A VARIANT ON THE CIRCUIT EXCHANGE AXIOM

JAMES OXLEY AND SUIJIE WANG

Abstract. This note proves a symmetric version of the strong circuit elimination axiom for matroids and thereby gives a new symmetric axiom system for matroids in terms of their collections of circuits.

The matroid terminology used here will follow [2]. A *matroid* M consists of a finite set E and a collection $\mathscr{C}$ of nonempty pairwise incomparable subsets of E satisfying the following axiom.

(C3) *If C_1 and C_2 are distinct members of $\mathscr{C}$ and $e \in C_1 \cap C_2$, then $\mathscr{C}$ contains a member C_3 such that $C_3 \subseteq (C_1 \cup C_2) - e$.*

This axiom is called the *(weak) circuit elimination axiom*. The standard variant of this axiom, the *strong circuit elimination axiom*, is as follows.

(C3)$'$ *If C_1 and C_2 are members of $\mathscr{C}$ with $e \in C_1 \cap C_2$ and $e_1 \in C_1 - C_2$, then $\mathscr{C}$ contains a member C_3 such that $e_1 \in C_3 \subseteq (C_1 \cup C_2) - e$.*

It is natural to seek a more symmetric version of this in which C_3 can be found to contain designated elements e_1 of $C_1 - C_2$ and e_2 of $C_2 - C_1$ while avoiding the specified element e of $C_1 \cap C_2$. However, this strengthening of **(C3)**$'$ fails in general. For instance, let M be the matroid that is obtained from a 3-circuit $\{e_1, e_2, e\}$ by adding f_i in parallel to e_i for each i. Then $\{e_1, f_1, e\}$ and $\{e_2, f_2, e\}$ are circuits, C_1 and C_2, with e_1 and e_2 in $C_1 - C_2$ and $C_2 - C_1$, respectively. But $(C_1 \cup C_2) - e$ does not contain a circuit containing $\{e_1, e_2\}$. By adding an additional hypothesis, we are able to recover the desired symmetric variant of **(C3)**$'$.

Lemma 1. *The set $\mathscr{C}$ of circuits of a matroid M obeys the following.*

(C3)$''$ *Let C_1 and C_2 be members of $\mathscr{C}$ with $e_1 \in C_1 - C_2$ and $e_2 \in C_2 - C_1$. If $e \in C_1 \cap C_2$ and $(C_1 - e_1) \cup (C_2 - e_2)$ contains no member of $\mathscr{C}$, then $\mathscr{C}$ contains a member C_3 such that $\{e_1, e_2\} \subseteq C_3 \subseteq (C_1 \cup C_2) - e$.*

Furthermore, C_3 is the unique circuit of M contained in $(C_1 \cup C_2) - e$

Proof. Certainly $(C_1 \cup C_2) - e$ is dependent. Let C_3 be a circuit contained in this set. We shall show first that $\{e_1, e_2\} \subseteq C_3$. As $(C_1 - e_1) \cup (C_2 - e_2)$ is independent, we may assume that $e_1 \in C_3$. Suppose $e_2 \notin C_3$. Then

Date: May 27, 2019.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. circuit elimination, strong circuit elimination, matroid circuit axioms.

This work was supported by the National Natural Science Foundation of China (11871204).

$e_1 \in C_1 \cap C_3$ and $e \in C_3 - C_1$, so there is a circuit C_4 such that $C_4 \subseteq (C_1 \cup C_3) - e_1$. Thus $C_4 \subseteq (C_1 - e_1) \cup (C_2 - e_2)$, a contradiction. We deduce that $\{e_1, e_2\} \subseteq C_3$.

To see that C_3 is unique, suppose there is a second circuit C_3' contained in $(C_1 \cup C_2) - e$. Then $e_1 \in C_3 \cap C_3'$, so M has a circuit C_5 contained in $(C_3 \cup C_3') - e_1$. As C_5 is contained in $(C_1 \cup C_2) - e$, we deduce that $\{e_1, e_2\} \subseteq C_5$, a contradiction. Hence C_3 is indeed unique. □

The following theorem seems to give a new axiom system for matroids in terms of their circuits. For example, it is absent from the two standard reference books for the subject [2,3] and also does not appear in Brylawski's encyclopedic appendix of matroid cryptomorphisms [1].

Theorem 2. *A collection $\mathscr{C}$ of nonempty pairwise incomparable subsets of a finite set E is the set of circuits of a matroid on E if and only if $\mathscr{C}$ satisfies* **(C3)**$''$.

Proof. By the lemma, if $\mathscr{C}$ is the set of circuits of a matroid on E, then $\mathscr{C}$ satisfies **(C3)**$''$. Conversely, assume $\mathscr{C}$ satisfies **(C3)**$''$. Suppose C_1 and C_2 are distinct members of $\mathscr{C}$ with e in $C_1 \cap C_2$. Assume that **(C3)** fails for (C_1, C_2, e) and that $|C_1 \cup C_2|$ is a minimum among such triples. As the members of $\mathscr{C}$ are incomparable, there are elements e_1 and e_2 of $C_1 - C_2$ and $C_2 - C_1$, respectively. By **(C3)**$''$, $(C_1 - e_1) \cup (C_2 - e_2)$ must contain a member C_4 of $\mathscr{C}$, so $e \in C_4$. Then $e \in C_1 \cap C_4$ and $|C_1 \cup C_4| \leq |(C_1 \cup C_2) - e_2| < |C_1 \cup C_2|$, so $(C_1 \cup C_4) - e$, and hence $(C_1 \cup C_2) - e$, contains a member of $\mathscr{C}$, a contradiction. □

It is tempting to try to weaken **(C3)**$''$ to require only that $e_1 \in C_1$ and $e_2 \in C_2$. To see that this variant need not hold, consider the cycle matroid of the graph $K_{2,3}$ and let C_1 and C_2 be the circuits $\{e_1, a, e, e_2\}$ and $\{b, c, e, e_2\}$. Then $(C_1 - e_1) \cup (C_2 - e_2)$ does not contain a circuit. But, although $(C_1 \cup C_2) - e$ does contain a circuit, that circuit does not contain e_2.

References

[1] Brylawski, T., Appendix of matroid cryptomorphisms, in *Theory of Matroids* (ed. N. White), Cambridge University Press, Cambridge, 1986, pp. 298–312.
[2] Oxley, J., *Matroid Theory*, Second edition, Oxford University Press, New York, 1992.
[3] Welsh, D.J.A., *Matroid Theory*, Academic Press, London, 1976.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana, USA
E-mail address: oxley@math.lsu.edu

Institute of Mathematics, Hunan University, Changsha, Hunan, China
E-mail address: wangsuijie@hnu.edu.cn