A VARIANT ON THE CIRCUIT EXCHANGE AXIOM
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Abstract. This note proves a symmetric version of the strong circuit elimination axiom for matroids and thereby gives a new symmetric axiom system for matroids in terms of their collections of circuits.

The matroid terminology used here will follow [2]. A matroid $M$ consists of a finite set $E$ and a collection $\mathcal{C}$ of nonempty pairwise incomparable subsets of $E$ satisfying the following axiom.

(C3) If $C_1$ and $C_2$ are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then $\mathcal{C}$ contains a member $C_3$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

This axiom is called the (weak) circuit elimination axiom. The standard variant of this axiom, the strong circuit elimination axiom, is as follows.

(C3)' If $C_1$ and $C_2$ are members of $\mathcal{C}$ with $e \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$, then $\mathcal{C}$ contains a member $C_3$ such that $e_1 \in C_3 \subseteq (C_1 \cup C_2) - e$.

It is natural to seek a more symmetric version of this in which $C_3$ can be found to contain designated elements $e_1$ of $C_1 - C_2$ and $e_2$ of $C_2 - C_1$ while avoiding the specified element $e$ of $C_1 \cap C_2$. However, this strengthening of (C3)' fails in general. For instance, let $M$ be the matroid that is obtained from a 3-circuit $\{e_1, e_2, e\}$ by adding $f_i$ in parallel to $e_i$ for each $i$. Then $\{e_1, f_1, e\}$ and $\{e_2, f_2, e\}$ are circuits, $C_1$ and $C_2$, with $e_1$ and $e_2$ in $C_1 - C_2$ and $C_2 - C_1$, respectively. But $(C_1 \cup C_2) - e$ does not contain a circuit containing $\{e_1, e_2\}$. By adding an additional hypothesis, we are able to recover the desired symmetric variant of (C3)'.

Lemma 1. The set $\mathcal{C}$ of circuits of a matroid $M$ obeys the following.

(C3)'' Let $C_1$ and $C_2$ be members of $\mathcal{C}$ with $e_1 \in C_1 - C_2$ and $e_2 \in C_2 - C_1$. If $e \in C_1 \cap C_2$ and $(C_1 - e_1) \cup (C_2 - e_2)$ contains no member of $\mathcal{C}$, then $\mathcal{C}$ contains a member $C_3$ such that $\{e_1, e_2\} \subseteq C_3 \subseteq (C_1 \cup C_2) - e$.

Furthermore, $C_3$ is the unique circuit of $M$ contained in $(C_1 \cup C_2) - e$.

Proof. Certainly $(C_1 \cup C_2) - e$ is dependent. Let $C_3$ be a circuit contained in this set. We shall show first that $\{e_1, e_2\} \subseteq C_3$. As $(C_1 - e_1) \cup (C_2 - e_2)$ is independent, we may assume that $e_1 \in C_3$. Suppose $e_2 \notin C_3$. Then
e_1 \in C_1 \cap C_3 \text{ and } e \in C_3 - C_1, \text{ so there is a circuit } C_4 \text{ such that } C_4 \subseteq (C_1 \cup C_3) - e_1. \text{ Thus } C_4 \subseteq (C_1 - e_1) \cup (C_2 - e_2), \text{ a contradiction. We deduce that } \{e_1, e_2\} \subseteq C_3.

To see that C_3 is unique, suppose there is a second circuit C'_3 contained in (C_1 \cup C_2) - e. Then e_1 \in C_3 \cap C'_3, \text{ so } M \text{ has a circuit } C_5 \text{ contained in } (C_3 \cup C'_3) - e_1. \text{ As } C_5 \text{ is contained in } (C_1 \cup C_2) - e, \text{ we deduce that } \{e_1, e_2\} \subseteq C_5, \text{ a contradiction. Hence } C_3 \text{ is indeed unique.} \qed

The following theorem seems to give a new axiom system for matroids in terms of their circuits. For example, it is absent from the two standard reference books for the subject [2,3] and also does not appear in Brylawski’s encyclopedic appendix of matroid cryptomorphisms [1].

**Theorem 2.** A collection \( \mathcal{C} \) of nonempty pairwise incomparable subsets of a finite set \( E \) is the set of circuits of a matroid on \( E \) if and only if \( \mathcal{C} \) satisfies (C3)''.

**Proof.** By the lemma, if \( \mathcal{C} \) is the set of circuits of a matroid on \( E \), then \( \mathcal{C} \) satisfies (C3)''. Conversely, assume \( \mathcal{C} \) satisfies (C3)''. Suppose \( C_1 \) and \( C_2 \) are distinct members of \( \mathcal{C} \) with \( e \in C_1 \cap C_2 \). Assume that (C3) fails for \((C_1, C_2, e)\) and that \([C_1 \cup C_2]\) is a minimum among such triples. As the members of \( \mathcal{C} \) are incomparable, there are elements \( e_1 \) and \( e_2 \) of \( C_1 - C_2 \) and \( C_2 - C_1 \), respectively. By (C3)'', \((C_1 - e_1) \cup (C_2 - e_2)\) must contain a member \( C_4 \) of \( \mathcal{C} \), so \( e \in C_4 \). Then \( e \in C_1 \cap C_4 \) and \( |C_1 \cup C_4| \leq |(C_1 \cup C_2) - e_2| < |C_1 \cup C_2| \), so \((C_1 \cup C_4) - e\), and hence \((C_1 \cup C_2) - e\), contains a member of \( \mathcal{C} \), a contradiction. \qed

It is tempting to try to weaken (C3)'' to require only that \( e_1 \in C_1 \) and \( e_2 \in C_2 \). To see that this variant need not hold, consider the cycle matroid of the graph \( K_{2,3} \) and let \( C_1 \) and \( C_2 \) be the circuits \( \{e_1, a, e, e_2\} \) and \( \{b, c, e, e_2\} \). Then \((C_1 - e_1) \cup (C_2 - e_2)\) does not contain a circuit. But, although \((C_1 \cup C_2) - e\) does contain a circuit, that circuit does not contain \( e_2 \).

**References**


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