)\geq 4$. We use Figure 1 as a guide for the points that may exist in V. We consider which positions are filled, keeping in mind that V is the union of circuits and the union of cocircuits.

Suppose V has rank four and view V as a restriction of $Q=PG(3,2)$. Then $\mathrm{cl}_Q(P)\cap\mathrm{cl}_Q(L)$ is a point of Q, so we may suppose $e=z$. Furthermore, as $r(V)+r^*(V)-|V|=3$, we know that V contains, and therefore is, a cocircuit. Thus $|V|=6$. As V contains no triangles, $|(P\cup L)\cap\mathrm{cl}_Q(P)|\leq 4$, and $|(P\cup L)\cap\mathrm{cl}_Q(L)|\leq 2$. Thus $e\notin P\cup L$. Without loss of generality, the points in V are a,b,f,g,x, and y, and the result follows by Lemma 4.2 provided $e\in E(M)$.

We assume therefore that $e\notin E(M)$. We know that $V=\{x,y,a,b,f,g\}$. By orthogonality, without loss of generality, the three triangles of M containing x are $\{x,a,a'\},\{x,f,f'\}$, and $\{x,b,b'\}$. Thus M has as triangles each of $\{y,a',f\},\{y,a,f'\}$, and $\{y,b',g\}$. Hence M has no other triangles containing x or y. Thus the remaining triangles containing g must be in P, and so contain c and d. But then $\{a,b,c\}$ and $\{a,g,d\}$ are triangles of M, so a is in four triangles; a contradiction.

Suppose that $r(V)=5$. Then P and L are skew, and V is the union of two 6-cocircuits, C^* and D^*. By orthogonality, each of C^* and D^* contains at most four elements of P. Thus, by orthogonality, $|P|\leq 6$ so $|C^*\cup D^*|\leq 9$. Hence $|C^*\bigtriangleup D^*|=6$ and $|V|=9$. Then, without loss of generality, each of C^* and D^* meets P in four elements and L in two elements. The result now follows by Lemma 4.5. □

Lemma 4.7. *If M has a 6-element cocircuit $C^*=\{a,b,c,d,e,f\}$ where $\{a,b,c,d\}$ and $\{a,b,e,f\}$ are circuits, then $\mathrm{si}(M/x)$ is internally 4-connected for all x in C^*.*

Proof. By symmetric difference, $\{c,d,e,f\}$ is also a circuit. Thus C^* is the union of three disjoint pairs, $\{a,b\}$, $\{c,d\}$, and $\{e,f\}$ such that the union of any two of these pairs is a circuit. If one of these pairs is in a triangle with some element x, then each of the pairs is in a triangle with x and the lemma follows by Lemma 4.2. Thus we may assume that each of $\{a,c\}$ and $\{a,d\}$ is in a triangle. Hence so are $\{b,c\}$ and

$\{b, d\}$. Thus each of a, b, c and d is in exactly one triangle with an element of $\{e, f\}$. Hence e and f cannot both be in exactly three triangles; a contradiction. □

Lemma 4.8. *Let (J, K) be an exact 4-separation of M with J closed. If $|K| \leq 6$, then K is a 6-cocircuit and $\mathrm{si}(M/k)$ is internally 4-connected for all k in K.*

Proof. We have $r(K) + r^*(K) - |K| = 3$ and $|K| \geq 4$. If $|K| = 4$, then K is a cocircuit; a contradiction. Thus $|K| \geq 5$. Since K is a union of cocircuits each of which has even cardinality, it follows that $|K| \geq 6$. Hence K is a 6-cocircuit. Thus $r(K) = 4$ so K contains two circuits such that they and their symmetric difference have even cardinality. Hence K is the union of two 4-circuits that meet in exactly two elements and the result follows by Lemma 4.6. □

5. The proof of the main result

The next lemma essentially completes the proof of Theorem 1.1.

Lemma 5.1. *Let M be an internally 4-connected binary matroid in which every element is in exactly three triangles. Suppose M has no 4-cocircuits. Then M has at least six elements e such that $\mathrm{si}(M/e)$ is internally 4-connected.*

Proof. By Lemma 3.2, we know that $|E(M)| \geq 14$. Assume that the lemma fails. By Lemma 4.4, M has a non-sequential 4-separation (X, Y) where X is minimal. Then Y is fully closed. By Lemma 4.8, $|X| \geq 7$ and X contains an element α such that $\mathrm{si}(M/\alpha)$ is not internally 4-connected. Let $\{\alpha, f_i, g_i\}$ be a triangle for all $i \in \{1, 2, 3\}$. Now $M' = \mathrm{si}(M/\alpha) = M/\alpha \backslash f_1, f_2, f_3$ is not internally 4-connected. By Lemma 2.1, it is 3-connected. Take a $(4, 3)$-violator (U', V') in M'. Then $|U'|, |V'| \geq 4$. Hence $r_{M/\alpha}(U')$ and $r_{M/\alpha}(V')$ exceed two. Add f_i to the side containing g_i for all $i \in \{1, 2, 3\}$ to obtain (U'', V''). Then both $(U'' \cup \alpha, V'')$ and $(U'', V'' \cup \alpha)$ are exact 4-separations of M. Since $\alpha \in \mathrm{cl}(U'')$ and $\alpha \in \mathrm{cl}(V'')$, we deduce that $r_M(U'') \geq 4$ and $r_M(V'') \geq 4$. Moreover, by Lemma 4.3, both $(U'' \cup \alpha, V'')$ and $(U'', V'' \cup \alpha)$ are non-sequential. Without loss of generality, we may assume that $r(U'' \cap X) \geq r(V'' \cap X)$ and, when equality holds, $|U'' \cap X| \geq |V'' \cap X|$. Let $(U, V) = (\mathrm{cl}(U''), V'' - \mathrm{cl}(U''))$. Then

5.1.1. *$r_M(U \cap X) \geq r_M(V \cap X)$, and, when equality holds, $|U \cap X| > |V \cap X|$.*

We show next that

5.1.2. *$X \cap U, X \cap V, Y \cap U$, and $Y \cap V$ are all non-empty.*

As $\alpha \in X \cap U$, the first set is not empty. If the second is empty, then, as α is in the closure of $V = V \cap Y$, we can move α to Y to get $(X - \alpha, Y \cup \alpha)$ as a non-sequential 4-separation of M; a contradiction to our choice of (X, Y). If the third is empty, then $U = X \cap U$, and $(X \cap U, Y \cup V)$ contradicts our choice of (X, Y). Likewise, if the fourth set is empty, then $V = X \cap V$, and $(X \cap V, Y \cup U)$ violates our choice of (X, Y). This completes our proof of 5.1.2.

By submodularity of the connectivity function, $\lambda_M(X \cup U) + \lambda_M(X \cap U) \leq \lambda_M(X) + \lambda_M(U) = 3 + 3$. We now break the rest of the argument into the following two cases, which we shall then consecutively eliminate.

(A) $\lambda(X \cap U) \geq 4$ and $\lambda(X \cup U) = \lambda(Y \cap V) \leq 2$; or
(B) $\lambda(X \cap U) \leq 3$.

5.1.3. *(A) does not hold.*

Suppose that (A) holds. As M is internally 4-connected, $Y \cap V$ is a triangle or a triad, or it contains at most two elements. Clearly, this set is not a triad. Suppose $\lambda(X \cap V) \geq 4$. Then, by submodularity again, $\lambda(Y \cap U) \leq 2$, so $|Y \cap U| \leq 3$. Then $|Y| \leq 6$, so Y contains and hence is a cocircuit. As this cocircuit cannot contain a triangle, $|Y \cap V| \leq 2$, so $|Y| \leq 5$; a contradiction. Thus $\lambda(X \cap V) \leq 3$. If $\lambda(X \cap V) \leq 2$, then $X \cap V$ is contained in a triangle, so V is contained in the union of two lines; a contradiction as V contains a cocircuit that must have six elements and so contain a triangle. We deduce that $\lambda(X \cap V) = 3$. Hence $X \cap V \subseteq \mathrm{fcl}(Y \cup U)$. Lemma 4.3 implies that $X \cap V$ has rank at most three. Thus V is contained in the union of a line L and a plane P. It now follows by Lemma 4.8 that 5.1.3 holds.

Next we show that

5.1.4. *(B) does not hold.*

Assume that (B) holds. Since $\lambda(X \cap U) \leq 3$ and $X \cap U$ is properly contained in X, either $X \cap U \subseteq \mathrm{fcl}(Y \cup V))$, or $\lambda(X \cap U) \leq 2$. It follows using Lemma 4.3 that $r(X \cap U) \leq 3$. Thus, by 5.1.1, $r(X \cap V) \leq 3$. If $r(X \cap V) \leq 2$, then X is contained in the union of a plane and a line. Then, arguing as in (A), it follows that $|X| = 6$ or $|X| = 9$ and $\mathrm{si}(M/x)$ is internally 4-connected for all x in X. Each alternative gives a contradiction. Thus, by 5.1.1, $r(X \cap V) = 3 = r(X \cap U)$ and $|X \cap V| < |X \cap U| \leq 7$. Hence $4 \leq r(X) \leq 6$.

Now view M as a restriction of $Q = PG(r-1, 2)$, where $r = r(M)$. As (X, Y) is an exact 4-separation, $\mathrm{cl}_Q(X) \cap \mathrm{cl}_Q(Y)$ is a plane P of Q. Because Y is fully closed, no element of X is in P. It follows by orthogonality, since X is a union of cocircuits of M, that each triangle that meets an element of X is either contained in X or contains exactly two elements of X with the third element being in P.

We now show that

5.1.5. $r(X) \in \{5, 6\}$.

Suppose not. Then $r(X) = 4$ and $X \subseteq \mathrm{cl}_Q(X) - P$. So X is contained in an $AG(3, 2)$-restriction of M. As X is a cocircuit, $|X| = 6$ or $|X| = 8$. Since $|X \cap U| \neq |X \cap V|$ and each is at least three, it follows that $|X| = 8$. To have a triangle meeting X, there must be an element y of Y in P. But y is the tip of a binary spike in $X \cup y$ so it is in at least four triangles; a contradiction.

We show next that

5.1.6. $r(X) = 5$.

Suppose not. Then $r(X) = 6$. As $r(X \cap U) = r(X \cap V) = 3$, we deduce that $\mathrm{cl}_Q(X \cap U) \cap \mathrm{cl}_Q(X \cap V) = \emptyset$, where we recall that $Q = PG(r-1, 2)$ and $P = \mathrm{cl}_Q(X) \cap \mathrm{cl}_Q(Y)$.

Suppose $\mathrm{cl}_Q(X \cap V)$ meets P. As $3 = \lambda(X) = r(X) + r(Y) - r(M)$, we know that $r(Y) = r(M) - 3$. Then $\mathrm{cl}_M(Y \cup (X \cap V))$ is a flat with rank at most $r(M) - 1$. Hence its complement, which is contained in $X \cap U$, contains a cocircuit. But this cocircuit contains at least six elements by Lemma 4.1, so it contains a triangle in $X \cap U$. We deduce that $\mathrm{cl}_Q(X \cap V)$ avoids P. By symmetry, so does $\mathrm{cl}_Q(X \cap U)$. It follows that each triangle that meets X is either contained in $X \cap U$ or $X \cap V$, or contains an element of each of $X \cap U, X \cap V$, and P. If $|X \cap U| = 7$, then $M|(X \cap U) \cong F_7$-restriction, so each element in $X \cap U$ is in three triangles contained in $X \cap U$. Then each element in $X \cap V$ is contained in three triangles in $X \cap V$, so $M|(X \cap V) \cong F_7$, and $|X \cap U| = |X \cap V|$; a contradiction to 5.1.1. Thus $|X \cap U| \leq 6$ and 5.1.1 implies that $|X \cap V| \leq 5$. Thus $X \cap V$ contains an element v that is in at most one triangle in $X \cap V$. Hence v is in triangles $\{v, u_1, p_1\}$ and $\{v, u_2, p_2\}$ for some u_1 and u_2 in $X \cap U$, and p_1 and p_2 in P. Take u_3 in $X \cap U$ such that $\{u_1, u_2, u_3\}$ is a basis for $X \cap U$. Then $\mathrm{cl}(Y \cup \{v, u_3\})$ is a flat of rank at most $r(M) - 1$ whose complement, which is contained in $X \cap V$, contains a cocircuit. This cocircuit has at most five elements; a contradiction to Lemma 4.1. Hence 5.1.6 holds.

We now know that $r(X) = 5$. It follows, since $r(X \cap U) = r(X \cap V) = 3$, that $\mathrm{cl}_Q(X \cap U) \cap \mathrm{cl}_Q(X \cap V)$ is a point p of Q. Moreover, $r(Y) = r(M) - 2$, so $r(\mathrm{cl}_Q(Y) \cap \mathrm{cl}_Q(X \cap U)) = 1$ since $r(\mathrm{cl}_Q(Y \cup (X \cap U))) = r(M)$, otherwise $X \cap V$ contains a cocircuit of M that either has fewer than six elements or contains a triangle. Similarly, $r(\mathrm{cl}_Q(Y) \cap \mathrm{cl}_Q(X \cap V)) = 1$.

The following is an immediate consequence of the fact that U is closed.

5.1.7. *If $p \in X$, then $p \in X \cap U$.*

Let $\mathrm{cl}_Q(Y) \cap \mathrm{cl}_Q(X \cap U) = \{s\}$ and $\mathrm{cl}_Q(Y) \cap \mathrm{cl}_Q(X \cap V) = \{t\}$. Neither s nor t is in X, so

$$|X \cap U| \leq 6.$$

Hence $|X \cap V| \leq |X \cap U| - 1 \leq 5$. Recall that $|X| \geq 9$. As $|X \cap U| \geq |X \cap V|$, it follows that $|X \cap U| \geq 5$. Hence

5.1.8. $|X \cap U| \in \{5, 6\}$.

Call a triangle of M *special* if it contains an element of $X \cap U$, an element of $X \cap V$, and an element of P. Construct a bipartite graph H with vertex classes $X \cap U$ and $X \cap V$ with uv being an edge, where $u \in X \cap U$ and $v \in X \cap V$, precisely when $\{u, v\}$ is contained in a special triangle. Clearly

$$\sum_{u \in X \cap U} d_H(u) = \sum_{v \in X \cap V} d_H(v). \tag{1}$$

Next we show the following.

5.1.9. *Every vertex x of $V(H) - \{p\}$ has its degree in $\{1, 2\}$.*

Let $\{X', X''\} = \{X \cap U, X \cap V\}$ and take $x \in X'$ such that $x \neq p$. Let x'' be the element of $\mathrm{cl}_Q(X'') \cap P$. Thus $x'' \in \{s, t\}$. Clearly $d_H(x) \leq 3$. Assume $d_H(x) = 3$. Then $\mathrm{cl}_Q(Y \cup x)$ contains x', at least three distinct elements of X'', and x''. Thus $\mathrm{cl}_Q(Y \cup x)$ contains X''. Hence $E(M) - \mathrm{cl}_M(Y \cup x)$ contains at most five elements of M; a contradiction to the fact that every cocircuit of M has at least six elements. Thus $d_H(x) < 3$.

Next suppose that $d_H(x) = 0$. Then all three triangles containing x are contained in $\mathrm{cl}_M(X')$. Thus $M|\mathrm{cl}_M(X') \cong F_7$. Hence, for $z \in X'' - \mathrm{cl}_M(X')$, the three triangles containing z are contained in $\mathrm{cl}_M(X'')$. Thus $M|\mathrm{cl}_M(X'') \cong F_7$. Hence $\mathrm{cl}_M(X') \cap \mathrm{cl}_M(X'')$ contains a point of M that is in six triangles; a contradiction. Thus 5.1.9 holds.

Now either

(i) $s = t = p$; or
(ii) s, t, and p are distinct.

Suppose that (i) holds. Assume first that $p \notin Y$. By 5.1.9, for $W \in \{U, V\}$, every element of $M|(X \cap W)$ is in a triangle contained in $X \cap W$. Thus either

$M|(X \cap W) \cong M(K_4)$ and $\sum_{w \in X \cap W} d_H(w) = 6$; or $M|(X \cap W) \cong M(K_4 \backslash e)$ and $\sum_{w \in X \cap W} d_H(w) = 9$. Since $|X \cap U| > |X \cap V|$, we obtain a contradiction using (1). Thus $p \in Y$.

As $|X \cap U| \in \{5, 6\}$ by 5.1.8, we see that $|X \cap U| = 5$, otherwise $M|((X \cap U) \cup p) \cong F_7$, and $d_H(x) = 0$ for every $x \in X \cap V$; a contradiction to 5.1.9. We deduce that $M|((X \cap U) \cup p) \cong M(K_4)$, and $5 = \sum_{u \in X \cap U} d_H(u)$. Now p is in two triangles in $(X \cap U) \cup p$. Thus, of the three triangles in $\text{cl}_Q(X \cap V)$ containing p, at most one contains two elements of $X \cap V$. Hence, using 5.1.9, we see that $M|\text{cl}_M(X \cap V)$ comprises two triangles with a single element, not p, in common. Thus $\sum_{v \in X \cap V} d_H(v) = 7$; a contradiction to (1). Therefore (i) does not hold.

We now know that s, t, and p are distinct. We show next that

5.1.10. $p \in X$.

Suppose $p \notin X$. Then $|X \cap U| = 5$ so $|X \cap V| = 4$. Thus $\sum_{u \in X \cap U} d_H(u)$ is five when $s \in Y$ and is nine otherwise. By 5.1.9, $d_H(v) < 3$ for each $v \in X \cap V$, so $t \in Y$. Then $\sum_{v \in X \cap V} d_H(v)$ is eight or seven depending on whether $M|(X \cap V)$ is $U_{3,4}$ or $U_{2,3} \oplus U_{1,1}$. Thus, by (1), we have a contradiction. Hence 5.1.10 holds.

Suppose $|X \cap U| = 6$. Then $s \notin Y$, otherwise there is an element of $(X \cap U) - p$ with degree zero in H; a contradiction to 5.1.9. Then $\sum_{u \in X \cap U} d_H(u) = 6$. Suppose $t \in Y$. If the line through $\{p, t\}$ contains a third point of M, say q, then each of the other two lines through p in $\text{cl}_Q(X \cap V)$ contains at most one point of M. Thus $|X \cap V| = 3$ and, as $r(X \cap V) = 3$, we see that $\{p, q, t\}$ is the unique triangle in $M|\text{cl}_M(X \cap V)$ containing q. As this triangle is special, it follows that $d_H(q) = 3$; a contradiction to 5.1.9. Evidently the line through $\{p, t\}$ does not contain a third point of M. We deduce that $M|\text{cl}_M(X \cap V)$ comprises two triangles that have one element, not p or t, in common. Then $\sum_{v \in X \cap V} d_H(v) = 5$; a contradiction. We deduce that $t \notin Y$. Then exactly one of the lines in $\text{cl}_M(X \cap V)$ through p contains exactly three points. As no point of $X \cap V$ has degree three in H, it follows that $M|\text{cl}_M(X \cap V)$ comprises two triangles with a point, not p, in common. As $p \notin X \cap V$, it follows that $\sum_{v \in X \cap V} d_H(v) = 7$; a contradiction. Hence $|X \cap U| \neq 6$.

It remains to consider the case that $|X \cap U| = 5$ and $|X \cap V| = 4$. Then $\sum_{u \in X \cap U} d_H(u)$ is five or nine depending on whether or not s is in Y. From 5.1.10, $p \in X$. Thus $M|\,[(X \cap V) \cup p]$ consists of two three-point lines meeting in a point z. If $z = p$, then $\sum_{v \in X \cap V} d_H(v)$ is four or eight, depending on whether or not t is in Y; a contradiction. Hence $z \neq p$. Thus the third element on the line containing $\{p, t\}$ is in X. Again $\sum_{v \in X \cap V} d_H(v)$ is seven, if $t \notin Y$, or four, if $t \in Y$; a contradiction to (1). We conclude that 5.1.4 holds and the lemma follows. □

It is now straightforward to complete the proof of our main result.

Proof of Theorem 1.1. If M has a 4-cocircuit, then the result follows by Lemma 4.1. If M has no 4-cocircuits, then the theorem follows by Lemma 5.1. □

6. A (non)-extension

It is natural to ask whether, for an internally 4-connected binary matroid M with every element in exactly three triangles, $\text{si}(M/e)$ is internally 4-connected for every element e. We now describe an example where this is not the case.

Begin with $K_{3,3}$ having vertex classes $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Form the graph G by adjoining three new vertices u, v, and w, each adjacent to all of

$a_1, a_2, a_3, b_1, b_2,$ and b_3 but not to each other. The vertex-edge incidence matrix of G is the matrix A shown below.

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \\ u \\ v \\ w \end{array}
\begin{pmatrix}
1&1&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1
\end{pmatrix}$$

Then $M(G)$ is an internally 4-connected matroid in which every element is in exactly three triangles. Now adjoin the matrix B to A where B is shown below.

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\begin{pmatrix}
1&0&1&1&1&0\\
1&0&1&1&1&0\\
1&0&1&1&1&0\\
1&1&0&0&1&1\\
0&1&1&1&0&1\\
0&1&1&1&0&1\\
1&1&0&0&1&1\\
1&0&0&1&0&1\\
0&0&1&0&1&1
\end{pmatrix}$$

The matroid N that is represented by $[A|B]$ has each element in $M(G)$ in exactly three triangles, and each element of $\{a, b, c, d, e, f\}$ is in exactly two triangles. To see this, observe that $N|\{a, b, c, d, e, f\} \cong M(K_4)$. Moreover, no element of $M(G)$ lies on a line with two elements of $\{a, b, c, d, e, f\}$ and it is straightforward to check that no element of $\{a, b, c, d, e, f\}$ is in a triangle with two elements of $M(G)$.

Now take the generalized parallel connection of $M(K_5)$ and N across $\{a, b, c, d, e, f\}$ to get an internally 4-connected binary matroid M in which every element is in exactly three triangles. Evidently $\mathrm{si}(M/z)$ is not internally 4-connected for all z in $\{a, b, c, d, e, f\}$.

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